HYBRIDIZATION AND POSTPROCESSING IN FINITE ELEMENT EXTERIOR CALCULUS

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Abstract. We hybridize the methods of finite element exterior calculus for the Hodge–Laplace problem on differential k-forms in $\mathbb{R}^n$. In the cases $k = 0$ and $k = n$, we recover well-known primal and mixed hybrid methods for the scalar Poisson equation, while for $0 < k < n$, we obtain new hybrid finite element methods, including methods for the vector Poisson equation in $n = 2$ and $n = 3$ dimensions. We also generalize Stenberg postprocessing from $k = n$ to arbitrary k, proving new superconvergence estimates. Finally, we discuss how this hybridization framework may be extended to include nonconforming and hybridizable discontinuous Galerkin methods.

1. Introduction

Finite element exterior calculus (FEEC) is a powerful framework that unifies the analysis of several families of conforming finite element methods for problems involving Laplace-type operators (Arnold, Falk, and Winther [4, 5], Arnold [2]). These include the classic "continuous Galerkin" Lagrange finite element method and the Raviart–Thomas (RT) [38] and Brezzi–Douglas–Marini (BDM) [8] mixed methods for the scalar Poisson equation, as well as mixed methods based on Nédélec elements [33, 34] for the 2- and 3-dimensional vector Poisson equation. In FEEC, these are all seen as finite element methods for the Hodge–Laplace operator on differential k-forms in $\mathbb{R}^n$, where scalar fields are identified with 0- and n-forms and vector fields with 1- and $(n-1)$-forms.

In this paper, we hybridize FEEC for arbitrary dimension n and form degree k. That is, we construct hybrid finite element methods using discontinuous spaces of differential forms, enforcing continuity and boundary conditions using Lagrange multipliers on the element boundaries. The solutions agree with those of the original, non-hybrid FEEC methods, and the Lagrange multipliers are seen to correspond to weak tangential and normal traces. This hybrid formulation enables static condensation: since only the Lagrange multipliers are globally coupled, the remaining internal degrees of freedom can be eliminated using an efficient local procedure, and the resulting Schur complement system can be substantially smaller than the original one. We also present a generalization of Stenberg postprocessing [40], which for $0 < k < n$ is shown to give new improved estimates.

The special cases $k = 0$ and $k = n$ are shown to recover known results on hybridization and postprocessing for the scalar Poisson equation. In particular, the case $k = n$ corresponds to the hybridized RT [3] and BDM [8] methods, and the postprocessing procedure is precisely that of Stenberg [40]. The case $k = 0$ corresponds to the more recent hybridization of the continuous Galerkin method by Cockburn, Gopalakrishnan, and Wang [19]. The hybrid and postprocessing schemes in the remaining cases $0 < k < n$ are new and, to the best of our knowledge, have not appeared in the literature even for the vector Poisson equation when $n = 2$ or $n = 3$. In particular, the hybridization of Nédélec edge elements is different from that in Cockburn and Gopalakrishnan [17]: here, the Lagrange multipliers are simply traces of standard elements, rather than living in a space of "jumps." We expect these new methods to be especially useful in computational electromagnetics, where Nédélec elements are ubiquitous and the differential forms point of view has provided significant insight (cf. Hiptmair [26]).

While we restrict our attention primarily to hybrid methods for conforming simplicial meshes, we remark that the framework developed here has the potential to be applied to other types of domain decomposition methods, including methods on cubical meshes, nonconforming meshes, mortar methods, etc. We also discuss briefly how the unified hybridization framework of Cockburn, Gopalakrishnan, and Lazarov [18], which includes hybridizable discontinuous Galerkin (HDG) methods, may also be generalized to the Hodge–Laplace problem for $0 < k < n$.

1.1. **Why hybridize?** There are several theoretical and practical benefits of hybridization:

- *additional information about solutions*: The Lagrange multiplier functions often correspond to weak boundary traces of solution components, even though the numerical solution may not be regular enough for a trace to exist in the usual sense (e.g., the trace of an L^2 function or normal derivative of an H^1 function).
- *static condensation*: Degrees of freedom for discontinuous function spaces can be locally eliminated. The resulting Schur complement only involves boundary degrees of freedom for the Lagrange multipliers, so it can be substantially smaller than the original global problem.
- *local postprocessing and superconvergence*: The numerical solution may be efficiently "post-processed" by using the boundary traces to solve a local problem on each element, resulting in an improved approximation compared to the original solution.

Seminal work on hybridization of mixed finite element methods was done by Fraeijs de Veubeke [21]. For the scalar Poisson equation, the RT method was hybridized in this manner by Arnold and Brezzi [3], who introduced the notion of postprocessing. Hybridization and postprocessing were also discussed in the original paper introducing the BDM method [8], and an interesting characterization of the Lagrange multipliers for the hybridized RT and BDM methods appears in Cockburn and Gopalakrishnan [16]. A refined local postprocessing procedure for mixed methods, which can be applied with or without hybridization, was given by Stenberg [40]; see also Gastaldi and Nochetto [22], who discovered this independently (cf. [22, eqs. 4.14–4.15]), as well as Bramble and Xu [7].

More recently, Cockburn, Gopalakrishnan, and Wang [19] hybridized the continuous Galerkin method, using an approach similar to the "three-field domain decomposition method" of Brezzi and Marini [9], and showed that static condensation yields the same condensed system as that obtained by the original, non-hybrid static condensation procedure of Guyan [25]. Even more recently, Cockburn, Gopalakrishnan, and Lazarov [18] introduced an important unified hybridization framework that includes the above methods, as well as nonconforming and HDG methods, for the scalar Poisson equation. A survey of historical and recent developments appears in [15].

1.2. **Organization of the paper.** The paper is organized as follows:

- Section 2 recalls the basic machinery and terminology of differential forms, the Hodge–Laplace problem, and FEEC. This includes a discussion of tangential and normal traces, which play an important role throughout the paper, in Sections 2.3 and 2.4.
- Section 3 presents a domain decomposition of the Hodge–Laplace problem. The variational form of this problem involves broken spaces of differential forms, along with boundary traces that act as Lagrange multipliers enforcing interelement continuity and boundary conditions.
- Section 4 develops hybrid finite element methods for the Hodge–Laplace problem, based on the domain-decomposed variational principle from the previous section. We prove that these are hybridized versions of the FEEC methods, show how static condensation can be used to reduce the size of the global system, and develop error estimates for the hybrid variables.
- Section 5 generalizes the postprocessing procedure of Stenberg [40] from $k = n$ to arbitrary k. This procedure only uses the statically condensed variables, so it can be applied immediately after solving the condensed system, or it can be applied to solutions obtained by ordinary finite element methods without hybridization. In addition to known superconvergence results for $k = n$, we give new improved error estimates for $k < n$.
- Section 6 gives concrete illustrations of the hybrid and postprocessing methods when $n = 3$, using the language of vector calculus and classic families of finite elements.
- Section 7 presents numerical experiments, confirming the error estimates of Sections 4 and 5.
- Finally, Section 8 presents an extension of the framework of Cockburn, Gopalakrishnan, and Lazarov [18], whereas the previous sections only address conforming methods. This lays the groundwork for hybridization of nonconforming and discontinuous Galerkin methods for FEEC, although we postpone the analysis of such methods for future work.

2. Background: differential forms and finite element exterior calculus

In this section, we quickly recall the exterior calculus of differential forms, the Hodge–Laplace problem, and FEEC, in order to lay the foundation and fix the notation for the subsequent sections. We refer to Arnold, Falk, and Winther [4, 5], Arnold [2], and references therein for a comprehensive treatment. We also discuss tangential and normal traces of differential forms, which will play an important role in domain decomposition and hybridization. Our treatment of these traces follows that in Weck [41] (see also Kurz and Auchmann [28]), which extended work of Buffa and Ciarlet [11, 12], Buffa, Costabel, and Sheen [13] in $\mathbb{R}^3$. Throughout the discussion, we relate these ideas to those of vector calculus on a domain $\Omega \subset \mathbb{R}^3$, where differential forms can be identified with scalar and vector "proxy" fields.

2.1. **Exterior algebra.** Given a real n-dimensional vector space V, let $\operatorname{Alt}^k V$ denote the space of k-linear forms $\alpha\colon V \times \cdots \times V \to \mathbb{R}$ that are alternating, i.e., totally antisymmetric. We have $\operatorname{Alt}^0 V = \mathbb{R}$, $\operatorname{Alt}^1 V = V^*$, and $\operatorname{Alt}^k V = 0$ whenever $k > n$.

Given $\alpha \in \operatorname{Alt}^k V$ and $\beta \in \operatorname{Alt}^\ell V$, the *exterior product* (or *wedge product*) $\alpha \wedge \beta \in \operatorname{Alt}^{k+\ell} V$ is defined by

$$(\alpha \wedge \beta)(v_1, \ldots, v_{k+\ell}) \coloneqq \sum_{\sigma \in S_{k,\ell}} (\operatorname{sign} \sigma) \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}),$$

where $S_{k,\ell}$ is the space of (k,ℓ)-shuffles, i.e., permutations σ with $\sigma(1) < \ldots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+\ell)$. If $\{e_1, \ldots, e_n\}$ is a basis for V and $\{e^1, \ldots, e^n\}$ the dual basis for V^*, then $\{e^{i_1} \wedge \cdots \wedge e^{i_k} : 1 \le i_1 < \cdots < i_k \le n\}$ forms a basis for $\operatorname{Alt}^k V$. When $V = \mathbb{R}^n$, we take the standard coordinate basis and follow the convention of denoting the ith dual basis vector by $\mathrm{d}x^i$.

Given $v \in V$ and $\alpha \in \operatorname{Alt}^k V$ with $k > 0$, the *interior product* (or *contraction*) $\iota_v \alpha \in \operatorname{Alt}^{k-1} V$ is the form defined by

$$(\iota_v \alpha)(v_1, \ldots, v_{k-1}) \coloneqq \alpha(v, v_1, \ldots, v_{k-1}).$$

This is also sometimes written as $v \lrcorner \alpha$. By convention, $\iota_v \alpha$ vanishes when $k = 0$.

A linear map $A\colon V \to W$ induces a map $A^*\colon \operatorname{Alt}^k W \to \operatorname{Alt}^k V$, defined by

$$(A^* \alpha)(v_1, \ldots, v_k) \coloneqq \alpha(Av_1, \ldots Av_k),$$

which is called the *pullback* of $\alpha \in \operatorname{Alt}^k W$ by A. In particular, if A is the inclusion of a subspace $V \subset W$, then $A^* \alpha$ is just the restriction of α to V.

An inner product $(\cdot, \cdot)$ on V also induces one on $\operatorname{Alt}^k V$. If $\{e_1, \ldots, e_n\}$ is any orthonormal basis for V, and $\alpha, \beta \in \operatorname{Alt}^k V$, then

$$(\alpha, \beta) \coloneqq \sum_{1 \le i_1 < \cdots < i_k \le n} \alpha(e_{i_1}, \ldots, e_{i_k}) \beta(e_{i_1}, \ldots, e_{i_k}),$$

so that $\{e^{i_1} \wedge \cdots \wedge e^{i_k} : 1 \le i_1 < \cdots < i_k \le n\}$ is an orthonormal basis for $\operatorname{Alt}^k V$. The inner product also gives an isomorphism $\flat\colon V \to V^*$, $v \mapsto v^\flat \coloneqq (v, \cdot)$, whose inverse is denoted $\sharp\colon V^* \to V$.

An ordering of the orthonormal basis defines a *volume form* $\operatorname{vol} = e^1 \wedge \cdots \wedge e^n \in \operatorname{Alt}^n V$, which is unique up to sign and determines an orientation on V. This gives an isometric isomorphism $\star\colon \operatorname{Alt}^k V \to \operatorname{Alt}^{n-k} V$, called the *Hodge star* operator, defined by

$$\alpha \wedge \star \beta = (\alpha, \beta) \operatorname{vol}$$

for $\alpha, \beta \in \operatorname{Alt}^k V$. In particular,

$$\star(e^{\sigma(1)} \wedge \cdots \wedge e^{\sigma(k)}) = (\operatorname{sign} \sigma)\, e^{\sigma(k+1)} \wedge \cdots \wedge e^{\sigma(n)},$$

for any permutation σ, from which it can be seen that $\star^{-1} = (-1)^{k(n-k)} \star$.

When $V = \mathbb{R}^3$, alternating forms can be identified with scalar and vector "proxies." Specifically, a scalar $c \in \mathbb{R}$ corresponds to either

$$c \in \operatorname{Alt}^0 \mathbb{R}^3 \quad \text{or} \quad \star c = c\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \in \operatorname{Alt}^3 \mathbb{R}^3,$$

while a vector $v \in \mathbb{R}^3$ corresponds to either

$$v^\flat = v_1\, \mathrm{d}x^1 + v_2\, \mathrm{d}x^2 + v_3\, \mathrm{d}x^3 \in \operatorname{Alt}^1 \mathbb{R}^3 \quad \text{or} \quad \star v^\flat = v^1\, \mathrm{d}x^2 \wedge \mathrm{d}x^3 - v^2\, \mathrm{d}x^1 \wedge \mathrm{d}x^3 + v^3\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \in \operatorname{Alt}^2 \mathbb{R}^3.$$

We then have the following correspondences of algebraic operations for $V = \mathbb{R}^3$:

- The exterior product of a 0-form with any other k-form corresponds to scalar multiplication, that of two 1-forms corresponds to the cross product, and that of a 1-form with a 2-form corresponds to the dot product.
- The interior product of $w \in \mathbb{R}^3$ with a 1-form corresponds to the dot product $v \mapsto v \cdot w$, that with a 2-form corresponds to the cross product $v \mapsto v \times w$, and that with a 3-form corresponds to vector multiplication $c \mapsto cw$.
- The inner product on 0-forms and 3-forms corresponds to scalar multiplication, while that on 1-forms and 2-forms corresponds to the dot product.
- The Hodge star corresponds to the identity, since it simply exchanges the two representations of a scalar or vector.

More generally, when $V = \mathbb{R}^n$, we can still identify scalars with $\operatorname{Alt}^0 \mathbb{R}^n \cong \operatorname{Alt}^n \mathbb{R}^n$ and vectors with $\operatorname{Alt}^1 \mathbb{R}^n \cong \operatorname{Alt}^{n-1} \mathbb{R}^n$, and the correspondences with scalar/vector multiplication and the dot product still hold. Note that care must be taken when $n = 2$, since then $n - 1 = 1$, so there are two distinct ways to identify a vector with a 1-form.

2.2. **Exterior calculus of differential forms.** Let Ω be an n-dimensional smooth manifold, possibly with boundary, and denote by $\Lambda^k(\Omega)$ the space of smooth *differential k-forms* (or just *k-forms*) on Ω. A k-form $\alpha \in \Lambda^k(\Omega)$ is a smooth function mapping $x \in \Omega \mapsto \alpha_x \in \operatorname{Alt}^k T_x\Omega$, where $T_x\Omega$ is the space of tangent vectors to Ω at x. For our purposes, generally $\Omega \subset \mathbb{R}^n$, so a k-form is just a function $\Omega \to \operatorname{Alt}^k \mathbb{R}^n$. In coordinates,

$$\alpha = \alpha_{i_1 \cdots i_k}\, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k},$$

where $\alpha_{i_1 \cdots i_k} \colon \Omega \to \mathbb{R}$ are coefficient functions. We use Einstein index notation, so this expression has an implied sum over $1 \leq i_1 < \cdots < i_k \leq n$.

The operations of exterior algebra from Section 2.1 are extended to differential forms by applying them pointwise. If $\alpha \in \Lambda^k(\Omega)$ and $\beta \in \Lambda^\ell(\Omega)$, then $\alpha \wedge \beta \in \Lambda^{k+\ell}(\Omega)$ is defined by taking $\alpha_x \wedge \beta_x$ at each $x \in \Omega$. Similarly, if $v \colon x \mapsto v_x \in T_x\Omega$ is a smooth vector field, then $\iota_v \alpha \in \Lambda^{k-1}(\Omega)$ is given by $\iota_{v_x} \alpha_x$ at each $x \in \Omega$. If $\phi \colon \Omega \to \Omega'$ is a smooth map, then the pullback $\phi^* \colon \Lambda^k(\Omega') \to \Lambda^k(\Omega)$ is defined by taking the pullback of the derivative $D\phi(x)$ at each $x \in \Omega$. If Ω is equipped with a Riemannian metric, which defines an inner product $(\cdot,\cdot)_x$ on $T_x\Omega$ for each $x \in \Omega$, then $\flat$ maps vector fields to 1-forms, and $\sharp$ maps 1-forms to vector fields. If Ω is also oriented, so that it has a Riemannian volume form $\operatorname{vol} \in \Lambda^n(\Omega)$, then $\star \colon \Lambda^k(\Omega) \to \Lambda^{n-k}(\Omega)$ satisfies $\alpha \wedge \star\beta = (\alpha, \beta) \operatorname{vol}$ for $\alpha, \beta \in \Lambda^k(\Omega)$, which means that $(\alpha \wedge \star\beta)_x = (\alpha_x, \beta_x)_x \operatorname{vol}_x$ at each $x \in \Omega$. Using this, we can define the L^2 inner product,

$$(\alpha, \beta)_\Omega \coloneqq \int_\Omega \alpha \wedge \star\beta,$$

for α, β with compact support.

In addition to these algebraic operations, exterior calculus also has differential operators. The most fundamental of these is the *exterior derivative* $\mathrm{d} \colon \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$, given in coordinates by

$$\mathrm{d}(\alpha_{i_1 \cdots i_k}\, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}) = \frac{\partial \alpha_{i_1 \cdots i_k}}{\partial x^i}\, \mathrm{d}x^i \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k},$$

where the sum is over $1 \leq i \leq n$ and $1 \leq i_1 < \cdots < i_k \leq n$. This has the fundamental property that $\mathrm{d}\mathrm{d} = 0$, so that

$$0 \to \Lambda^0(\Omega) \xrightarrow{\mathrm{d}} \Lambda^1(\Omega) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Lambda^n(\Omega) \to 0$$

is a differential complex (i.e., the composition of any two arrows is zero), called the *de Rham complex*. The exterior derivative also satisfies the Leibniz rule $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta$ for $\alpha \in \Lambda^k(\Omega)$, $\beta \in \Lambda^\ell(\Omega)$.

The *codifferential* $\delta \colon \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$ is defined by $\delta \coloneqq (-1)^k \star^{-1} \mathrm{d} \star$. This definition is chosen so that, when $\alpha \in \Lambda^{k-1}(\Omega)$ and $\beta \in \Lambda^k(\Omega)$, the Leibniz rule implies

$$\mathrm{d}(\alpha \wedge \star\beta) = \mathrm{d}\alpha \wedge \star\beta - \alpha \wedge \star\delta\beta. \tag{1}$$

It is immediate that $\delta\delta = 0$, so

$$0 \leftarrow \Lambda^0(\Omega) \xleftarrow{\delta} \Lambda^1(\Omega) \xleftarrow{\delta} \cdots \xleftarrow{\delta} \Lambda^n(\Omega) \leftarrow 0$$

is also a differential complex.

Finally, the *Hodge–Laplace operator* $L \colon \Lambda^k(\Omega) \to \Lambda^k(\Omega)$ is defined by $L \coloneqq \mathrm{d}\delta + \delta\mathrm{d}$. If we combine the differential complexes associated to d and δ into a single diagram,

$$0 \rightleftarrows \Lambda^0(\Omega) \underset{\delta}{\overset{\mathrm{d}}{\rightleftarrows}} \cdots \underset{\delta}{\overset{\mathrm{d}}{\rightleftarrows}} \Lambda^{k-1}(\Omega) \underset{\delta}{\overset{\mathrm{d}}{\rightleftarrows}} \Lambda^k(\Omega) \underset{\delta}{\overset{\mathrm{d}}{\rightleftarrows}} \Lambda^{k+1}(\Omega) \underset{\delta}{\overset{\mathrm{d}}{\rightleftarrows}} \cdots \underset{\delta}{\overset{\mathrm{d}}{\rightleftarrows}} \Lambda^n(\Omega) \rightleftarrows 0,$$

then L is the sum of the two ways to compose a left arrow with a right arrow.

When $\Omega \subset \mathbb{R}^3$, we can identify scalar fields with 0-forms and 3-forms and vector fields with 1-forms and 2-forms, using the same correspondence as in Section 2.1, and the correspondence of algebraic operations extends pointwise. The operators d and δ correspond to the standard vector calculus operators of gradient, curl, and divergence, as shown in the following diagram:

$$\begin{tikzcd}
0 \arrow[r, shift left] & \Lambda^0(\Omega) \arrow[l, shift left] \arrow[r, shift left, "\mathrm{d}"] \arrow[d, phantom, "\cong" rotate=-90] & \Lambda^1(\Omega) \arrow[l, shift left, "\delta"] \arrow[r, shift left, "\mathrm{d}"] \arrow[d, phantom, "\cong" rotate=-90] & \Lambda^2(\Omega) \arrow[l, shift left, "\delta"] \arrow[r, shift left, "\mathrm{d}"] \arrow[d, phantom, "\cong" rotate=-90] & \Lambda^3(\Omega) \arrow[l, shift left, "\delta"] \arrow[r, shift left] \arrow[d, phantom, "\cong" rotate=-90] & 0 \arrow[l, shift left] \\
0 \arrow[r, shift left] & C^\infty(\Omega) \arrow[l, shift left] \arrow[r, shift left, "\mathrm{grad}"] & C^\infty(\Omega, \mathbb{R}^3) \arrow[l, shift left, "-\mathrm{div}"] \arrow[r, shift left, "\mathrm{curl}"] & C^\infty(\Omega, \mathbb{R}^3) \arrow[l, shift left, "\mathrm{curl}"] \arrow[r, shift left, "\mathrm{div}"] & C^\infty(\Omega) \arrow[l, shift left, "-\mathrm{grad}"] \arrow[r, shift left] & 0. \arrow[l, shift left]
\end{tikzcd}$$

The Hodge–Laplace operator therefore corresponds to the negative Laplacian $-\Delta$, which is $-\operatorname{div}\operatorname{grad}$ for scalar fields and $\operatorname{curl}\operatorname{curl} - \operatorname{grad}\operatorname{div}$ for vector fields.

More generally, when $\Omega \subset \mathbb{R}^n$, we can still identify scalar fields with 0- and n-forms and vector fields with 1- and $(n-1)$-forms; we have the correspondences

$$\begin{tikzcd}
0 \arrow[r, shift left] & \Lambda^0(\Omega) \arrow[l, shift left] \arrow[r, shift left, "\mathrm{d}"] \arrow[d, phantom, "\cong" rotate=-90] & \Lambda^1(\Omega) \arrow[l, shift left, "\delta"] \arrow[d, phantom, "\cong" rotate=-90] & \cdots & \Lambda^{n-1}(\Omega) \arrow[r, shift left, "\mathrm{d}"] \arrow[d, phantom, "\cong" rotate=-90] & \Lambda^n(\Omega) \arrow[l, shift left, "\delta"] \arrow[r, shift left] \arrow[d, phantom, "\cong" rotate=-90] & 0 \arrow[l, shift left] \\
0 \arrow[r, shift left] & C^\infty(\Omega) \arrow[l, shift left] \arrow[r, shift left, "\mathrm{grad}"] & C^\infty(\Omega, \mathbb{R}^n) \arrow[l, shift left, "-\mathrm{div}"] & \cdots & C^\infty(\Omega, \mathbb{R}^n) \arrow[r, shift left, "\mathrm{div}"] & C^\infty(\Omega) \arrow[l, shift left, "-\mathrm{grad}"] \arrow[r, shift left] & 0; \arrow[l, shift left]
\end{tikzcd}$$

and L still corresponds to $-\Delta$. Again, it is important to be careful when $n = 2$, since there are two distinct ways to identify a vector field with a 1-form.

2.3. Tangential and normal traces.

If Ω has smooth boundary[1] $\partial\Omega$, the *trace map* $\operatorname{tr} \colon \Lambda^k(\Omega) \to \Lambda^k(\partial\Omega)$ is defined to be the pullback of k-forms by the inclusion $\partial\Omega \hookrightarrow \Omega$. Since $\operatorname{tr}\alpha \in \Lambda^k(\partial\Omega)$ is just the restriction of $\alpha \in \Lambda^k(\Omega)$ to vectors tangent to $\partial\Omega$, we refer to $\operatorname{tr}\alpha$ as the *tangential trace* of α and also use the notation $\alpha^{\tan} \coloneqq \operatorname{tr}\alpha$. At $x \in \partial\Omega$, note that α_x has $\binom{n}{k}$ components, while $\alpha_x^{\tan}$ only has $\binom{n-1}{k}$ components.

[1]The generalization to non-smooth Lipschitz boundaries (e.g., polyhedral domains) is discussed in the next section.

If Ω is oriented and Riemannian, then we may also define the *normal trace* $\alpha^{\mathrm{nor}} \in \Lambda^{k-1}(\partial\Omega)$, which contains the remaining $\binom{n-1}{k-1}$ pieces of information about α at the boundary. Let $\langle \cdot,\cdot \rangle$ denote the induced metric on $\partial\Omega$, obtained by restricting $(\cdot,\cdot)$ to vectors tangent to the boundary. If $\mathbf{n}$ is the outer normal vector field to $\partial\Omega$, then $\iota_{\mathbf{n}}\mathrm{vol} \in \Lambda^{n-1}(\partial\Omega)$ is the induced Riemannian volume form on $\partial\Omega$ [30, Corollary 10.40], and we denote the associated Hodge star on $\partial\Omega$ by $\widehat{\star}$ and the L^2 inner product by $\langle \cdot,\cdot \rangle_{\partial\Omega}$.

Definition 2.1 (tangential and normal traces)**.** Given $\alpha \in \Lambda^k(\Omega)$,

$$\alpha^{\mathrm{tan}} \coloneqq \operatorname{tr}\alpha \in \Lambda^k(\partial\Omega), \qquad \alpha^{\mathrm{nor}} \coloneqq \widehat{\star}^{-1} \operatorname{tr} \star\alpha \in \Lambda^{k-1}(\partial\Omega).$$

Stokes' theorem states that, when $\alpha \in \Lambda^{n-1}(\Omega)$ has compact support, $\int_{\partial\Omega} \operatorname{tr}\alpha = \int_\Omega \mathrm{d}\alpha$ [30, Theorem 10.23]. From this and the Leibniz rule for the exterior derivative, we get an integration by parts formula, which can be conveniently expressed in terms of tangential and normal traces and the L^2 inner products on $\partial\Omega$ and Ω. The following result is standard, but the proof is short and illuminates the definition of the normal trace.

Proposition 2.2. *Let $\alpha \in \Lambda^{k-1}(\Omega)$, $\beta \in \Lambda^k(\Omega)$, such that $\alpha \wedge \star\beta$ has compact support. Then we have the integration by parts formula*

$$\langle \alpha^{\mathrm{tan}}, \beta^{\mathrm{nor}} \rangle_{\partial\Omega} = (\mathrm{d}\alpha, \beta)_\Omega - (\alpha, \delta\beta)_\Omega. \tag{2}$$

Proof. Using the definitions of α^{tan} and β^{nor}, we calculate

$$\langle \alpha^{\mathrm{tan}}, \beta^{\mathrm{nor}} \rangle_{\partial\Omega} = \int_{\partial\Omega} \alpha^{\mathrm{tan}} \wedge \widehat{\star}\beta^{\mathrm{nor}} = \int_{\partial\Omega} \operatorname{tr}\alpha \wedge \operatorname{tr}\star\beta = \int_{\partial\Omega} \operatorname{tr}(\alpha \wedge \star\beta) = \int_\Omega \mathrm{d}(\alpha \wedge \star\beta),$$

where the last step uses Stokes' theorem. Applying (1) completes the proof. $\square$

Remark 2.3. An equivalent approach to tangential and normal traces begins by augmenting an orthonormal basis $\{e_1, \ldots, e_{n-1}\}$ of $T_x\partial\Omega$ by $\mathbf{n}_x$ to get an orthonormal basis of $T_x\Omega$ at each $x \in \partial\Omega$. It can then be seen that

$$\alpha|_{\partial\Omega} = \alpha^{\mathrm{tan}}_{i_1\cdots i_k}\, e^{i_1} \wedge \cdots \wedge e^{i_k} + \alpha^{\mathrm{nor}}_{i_1\cdots i_{k-1}}\, \mathbf{n}^\flat \wedge e^{i_1} \wedge \cdots \wedge e^{i_{k-1}},$$

which splits $\alpha|_{\partial\Omega}$ into components excluding $\mathbf{n}^\flat$ and those including $\mathbf{n}^\flat$. To get a coordinate-free version of this, we use the Leibniz rule for contraction to write

$$\iota_{\mathbf{n}}(\mathbf{n}^\flat \wedge \alpha) = (\iota_{\mathbf{n}}\mathbf{n}^\flat)\alpha - \mathbf{n}^\flat \wedge (\iota_{\mathbf{n}}\alpha) = \alpha|_{\partial\Omega} - \mathbf{n}^\flat \wedge (\iota_{\mathbf{n}}\alpha),$$

which rearranges to

$$\alpha|_{\partial\Omega} = \iota_{\mathbf{n}}(\mathbf{n}^\flat \wedge \alpha) + \mathbf{n}^\flat \wedge (\iota_{\mathbf{n}}\alpha).$$

Therefore, we may identify α^{tan} with $\iota_{\mathbf{n}}(\mathbf{n}^\flat \wedge \alpha)$ and α^{nor} with $\iota_{\mathbf{n}}\alpha$, and

$$\begin{aligned} \langle \alpha^{\mathrm{tan}}, \beta^{\mathrm{nor}} \rangle &= \big(\iota_{\mathbf{n}}(\mathbf{n}^\flat \wedge \alpha), \iota_{\mathbf{n}}\beta\big) = (\mathbf{n}^\flat \wedge \alpha, \beta) \\ &= (\mathbf{n}^\flat \wedge \alpha, \mathbf{n}^\flat \wedge \iota_{\mathbf{n}}\beta) = (\alpha, \iota_{\mathbf{n}}\beta), \end{aligned}$$

which gives several equivalent ways to express the left-hand side of (2).

When $\Omega \subset \mathbb{R}^3$, the correspondence of these traces to scalar and vector proxy fields is given in Table 1, using Remark 2.3 and the proxy operations for $\iota_{\mathbf{n}}$ and $\mathbf{n}^\flat \wedge$. Equation (2) then gives the familiar integration by parts formulas of vector calculus. More generally, when $\Omega \subset \mathbb{R}^n$, all but one of the expressions in Table 1 generalize to $k = 0, 1, n-1, n$. The lone exception is the normal trace of an $(n-1)$-form, which is an $(n-2)$-form and therefore does not correspond to a scalar or vector field when $n > 3$.

k	proxy field	tangential trace	normal trace
0	$\varphi \in C^\infty(\Omega)$	$\varphi\rvert_{\partial\Omega}$	0
1	$v \in C^\infty(\Omega, \mathbb{R}^3)$	$v\rvert_{\partial\Omega} - (v \cdot \mathbf{n})\mathbf{n}$	$v \cdot \mathbf{n}$
2	$w \in C^\infty(\Omega, \mathbb{R}^3)$	$(w \cdot \mathbf{n})\mathbf{n}$	$w \times \mathbf{n}$
3	$\psi \in C^\infty(\Omega)$	0	$\psi\mathbf{n}$

TABLE 1. Tangential and normal traces of differential forms on $\Omega \subset \mathbb{R}^3$, in terms of scalar and vector proxy fields.

2.4. **Hilbert complexes of differential forms.** We now recall the Hilbert spaces of differential forms and weak differential operators that form the functional analytic foundation of FEEC. Henceforth, we restrict attention to the case where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, although many of the results also hold on more general classes of manifolds (e.g., compact oriented Riemannian manifolds with Lipschitz boundary).

Let $C_0^\infty \Lambda^k(\Omega)$ be the space of smooth k-forms with compact support in the interior of Ω, and define $L^2\Lambda^k(\Omega)$ to be the completion of $C_0^\infty \Lambda^k(\Omega)$ with respect to the L^2 norm $\lVert\cdot\rVert_\Omega$. Following Arnold, Falk, and Winther [5], Arnold [2], one says that $v \in L^2\Lambda^k(\Omega)$ has the *weak exterior derivative* $\mathrm{d}v \in L^2\Lambda^{k+1}(\Omega)$ if

$$(\mathrm{d}v, \eta)_\Omega = (v, \delta\eta)_\Omega, \quad \forall \eta \in C_0^\infty \Lambda^{k+1}(\Omega),$$

which agrees with the integration by parts formula (2) since η vanishes on $\partial\Omega$. Since this is the adjoint of the densely defined operator $\delta \colon C_0^\infty \Lambda^{k+1}(\Omega) \to C_0^\infty \Lambda^k(\Omega)$, it follows that d is a closed, densely defined operator with domain

$$H\Lambda^k(\Omega) := \bigl\{ v \in L^2\Lambda^k(\Omega) : \mathrm{d}v \in L^2\Lambda^{k+1}(\Omega) \bigr\},$$

which is a Hilbert space with the graph inner product $(v, w)_{H\Lambda^k(\Omega)} := (v, w)_\Omega + (\mathrm{d}v, \mathrm{d}w)_\Omega$. This gives a *Hilbert complex* of differential forms,

$$0 \to H\Lambda^0(\Omega) \xrightarrow{\mathrm{d}} H\Lambda^1(\Omega) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} H\Lambda^n(\Omega) \to 0,$$

where $\mathrm{d}\mathrm{d} = 0$. Similarly, $v \in L^2\Lambda^k(\Omega)$ has the *weak codifferential* $\delta v \in L^2\Lambda^{k-1}(\Omega)$ if

$$(\tau, \delta v)_\Omega = (\mathrm{d}\tau, v)_\Omega, \quad \forall \tau \in C_0^\infty \Lambda^{k-1}(\Omega),$$

so δ is a closed, densely defined operator with domain

$$H^*\Lambda^k(\Omega) := \bigl\{ v \in L^2\Lambda^k(\Omega) : \delta v \in L^2\Lambda^{k-1}(\Omega) \bigr\} = \star H\Lambda^{n-k}(\Omega).$$

This is a Hilbert space with the graph inner product $(v, w)_{H^*\Lambda^k(\Omega)} := (v, w)_\Omega + (\delta v, \delta w)_\Omega$, and again we have a Hilbert complex, since $\delta\delta = 0$.

Weck [41] showed that it is possible to extend the tangential trace to $H\Lambda^k(\Omega)$ and normal trace to $H^*\Lambda^k(\Omega)$, such that a weak version of the integration by parts formula (2) holds:

$$\langle \tau^{\tan}, v^{\mathrm{nor}} \rangle_{\partial\Omega} = (\mathrm{d}\tau, v)_\Omega - (\tau, \delta v)_\Omega, \quad \forall \tau \in H\Lambda^{k-1}(\Omega),\ v \in H^*\Lambda^k(\Omega).$$

Here, $\tau^{\tan}$ and v^{nor} generally live in subspaces of $H^{-1/2}\Lambda^{k-1}(\partial\Omega)$, not necessarily in $L^2\Lambda^{k-1}(\partial\Omega)$, so $\langle \cdot, \cdot \rangle_{\partial\Omega}$ should be interpreted as a duality pairing rather than the L^2 inner product. See Kurz and Auchmann [28] for an excellent account of Weck's results and some concrete applications to electromagnetics. Mitrea, Mitrea, and Shaw [32] obtain similar results by extending the approach described in Remark 2.3.

We then define the subspaces

$$\mathring{H}\Lambda^k(\Omega) := \{v \in H\Lambda^k(\Omega) : v^{\tan} = 0\},$$
$$\mathring{H}^*\Lambda^k(\Omega) := \{v \in H^*\Lambda^k(\Omega) : v^{\mathrm{nor}} = 0\} = \star\mathring{H}\Lambda^{n-k}(\Omega),$$

which are seen to be closed in $H\Lambda^k(\Omega)$ and $H^*\Lambda^k(\Omega)$, respectively. Brüning and Lesch [10], Arnold [2] construct the spaces $\mathring{H}\Lambda^k(\Omega)$ and $\mathring{H}^*\Lambda^k(\Omega)$ in a different but equivalent way: by completing $C_0^\infty\Lambda^k(\Omega)$ with respect to the graph norms $\|\cdot\|_{H\Lambda^k(\Omega)}$ and $\|\cdot\|_{H^*\Lambda^k(\Omega)}$. This means that $\mathring{H}\Lambda^k(\Omega)$ is the domain of the *minimal* closed extension (i.e., the closure) of $\mathrm{d}\colon C_0^\infty\Lambda^k(\Omega) \to C_0^\infty\Lambda^{k+1}(\Omega)$, while $H\Lambda^k(\Omega)$ is the domain of the *maximal* closed extension, and likewise for $\mathring{H}^*\Lambda^k(\Omega)$ and $H^*\Lambda^k(\Omega)$.

Remark 2.4. More generally, any closed extension of d resulting in a Hilbert complex $\mathring{H}\Lambda^k(\Omega) \subset V^k \subset H\Lambda^k(\Omega)$ is called a choice of *ideal boundary conditions* [10]. For example, one may take a suitably nice decomposition of $\partial\Omega$ into two pieces, $\Gamma^{\tan}$ and Γ^{nor}, and let $V^k := \{v \in H\Lambda^k(\Omega) : v^{\tan}|_{\Gamma^{\tan}} = 0\}$. For an analysis of these mixed boundary conditions (including what qualifies as a "suitably nice decomposition"), see Jakab, Mitrea, and Mitrea [27], Gol'dshtein, Mitrea, and Mitrea [24].

When $\Omega \subset \mathbb{R}^3$, we have the correspondences

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H\Lambda^0(\Omega) & \xrightarrow{\mathrm{d}} & H\Lambda^1(\Omega) & \xrightarrow{\mathrm{d}} & H\Lambda^2(\Omega) & \xrightarrow{\mathrm{d}} & H\Lambda^3(\Omega) & \longrightarrow & 0 \\ & & \wr\Vert & & \wr\Vert & & \wr\Vert & & \wr\Vert & & \\ 0 & \longrightarrow & H^1(\Omega) & \xrightarrow{\mathrm{grad}} & H(\mathrm{curl};\Omega) & \xrightarrow{\mathrm{curl}} & H(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) & \longrightarrow & 0, \end{array}$$

and

$$\begin{array}{ccccccccccc} 0 & \longleftarrow & H^*\Lambda^0(\Omega) & \xleftarrow{\delta} & H^*\Lambda^1(\Omega) & \xleftarrow{\delta} & H^*\Lambda^2(\Omega) & \xleftarrow{\delta} & H^*\Lambda^3(\Omega) & \longleftarrow & 0 \\ & & \wr\Vert & & \wr\Vert & & \wr\Vert & & \wr\Vert & & \\ 0 & \longleftarrow & L^2(\Omega) & \xleftarrow{-\mathrm{div}} & H(\mathrm{div};\Omega) & \xleftarrow{\mathrm{curl}} & H(\mathrm{curl};\Omega) & \xleftarrow{-\mathrm{grad}} & H^1(\Omega) & \longleftarrow & 0, \end{array}$$

and similarly for the $\mathring{H}\Lambda$ and $\mathring{H}^*\Lambda$ spaces. The weak tangential and normal traces and the duality pairing $\langle\cdot,\cdot\rangle_{\partial\Omega}$ correspond to those described in Buffa and Ciarlet [11, 12], Buffa, Costabel, and Sheen [13]. More generally, when $\Omega \subset \mathbb{R}^n$, we can still identify $H\Lambda^0(\Omega) \cong H^1(\Omega) \cong H^*\Lambda^n(\Omega)$, $H\Lambda^n(\Omega) \cong L^2(\Omega) \cong H^*\Lambda^0(\Omega)$, and $H\Lambda^{n-1}(\Omega) \cong H(\mathrm{div};\Omega) \cong H^*\Lambda^1(\Omega)$, and similarly for the $\mathring{H}\Lambda$ and $\mathring{H}^*\Lambda$ spaces.

2.5. **The Hodge decomposition and Poincaré inequality.** Although much of the following analysis applies to more general Hilbert complexes, we focus our attention on

$$0 \to H\Lambda^0(\Omega) \xrightarrow{\mathrm{d}} H\Lambda^1(\Omega) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} H\Lambda^n(\Omega) \to 0.$$

The operators d satisfy a compactness property, as shown by Picard [36], and in particular they are Fredholm and thus have closed range. Define

$$\mathfrak{B}^k := \{\mathrm{d}\tau : \tau \in H\Lambda^{k-1}(\Omega)\}, \qquad \mathfrak{Z}^k := \{v \in H\Lambda^k(\Omega) : \mathrm{d}v = 0\}, \qquad \mathfrak{H}^k := \mathfrak{Z}^k \cap \mathfrak{B}^{k\perp},$$

which are the subspaces of *exact*, *closed*, and *harmonic* k-forms in $L^2\Lambda^k(\Omega)$. It follows that

$$L^2\Lambda^k(\Omega) = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp},$$

which is an L^2-orthogonal decomposition called the *Hodge decomposition*. By Banach's closed range theorem and the adjointness of d and δ, we may also write

$$\mathfrak{B}^{k\perp} = \{v \in \mathring{H}^*\Lambda^k(\Omega) : \delta v = 0\} =: \mathring{\mathfrak{Z}}^*_k, \qquad \mathfrak{Z}^{k\perp} = \{\delta\eta : \eta \in \mathring{H}^*\Lambda^{k+1}(\Omega)\} =: \mathring{\mathfrak{B}}^*_k,$$

called *coclosed* and *coexact* k-forms. This implies

$$\mathfrak{H}^k = \mathfrak{Z}^k \cap \mathring{\mathfrak{Z}}^*_k = \{v \in H\Lambda^k(\Omega) \cap \mathring{H}^*\Lambda^k(\Omega) : \mathrm{d}v = 0,\, \delta v = 0\},$$

which is an equivalent characterization of harmonic forms. Hence, the Hodge decomposition of a k-form is a sum of exact, harmonic (closed and coclosed), and coexact k-forms.

Finally, since d is an $H\Lambda$-bounded isomorphism between $H\Lambda^k(\Omega) \cap \mathfrak{Z}^{k\perp}$ and $\mathfrak{B}^{k+1}$, Banach's bounded inverse theorem implies that there exists a constant $c_P(\Omega)$ such that

$$\|v\|_\Omega \leq c_P(\Omega)\|\mathrm{d}v\|_\Omega, \quad \forall v \in H\Lambda^k(\Omega) \cap \mathfrak{Z}^{k\perp},$$

which we call the *Poincaré inequality.* Note that Arnold, Falk, and Winther [5], Arnold [2] write the Poincaré inequality differently, using the $\|\cdot\|_{H\Lambda^k(\Omega)}$ norm, so that the constant is $\sqrt{1 + c_P(\Omega)^2}$. However, the form we have chosen is more convenient for scaling arguments that we will apply later.

For scalar fields, the $k = 0$ case gives the decomposition of $L^2(\Omega)$ into constant functions and functions with average zero on each connected component of Ω. When Ω is connected, the Poincaré inequality is therefore the classical one,

$$\|v\|_\Omega \leq c_P(\Omega)\|\operatorname{grad} v\|_\Omega, \quad \forall v \in H^1(\Omega) : \int_\Omega v = 0.$$

When $k = n$, we have the trivial decomposition of scalar proxies $L^2(\Omega) \cong \mathfrak{B}^n$, and the Poincaré inequality is also trivial. If $\Omega \subset \mathbb{R}^3$ has no holes, then the Hodge decomposition of 1- and 2-forms corresponds to the *Helmholtz decomposition* of vector fields into irrotational and solenoidal components; otherwise there are nontrivial harmonic 1- and 2-forms, corresponding to vector fields that are both divergence-free and curl-free with vanishing tangential/normal trace.

2.6. **The Hodge–Laplace problem.** Recall the Hodge–Laplace operator $L \coloneqq \mathrm{d}\delta + \delta\mathrm{d}$ on k-forms, which we can now interpret in a weak sense. Given $f \in L^2\Lambda^k(\Omega)$, we wish to solve the following problem: Find $u \in \mathfrak{H}^{k\perp}$, $p \in \mathfrak{H}^k$, such that

$$\begin{aligned} Lu + p = f &\quad \text{in } \Omega, \\ u^{\mathrm{nor}} = 0,\ (\mathrm{d}u)^{\mathrm{nor}} = 0, &\quad \text{on } \partial\Omega. \end{aligned}$$

The solution gives the Hodge decomposition $f = \mathrm{d}\sigma + p + \delta\rho$, where $\sigma = \delta u$ and $\rho = \mathrm{d}u$.

FEEC is based on the following mixed formulation of the Hodge–Laplace problem: Find $\sigma \in H\Lambda^{k-1}(\Omega)$, $u \in H\Lambda^k(\Omega)$, $p \in \mathfrak{H}^k$ such that

$$(\sigma, \tau)_\Omega - (u, \mathrm{d}\tau)_\Omega = 0, \qquad \forall \tau \in H\Lambda^{k-1}(\Omega), \tag{3a}$$

$$(\mathrm{d}\sigma, v)_\Omega + (\mathrm{d}u, \mathrm{d}v)_\Omega + (p, v)_\Omega = (f, v)_\Omega, \quad \forall v \in H\Lambda^k(\Omega), \tag{3b}$$

$$(u, q)_\Omega = 0, \qquad \forall q \in \mathfrak{H}^k, \tag{3c}$$

where both boundary conditions are now natural. More generally, nonvanishing natural boundary conditions may be imposed by adding $\langle \cdot, \cdot \rangle_{\partial\Omega}$ terms on the right-hand side. The well-posedness of this mixed formulation is proved in Arnold, Falk, and Winther [4, Theorem 7.2] and generalized to abstract Hilbert complexes in Arnold, Falk, and Winther [5, Theorem 3.2].

Remark 2.5. Instead of natural boundary conditions, one may impose essential boundary conditions $\sigma^{\tan} = 0$ and $u^{\tan} = 0$ by taking the test and trial functions from $\mathring{H}\Lambda^{k-1}(\Omega)$, $\mathring{H}\Lambda^k(\Omega)$, $\mathring{\mathfrak{H}}^k$, cf. [5, Section 6.2]. This may be generalized to nonvanishing $\sigma^{\tan}$ and $u^{\tan}$ via a standard extension argument: if $\chi \in H\Lambda^{k-1}(\Omega)$ and $w \in H\Lambda^k(\Omega)$ are extensions of the prescribed tangential traces, then $\mathring{\sigma} = \sigma - \chi \in \mathring{H}\Lambda^{k-1}(\Omega)$, $\mathring{u} = u - w \in \mathring{H}\Lambda^k(\Omega)$, $p \in \mathring{\mathfrak{H}}^k$ satisfy

$$\begin{aligned} (\mathring{\sigma}, \tau)_\Omega - (\mathring{u}, \mathrm{d}\tau)_\Omega &= -(\chi, \tau)_\Omega + (w, \mathrm{d}\tau)_\Omega, && \forall \tau \in \mathring{H}\Lambda^{k-1}(\Omega), \\ (\mathrm{d}\mathring{\sigma}, v)_\Omega + (\mathrm{d}\mathring{u}, \mathrm{d}v)_\Omega + (p, v)_\Omega &= (f, v)_\Omega - (\mathrm{d}\chi, v)_\Omega - (\mathrm{d}w, \mathrm{d}v)_\Omega, && \forall v \in \mathring{H}\Lambda^k(\Omega), \\ (u, q)_\Omega &= -(w, q)_\Omega, && \forall q \in \mathring{\mathfrak{H}}^k, \end{aligned}$$

so well-posedness follows from the inf-sup condition. We may also impose other ideal boundary conditions $\mathring{H}\Lambda(\Omega) \subset V \subset H\Lambda(\Omega)$, as discussed in Remark 2.4. For example, mixed boundary conditions are essential for $\sigma^{\tan}$, $u^{\tan}$ on $\Gamma^{\tan}$ and natural for u^{nor}, $(\mathrm{d}u)^{\mathrm{nor}}$ on Γ^{nor}.

When $\Omega \subset \mathbb{R}^3$ is contractible, so that $\mathfrak{H}^0 \cong \mathbb{R}$ and $\mathfrak{H}^k$ is trivial otherwise, we have the following in terms of scalar and vector proxies. The case $k = 0$ corresponds to the primal formulation of the Neumann problem: Find $u \in H^1(\Omega)$, $p \in \mathbb{R}$, such that

$$\begin{aligned} (\operatorname{grad} u, \operatorname{grad} v)_\Omega + (p, v)_\Omega &= (f, v)_\Omega, && \forall v \in H^1(\Omega), \\ (u, q)_\Omega &= 0, && \forall q \in \mathbb{R}. \end{aligned}$$

Taking $v = 1$ implies that $(p, 1)_\Omega = (f, 1)_\Omega$, so p is simply the average of f, while the second equation says that $(u, 1)_\Omega = 0$, so u has average zero. The case $k = 1$ corresponds to the following mixed formulation of the vector Poisson equation with $u \cdot \mathbf{n} = 0$ and $\operatorname{curl} u \times \mathbf{n} = 0$: Find $\sigma \in H^1(\Omega)$, $u \in H(\operatorname{curl}; \Omega)$ such that

$$\begin{aligned} (\sigma, \tau)_\Omega - (u, \operatorname{grad} \tau)_\Omega &= 0, && \forall \tau \in H^1(\Omega), \\ (\operatorname{grad} \sigma, v)_\Omega + (\operatorname{curl} u, \operatorname{curl} v)_\Omega &= (f, v)_\Omega, && \forall v \in H(\operatorname{curl}; \Omega). \end{aligned}$$

The case $k = 2$ corresponds to the mixed formulation of the vector Poisson equation with $u \times \mathbf{n} = 0$ and $\operatorname{div} u|_{\partial\Omega} = 0$: Find $\sigma \in H(\operatorname{curl}; \Omega)$, $u \in H(\operatorname{div}; \Omega)$ such that

$$\begin{aligned} (\sigma, \tau)_\Omega - (u, \operatorname{curl} \tau)_\Omega &= 0, && \forall \tau \in H(\operatorname{curl}; \Omega), \\ (\operatorname{curl} \sigma, v)_\Omega + (\operatorname{div} u, \operatorname{div} v)_\Omega &= (f, v)_\Omega, && \forall v \in H(\operatorname{div}; \Omega). \end{aligned}$$

Finally, the case $k = 3$ (more generally, $k = n$) corresponds to the mixed formulation of the Dirichlet problem: Find $\sigma \in H(\operatorname{div}; \Omega)$, $u \in L^2(\Omega)$ such that

$$\begin{aligned} (\sigma, \tau)_\Omega - (u, \operatorname{div} \tau)_\Omega &= 0, && \forall \tau \in H(\operatorname{div}; \Omega), \\ (\operatorname{div} \sigma, v)_\Omega &= (f, v)_\Omega, && \forall v \in L^2(\Omega). \end{aligned}$$

2.7. **Finite element exterior calculus.** Just as the Galerkin method approximates problems on infinite-dimensional Hilbert spaces by restricting to finite-dimensional subspaces, FEEC approximates problems on infinite-dimensional Hilbert *complexes* by restricting to finite-dimensional *subcomplexes*.

A subcomplex $V_h \subset H\Lambda(\Omega)$ is a sequence of (here, finite-dimensional) subspaces $V_h^k \subset H\Lambda^k(\Omega)$ that is closed with respect to d, i.e., $\mathrm{d}V_h^k \subset V_h^{k+1}$. Just as in Section 2.5, we have subspaces

$$\mathfrak{B}_h^k \coloneqq \{\mathrm{d}\tau_h : \tau_h \in V_h^{k-1}\}, \qquad \mathfrak{Z}_h^k \coloneqq \{v_h \in V_h^k : \mathrm{d}v_h = 0\}, \qquad \mathfrak{H}_h^k \coloneqq \mathfrak{Z}_h^k \cap \mathfrak{B}_h^{k\perp},$$

along with a discrete Hodge decomposition $V_h^k = \mathfrak{B}_h^k \oplus \mathfrak{H}_h^k \oplus \mathfrak{Z}_h^{k\perp}$ and discrete Poincaré inequality. Note that the subcomplex assumption implies $\mathfrak{B}_h^k \subset \mathfrak{B}^k$ and $\mathfrak{Z}_h^k \subset \mathfrak{Z}^k$, although in general $\mathfrak{H}_h^k \not\subset \mathfrak{H}^k$ and $\mathfrak{Z}_h^{k\perp} \not\subset \mathfrak{Z}^{k\perp}$. An additional key assumption in the analysis (but not implementation) of FEEC is the existence of *bounded commuting projections* $\pi_h^k \colon H\Lambda^k(\Omega) \to V_h^k$, which among other uses gives control of the discrete Poincaré constant in terms of $c_P(\Omega)$.

In FEEC, one then approximates the Hodge–Laplace problem (3) by the following finite-dimensional variational problem: Find $\sigma_h \in V_h^{k-1}$, $u_h \in V_h^k$, $p_h \in \mathfrak{H}_h^k$ such that

$$(\sigma_h, \tau_h)_\Omega - (u_h, \mathrm{d}\tau_h)_\Omega = 0, \qquad \forall \tau_h \in V_h^{k-1}, \tag{4a}$$

$$(\mathrm{d}\sigma_h, v_h)_\Omega + (\mathrm{d}u_h, \mathrm{d}v_h)_\Omega + (p_h, v_h)_\Omega = (f, v_h)_\Omega, \qquad \forall v_h \in V_h^k, \tag{4b}$$

$$(u_h, q_h)_\Omega = 0, \qquad \forall q_h \in \mathfrak{H}_h^k. \tag{4c}$$

Arnold, Falk, and Winther [4, 5] establish stability and convergence for this problem, proving quasi-optimal error estimates in the $H\Lambda$-norm and improved L^2-error estimates under additional regularity assumptions using the aforementioned compactness property. (In [5], much of this analysis takes place in the setting of abstract Hilbert complexes.) As in Remark 2.5, we may instead take essential boundary conditions for $\sigma_h^{\tan}$ and $u_h^{\tan}$. Licht [31] has recently extended the analysis of FEEC to mixed boundary conditions, including the construction of bounded commuting projections.

One more essential ingredient of FEEC is the construction of finite elements for the spaces V_h^k. Suppose that $\Omega \subset \mathbb{R}^n$ is polyhedral, and let $\mathcal{T}_h$ be a triangulation of Ω by n-simplices $K \in \mathcal{T}_h$. Arnold, Falk, and Winther [4, 5] construct two families of piecewise-polynomial differential forms, called $\mathcal{P}_r\Lambda$ and $\mathcal{P}_r^-\Lambda$, which we will sometimes refer to collectively as $\mathcal{P}_r^{\pm}\Lambda$. Let $\mathcal{P}_r(K)$ denote the space of polynomials in $(x^1,\ldots,x^n) \in K$ of degree $\leq r$, and define $\mathcal{P}_r\Lambda^k(K)$ to be the space of polynomial k-forms on K,

$$\mathcal{P}_r\Lambda^k(K) \coloneqq \bigl\{ v_{i_1\cdots i_k}\, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} : v_{i_1\ldots i_k} \in \mathcal{P}_r(K) \bigr\}.$$

The space $\mathcal{P}_r^-\Lambda^k(K)$ has the slightly more technical definition,

$$\mathcal{P}_r^-\Lambda^k(K) \coloneqq \mathcal{P}_{r-1}\Lambda^k(K) + \kappa \mathcal{P}_{r-1}\Lambda^{k+1}(K),$$

where κ is the *Koszul differential*, defined to be contraction with the vector field $x^i e_i$. We then define the $H\Lambda^k(\Omega)$-conforming finite element spaces

$$\mathcal{P}_r^{\pm}\Lambda^k(\mathcal{T}_h) \coloneqq \bigl\{ v \in H\Lambda^k(\Omega) : v|_K \in \mathcal{P}_r^{\pm}\Lambda^k(K),\ \forall K \in \mathcal{T}_h \bigr\}.$$

Arnold, Falk, and Winther [4, 5] show that any of the pairs of spaces

$$V_h^{k-1} = \mathcal{P}_{r+1}^{\pm}\Lambda^{k-1}(\mathcal{T}_h), \qquad V_h^k = \left\{ \begin{matrix} \mathcal{P}_r\Lambda^k(\mathcal{T}_h) \text{ (if } r \geq 1) \\ \text{or} \\ \mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T}_h) \end{matrix} \right\}, \tag{5}$$

results in a subcomplex for the problem (4) satisfying the needed analytical assumptions. When $n = 3$, the $\mathcal{P}_r^{\pm}\Lambda$ elements unify various families of finite elements for scalar and vector fields: the $\mathcal{P}_r^{\pm}\Lambda^0$ elements are continuous Lagrange finite elements, $\mathcal{P}_r^{\pm}\Lambda^1$ are Nédélec edge elements of the first ($-$, [33]) and second ($+$, [34]) kinds, $\mathcal{P}_r^{\pm}\Lambda^2$ are Nédélec face elements of the first ($-$) and second ($+$) kinds, and $\mathcal{P}_r^{\pm}\Lambda^3$ are discontinuous Lagrange elements.

For the finite element subcomplexes (5), the L^2 error estimates [5, Theorem 3.11] take the form

$$\begin{aligned}
\bigl\lVert \mathrm{d}(\sigma - \sigma_h) \bigr\rVert_\Omega &\lesssim E(\mathrm{d}\sigma), \\
\lVert \sigma - \sigma_h \rVert_\Omega &\lesssim E(\sigma) + hE(\mathrm{d}\sigma), \\
\lVert p - p_h \rVert_\Omega &\lesssim E(p) + h^{r+1}E(\mathrm{d}\sigma), \\
\bigl\lVert \mathrm{d}(u - u_h) \bigr\rVert_\Omega &\lesssim E(\mathrm{d}u) + h\bigl[E(\mathrm{d}\sigma) + E(p)\bigr], \\
\lVert u - u_h \rVert_\Omega &\lesssim E(u) + h\bigl[E(\mathrm{d}u) + E(\sigma)\bigr] + h^{\min(2,r+1)}\bigl[E(\mathrm{d}\sigma) + E(p)\bigr],
\end{aligned}$$

where $E(v) \coloneqq \inf_{v_h \in V_h} \lVert v - v_h \rVert_\Omega$ and h is the maximum diameter of a simplex $K \in \mathcal{T}_h$. Hence, approximating (3) by the FEEC method (4) gives the best rate of convergence allowed by the regularity of the exact solution and the polynomial degree of the finite element spaces. We will frequently apply [5, Theorem 3.11] in this form, along with a corresponding version on the $H^*\Lambda$ complex for the postprocessing analysis in Section 5.2.

3. Domain decomposition of the Hodge–Laplace problem

This section presents a domain decomposition of the Hodge–Laplace problem, where the bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ is partitioned into Lipschitz subdomains $K \in \mathcal{T}_h$ with non-overlapping interiors. This will be the foundation for the hybrid methods in Section 4, where Ω is polyhedral and $K \in \mathcal{T}_h$ are individual elements of a conforming mesh. However, the results of this section also apply to more general types of domain decomposition. Our approach generalizes the three-field domain decomposition method of Brezzi and Marini [9] for the scalar Poisson equation, which is the case $k = 0$.

3.1. **Decomposition of Hilbert complexes of differential forms.** Define the broken spaces of differential forms

$$H\Lambda^k(\mathcal{T}_h) \coloneqq \prod_{K\in\mathcal{T}_h} H\Lambda^k(K), \qquad H^*\Lambda^k(\mathcal{T}_h) \coloneqq \prod_{K\in\mathcal{T}_h} H^*\Lambda^k(K).$$

As product spaces, these naturally inherit the inner products

$$(\cdot,\cdot)_{\mathcal{T}_h} \coloneqq \sum_{K\in\mathcal{T}_h} (\cdot,\cdot)_K, \quad (\cdot,\cdot)_{H\Lambda^k(\mathcal{T}_h)} \coloneqq \sum_{K\in\mathcal{T}_h} (\cdot,\cdot)_{H\Lambda^k(K)}, \quad (\cdot,\cdot)_{H^*\Lambda^k(\mathcal{T}_h)} \coloneqq \sum_{K\in\mathcal{T}_h} (\cdot,\cdot)_{H^*\Lambda^k(K)}.$$

We can then define $\mathrm{d}\colon H\Lambda^k(\mathcal{T}_h) \to H\Lambda^{k+1}(\mathcal{T}_h)$ to be $\mathrm{d}|_{H\Lambda^k(K)}$ on each $K \in \mathcal{T}_h$, and likewise for $\delta\colon H^*\Lambda^k(\mathcal{T}_h) \to H^*\Lambda^{k-1}(\mathcal{T}_h)$. These broken Hilbert complexes are simply the $H\Lambda$ and $H^*\Lambda$ complexes for the disjoint union $\bigsqcup_{K\in\mathcal{T}_h} K$.

For these broken spaces, we can define tangential and normal traces on $\partial\mathcal{T}_h \coloneqq \bigsqcup_{K\in\mathcal{T}_h} \partial K$ by taking the trace on ∂K for each $K \in \mathcal{T}_h$. Defining the pairing $\langle\cdot,\cdot\rangle_{\partial\mathcal{T}_h} \coloneqq \sum_{K\in\mathcal{T}_h} \langle\cdot,\cdot\rangle_{\partial K}$, we immediately get the integration by parts formula

$$\langle \tau^{\tan}, v^{\mathrm{nor}}\rangle_{\partial\mathcal{T}_h} = (\mathrm{d}\tau, v)_{\mathcal{T}_h} - (\tau, \delta v)_{\mathcal{T}_h}, \quad \forall \tau \in H\Lambda^{k-1}(\mathcal{T}_h),\ v \in H^*\Lambda^k(\mathcal{T}_h),$$

simply by summing the integration by parts formulas for each $K \in \mathcal{T}_h$. Note that, if $e = \partial K^+ \cap \partial K^-$ is the interface between $K^\pm \in \mathcal{T}_h$, then e appears twice in the disjoint union $\partial\mathcal{T}_h$: once as part of ∂K^+, and a second time as part of ∂K^-. The traces of broken differential forms can therefore be seen as "double valued," since there is no continuity imposed at interfaces between subdomains.

There are natural inclusions $H\Lambda^k(\Omega) \hookrightarrow H\Lambda^k(\mathcal{T}_h)$ and $H^*\Lambda^k(\Omega) \hookrightarrow H^*\Lambda^k(\mathcal{T}_h)$, which are defined by restriction to each $K \in \mathcal{T}_h$. The next result characterizes these subspaces of "unbroken" differential forms, generalizing some classic results on domain decomposition of H^1, $H(\mathrm{curl})$, and $H(\mathrm{div})$ spaces (cf. Propositions 2.1.1–2.1.3 of Boffi, Brezzi, and Fortin [6]). In a weak sense, it says that unbroken differential forms are precisely those with "single valued" tangential or normal traces on $\partial\mathcal{T}_h$.

Proposition 3.1. *If $\mathcal{T}_h$ is a decomposition of Ω into Lipschitz subdomains, then*

$$\begin{aligned}
H\Lambda^k(\Omega) &= \bigl\{ v \in H\Lambda^k(\mathcal{T}_h) : \langle v^{\tan}, \eta^{\mathrm{nor}}\rangle_{\partial\mathcal{T}_h} = 0,\ \forall \eta \in \mathring{H}^*\Lambda^{k+1}(\Omega)\bigr\},\\
\mathring{H}\Lambda^k(\Omega) &= \bigl\{ v \in H\Lambda^k(\mathcal{T}_h) : \langle v^{\tan}, \eta^{\mathrm{nor}}\rangle_{\partial\mathcal{T}_h} = 0,\ \forall \eta \in H^*\Lambda^{k+1}(\Omega)\bigr\},\\
H^*\Lambda^k(\Omega) &= \bigl\{ v \in H^*\Lambda^k(\mathcal{T}_h) : \langle \tau^{\tan}, v^{\mathrm{nor}}\rangle_{\partial\mathcal{T}_h} = 0,\ \forall \tau \in \mathring{H}\Lambda^{k-1}(\Omega)\bigr\},\\
\mathring{H}^*\Lambda^k(\Omega) &= \bigl\{ v \in H^*\Lambda^k(\mathcal{T}_h) : \langle \tau^{\tan}, v^{\mathrm{nor}}\rangle_{\partial\mathcal{T}_h} = 0,\ \forall \tau \in H\Lambda^{k-1}(\Omega)\bigr\}.
\end{aligned}$$

Proof. These four identities are proved using essentially the same argument, so we give only a proof of the first. If $v \in H\Lambda^k(\Omega)$, then for all $\eta \in \mathring{H}^*\Lambda^{k+1}(\Omega)$,

$$\langle v^{\tan}, \eta^{\mathrm{nor}}\rangle_{\partial\mathcal{T}_h} = (\mathrm{d}v, \eta)_{\mathcal{T}_h} - (v, \delta\eta)_{\mathcal{T}_h} = (\mathrm{d}v, \eta)_\Omega - (v, \delta\eta)_\Omega = \langle v^{\tan}, \eta^{\mathrm{nor}}\rangle_{\partial\Omega} = 0.$$

Conversely, suppose that $v \in H\Lambda^k(\mathcal{T}_h) \subset L^2\Lambda^k(\mathcal{T}_h) \cong L^2\Lambda^k(\Omega)$ satisfies $\langle v^{\tan}, \eta^{\mathrm{nor}}\rangle_{\partial\mathcal{T}_h} = 0$ for all $\eta \in \mathring{H}^*\Lambda^{k+1}(\Omega)$. Then, using integration by parts and Cauchy–Schwarz,

$$(v, \delta\eta)_\Omega = (v, \delta\eta)_{\mathcal{T}_h} = (\mathrm{d}v, \eta)_{\mathcal{T}_h} \le \lVert \mathrm{d}v\rVert_{\mathcal{T}_h} \lVert \eta\rVert_{\mathcal{T}_h} = \lVert \mathrm{d}v\rVert_{\mathcal{T}_h} \lVert \eta\rVert_\Omega.$$

In particular, this holds for $\eta \in C_0^\infty\Lambda^{k+1}(\Omega)$, implying $\mathrm{d}v \in L^2\Lambda^{k+1}(\Omega)$ and hence $v \in H\Lambda^k(\Omega)$. $\square$

3.2. **Decomposition of the Hodge–Laplace problem.** Before introducing the domain decomposition of (3), we first provide some motivating intuition. For each $K \in \mathcal{T}_h$, observe that σ and u solve the local problem

$$(\sigma,\tau)_K - (u, \mathrm{d}\tau)_K = 0, \qquad \forall \tau \in \mathring{H}\Lambda^{k-1}(K), \tag{6a}$$

$$(\mathrm{d}\sigma, v)_K + (\mathrm{d}u, \mathrm{d}v)_K = (f - p, v)_K, \quad \forall v \in \mathring{H}\Lambda^k(K), \tag{6b}$$

with essential boundary conditions $\sigma^{\tan}$ and $u^{\tan}$. However, if the space of *local harmonic forms* $\mathring{\mathfrak{H}}^k(K)$ is nontrivial, then this local problem is not well-posed. (When $K \in \mathcal{T}_h$ are contractible, as with simplices in a triangulation, this is only an issue for $k = n$; see Section 3.4.) Therefore, we include an additional local variable $\overline{p} \in \mathring{\mathfrak{H}}^k(K)$ and solve

$$(\sigma, \tau)_K - (u, \mathrm{d}\tau)_K = 0, \qquad \forall \tau \in \mathring{H}\Lambda^{k-1}(K), \tag{7a}$$

$$(\mathrm{d}\sigma, v)_K + (\mathrm{d}u, \mathrm{d}v)_K + (\overline{p}, v)_K = (f - p, v)_K, \quad \forall v \in \mathring{H}\Lambda^k(K), \tag{7b}$$

$$(u, \overline{q})_K = (\overline{u}, \overline{q})_K, \qquad \forall \overline{q} \in \mathring{\mathfrak{H}}^k(K), \tag{7c}$$

where $\overline{u}$ is the projection of u onto $\mathring{\mathfrak{H}}^k(K)$. Following Remark 2.5, these local solvers are well-posed for any right-hand side and tangential traces $\sigma^{\tan}$, $u^{\tan}$.

We now allow the tangential traces $\widehat{\sigma}^{\tan}$, $\widehat{u}^{\tan}$ to be independent variables and impose the constraints $\sigma^{\tan} = \widehat{\sigma}^{\tan}$, $u^{\tan} = \widehat{u}^{\tan}$ using Lagrange multipliers $\widehat{u}^{\mathrm{nor}}$, $\widehat{\rho}^{\mathrm{nor}}$, which will turn out to be the normal traces of u and $\rho = \mathrm{d}u$. Define the spaces

$$W^k := H\Lambda^k(\mathcal{T}_h), \qquad \overline{\mathfrak{H}}^k := \prod_{K \in \mathcal{T}_h} \mathring{\mathfrak{H}}^k(K),$$

$$\widehat{W}^{k,\mathrm{nor}} := \{\eta^{\mathrm{nor}} : \eta \in H^*\Lambda^{k+1}(\mathcal{T}_h)\}, \qquad \widehat{V}^{k,\tan} := \{v^{\tan} : v \in H\Lambda^k(\Omega)\}.$$

Note that $\widehat{V}^{k,\tan}$ consists of "single valued" traces from the unbroken space $H\Lambda^k(\Omega)$, whereas the other three spaces contain broken k-forms. Consider the variational problem: Find

$$\text{(local variables)} \quad \sigma \in W^{k-1}, \quad u \in W^k, \quad \overline{p} \in \overline{\mathfrak{H}}^k, \quad \widehat{u}^{\mathrm{nor}} \in \widehat{W}^{k-1,\mathrm{nor}}, \quad \widehat{\rho}^{\mathrm{nor}} \in \widehat{W}^{k,\mathrm{nor}},$$

$$\text{(global variables)} \quad p \in \mathfrak{H}^k, \quad \overline{u} \in \overline{\mathfrak{H}}^k, \quad \widehat{\sigma}^{\tan} \in \widehat{V}^{k-1,\tan}, \quad \widehat{u}^{\tan} \in \widehat{V}^{k,\tan},$$

satisfying

$$(\sigma, \tau)_{\mathcal{T}_h} - (u, \mathrm{d}\tau)_{\mathcal{T}_h} + \langle \widehat{u}^{\mathrm{nor}}, \tau^{\tan} \rangle_{\partial\mathcal{T}_h} = 0, \qquad \forall \tau \in W^{k-1}, \tag{8a}$$

$$(\mathrm{d}\sigma, v)_{\mathcal{T}_h} + (\mathrm{d}u, \mathrm{d}v)_{\mathcal{T}_h} + (\overline{p} + p, v)_{\mathcal{T}_h} - \langle \widehat{\rho}^{\mathrm{nor}}, v^{\tan} \rangle_{\partial\mathcal{T}_h} = (f, v)_{\mathcal{T}_h}, \qquad \forall v \in W^k, \tag{8b}$$

$$(\overline{u} - u, \overline{q})_{\mathcal{T}_h} = 0, \qquad \forall \overline{q} \in \overline{\mathfrak{H}}^k, \tag{8c}$$

$$\langle \widehat{\sigma}^{\tan} - \sigma^{\tan}, \widehat{v}^{\mathrm{nor}} \rangle_{\partial\mathcal{T}_h} = 0, \qquad \forall \widehat{v}^{\mathrm{nor}} \in \widehat{W}^{k-1,\mathrm{nor}}, \tag{8d}$$

$$\langle \widehat{u}^{\tan} - u^{\tan}, \widehat{\eta}^{\mathrm{nor}} \rangle_{\partial\mathcal{T}_h} = 0, \qquad \forall \widehat{\eta}^{\mathrm{nor}} \in \widehat{W}^{k,\mathrm{nor}}, \tag{8e}$$

$$(u, q)_{\mathcal{T}_h} = 0, \qquad \forall q \in \mathfrak{H}^k, \tag{8f}$$

$$(\overline{p}, \overline{v})_{\mathcal{T}_h} = 0, \qquad \forall \overline{v} \in \overline{\mathfrak{H}}^k, \tag{8g}$$

$$\langle \widehat{u}^{\mathrm{nor}}, \widehat{\tau}^{\tan} \rangle_{\partial\mathcal{T}_h} = 0, \qquad \forall \widehat{\tau}^{\tan} \in \widehat{V}^{k-1,\tan}, \tag{8h}$$

$$\langle \widehat{\rho}^{\mathrm{nor}}, \widehat{v}^{\tan} \rangle_{\partial\mathcal{T}_h} = 0, \qquad \forall \widehat{v}^{\tan} \in \widehat{V}^{k,\tan}. \tag{8i}$$

Remark 3.2. Given values for the global variables, notice that (8a)–(8e) simply amounts to solving the local problem (7) on each $K \in \mathcal{T}_h$. Therefore, although this formulation appears at first to be much larger than (3), in practice, a Schur complement approach can be used to eliminate the local variables, which can result in a smaller global system. This is the idea behind static condensation, which is discussed further in Section 4.2

We now prove that this is indeed a domain decomposition of the Hodge–Laplace problem (3), which in particular implies well-posedness of (8).

Theorem 3.3. *The following are equivalent:*

- $(\sigma, u, \overline{p}, \widehat{u}^{\mathrm{nor}}, \widehat{\rho}^{\mathrm{nor}}, p, \overline{u}, \widehat{\sigma}^{\tan}, \widehat{u}^{\tan})$ *is a solution to* (8).

- *(σ, u, p) is a solution to (3), and furthermore, $\overline{p} = 0$, $\widehat{u}^{\mathrm{nor}} = u^{\mathrm{nor}}$, $\widehat{\rho}^{\mathrm{nor}} = (\mathrm{d}u)^{\mathrm{nor}}$, $\overline{u}$ is the projection of u onto $\overline{\mathfrak{H}}^k$, $\widehat{\sigma}^{\tan} = \sigma^{\tan}$, and $\widehat{u}^{\tan} = u^{\tan}$.*

Proof. Suppose we have a solution to (8). The claimed equalities are immediate from the variational problem, so it remains only to show that (σ, u, p) solves (3). Since $\sigma^{\tan} = \widehat{\sigma}^{\tan}$ and $u^{\tan} = \widehat{u}^{\tan}$, Proposition 3.1 implies that $\sigma \in H\Lambda^{k-1}(\Omega)$ and $u \in H\Lambda^k(\Omega)$. Therefore, taking test functions $\tau \in H\Lambda^{k-1}(\Omega)$ and $v \in H\Lambda^k(\Omega)$ in (8a)–(8b), the normal trace terms vanish by (8h)–(8i), and we obtain (3a)–(3b). Finally, (10f) is the same as (3c), which proves the forward direction.

Conversely, given a solution (σ, u, p) to (3), it is immediate that (8a)–(8g) hold. For the remaining two equations, first observe that combining (3a) and (8a) gives $\langle \widehat{u}^{\mathrm{nor}}, \tau^{\tan} \rangle_{\partial \mathcal{T}_h} = 0$ for $\tau \in H\Lambda^{k-1}(\Omega)$, which implies (8h). Similarly, combining (3b) and (8b) gives $\langle \widehat{\rho}^{\mathrm{nor}}, v^{\tan} \rangle_{\partial \mathcal{T}_h} = 0$ for $v \in H\Lambda^k(\Omega)$, which implies (8i). $\square$

For the last step of the proof, we could instead have used that (3a) gives $u \in \mathring{H}^*\Lambda^k(\Omega)$ and (3b) gives $\mathrm{d}u \in \mathring{H}^*\Lambda^{k+1}(\Omega)$, applying Proposition 3.1 to conclude that their normal traces satisfy (8h)–(8i). However, as we will see, the variational argument above generalizes more readily to the hybridization of FEEC in Section 4.

Remark 3.4. Although the domain decomposition is presented above for $H\Lambda(\Omega)$ with natural boundary conditions on $\partial\Omega$, it is easily generalized to $\mathring{H}\Lambda(\Omega)$ or other ideal boundary conditions $\mathring{H}\Lambda(\Omega) \subset V \subset H\Lambda(\Omega)$, as in Remark 2.5. In this case, the broken spaces are unchanged, and we take the unbroken tangential traces and harmonic forms to be those from the complex V.

For nonvanishing essential boundary conditions on $\partial\Omega$, it is straightforward to adapt the extension approach of Remark 2.5 to $\widehat{\sigma}^{\tan}$ and $\widehat{u}^{\tan}$: take extensions χ and w of the prescribed boundary values, so that $\widehat{\sigma}^{\tan} - \chi^{\tan} \in \widehat{V}^{k-1,\tan}$ and $\widehat{u}^{\tan} - w^{\tan} \in \widehat{V}^{k,\tan}$. Nonvanishing natural boundary conditions on $\widehat{u}^{\mathrm{nor}}$ and $\widehat{\rho}^{\mathrm{nor}}$ are even easier: we simply introduce appropriate $\langle \cdot, \cdot \rangle_{\partial\Omega}$ terms on the right-hand sides of (8h)–(8i).

3.3. **The domain-decomposed Hodge–Laplace problem as a saddle point problem.** We now show that the problem (8) can be written as a saddle point problem, thereby relating it to the standard theory of mixed and hybrid finite element methods, cf. Boffi, Brezzi, and Fortin [6].

Observe that solutions to (8) correspond to critical points of the quadratic functional

$$\begin{aligned} J(\sigma, u, \overline{p}, \widehat{u}^{\mathrm{nor}}, \widehat{\rho}^{\mathrm{nor}}, p, \overline{u}, \widehat{\sigma}^{\tan}, \widehat{u}^{\tan}) &\coloneqq -\frac{1}{2}\|\sigma\|_{\mathcal{T}_h}^2 + (\mathrm{d}\sigma, u)_{\mathcal{T}_h} + \frac{1}{2}\|\mathrm{d}u\|_{\mathcal{T}_h}^2 + (u, p)_{\mathcal{T}_h} - (f, u)_{\mathcal{T}_h} \\ &\quad + \langle \widehat{\sigma}^{\tan} - \sigma^{\tan}, \widehat{u}^{\mathrm{nor}} \rangle_{\partial\mathcal{T}_h} + \langle \widehat{u}^{\tan} - u^{\tan}, \widehat{\rho}^{\mathrm{nor}} \rangle_{\partial\mathcal{T}_h} + (u - \overline{u}, \overline{p})_{\mathcal{T}_h}. \end{aligned}$$

To write the problem in another standard form, define the bilinear forms

$$\begin{aligned} a\big((\sigma, u, \overline{p}, \widehat{u}^{\mathrm{nor}}, \widehat{\rho}^{\mathrm{nor}}), (\tau, v, \overline{q}, \widehat{v}^{\mathrm{nor}}, \widehat{\eta}^{\mathrm{nor}})\big) &\coloneqq -(\sigma, \tau)_{\mathcal{T}_h} + (u, \mathrm{d}\tau)_{\mathcal{T}_h} - \langle \widehat{u}^{\mathrm{nor}}, \tau^{\tan} \rangle_{\partial\mathcal{T}_h} \\ &\quad + (\mathrm{d}\sigma, v)_{\mathcal{T}_h} + (\mathrm{d}u, \mathrm{d}v)_{\mathcal{T}_h} + (\overline{p}, v)_{\mathcal{T}_h} - \langle \widehat{\rho}^{\mathrm{nor}}, v^{\tan} \rangle_{\partial\mathcal{T}_h} \\ &\quad + (u, \overline{q})_{\mathcal{T}_h} - \langle \sigma^{\tan}, \widehat{v}^{\mathrm{nor}} \rangle_{\partial\mathcal{T}_h} - \langle u^{\tan}, \widehat{\rho}^{\mathrm{nor}} \rangle_{\partial\mathcal{T}_h}, \\ b\big((\sigma, u, \overline{p}, \widehat{u}^{\mathrm{nor}}, \widehat{\rho}^{\mathrm{nor}}), (q, \overline{v}, \widehat{\tau}^{\tan}, \widehat{v}^{\tan})\big) &\coloneqq (u, q)_{\mathcal{T}_h} - (\overline{p}, \overline{v})_{\mathcal{T}_h} + \langle \widehat{u}^{\mathrm{nor}}, \widehat{\tau}^{\tan} \rangle_{\partial\mathcal{T}_h} + \langle \widehat{\rho}^{\mathrm{nor}}, \widehat{v}^{\tan} \rangle_{\partial\mathcal{T}_h}, \end{aligned}$$

where we have chosen the signs so that $a(\cdot, \cdot)$ is symmetric. Then (8) becomes

$$\begin{aligned} a\big((\sigma, u, \overline{p}, \widehat{u}^{\mathrm{nor}}, \widehat{\rho}^{\mathrm{nor}}), (\tau, v, \overline{q}, \widehat{v}^{\mathrm{nor}}, \widehat{\eta}^{\mathrm{nor}})\big) + b\big((\tau, v, \overline{q}, \widehat{v}^{\mathrm{nor}}, \widehat{\eta}^{\mathrm{nor}}), (p, \overline{u}, \widehat{\sigma}^{\tan}, \widehat{u}^{\tan})\big) &= (f, v)_{\mathcal{T}_h}, \\ b\big((\sigma, u, \overline{p}, \widehat{u}^{\mathrm{nor}}, \widehat{\rho}^{\mathrm{nor}}), (q, \overline{v}, \widehat{\tau}^{\tan}, \widehat{v}^{\tan})\big) &= 0, \end{aligned}$$

which has the abstract form of a saddle point problem,

$$\begin{aligned} a(x, x') + b(x', y) &= F(x'), \quad \forall x' \in X, \\ b(x, y') &= G(y'), \quad \forall y' \in Y. \end{aligned}$$

Here, X is the space of local variables and Y is the space of global variables, so $a(\cdot,\cdot)$ corresponds to the local solvers and $b(\cdot,\cdot)$ to the coupling between local and global variables. This form of the problem will be useful for describing the procedure of static condensation in Section 4.2.

3.4. **Simplified cases.** In the general setting, the domain-decomposed variational problem (8) contains as many as 9 fields. However, there are several important cases where the problem simplifies substantially, as some of the fields become trivial or may be eliminated.

3.4.1. *The case $k = 0$.* In this case, recall that the Hodge–Laplace problem corresponds to the scalar Poisson equation with Neumann boundary conditions. The local harmonic forms are trivial, as are the spaces of $(k-1)$-forms, and the global harmonic forms are constant on each connected component. Identifying 0- and 1-forms with their scalar and vector proxies, (8) reduces to

$$\begin{aligned}
(\operatorname{grad} u, \operatorname{grad} v)_{\mathcal{T}_h} + (p, v)_{\mathcal{T}_h} - \langle \widehat{\rho}^{\mathrm{nor}}, v \rangle_{\partial\mathcal{T}_h} &= (f, v)_{\mathcal{T}_h}, && \forall v \in H^1(\mathcal{T}_h),\\
\langle \widehat{u}^{\tan} - u, \widehat{\eta}^{\mathrm{nor}} \rangle_{\partial\mathcal{T}_h} &= 0, && \forall \widehat{\eta}^{\mathrm{nor}} \in H^{-1/2}(\partial\mathcal{T}_h),\\
(u, q)_{\mathcal{T}_h} &= 0, && \forall q \in \mathfrak{H}^0,\\
\langle \widehat{\rho}^{\mathrm{nor}}, \widehat{v}^{\tan} \rangle_{\partial\mathcal{T}_h} &= 0, && \forall \widehat{v}^{\tan} \in H^1(\Omega)|_{\partial\mathcal{T}_h},
\end{aligned}$$

If we instead impose Dirichlet boundary conditions, then there are no global harmonic forms, so

$$\begin{aligned}
(\operatorname{grad} u, \operatorname{grad} v)_{\mathcal{T}_h} - \langle \widehat{\rho}^{\mathrm{nor}}, v \rangle_{\partial\mathcal{T}_h} &= (f, v)_{\mathcal{T}_h}, && \forall v \in H^1(\mathcal{T}_h),\\
\langle \widehat{u}^{\tan} - u, \widehat{\eta}^{\mathrm{nor}} \rangle_{\partial\mathcal{T}_h} &= 0, && \forall \widehat{\eta}^{\mathrm{nor}} \in H^{-1/2}(\partial\mathcal{T}_h),\\
\langle \widehat{\rho}^{\mathrm{nor}}, \widehat{v}^{\tan} \rangle_{\partial\mathcal{T}_h} &= 0, && \forall \widehat{v}^{\tan} \in \mathring{H}^1(\Omega)|_{\partial\mathcal{T}_h},
\end{aligned}$$

which is the *three-field domain decomposition method* of Brezzi and Marini [9]. Theorem 1 of [9] becomes a special case of Theorem 3.3: in particular, u satisfies Poisson's equation, $\widehat{u}^{\tan} = u|_{\partial\mathcal{T}_h}$, and $\widehat{\rho}^{\mathrm{nor}} = \operatorname{grad} u \cdot \mathbf{n}$. This domain decomposition forms the foundation for the hybridized continuous Galerkin method of Cockburn, Gopalakrishnan, and Wang [19].

3.4.2. *The case $k < n$ with contractible subdomains.* If the subdomains $K \in \mathcal{T}_h$ are contractible (e.g., simplices in a triangulation), then the local harmonic k-forms are trivial for $k < n$. In this case, (8) reduces to

$$\begin{aligned}
(\sigma, \tau)_{\mathcal{T}_h} - (u, \mathrm{d}\tau)_{\mathcal{T}_h} + \langle \widehat{u}^{\mathrm{nor}}, \tau^{\tan} \rangle_{\partial\mathcal{T}_h} &= 0, && \forall \tau \in W^{k-1},\\
(\mathrm{d}\sigma, v)_{\mathcal{T}_h} + (\mathrm{d}u, \mathrm{d}v)_{\mathcal{T}_h} + (p, v) - \langle \widehat{\rho}^{\mathrm{nor}}, v^{\tan} \rangle_{\partial\mathcal{T}_h} &= (f, v)_{\mathcal{T}_h}, && \forall v \in W^k,\\
\langle \widehat{\sigma}^{\tan} - \sigma^{\tan}, \widehat{v}^{\mathrm{nor}} \rangle_{\partial\mathcal{T}_h} &= 0, && \forall \widehat{v}^{\mathrm{nor}} \in \widehat{W}^{k-1,\mathrm{nor}},\\
\langle \widehat{u}^{\tan} - u^{\tan}, \widehat{\eta}^{\mathrm{nor}} \rangle_{\partial\mathcal{T}_h} &= 0, && \forall \widehat{\eta}^{\mathrm{nor}} \in \widehat{W}^{k,\mathrm{nor}},\\
(u, q)_{\mathcal{T}_h} &= 0, && \forall q \in \mathfrak{H}^k,\\
\langle \widehat{u}^{\mathrm{nor}}, \widehat{\tau}^{\tan} \rangle_{\partial\mathcal{T}_h} &= 0, && \forall \widehat{\tau}^{\tan} \in \widehat{V}^{k-1,\tan},\\
\langle \widehat{\rho}^{\mathrm{nor}}, \widehat{v}^{\tan} \rangle_{\partial\mathcal{T}_h} &= 0, && \forall \widehat{v}^{\tan} \in \widehat{V}^{k,\tan},
\end{aligned}$$

so that the local solvers are equivalent to (6). If $\mathfrak{H}^k$ is also trivial (e.g., the domain Ω is contractible), then we obtain a six-field domain decomposition method: there are twice as many fields as in the three-field decomposition, since there are nontrivial $(k-1)$-forms as well as k-forms.

3.4.3. *The case $k = n$ with connected subdomains.* In this case, recall that the Hodge–Laplace problem corresponds to the scalar Poisson equation with Dirichlet boundary conditions. If the subdomains $K \in \mathcal{T}_h$ are connected (again, as with simplices in a triangulation), then $\mathring{\mathfrak{H}}^n(K) \cong \mathbb{R}$, so $\overline{\mathfrak{H}}^n \cong \mathbb{R}^{\mathcal{T}_h}$ consists of piecewise constants. In terms of scalar and vector proxies, (8) becomes

$$
\begin{aligned}
(\sigma, \tau)_{\mathcal{T}_h} - (u, \operatorname{div} \tau)_{\mathcal{T}_h} + \langle \widehat{u}^{\mathrm{nor}}, \tau \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \tau \in H(\operatorname{div}; \mathcal{T}_h),\\
(\operatorname{div} \sigma, v)_{\mathcal{T}_h} + (\overline{p}, v)_{\mathcal{T}_h} &= (f, v)_{\mathcal{T}_h}, && \forall v \in L^2(\mathcal{T}_h),\\
(\overline{u} - u, \overline{q})_{\mathcal{T}_h} &= 0, && \forall \overline{q} \in \mathbb{R}^{\mathcal{T}_h},\\
\langle \widehat{\sigma}^{\tan} - \sigma, \widehat{v}^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{v}^{\mathrm{nor}} \in H^1(\mathcal{T}_h)\mathbf{n},\\
(\overline{p}, \overline{v})_{\mathcal{T}_h} &= 0, && \forall \overline{v} \in \mathbb{R}^{\mathcal{T}_h},\\
\langle \widehat{u}^{\mathrm{nor}}, \widehat{\tau}^{\tan} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{\tau}^{\tan} \in \big(H(\operatorname{div}; \Omega) \cdot \mathbf{n}\big)\mathbf{n}.
\end{aligned}
$$

As we now show, this is equivalent to a domain decomposition of the mixed formulation for Poisson's equation, presented in Cockburn [15, Section 5], which uses local Neumann solvers. Observe that the local solvers (7) can be written as

$$(\sigma, \tau)_K - (u, \operatorname{div} \tau)_K = 0, \qquad \forall \tau \in \mathring{H}(\operatorname{div}; K), \tag{9a}$$

$$(\operatorname{div} \sigma, v)_K + (\overline{p}, v)_K = (f, v)_K, \quad \forall v \in L^2(K), \tag{9b}$$

$$(u, \overline{q})_K = (\overline{u}, \overline{q})_K, \quad \forall \overline{q} \in \mathbb{R}, \tag{9c}$$

subject to the Neumann conditions $\sigma \cdot \mathbf{n} = \widehat{\sigma}^{\tan} \cdot \mathbf{n}$ on ∂K. Equation (9c) says that $(u, 1)_K = (\overline{u}, 1)_K$, i.e., $\overline{u}|_K$ is the average of u over K. Next, taking $v = 1$ in (9b) implies

$$\langle \widehat{\sigma}^{\tan}, \mathbf{n} \rangle_{\partial K} + (\overline{p}, 1)_K = (f, 1)_K,$$

and hence (9b) can be rewritten as

$$\operatorname{div} \sigma = f + \big[\langle \widehat{\sigma}^{\tan}, \mathbf{n} \rangle_{\partial K} - (f, 1)_K\big]/|K| \quad \text{on } K,$$

which is precisely the description of the local solvers in Cockburn [15, Section 5].

3.5. **Remarks on an alternative domain decomposition.** For the mixed scalar Poisson equation, Arnold and Brezzi [3] give an alternative domain decomposition that uses natural Dirichlet conditions rather than essential Neumann conditions for the local solvers when $k = n$. We briefly present a generalization of this approach to the Hodge–Laplace problem and explain why it will *not* give us a hybridization of FEEC when $k < n$.

Define the spaces of unbroken normal traces $\widehat{V}^{k,\mathrm{nor}} \coloneqq \{\eta^{\mathrm{nor}} : \eta \in \mathring{H}^* \Lambda^{k+1}(\Omega)\}$, and observe that (8h)–(8i) imply that $\widehat{u}^{\mathrm{nor}} \in \widehat{V}^{k-1,\mathrm{nor}}$ and $\widehat{\rho}^{\mathrm{nor}} \in \widehat{V}^{k,\mathrm{nor}}$, by Proposition 3.1. Likewise, taking the test functions $\widehat{v}^{\mathrm{nor}}$ and $\widehat{\eta}^{\mathrm{nor}}$ from these spaces eliminates $\widehat{\sigma}^{\tan}$ and $\widehat{u}^{\tan}$ from (8d)–(8e). Since the local solvers now have natural boundary conditions, the appropriate local harmonic space is $\widetilde{\mathfrak{H}}^k \coloneqq \prod_{K \in \mathcal{T}_h} \mathfrak{H}^k(K)$. Hence, (8) simplifies to

$$(\sigma, \tau)_{\mathcal{T}_h} - (u, \mathrm{d}\tau)_{\mathcal{T}_h} + \langle \widehat{u}^{\mathrm{nor}}, \tau^{\tan} \rangle_{\partial \mathcal{T}_h} = 0, \qquad \forall \tau \in W^{k-1}, \tag{10a}$$

$$(\mathrm{d}\sigma, v)_{\mathcal{T}_h} + (\mathrm{d}u, \mathrm{d}v)_{\mathcal{T}_h} + (\overline{p} + p, v)_{\mathcal{T}_h} - \langle \widehat{\rho}^{\mathrm{nor}}, v^{\tan} \rangle_{\partial \mathcal{T}_h} = (f, v)_{\mathcal{T}_h}, \qquad \forall v \in W^k, \tag{10b}$$

$$(\overline{u} - u, \overline{q})_{\mathcal{T}_h} = 0, \qquad \forall \overline{q} \in \widetilde{\mathfrak{H}}^k, \tag{10c}$$

$$\langle \sigma^{\tan}, \widehat{v}^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} = 0, \qquad \forall \widehat{v}^{\mathrm{nor}} \in \widehat{V}^{k-1,\mathrm{nor}}, \tag{10d}$$

$$\langle u^{\tan}, \widehat{\eta}^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} = 0, \qquad \forall \widehat{\eta}^{\mathrm{nor}} \in \widehat{V}^{k,\mathrm{nor}}, \tag{10e}$$

$$(u, q)_{\mathcal{T}_h} = 0, \qquad \forall q \in \mathfrak{H}^k, \tag{10f}$$

$$(\overline{p}, \overline{v})_{\mathcal{T}_h} = 0, \qquad \forall \overline{v} \in \widetilde{\mathfrak{H}}^k, \tag{10g}$$

which still has four global variables but only three local variables instead of five. In the case $k = n$, the local harmonic forms with Dirichlet boundary conditions are trivial, so this becomes

$$\begin{aligned} (\sigma, \tau)_{\mathcal{T}_h} - (u, \operatorname{div} \tau)_{\mathcal{T}_h} + \langle \widehat{u}^{\mathrm{nor}}, \tau \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \tau \in H(\operatorname{div}; \mathcal{T}_h), \\ (\operatorname{div} \sigma, v)_{\mathcal{T}_h} &= (f, v)_{\mathcal{T}_h}, && \forall v \in L^2(\mathcal{T}_h), \\ \langle \sigma, \widehat{v}^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{v}^{\mathrm{nor}} \in \mathring{H}^1(\Omega)\mathbf{n}, \end{aligned}$$

which is precisely the domain decomposition used by Arnold and Brezzi [3].

However, when $k < n$, using (10) will generally result in a *nonconforming* finite element method. Finite element discretization of (10d)–(10e) will only give that $\sigma_h^{\tan}$ and $u_h^{\tan}$ are *weakly* single-valued, in a Galerkin sense, so that σ_h and u_h are not necessarily $H\Lambda(\Omega)$-conforming. For example, when $k = 0$ with Dirichlet boundary conditions on $\partial\Omega$, we get

$$\begin{aligned} (\operatorname{grad} u, \operatorname{grad} v)_{\mathcal{T}_h} + (\overline{p}, v)_{\mathcal{T}_h} - \langle \widehat{\rho}^{\mathrm{nor}}, v \rangle_{\partial \mathcal{T}_h} &= (f, v)_{\mathcal{T}_h}, && \forall v \in H^1(\mathcal{T}_h), \\ (\overline{u} - u, \overline{q})_{\mathcal{T}_h} &= 0, && \forall \overline{q} \in \widetilde{\mathfrak{H}}^0, \\ \langle u, \widehat{\eta}^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{\eta}^{\mathrm{nor}} \in H(\operatorname{div}; \Omega) \cdot \mathbf{n}, \\ (\overline{p}, \overline{v})_{\mathcal{T}_h} &= 0, && \forall \overline{v} \in \widetilde{\mathfrak{H}}^0, \end{aligned}$$

where taking finite element subspaces of $H^1(\mathcal{T}_h)$ and $H(\operatorname{div}; \Omega)$ famously results in a primal hybrid nonconforming method, cf. Raviart and Thomas [39].[2]

Therefore, for the purposes of hybridizing the *conforming* methods of FEEC, we return our attention to the domain decomposition (8). Later, in Section 8, we will present a generalization that includes both approaches, as well as the HDG methods of Cockburn, Gopalakrishnan, and Lazarov [18].

4. Hybrid methods and static condensation

In this section, we present a hybridization of the FEEC methods of Section 2.7 for the Hodge–Laplace problem, based on the domain-decomposed variational principle (8). We then perform static condensation of these methods, using the local solvers to efficiently reduce the system to a smaller one involving only the global variables. This condensed system is shown to be as small or smaller than that for standard FEEC without hybridization, and we prove an explicit formula for the number of reduced degrees of freedom. Finally, we prove error estimates for the hybrid variables, which approximate tangential and normal traces.

4.1. **Hybridized FEEC methods.** For each $K \in \mathcal{T}_h$, let $W_h(K) \subset H\Lambda(K)$ be a finite-dimensional subcomplex, so that

$$W_h \coloneqq \prod_{K \in \mathcal{T}_h} W_h(K), \qquad V_h \coloneqq V \cap W_h,$$

are respectively subcomplexes of $W = H\Lambda(\mathcal{T}_h)$ and $V = H\Lambda(\Omega)$.[3] Let $\overline{\mathfrak{H}}_h^k \coloneqq \prod_{K \in \mathcal{T}_h} \mathring{\mathfrak{H}}_h^k(K)$, where $\mathring{\mathfrak{H}}_h^k(K)$ is the space of local harmonic k-forms in $\mathring{W}_h^k(K)$, and let $\mathfrak{H}_h^k$ be the space of global harmonic k-forms in V_h^k. Next, we define broken and unbroken tangential traces,

$$\widehat{W}_h^{k,\tan} \coloneqq \{ v_h^{\tan} : v_h \in W_h^k \}, \qquad \widehat{V}_h^{k,\tan} \coloneqq \{ v_h^{\tan} : v_h \in V_h^k \} = \widehat{V}^{k,\tan} \cap \widehat{W}_h^{k,\tan},$$

and take $\widehat{W}_h^{k,\mathrm{nor}} \coloneqq (\widehat{W}_h^{k,\tan})^*$. Since $\langle \cdot, \cdot \rangle_{\partial \mathcal{T}_h}$ is a duality pairing, we use this same notation for the pairing of $\widehat{W}_h^{k,\tan}$ with its dual space $\widehat{W}_h^{k,\mathrm{nor}}$.

[2] The more familiar expression of this method eliminates $\overline{u}$ and $\overline{p} = 0$. Although this results in a well-posed global system, the local Neumann solvers are then no longer well-posed.

[3] As in Remark 3.4, the arguments readily generalize to $V = \mathring{H}\Lambda(\Omega)$ or other choices of ideal boundary conditions.

Example 4.1 (decomposition of $\mathcal{P}_r^{\pm}\Lambda$ elements)**.** If $\mathcal{T}_h$ is a conforming simplicial mesh and $W_h^k(K) = \mathcal{P}_r^{\pm}\Lambda^k(K)$ for each $K \in \mathcal{T}_h$, then $V_h^k = \mathcal{P}_r^{\pm}\Lambda^k(\mathcal{T}_h)$. Since simplices are contractible, the local harmonic forms are trivial for $k < n$ and piecewise constants for $k = n$, and the global harmonic forms $\mathfrak{H}_h^k$ are as in Section 2.7.

For each $K \in \mathcal{T}_h$, the broken trace space $\widehat{W}_h^{k,\mathrm{tan}}$ contains tangential traces of $\mathcal{P}_r^{\pm}\Lambda^k(K)$, so the degrees of freedom are just those living on ∂K. Since this is a broken space, the degrees of freedom need not match on interior facets $e = \partial K^+ \cap \partial K^-$. By contrast, $\widehat{V}_h^{k,\mathrm{tan}}$ contains tangential traces from the unbroken space $\mathcal{P}_r^{\pm}\Lambda^k(\mathcal{T}_h)$, so the degrees of freedom are single-valued. Finally, we can use duality to identify $\widehat{W}_h^{k,\mathrm{nor}}$ with the degrees of freedom for $\widehat{W}_h^{k,\mathrm{tan}}$. Since these tangential traces are piecewise polynomial and thus in $L^2(\partial\mathcal{T}_h)$, for implementation we may simply take $\widehat{W}_h^{k,\mathrm{nor}} = \widehat{W}_h^{k,\mathrm{tan}}$ where $\langle\cdot,\cdot\rangle_{\partial\mathcal{T}_h}$ is the L^2 inner product.

Now that we have defined these finite-dimensional subspaces, we may consider the following finite-dimensional version of the domain-decomposed variational problem (8): Find

$$\text{(local variables)} \quad \sigma_h \in W_h^{k-1}, \quad u_h \in W_h^k, \quad \overline{p}_h \in \overline{\mathfrak{H}}_h^k, \quad \widehat{u}_h^{\mathrm{nor}} \in \widehat{W}_h^{k-1,\mathrm{nor}}, \quad \widehat{\rho}_h^{\mathrm{nor}} \in \widehat{W}_h^{k,\mathrm{nor}},$$

$$\text{(global variables)} \quad p_h \in \mathfrak{H}_h^k, \quad \overline{u}_h \in \overline{\mathfrak{H}}_h^k, \quad \widehat{\sigma}_h^{\mathrm{tan}} \in \widehat{V}_h^{k-1,\mathrm{tan}}, \quad \widehat{u}_h^{\mathrm{tan}} \in \widehat{V}_h^{k,\mathrm{tan}},$$

satisfying

$$(\sigma_h,\tau_h)_{\mathcal{T}_h} - (u_h,\mathrm{d}\tau_h)_{\mathcal{T}_h} + \langle \widehat{u}_h^{\mathrm{nor}}, \tau_h^{\mathrm{tan}}\rangle_{\partial\mathcal{T}_h} = 0, \quad \forall \tau_h \in W_h^{k-1}, \tag{11a}$$

$$(\mathrm{d}\sigma_h, v_h)_{\mathcal{T}_h} + (\mathrm{d}u_h,\mathrm{d}v_h)_{\mathcal{T}_h} + (\overline{p}_h + p_h, v_h)_{\mathcal{T}_h} - \langle \widehat{\rho}_h^{\mathrm{nor}}, v_h^{\mathrm{tan}}\rangle_{\partial\mathcal{T}_h} = (f, v_h)_{\mathcal{T}_h}, \quad \forall v_h \in W_h^k, \tag{11b}$$

$$(\overline{u}_h - u_h, \overline{q}_h)_{\mathcal{T}_h} = 0, \quad \forall \overline{q}_h \in \overline{\mathfrak{H}}_h^k, \tag{11c}$$

$$\langle \widehat{\sigma}_h^{\mathrm{tan}} - \sigma_h^{\mathrm{tan}}, \widehat{v}_h^{\mathrm{nor}}\rangle_{\partial\mathcal{T}_h} = 0, \quad \forall \widehat{v}_h^{\mathrm{nor}} \in \widehat{W}_h^{k-1,\mathrm{nor}}, \tag{11d}$$

$$\langle \widehat{u}_h^{\mathrm{tan}} - u_h^{\mathrm{tan}}, \widehat{\eta}_h^{\mathrm{nor}}\rangle_{\partial\mathcal{T}_h} = 0, \quad \forall \widehat{\eta}_h^{\mathrm{nor}} \in \widehat{W}_h^{k,\mathrm{nor}}, \tag{11e}$$

$$(u_h, q_h)_{\mathcal{T}_h} = 0, \quad \forall q_h \in \mathfrak{H}_h^k, \tag{11f}$$

$$(\overline{p}_h, \overline{v}_h)_{\mathcal{T}_h} = 0, \quad \forall \overline{v}_h \in \overline{\mathfrak{H}}_h^k, \tag{11g}$$

$$\langle \widehat{u}_h^{\mathrm{nor}}, \widehat{\tau}_h^{\mathrm{tan}}\rangle_{\partial\mathcal{T}_h} = 0, \quad \forall \widehat{\tau}_h^{\mathrm{tan}} \in \widehat{V}_h^{k-1,\mathrm{tan}}, \tag{11h}$$

$$\langle \widehat{\rho}_h^{\mathrm{nor}}, \widehat{v}_h^{\mathrm{tan}}\rangle_{\partial\mathcal{T}_h} = 0, \quad \forall \widehat{v}_h^{\mathrm{tan}} \in \widehat{V}_h^{k,\mathrm{tan}}. \tag{11i}$$

Given values for the global variables, (11a)–(11e) amounts to solving the local FEEC problems

$$(\sigma_h,\tau_h)_K - (u_h,\mathrm{d}\tau_h)_K = 0, \quad \forall \tau_h \in \mathring{W}_h^{k-1}(K), \tag{12a}$$

$$(\mathrm{d}\sigma_h, v_h)_K + (\mathrm{d}u_h,\mathrm{d}v_h)_K + (\overline{p}_h, v_h)_K = (f - p_h, v_h)_K, \quad \forall v_h \in \mathring{W}_h^k(K), \tag{12b}$$

$$(u_h, \overline{q}_h)_K = (\overline{u}_h, \overline{q}_h)_K, \quad \forall \overline{q}_h \in \mathring{\mathfrak{H}}_h^k(K), \tag{12c}$$

with essential tangential boundary conditions $\sigma_h^{\mathrm{tan}} = \widehat{\sigma}_h^{\mathrm{tan}}$ and $u_h^{\mathrm{tan}} = \widehat{u}_h^{\mathrm{tan}}$.

The following result shows that this is indeed a hybridization of the global FEEC problem (4), which in particular implies well-posedness of (11). The proof is quite similar to Theorem 3.3, but there are two important distinctions. First, $\widehat{u}_h^{\mathrm{nor}}$ and $\widehat{\rho}_h^{\mathrm{nor}}$ generally *do not* equal the normal traces of u_h and $\rho_h = \mathrm{d}u_h$, except weakly, in a Galerkin sense. Furthermore, a crucial role is played by the specific choice of broken tangential and normal trace spaces above, particularly the fact that they are in duality with respect to $\langle\cdot,\cdot\rangle_{\partial\mathcal{T}_h}$.

Theorem 4.2. *The following are equivalent:*

- $(\sigma_h, u_h, \overline{p}_h, \widehat{u}_h^{\mathrm{nor}}, \widehat{\rho}_h^{\mathrm{nor}}, p_h, \overline{u}_h, \widehat{\sigma}_h^{\mathrm{tan}}, \widehat{u}_h^{\mathrm{tan}})$ *is a solution to* (11).
- (σ_h, u_h, p_h) *is a solution to* (4)*, and furthermore,* $\overline{p}_h = 0$*,* $\widehat{u}_h^{\mathrm{nor}}$ *and* $\widehat{\rho}_h^{\mathrm{nor}}$ *are uniquely determined by* (11a)–(11b)*,* $\overline{u}_h$ *is the projection of* u_h *onto* $\overline{\mathfrak{H}}_h^k$*,* $\widehat{\sigma}_h^{\mathrm{tan}} = \sigma_h^{\mathrm{tan}}$*, and* $\widehat{u}_h^{\mathrm{tan}} = u_h^{\mathrm{tan}}$.

Proof. Suppose we have a solution to (11). The claimed equalities are immediate from the variational problem, with uniqueness of the broken tangential and normal traces following from the fact that these spaces are in duality with respect to $\langle \cdot, \cdot \rangle_{\partial \mathcal{T}_h}$, so it remains only to show that (σ_h, u_h, p_h) solves (4). Since $\sigma_h^{\tan} = \widehat{\sigma}_h^{\tan}$ and $u_h^{\tan} = \widehat{u}_h^{\tan}$, Proposition 3.1 implies that $\sigma_h \in V_h^{k-1}$ and $u_h \in V_h^k$. Taking $\tau_h \in V_h^{k-1}$ and $v_h \in V_h^k$ in (11a)–(11b), the normal trace terms vanish by (11h)–(11i), and we obtain (4a)–(4b). Finally, (11f) is the same as (4c), which proves the forward direction.

Conversely, given a solution (σ_h, u_h, p_h) to (4), it is immediate that (11a)–(11g) hold, again using the fact that $\langle \cdot, \cdot \rangle_{\partial \mathcal{T}_h}$ is a dual pairing to get uniqueness of the broken tangential and normal traces. For the remaining two equations, first observe that combining (4a) and (11a) gives $\langle \widehat{u}_h^{\mathrm{nor}}, \tau_h^{\tan} \rangle_{\partial \mathcal{T}_h} = 0$ for $\tau_h \in V_h^{k-1}$, which implies (11h). Similarly, combining (4b) and (11b) gives $\langle \widehat{\rho}_h^{\mathrm{nor}}, v_h^{\tan} \rangle_{\partial \mathcal{T}_h} = 0$ for $v_h \in V_h^k$, which implies (11i). ∎

4.2. **Static condensation.** We next perform static condensation of the hybridized FEEC method (11), eliminating the local variables using the local solvers (12) and thereby obtaining a condensed system involving only the global variables. We present the condensed system both in a matrix-free variational form and as a matrix Schur complement, and we prove that this system is as small or smaller than the standard FEEC method (4) without hybridization.

4.2.1. *Matrix-free variational form of static condensation.* As we did in Section 3.3 for the infinite-dimensional problem, we may write the hybridized FEEC method (11) as a saddle point problem,

$$a(x_h, x_h') + b(x_h', y_h) = F(x_h'), \quad \forall x_h' \in X_h, \tag{13a}$$

$$b(x_h, y_h') = G(y_h'), \quad \forall y_h' \in Y_h. \tag{13b}$$

Since the local FEEC solvers (12) corresponding to $a(\cdot,\cdot)$ are well-posed, for any given F and y_h we can write the solution to (13a) as $x_h = \mathsf{X}_F + \mathsf{X}_{y_h}$, where

$$a(\mathsf{X}_F, x_h') = F(x_h'), \qquad a(\mathsf{X}_{y_h}, x_h') = -b(x_h', y_h), \qquad \forall x_h' \in X_h.$$

This is an efficient local computation that may be done element-by-element in parallel. Substituting this into (13b) gives a reduced problem involving only the global variables: Find $y_h \in Y_h$ satisfying

$$b(\mathsf{X}_{y_h}, y_n') = G(y_n') - b(\mathsf{X}_F, y_n'), \quad \forall y_h' \in Y_h.$$

This procedure of eliminating variables using local solvers is known as *static condensation.* Once the condensed system has been solved for the global variables, the local variables may be recovered element-by-element, if desired, using the local solvers.

In particular, for the hybridized FEEC method (11), we may use linearity to separate the influence of the individual components, writing $\mathsf{X}_F = \mathsf{X}_f$ and $\mathsf{X}_{y_h} = \mathsf{X}_{p_h} + \mathsf{X}_{\overline{u}_h} + \mathsf{X}_{\widehat{\sigma}_h^{\tan}} + \mathsf{X}_{\widehat{u}_h^{\tan}}$, where $\mathsf{X} = (\mathsf{\Sigma}, \mathsf{U}, \overline{\mathsf{P}}, \widehat{\mathsf{U}}^{\mathrm{nor}}, \widehat{\mathsf{R}}^{\mathrm{nor}})$. The condensed variational problem can then be written explicitly as: Find $p_h \in \mathfrak{H}_h^k$, $\overline{u}_h \in \overline{\mathfrak{H}}_h^k$, $\widehat{\sigma}_h^{\tan} \in \widehat{V}_h^{k-1,\tan}$, $\widehat{u}_h^{\tan} \in \widehat{V}_h^{k,\tan}$ satisfying

$$(\mathsf{U}_{p_h} + \mathsf{U}_{\overline{u}_h} + \mathsf{U}_{\widehat{\sigma}_h^{\tan}} + \mathsf{U}_{\widehat{u}_h^{\tan}}, q_h)_{\mathcal{T}_h} = -(\mathsf{U}_f, q_h)_{\mathcal{T}_h}, \qquad \forall q_h \in \mathfrak{H}_h^k, \tag{14a}$$

$$(\overline{\mathsf{P}}_{p_h} + \overline{\mathsf{P}}_{\overline{u}_h} + \overline{\mathsf{P}}_{\widehat{\sigma}_h^{\tan}} + \overline{\mathsf{P}}_{\widehat{u}_h^{\tan}}, \overline{v}_h)_{\mathcal{T}_h} = -(\overline{\mathsf{P}}_f, \overline{v}_h)_{\mathcal{T}_h}, \qquad \forall \overline{v}_h \in \overline{\mathfrak{H}}_h^k, \tag{14b}$$

$$\langle \widehat{\mathsf{U}}_{p_h}^{\mathrm{nor}} + \widehat{\mathsf{U}}_{\overline{u}_h}^{\mathrm{nor}} + \widehat{\mathsf{U}}_{\widehat{\sigma}_h^{\tan}}^{\mathrm{nor}} + \widehat{\mathsf{U}}_{\widehat{u}_h^{\tan}}^{\mathrm{nor}}, \widehat{\tau}_h^{\tan} \rangle_{\partial \mathcal{T}_h} = -\langle \widehat{\mathsf{U}}_f^{\mathrm{nor}}, \widehat{\tau}_h^{\tan} \rangle_{\partial \mathcal{T}_h}, \quad \forall \widehat{\tau}_h^{\tan} \in \widehat{V}_h^{k-1,\tan}, \tag{14c}$$

$$\langle \widehat{\mathsf{R}}_{p_h}^{\mathrm{nor}} + \widehat{\mathsf{R}}_{\overline{u}_h}^{\mathrm{nor}} + \widehat{\mathsf{R}}_{\widehat{\sigma}_h^{\tan}}^{\mathrm{nor}} + \widehat{\mathsf{R}}_{\widehat{u}_h^{\tan}}^{\mathrm{nor}}, \widehat{v}_h^{\tan} \rangle_{\partial \mathcal{T}_h} = -\langle \widehat{\mathsf{R}}_f^{\mathrm{nor}}, \widehat{v}_h^{\tan} \rangle_{\partial \mathcal{T}_h}, \quad \forall \widehat{v}_h^{\tan} \in \widehat{V}_h^{k,\tan}. \tag{14d}$$

4.2.2. *The condensed stiffness matrix as a Schur complement.* Given a finite element basis, the saddle point problem (13) can be written in the block-matrix form

$$\begin{bmatrix} A & B^T \\ B & \end{bmatrix} \begin{bmatrix} x_h \\ y_h \end{bmatrix} = \begin{bmatrix} F_h \\ G_h \end{bmatrix}.$$

Since the matrix A corresponds to the local solvers (12), it has a block-diagonal structure, with blocks corresponding to each $K \in \mathcal{T}_h$, and can therefore be inverted efficiently block-by-block. Given F and y_h, we can locally solve

$$A\mathsf{X}_F = F_h, \qquad A\mathsf{X}_{y_h} = -B^T y_h \quad \Longrightarrow \quad x_h = \mathsf{X}_F + \mathsf{X}_{y_h} = A^{-1}F_h - A^{-1}B^T y_h.$$

Substituting this expression into $Bx_h = G_h$ gives the condensed system

$$-BA^{-1}B^T y_h = G_h - BA^{-1}F_h,$$

where $-BA^{-1}B^T$ is the *Schur complement* of the original stiffness matrix $\left[\begin{smallmatrix} A & B^T \\ B & \end{smallmatrix}\right]$. To separate the influence of individual components of the local FEEC solvers, we can write

$$\begin{bmatrix} \Sigma_f \\ \mathsf{U}_f \\ \overline{\mathsf{P}}_f \\ \widehat{\mathsf{U}}_f^{\mathrm{nor}} \\ \widehat{\mathsf{R}}_f^{\mathrm{nor}} \end{bmatrix} = A^{-1} \begin{bmatrix} 0 \\ f_h \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} \Sigma_{p_h} \\ \mathsf{U}_{p_h} \\ \overline{\mathsf{P}}_{p_h} \\ \widehat{\mathsf{U}}_{p_h}^{\mathrm{nor}} \\ \widehat{\mathsf{R}}_{p_h}^{\mathrm{nor}} \end{bmatrix} = -A^{-1}B^T \begin{bmatrix} p_h \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} \Sigma_{\overline{u}_h} \\ \mathsf{U}_{\overline{u}_h} \\ \overline{\mathsf{P}}_{\overline{u}_h} \\ \widehat{\mathsf{U}}_{\overline{u}_h}^{\mathrm{nor}} \\ \widehat{\mathsf{R}}_{\overline{u}_h}^{\mathrm{nor}} \end{bmatrix} = -A^{-1}B^T \begin{bmatrix} 0 \\ \overline{u}_h \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} \Sigma_{\widehat{\sigma}_h^{\tan}} \\ \mathsf{U}_{\widehat{\sigma}_h^{\tan}} \\ \overline{\mathsf{P}}_{\widehat{\sigma}_h^{\tan}} \\ \widehat{\mathsf{U}}_{\widehat{\sigma}_h^{\tan}}^{\mathrm{nor}} \\ \widehat{\mathsf{R}}_{\widehat{\sigma}_h^{\tan}}^{\mathrm{nor}} \end{bmatrix} = -A^{-1}B^T \begin{bmatrix} 0 \\ 0 \\ \widehat{\sigma}_h^{\tan} \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} \Sigma_{\widehat{u}_h^{\tan}} \\ \mathsf{U}_{\widehat{u}_h^{\tan}} \\ \overline{\mathsf{P}}_{\widehat{u}_h^{\tan}} \\ \widehat{\mathsf{U}}_{\widehat{u}_h^{\tan}}^{\mathrm{nor}} \\ \widehat{\mathsf{R}}_{\widehat{u}_h^{\tan}}^{\mathrm{nor}} \end{bmatrix} = -A^{-1}B^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ \widehat{u}_h^{\tan} \end{bmatrix},$$

so that the condensed system corresponds to (14), and in particular, the Schur complement $-BA^{-1}B^T$ is precisely the stiffness matrix for (14).

Remark 4.3. The classical static condensation technique of Guyan [25] did not use hybridization, but simply partitioned the matrix system into blocks corresponding to internal and facet degrees of freedom, then applied the Schur complement approach above to eliminate the interior degrees of freedom. A similar approach has been applied to edge elements for Maxwell's equations, as discussed in the survey by Ledger and Morgan [29, Section 4.5]. The discovery of the relationship between Guyan's static condensation and hybridization is more recent, cf. Cockburn [15].

4.2.3. *Reduced degrees of freedom.* The next result proves that in full generality—without assumptions on the topology of $K \in \mathcal{T}_h$ or the elements used—the condensed system (14) is as small or smaller than the standard FEEC system (4) without hybridization. Since the space $\mathfrak{H}_h^k$ appears in both systems, it suffices to compare $\dim \overline{\mathfrak{H}}_h^k + \dim \widehat{V}_h^{k-1,\tan} + \dim \widehat{V}_h^{k,\tan}$ (condensed) with $\dim V_h^{k-1} + \dim V_h^k$ (standard FEEC).

Theorem 4.4. *We have the equality*

$$\begin{aligned} &(\dim V_h^{k-1} + \dim V_h^k) - (\dim \overline{\mathfrak{H}}_h^k + \dim \widehat{V}_h^{k-1,\tan} + \dim \widehat{V}_h^{k,\tan}) \\ &\qquad = \sum_{K \in \mathcal{T}_h} \bigl(\dim \mathring{W}_h^{k-1}(K) + \dim \mathring{\mathfrak{B}}_h^k(K) + \dim \mathring{\mathfrak{Z}}_h^{k\perp}(K) \bigr). \end{aligned} \tag{15}$$

Consequently, the size of the hybridized and condensed FEEC system (14) *is always less than or equal to that of the standard FEEC system* (4), *with equality if and only if* $\mathring{W}_h^{k-1}(K)$ *is trivial and* $\mathring{W}_h^k(K) = \mathring{\mathfrak{H}}_h^k(K)$ *for all* $K \in \mathcal{T}_h$.

Proof. By definition, $\widehat{V}_h^{k,\tan}$ is the image of V_h^k under the tangential trace map. Therefore, the rank-nullity theorem implies that their dimensions differ by the dimension of the kernel, i.e.,

$$\dim V_h^k - \dim \widehat{V}_h^{k,\tan} = \dim\{v_h \in V_h^k : v_h^{\tan} = 0\} = \dim \prod_{K \in \mathcal{T}_h} \mathring{W}_h^k(K) = \sum_{K \in \mathcal{T}_h} \dim \mathring{W}_h^k(K).$$

Applying the discrete Hodge decomposition to each $\mathring{W}_h^k(K)$ and using $\overline{\mathfrak{H}}_h^k \coloneqq \prod_{K \in \mathcal{T}_h} \mathring{\mathfrak{H}}_h^k(K)$ gives

$$\sum_{K \in \mathcal{T}_h} \dim \mathring{W}_h^k(K) = \dim \overline{\mathfrak{H}}_h^k + \sum_{K \in \mathcal{T}_h} \bigl(\dim \mathring{\mathfrak{B}}_h^k(K) + \dim \mathring{\mathfrak{Z}}_h^{k\perp}(K) \bigr).$$

Combining this with the previous expression and the corresponding one for $\dim V_h^{k-1} - \dim \widehat{V}_h^{k-1,\tan}$ implies (15), which completes the proof. $\square$

We now give an explicit count of the reduced degrees of freedom when $\mathcal{T}_h$ is a simplicial mesh and $\mathcal{P}_r^{\pm}\Lambda$ elements are used. Arnold, Falk, and Winther [4, Sections 4.5–4.6] show that for $r \geq 1$,

$$\dim \mathring{\mathcal{P}}_r \Lambda^k(K) = \binom{r-1}{n-k}\binom{r+k}{k}, \qquad \dim \mathring{\mathcal{P}}_r^- \Lambda^k(K) = \binom{n}{k}\binom{r+k-1}{n},$$

with the convention that $\binom{a}{b} = 0$ when $b < 0$ or $b > a$. Applying these formulas to the stable pairs of spaces for FEEC given in (5), we get

$$\dim \mathring{\mathcal{P}}_{r+1} \Lambda^{k-1}(K) = \binom{r}{n-k+1}\binom{r+k}{k-1}, \qquad \dim \mathring{\mathcal{P}}_r \Lambda^k(K) = \binom{r-1}{n-k}\binom{r+k}{k} \quad (\text{if } r \geq 1),$$
$$\dim \mathring{\mathcal{P}}_{r+1}^- \Lambda^{k-1}(K) = \binom{n}{k-1}\binom{r+k-1}{n}, \quad \dim \mathring{\mathcal{P}}_{r+1}^- \Lambda^k(K) = \binom{n}{k}\binom{r+k}{n}.$$

For each $K \in \mathcal{T}_h$, these formulas count the number of internal degrees of freedom, which are precisely the ones eliminated by static condensation.

Since simplices are contractible, the local harmonic spaces are trivial, except for $\mathring{\mathfrak{H}}_h^n(K) \cong \mathbb{R}$. When $k = n$, static condensation *introduces* one global degree of freedom per simplex, so in this case, the number of degrees of freedom is reduced if and only if $r \geq 1$. When $r = 0$ (i.e., the lowest-order RT and BDM methods), the degrees of freedom for u_h are simply replaced by those for $\overline{u}_h$.

By checking when the spaces above have dimension greater than zero, we immediately obtain the following corollary to Theorem 4.4.

Corollary 4.5. *Let $\mathcal{T}_h$ be a simplicial mesh and V_h^{k-1}, V_h^k be one of the stable pairs in* (5)*. The hybridized and condensed FEEC system* (14) *is strictly smaller than the standard FEEC system* (4) *if and only if $r \geq 1$ and either*

- $V_h^k = \mathcal{P}_r \Lambda^k(\mathcal{T}_h)$ *with $r \geq n-k+1$, or*
- $V_h^k = \mathcal{P}_{r+1}^- \Lambda^k(\mathcal{T}_h)$ *with $r \geq n-k$.*

4.3. **Error estimates for the hybrid variables.** Let $\{\mathcal{T}_h\}$ be a shape-regular (but not necessarily quasi-uniform) family of simplicial meshes of Ω, where h_K denotes the diameter of $K \in \mathcal{T}_h$ and $h \coloneqq \max_{K \in \mathcal{T}_h} h_K$. We assume again that V_h^{k-1}, V_h^k is one of the stable pairs (5).

Error estimates are already known for σ, u, p (Arnold, Falk, and Winther [4, 5]), and for $\overline{u}$ when $k = n$ (Douglas and Roberts [20], Brezzi, Douglas, and Marini [8]), so it only remains to prove estimates for the tangential and normal traces. As in [5], we assume that the exact solution satisfies an elliptic regularity estimate of the form

$$\lVert u \rVert_{t+2,\Omega} + \lVert p \rVert_{t+2,\Omega} + \lVert \mathrm{d}u \rVert_{t+1,\Omega} + \lVert \sigma \rVert_{t+1,\Omega} + \lVert \mathrm{d}\sigma \rVert_{t,\Omega} \lesssim \lVert f \rVert_{t,\Omega},$$

for $0 \leq t \leq t_{\max}$, where $\lVert\cdot\rVert_{t,\Omega}$ denotes the H^t norm on Ω. In particular, this means that the traces are in L^2 for each $K \in \mathcal{T}_h$, so the duality pairing $\langle \cdot, \cdot \rangle_{\partial K}$ agrees with the L^2 inner product, and we can estimate the errors using $\|\cdot\|_{\partial K}$ rather than a weaker norm. For convenience, we define the scaled norms $|||\cdot|||_{\partial K} \coloneqq h_K^{1/2} \|\cdot\|_{\partial K}$ and $|||\cdot|||_{\partial \mathcal{T}_h} \coloneqq \bigl(\sum_{K \in \mathcal{T}_h} |||\cdot|||_{\partial K}^2 \bigr)^{1/2}$.

The tangential traces are fairly straightforward, since $\widehat{\sigma}_h^{\tan} = \sigma_h^{\tan}$ and $\widehat{u}_h^{\tan} = u_h^{\tan}$. The next proposition shows that these converge to $\sigma^{\tan}$ and $u^{\tan}$ with the same order as $\sigma_h \to \sigma$ and $u_h \to u$ in L^2, which is the optimal order allowed by elliptic regularity and the polynomial degree.

Theorem 4.6. *For each $K \in \mathcal{T}_h$ and $0 \leq s \leq t_{\max}$, we have*

$$|||\sigma^{\tan} - \sigma_h^{\tan}|||_{\partial K} \lesssim \|\sigma - \sigma_h\|_K + h_K^{s+1} |\sigma|_{s+1,K}, \text{ if } \begin{cases} s \leq r+1, & W_h^{k-1}(K) = \mathcal{P}_{r+1}\Lambda^{k-1}(K), \\ s \leq r, & W_h^{k-1}(K) = \mathcal{P}_{r+1}^-\Lambda^{k-1}(K), \end{cases}$$

$$|||u^{\tan} - u_h^{\tan}|||_{\partial K} \lesssim \|u - u_h\|_K + \begin{cases} h_K |u|_{1,K}, & \text{if } W_h^k(K) = \mathcal{P}_1^-\Lambda^k(K), \\ h_K^{s+2} |u|_{s+2,K}, & \text{if } s \leq r-1, \text{ otherwise.} \end{cases}$$

Consequently,

$$|||\sigma^{\tan} - \sigma_h^{\tan}|||_{\partial \mathcal{T}_h} \lesssim \|\sigma - \sigma_h\|_\Omega + h^{s+1} |\sigma|_{s+1,\Omega}, \text{ if } \begin{cases} s \leq r+1, & V_h^{k-1} = \mathcal{P}_{r+1}\Lambda^{k-1}(\mathcal{T}_h), \\ s \leq r, & V_h^{k-1} = \mathcal{P}_{r+1}^-\Lambda^{k-1}(\mathcal{T}_h), \end{cases}$$

$$|||u^{\tan} - u_h^{\tan}|||_{\partial \mathcal{T}_h} \lesssim \|u - u_h\|_\Omega + \begin{cases} h |u|_{1,\Omega}, & \text{if } V_h^k = \mathcal{P}_1^-\Lambda^k(\mathcal{T}_h), \\ h^{s+2} |u|_{s+2,\Omega}, & \text{if } s \leq r-1, \text{ otherwise.} \end{cases}$$

Proof. We begin by writing $|||\sigma^{\tan} - \sigma_h^{\tan}|||_{\partial K} \leq |||\sigma^{\tan} - \tau_h^{\tan}|||_{\partial K} + |||\tau_h^{\tan} - \sigma_h^{\tan}|||_{\partial K}$ for $\tau_h \in W_h^{k-1}(K)$. For the first term, the trace theorem gives

$$|||\sigma^{\tan} - \tau_h^{\tan}|||_{\partial K} \lesssim \|\sigma - \tau_h\|_K + h_K |\sigma - \tau_h|_{1,K}.$$

For the second, a scaling argument gives

$$|||\tau_h^{\tan} - \sigma_h^{\tan}|||_{\partial K} \lesssim \|\tau_h - \sigma_h\|_K \leq \|\sigma - \tau_h\|_K + \|\sigma - \sigma_h\|_K.$$

Combining these and applying the Bramble–Hilbert lemma completes the proof of the first estimate. The corresponding estimate for $|||u^{\tan} - u_h^{\tan}|||_{\partial K}$ is essentially identical, and the $|||\cdot|||_{\partial \mathcal{T}_h}$ estimates follow immediately from the $|||\cdot|||_{\partial K}$ estimates. ☐

Remark 4.7. Given sufficient elliptic regularity, combining these estimates with those of Arnold, Falk, and Winther [5] gives

$$\Bigl(\|\sigma - \sigma_h\|_\Omega^2 + |||\sigma^{\tan} - \sigma_h^{\tan}|||_{\partial \mathcal{T}_h}^2 \Bigr)^{1/2} \lesssim \begin{cases} h^{r+2} \|f\|_{r+1,\Omega}, & \text{if } V_h^{k-1} = \mathcal{P}_{r+1}\Lambda^{k-1}(\mathcal{T}_h), \\ h^{r+1} \|f\|_{r,\Omega}, & \text{if } V_h^{k-1} = \mathcal{P}_{r+1}^-\Lambda^{k-1}(\mathcal{T}_h), \end{cases}$$

$$\Bigl(\|u - u_h\|_\Omega^2 + |||u^{\tan} - u_h^{\tan}|||_{\partial \mathcal{T}_h}^2 \Bigr)^{1/2} \lesssim \begin{cases} h \|f\|_\Omega, & \text{if } V_h^k = \mathcal{P}_1^-\Lambda^k(\mathcal{T}_h), \\ h^{r+1} \|f\|_{r-1,\Omega}, & \text{otherwise,} \end{cases}$$

which can be interpreted as mesh-dependent norm estimates for the standard FEEC method.

We next give estimates for the normal traces, generalizing an argument of Arnold and Brezzi [3] for the hybridized RT method. Recall that $\widehat{u}_h^{\mathrm{nor}} \in (\widehat{W}_h^{k-1,\tan})^*$ and $\widehat{\rho}_h^{\mathrm{nor}} \in (\widehat{W}_h^{k,\tan})^*$, so we compare them to the natural projections $\widehat{P}_h u^{\mathrm{nor}} \in (\widehat{W}_h^{k-1,\tan})^*$ and $\widehat{P}_h \rho^{\mathrm{nor}} \in (\widehat{W}_h^{k,\tan})^*$ defined by

$$\langle \widehat{P}_h u^{\mathrm{nor}}, \widehat{\tau}_h^{\tan} \rangle_{\partial \mathcal{T}_h} = \langle u^{\mathrm{nor}}, \widehat{\tau}_h^{\tan} \rangle_{\partial \mathcal{T}_h}, \quad \forall \widehat{\tau}_h \in \widehat{W}_h^{k-1,\tan},$$
$$\langle \widehat{P}_h \rho^{\mathrm{nor}}, \widehat{v}_h^{\tan} \rangle_{\partial \mathcal{T}_h} = \langle \rho^{\mathrm{nor}}, \widehat{v}_h^{\tan} \rangle_{\partial \mathcal{T}_h}, \quad \forall \widehat{v}_h \in \widehat{W}_h^{k,\tan}.$$

If we simply identify $\widehat{u}_h^{\mathrm{nor}}$ with the corresponding element of $\widehat{W}_h^{k-1,\tan} \subset L^2\Lambda^{k-1}(\partial \mathcal{T}_h)$, we generally *do not* observe convergence to the unprojected u^{nor}, and likewise for $\widehat{\rho}_h^{\mathrm{nor}}$ and ρ^{nor}. The reason is that the identification of $\widehat{u}_h^{\mathrm{nor}}$ with an element of $L^2\Lambda^{k-1}(\partial \mathcal{T}_h)$ is only unique up to the annihilator $(\widehat{W}_h^{k-1,\tan})^\perp$.

Therefore, we should really measure the error in the quotient norm $L^2\Lambda^{k-1}(\partial\mathcal{T}_h)/(\widehat{W}_h^{k-1,\tan})^\perp$, which is equivalent to taking the projections above.

Theorem 4.8. *For each $K \in \mathcal{T}_h$, we have*

$$\begin{aligned}\vert\kern-0.25ex\vert\kern-0.25ex\vert \widehat{P}_h u^{\mathrm{nor}} - \widehat{u}_h^{\mathrm{nor}} \vert\kern-0.25ex\vert\kern-0.25ex\vert_{\partial K} &\lesssim \lVert P_h u - u_h \rVert_K + h_K \lVert \sigma - \sigma_h \rVert_K,\\ \vert\kern-0.25ex\vert\kern-0.25ex\vert \widehat{P}_h \rho^{\mathrm{nor}} - \widehat{\rho}_h^{\mathrm{nor}} \vert\kern-0.25ex\vert\kern-0.25ex\vert_{\partial K} &\lesssim \bigl\lVert P_h \mathrm{d}(u-u_h)\bigr\rVert_K + h_K\Bigl(\bigl\lVert \mathrm{d}(\sigma-\sigma_h)\bigr\rVert_K + \lVert p - p_h\rVert_K\Bigr),\end{aligned}$$

where P_h denotes L^2 projection onto W_h. Consequently,

$$\begin{aligned}\vert\kern-0.25ex\vert\kern-0.25ex\vert \widehat{P}_h u^{\mathrm{nor}} - \widehat{u}_h^{\mathrm{nor}} \vert\kern-0.25ex\vert\kern-0.25ex\vert_{\partial \mathcal{T}_h} &\lesssim \lVert P_h u - u_h \rVert_{\mathcal{T}_h} + h \lVert \sigma - \sigma_h \rVert_\Omega,\\ \vert\kern-0.25ex\vert\kern-0.25ex\vert \widehat{P}_h \rho^{\mathrm{nor}} - \widehat{\rho}_h^{\mathrm{nor}} \vert\kern-0.25ex\vert\kern-0.25ex\vert_{\partial \mathcal{T}_h} &\lesssim \bigl\lVert P_h \mathrm{d}(u-u_h)\bigr\rVert_{\mathcal{T}_h} + h\Bigl(\bigl\lVert \mathrm{d}(\sigma-\sigma_h)\bigr\rVert_\Omega + \lVert p - p_h\rVert_\Omega\Bigr).\end{aligned}$$

Proof. A scaling argument shows that each $\widehat{\tau}_h^{\tan} \in \widehat{W}_h^{k-1,\tan}(\partial K)$ has an extension $\tau_h \in W_h^{k-1}(K)$ with $\tau_h^{\tan} = \widehat{\tau}_h^{\tan}$ such that

$$\lVert \tau_h \rVert_K + h_K \lVert \mathrm{d}\tau_h \rVert_K \lesssim \vert\kern-0.25ex\vert\kern-0.25ex\vert \widehat{\tau}_h^{\tan} \vert\kern-0.25ex\vert\kern-0.25ex\vert_{\partial K}.$$

Therefore, subtracting (11a) from (8a), we get

$$\begin{aligned} h_K \langle \widehat{P}_h u^{\mathrm{nor}} - \widehat{u}_h^{\mathrm{nor}}, \widehat{\tau}_h^{\tan}\rangle_{\partial K} &= h_K \langle u^{\mathrm{nor}} - \widehat{u}_h^{\mathrm{nor}}, \tau_h^{\tan}\rangle_{\partial K}\\ &= h_K\Bigl[-(\sigma-\sigma_h,\tau_h)_K + (u-u_h,\mathrm{d}\tau_h)_K\Bigr]\\ &= h_K\Bigl[-(\sigma-\sigma_h,\tau_h)_K + (P_h u-u_h,\mathrm{d}\tau_h)_K\Bigr]\\ &\le \Bigl(h_K\lVert\sigma-\sigma_h\rVert_K + \lVert P_h u - u_h\rVert_K\Bigr)\Bigl(\lVert\tau_h\rVert_K + h_K\lVert \mathrm{d}\tau_h\rVert_K\Bigr)\\ &\lesssim \Bigl(h_K\lVert\sigma-\sigma_h\rVert_K + \lVert P_h u - u_h\rVert_K\Bigr)\vert\kern-0.25ex\vert\kern-0.25ex\vert \widehat{\tau}_h^{\tan} \vert\kern-0.25ex\vert\kern-0.25ex\vert_{\partial K}.\end{aligned}$$

Since $\langle \cdot,\cdot\rangle_{\partial K}$ agrees with the L^2 inner product,

$$\vert\kern-0.25ex\vert\kern-0.25ex\vert \widehat{P}_h u^{\mathrm{nor}} - \widehat{u}_h^{\mathrm{nor}} \vert\kern-0.25ex\vert\kern-0.25ex\vert_{\partial K} = h_K^{1/2} \sup_{\widehat{\tau}_h^{\tan}\neq 0} \frac{\langle \widehat{P}_h u^{\mathrm{nor}} - \widehat{u}_h^{\mathrm{nor}}, \widehat{\tau}_h^{\tan}\rangle_{\partial K}}{\lVert \widehat{\tau}_h^{\tan}\rVert_{\partial K}} = \sup_{\widehat{\tau}_h^{\tan}\neq 0} \frac{h_K\langle \widehat{P}_h u^{\mathrm{nor}} - \widehat{u}_h^{\mathrm{nor}}, \widehat{\tau}_h^{\tan}\rangle_{\partial K}}{\vert\kern-0.25ex\vert\kern-0.25ex\vert \widehat{\tau}_h^{\tan}\vert\kern-0.25ex\vert\kern-0.25ex\vert_{\partial K}},$$

which completes the proof of the first estimate. The estimate for $\vert\kern-0.25ex\vert\kern-0.25ex\vert \widehat{P}_h \rho^{\mathrm{nor}} - \widehat{\rho}_h^{\mathrm{nor}} \vert\kern-0.25ex\vert\kern-0.25ex\vert_{\partial K}$ is obtained similarly, and the $\vert\kern-0.25ex\vert\kern-0.25ex\vert\cdot\vert\kern-0.25ex\vert\kern-0.25ex\vert_{\partial\mathcal{T}_h}$ estimates again follow immediately from the $\vert\kern-0.25ex\vert\kern-0.25ex\vert\cdot\vert\kern-0.25ex\vert\kern-0.25ex\vert_{\partial K}$ estimates. □

For $k < n$, we generally cannot improve on $\lVert P_h u - u_h\rVert_{\mathcal{T}_h} \le \lVert u - u_h\rVert_\Omega$, so assuming sufficient elliptic regularity and applying the estimates from Arnold, Falk, and Winther [5] gives

$$\vert\kern-0.25ex\vert\kern-0.25ex\vert \widehat{P}_h u^{\mathrm{nor}} - \widehat{u}_h^{\mathrm{nor}} \vert\kern-0.25ex\vert\kern-0.25ex\vert_{\partial \mathcal{T}_h} \lesssim \begin{cases} h\lVert f\rVert_\Omega, & \text{if } V_h^k = \mathcal{P}_1^-\Lambda^k(\mathcal{T}_h),\\ h^{r+1}\lVert f\rVert_{r-1,\Omega}, & \text{otherwise,}\end{cases}$$

i.e., the convergence rate is the same as that for $u_h \to u$. When $k = n$, however, $\lVert P_h u - u_h\rVert_{\mathcal{T}_h}$ famously superconverges for the RT and BDM methods (Douglas and Roberts [20], Arnold and Brezzi [3], Brezzi, Douglas, and Marini [8]). In this case, we recover the superconvergence results of [3, 8] for the Lagrange multipliers:

$$\vert\kern-0.25ex\vert\kern-0.25ex\vert \widehat{P}_h u^{\mathrm{nor}} - \widehat{u}_h^{\mathrm{nor}} \vert\kern-0.25ex\vert\kern-0.25ex\vert_{\partial \mathcal{T}_h} \lesssim \begin{cases} h^2\lVert f\rVert_{1,\Omega}, & \text{if } r = 0,\\ h^{r+3}\lVert f\rVert_{r+1,\Omega}, & \text{if } r \ge 1,\ V_h^{n-1} = \mathcal{P}_{r+1}\Lambda^{n-1}(\mathcal{T}_h),\\ h^{r+2}\lVert f\rVert_{r,\Omega}, & \text{if } r \ge 1,\ V_h^{n-1} = \mathcal{P}_{r+1}^-\Lambda^{n-1}(\mathcal{T}_h).\end{cases}$$

From the perspective of FEEC, this occurs since $W_h^n = V_h^n = \mathfrak{B}_h^n$, so $\lVert P_h u - u_h\rVert_{\mathcal{T}_h} = \bigl\lVert P_{\mathfrak{B}_h}(u-u_h)\bigr\rVert_\Omega$, which superconverges according to [5, Lemma 3.13]. On the other hand, when $k < n$, the error is dominated by the nonvanishing $\mathfrak{Z}_h^{k\perp}$ component [5, Lemma 3.16], so there is no improvement.

Similarly, when $k < n-1$, we generally cannot do better than $\big\lVert P_h \mathrm{d}(u - u_h)\big\rVert_{\mathcal{T}_h} \leq \big\lVert \mathrm{d}(u-u_h)\big\rVert_\Omega$, so assuming sufficient elliptic regularity,

$$\lvert\lvert\lvert \widehat{P}_h \rho^{\mathrm{nor}} - \widehat{\rho}_h^{\mathrm{nor}} \rvert\rvert\rvert_{\partial\mathcal{T}_h} \lesssim \begin{cases} h^{r+1} \lVert f \rVert_{r,\Omega}, & \text{if } V_h^k = \mathcal{P}_{r+1}^- \Lambda^k(\mathcal{T}_h),\\ h^r \lVert f \rVert_{r-1,\Omega}, & \text{if } V_h^k = \mathcal{P}_r \Lambda^k(\mathcal{T}_h),\end{cases}$$

and the convergence rate is the same as that for $\mathrm{d}u_h \to \mathrm{d}u$. However, when $k = n-1$, we obtain superconvergence as a consequence of the following lemma (which holds for all k, not just $k = n-1$).

Lemma 4.9. *The FEEC solution* (4) *satisfies* $\big\lVert P_{\mathfrak{B}_h} \mathrm{d}(u-u_h)\big\rVert_\Omega \lesssim h\Big(\big\lVert \mathrm{d}(\sigma - \sigma_h)\big\rVert_\Omega + \lVert p - p_h \rVert_\Omega\Big)$.

Proof. The argument is similar to [5, Lemma 3.15]. Let $v_h \in \mathfrak{Z}_h^{k\perp}$ be such that $\mathrm{d}v_h = P_{\mathfrak{B}_h}\mathrm{d}(u-u_h)$, and take $v = P_{\mathfrak{Z}^\perp} v_h$. Since v is orthogonal to $\mathrm{d}(\sigma - \sigma_h)$ and $p - p_h$, subtracting (4b) from (3b) gives

$$\begin{aligned}\big\lVert P_{\mathfrak{B}_h}\mathrm{d}(u-u_h)\big\rVert_\Omega^2 &= \big(\mathrm{d}(u-u_h), \mathrm{d}v_h\big)_\Omega\\ &= \big(\mathrm{d}(\sigma-\sigma_h) + (p-p_h), v - v_h\big)_\Omega\\ &\leq \Big(\big\lVert \mathrm{d}(\sigma-\sigma_h)\big\rVert_\Omega + \lVert p - p_h\rVert_\Omega\Big)\lVert v - v_h \rVert_\Omega\\ &\lesssim h\Big(\big\lVert \mathrm{d}(\sigma-\sigma_h)\big\rVert_\Omega + \lVert p - p_h\rVert_\Omega\Big)\big\lVert P_{\mathfrak{B}_h}\mathrm{d}(u-u_h)\big\rVert_\Omega.\end{aligned}$$

The last step uses [5, Lemma 3.12], which says that $\lVert v - v_h\rVert_\Omega \lesssim h \lVert \mathrm{d}v_h\rVert_\Omega$. □

Corollary 4.10. *For $k = n-1$, we have the improved estimate*

$$\lvert\lvert\lvert \widehat{P}_h \rho^{\mathrm{nor}} - \widehat{\rho}_h^{\mathrm{nor}} \rvert\rvert\rvert_{\partial\mathcal{T}_h} \lesssim h\Big(\big\lVert \mathrm{d}(\sigma-\sigma_h)\big\rVert_\Omega + \lVert p - p_h\rVert_\Omega\Big).$$

In particular, when $f \in \mathring{\mathfrak{B}}_{n-1}^*$, we have $\widehat{\rho}_h^{\mathrm{nor}} = \widehat{P}_h \rho^{\mathrm{nor}}$ exactly.

Proof. Since $\big\lVert P_h \mathrm{d}(u-u_h)\big\rVert_{\mathcal{T}_h} = \big\lVert P_{\mathfrak{B}_h}\mathrm{d}(u-u_h)\big\rVert_\Omega$ when $k = n-1$, the improved estimate is immediate from Theorem 4.8 and Lemma 4.9. In particular, σ and p vanish when $f \in \mathring{\mathfrak{B}}_{n-1}^*$, so in that case the left-hand side is identically zero. □

Assuming sufficient elliptic regularity, this gives the superconvergent rates

$$\lvert\lvert\lvert \widehat{P}_h \rho^{\mathrm{nor}} - \widehat{\rho}_h^{\mathrm{nor}} \rvert\rvert\rvert_{\partial\mathcal{T}_h} \lesssim \begin{cases} 0, & \text{if } f \in \mathring{\mathfrak{B}}_{n-1}^*,\\ h^{r+2}\lVert f\rVert_{r+1,\Omega}, & \text{otherwise.}\end{cases}$$

5. Postprocessing

In this section, we introduce a local postprocessing procedure, which generalizes that of Stenberg [40] from $k = n$ to arbitrary k. We develop new error estimates for the postprocessed solution when $k < n$; in particular, postprocessing gives a superconvergent approximation ρ_h^* to $\mathrm{d}u$ for $k = n-1$, and $\delta\rho_h^*$ is an improved approximation to $\delta \mathrm{d}u$ for all k. Finally, we discuss how this analysis corresponds to that of Stenberg [40] in the case $k = n$, giving superconvergence of u_h^* to u.

5.1. **The postprocessing procedure.** To motivate the proposed procedure, recall that the exact local solver (7) corresponds to solving $Lu + \overline{p} = f - p$ such that $P_{\overline{\mathfrak{H}}} u = \overline{u}$, with tangential boundary conditions given by $\widehat{\sigma}^{\mathrm{tan}}$ and $\widehat{u}^{\mathrm{tan}}$. Instead of writing this as a variational problem on the $\mathring{H}\Lambda(K)$ complex, we can equivalently write it on the $H^*\Lambda(K)$ complex as

$$(\rho, \eta)_K - (u, \delta\eta)_K = \langle \widehat{u}^{\mathrm{tan}}, \eta^{\mathrm{nor}}\rangle_{\partial K}, \quad \forall \eta \in H^*\Lambda^{k+1}(K), \tag{16a}$$

$$(\delta\rho, v)_K + (\delta u, \delta v)_K + (\overline{p}, v)_K = (f - p, v)_K - \langle \widehat{\sigma}^{\mathrm{tan}}, v^{\mathrm{nor}}\rangle_{\partial K}, \quad \forall v \in H^*\Lambda^k(K), \tag{16b}$$

$$(u, \overline{q})_K = (\overline{u}, \overline{q})_K, \quad \forall \overline{q} \in \mathring{\mathfrak{H}}^k(K), \tag{16c}$$

where the tangential boundary conditions are now natural rather than essential. As before, we have $\sigma = \delta u$ and $\rho = \mathrm{d}u$

The postprocessing procedure is based on approximating (16) on a finite-dimensional subcomplex $W_h^*(K) \subset H^*\Lambda(K)$, meaning $\delta W_h^{*k+1}(K) \subset W_h^{*k}(K)$. Since $\star H^*\Lambda^k(K) = H\Lambda^{n-k}(K)$, an equivalent condition is that $\star W_h^*(K) \subset H\Lambda(K)$ is a subcomplex. Moreover, $\pi_h \colon H\Lambda(K) \to \star W_h^*(K)$ is a bounded commuting projection if and only if $\star^{-1}\pi_h\star \colon H^*\Lambda(K) \to W_h^*(K)$ is. For a simplicial mesh, we may therefore take

$$\star W_h^{*k+1}(K) = \mathcal{P}_{r^*+1}^{\pm}\Lambda^{n-k-1}(K), \qquad \star W_h^{*k}(K) = \begin{Bmatrix} \mathcal{P}_{r^*}\Lambda^{n-k}(K) \text{ (if } r^* \geq 1) \\ \text{or} \\ \mathcal{P}_{r^*+1}^{-}\Lambda^{n-k}(K) \end{Bmatrix}.$$

This is just the Hodge dual of the stable pairs (5) with k replaced by $n-k$ and r by r^*, so all of the results of Arnold, Falk, and Winther [5] apply immediately to the dual problem. We write the discrete Hodge decomposition for this complex as

$$W_h^{*k}(K) = \mathfrak{B}_h^{*k}(K) \oplus \mathfrak{H}_h^{*k}(K) \oplus \mathfrak{Z}_h^{*k\perp}(K).$$

When K is contractible (e.g., a simplex), we have $\mathfrak{H}_h^{*k}(K) = \mathring{\mathfrak{H}}^k(K)$, which is $\cong \mathbb{R}$ for $k = n$ and trivial otherwise.

We are now ready to define the postprocessing procedure on $K \in \mathcal{T}_h$: Find $\rho_h^* \in W_h^{*k+1}(K)$, $u_h^* \in W_h^{*k}(K)$, $\overline{p}_h^* \in \mathfrak{H}_h^{*k}(K)$ such that

$$(\rho_h^*, \eta_h)_K - (u_h^*, \delta\eta_h)_K = \langle \widehat{u}_h^{\tan}, \eta_h^{\mathrm{nor}} \rangle_{\partial K}, \qquad \forall \eta_h \in W_h^{*k+1}(K), \tag{17a}$$

$$(\delta\rho_h^*, v_h)_K + (\delta u_h^*, \delta v_h)_K + (\overline{p}_h^*, v_h)_K = (f - p_h, v_h)_K - \langle \widehat{\sigma}_h^{\tan}, v_h^{\mathrm{nor}} \rangle_{\partial K}, \quad \forall v_h \in W_h^{*k}(K), \tag{17b}$$

$$(u_h^*, \overline{q}_h)_K = (\overline{u}_h, \overline{q}_h)_K, \qquad \forall \overline{q}_h \in \mathfrak{H}_h^{*k}(K). \tag{17c}$$

Remark 5.1. The right-hand side only depends on the global variables p_h, $\overline{u}_h$, $\widehat{\sigma}_h^{\tan}$, $\widehat{u}_h^{\tan}$. Therefore, after we solve the statically condensed problem (14), this procedure can be used as an alternative to the local solvers (12) for recovering approximations to the local variables on $K \in \mathcal{T}_h$.

We can also apply postprocessing if FEEC is implemented using (4), without hybridization, since $\overline{u}_h = P_{\overline{\mathfrak{H}}_h} u_h$, $\widehat{\sigma}_h^{\tan} = \sigma_h^{\tan}$, and $\widehat{u}_h^{\tan} = u_h^{\tan}$. In the simplicial case, since $\mathfrak{H}_h^{*k}(K) = \mathring{\mathfrak{H}}_h^k(K)$, we can simply replace $\overline{u}_h$ by u_h on the right-hand side of (17c) without projecting.

Example 5.2 (Stenberg postprocessing)**.** When $k = n$ and $\mathcal{T}_h$ is a simplicial mesh, the space $W_h^{*n+1}(K)$ is trivial, $W_h^{*n}(K) \cong \mathcal{P}_{r^*}(K)$, and $\mathfrak{H}_h^{*n}(K) \cong \mathbb{R}$. Therefore, (17) becomes

$$\begin{aligned} (\operatorname{grad} u_h^*, \operatorname{grad} v_h)_K + (\overline{p}_h^*, v_h)_K &= (f, v_h)_K - \langle \widehat{\sigma}_h^{\tan}, v_h \mathbf{n} \rangle_{\partial K}, && \forall v_h \in \mathcal{P}_{r^*}(K), \\ (u_h^*, \overline{q}_h)_K &= (\overline{u}_h, \overline{q}_h)_K, && \forall \overline{q}_h \in \mathbb{R}, \end{aligned}$$

which coincides with Stenberg [40] postprocessing for the RT and BDM methods. Stenberg also considered a second form of postprocessing with $\overline{p}_h^*, \overline{q}_h \in \mathcal{P}_r(K)$, but we do not consider that here.

5.2. **Error estimates for $k < n$.** We now analyze this postprocessing procedure when, as before, $\{\mathcal{T}_h\}$ is a shape-regular family of simplicial meshes of Ω. We wish to determine the accuracy of the solution to the postprocessing problem (17), compared to that obtained using the local solvers (12).

The $k = n$ case has already been analyzed by Stenberg [40], so we restrict our attention to $k < n$. Since the local harmonic spaces are trivial, the exact solver (16) simplifies to

$$(\rho, \eta)_K - (u, \delta\eta)_K = \langle \widehat{u}^{\tan}, \eta^{\mathrm{nor}} \rangle_{\partial K}, \qquad \forall \eta \in H^*\Lambda^{k+1}(K), \tag{18a}$$

$$(\delta\rho, v)_K + (\delta u, \delta v)_K = (f - p, v)_K - \langle \widehat{\sigma}^{\tan}, v^{\mathrm{nor}} \rangle_{\partial K}, \quad \forall v \in H^*\Lambda^k(K), \tag{18b}$$

and the postprocessing problem (17) simplifies to

$$(\rho_h^*, \eta_h)_K - (u_h^*, \delta\eta_h)_K = \langle \widehat{u}_h^{\tan}, \eta_h^{\mathrm{nor}} \rangle_{\partial K}, \qquad \forall \eta_h \in W_h^{*k+1}(K), \tag{19a}$$

$$(\delta\rho_h^*, v_h)_K + (\delta u_h^*, \delta v_h)_K = (f - p_h, v_h)_K - \langle \widehat{\sigma}_h^{\tan}, v_h^{\mathrm{nor}} \rangle_{\partial K}, \quad \forall v_h \in W_h^{*k}(K). \tag{19b}$$

To aid in the analysis, we introduce the intermediate approximation $\widetilde{\rho}_h \in W_h^{*k+1}(K)$, $\widetilde{u}_h \in W_h^{*k}(K)$ such that

$$(\widetilde{\rho}_h, \eta_h)_K - (\widetilde{u}_h, \delta\eta_h)_K = \langle \widehat{u}^{\tan}, \eta_h^{\mathrm{nor}} \rangle_{\partial K}, \qquad \forall \eta_h \in W_h^{*k+1}(K), \tag{20a}$$

$$(\delta\widetilde{\rho}_h, v_h)_K + (\delta \widetilde{u}_h, \delta v_h)_K = (f - p, v_h)_K - \langle \widehat{\sigma}^{\tan}, v_h^{\mathrm{nor}} \rangle_{\partial K}, \quad \forall v_h \in W_h^{*k}(K), \tag{20b}$$

where the global variables on the right-hand side are the same as those in the exact solution (18). Note that (20) is just the FEEC approximation of (18) on the subcomplex $W_h^*(K) \subset H^*\Lambda(K)$, so the results of Arnold, Falk, and Winther [5] immediately give us estimates for $\rho - \widetilde{\rho}_h$ and $u - \widetilde{u}_h$. It therefore remains to analyze the difference between (19) and (20).

We want the postprocessed solution to be at least as good as the standard FEEC solution obtained from the local solvers (12). The following assumptions ensure that r^* is large enough for the $W_h^*(K)$ complex to approximate the exact solution as well as $W_h(K)$ does. If $f \perp \mathfrak{B}^k$, then $\sigma = 0$, so it is enough for $W_h^{*k}(K)$ to contain the same total space of polynomials as $W_h^k(K)$, i.e., $r^* \geq r$. Otherwise, in order to approximate $\sigma \neq 0$, we also need the stronger condition that $W_h^{*k-1}(K)$ contains the same total space of polynomials as $W_h^{k-1}(K)$.

Assumption A. Assume that we are in one of the following three cases:

(1) $f \perp \mathfrak{B}^k$ and $r^* \geq r$.

(2) $W_h^{k-1}(K) = \mathcal{P}_{r+1}\Lambda^{k-1}(K)$ and $\star W_h^{*k}(K) = \begin{cases} \mathcal{P}_{r^*}\Lambda^{n-k}(K), & r^* \geq r+2, \\ \mathcal{P}^-_{r^*+1}\Lambda^{n-k}(K), & r^* \geq r+1. \end{cases}$

(3) $W_h^{k-1}(K) = \mathcal{P}^-_{r+1}\Lambda^{k-1}(K)$ and $\star W_h^{*k}(K) = \begin{cases} \mathcal{P}_{r^*}\Lambda^{n-k}(K), & r^* \geq r+1, \\ \mathcal{P}^-_{r^*+1}\Lambda^{n-k}(K), & r^* \geq r. \end{cases}$

Our first result shows that $\delta\rho_h^*$ gives an improved approximation of $\delta\rho = \delta \mathrm{d}u$, compared to $\delta \mathrm{d}u_h$. In particular, when $f = \delta\rho \in \mathring{\mathfrak{B}}_k^*$, we can obtain an arbitrarily good approximation by taking the postprocessing degree r^* large enough.

Theorem 5.3. *For each $K \in \mathcal{T}_h$ and $0 \leq s \leq t_{\max}$, we have*

$$\left\|\delta(\rho - \widetilde{\rho}_h)\right\|_K \lesssim h_K^s \|f\|_{s,K}, \quad \text{if } s \leq r^* + 1,$$
$$\left\|\delta(\widetilde{\rho}_h - \rho_h^*)\right\|_K \leq \left\|\mathrm{d}(\sigma - \sigma_h)\right\|_K + \|p - p_h\|_K.$$

Consequently, if Assumption A holds, then

$$\left\|\delta(\rho - \rho_h^*)\right\|_{\mathcal{T}_h} \lesssim h^s \|f\|_{s,\Omega}, \quad \text{if } \begin{cases} s \leq r^* + 1, & f \in \mathring{\mathfrak{B}}_k^*, \\ s \leq r + 1, & \text{otherwise.} \end{cases}$$

Proof. The first estimate is immediate from [5, Theorem 3.11] applied to the problem (20). Next, subtracting (19b) from (20b) with $v_h \in \mathfrak{B}_h^{*k}(K)$ gives

$$\begin{aligned} \big(\delta(\widetilde{\rho}_h - \rho_h^*), v_h\big)_K &= (p_h - p, v_h)_K + \langle \widehat{\sigma}_h^{\tan} - \widehat{\sigma}^{\tan}, v_h^{\mathrm{nor}} \rangle_{\partial K} \\ &= (p_h - p, v_h)_K + \big(\mathrm{d}(\sigma_h - \sigma), v_h\big)_K \\ &\leq \Big(\left\|\mathrm{d}(\sigma - \sigma_h)\right\|_K + \|p - p_h\|_K\Big)\|v_h\|_K, \end{aligned}$$

and taking $v_h = \delta(\widetilde{\rho}_h - \rho_h^*)$ implies the second estimate. Finally, summing over $K \in \mathcal{T}_h$ and applying [5, Theorem 3.11] once more gives

$$\left\| \delta(\widetilde{\rho}_h - \rho_h^*) \right\|_{\mathcal{T}_h} \lesssim \begin{cases} 0, & \text{if } f \in \mathring{\mathfrak{B}}_k^*, \\ h^s \|f\|_{s,\Omega}, & \text{if } s \leq r+1, \text{ otherwise,} \end{cases}$$

so the last estimate follows by Assumption A and the triangle inequality. □

The next result says that, generically, δu_h^* approximates $\sigma = \delta u$ as well as σ_h does, but no better. In the case $f \in \mathring{\mathfrak{B}}_k^*$, when $\sigma = \sigma_h = 0$, we can make δu_h^* arbitrarily small by taking r^* large enough.

Theorem 5.4. *For each $K \in \mathcal{T}_h$ and $0 \leq s \leq t_{\max}$, we have*

$$\left\| \delta(u - \widetilde{u}_h) \right\|_K \lesssim h_K^{s+1} \|f\|_{s,K}, \quad \text{if } \begin{cases} s \leq r^* + 1, & f \in \mathring{\mathfrak{B}}_k^*, \\ s \leq r^*, & \star W_h^{*k}(K) = \mathcal{P}_{r^*+1}^- \Lambda^{n-k}(K), \\ s \leq r^* - 1, & \star W_h^{*k}(K) = \mathcal{P}_{r^*} \Lambda^{n-k}(K), \end{cases}$$

$$\left\| \delta(\widetilde{u}_h - u_h^*) \right\|_K \lesssim \|\sigma - \sigma_h\|_K + h_K \Bigl(\left\| \mathrm{d}(\sigma - \sigma_h) \right\|_K + \|p - p_h\|_K \Bigr).$$

Consequently, if Assumption A holds, then

$$\left\| \delta(u - u_h^*) \right\|_{\mathcal{T}_h} \lesssim h^{s+1} \|f\|_{s,\Omega}, \quad \text{if } \begin{cases} s \leq r^* + 1, & f \in \mathring{\mathfrak{B}}_k^*, \\ s \leq r+1, & V_h^{k-1} = \mathcal{P}_{r+1} \Lambda^{k-1}(\mathcal{T}_h), \\ s \leq r, & V_h^{k-1} = \mathcal{P}_{r+1}^- \Lambda^{k-1}(\mathcal{T}_h). \end{cases}$$

Proof. The first estimate is immediate from [5, Theorem 3.11]. Next, subtracting (19b) from (20b) with $v_h \in \mathfrak{Z}_h^{*k\perp}(K)$ gives

$$\begin{aligned} \bigl(\delta(\widetilde{u}_h - u_h^*), \delta v_h \bigr)_K &= (p_h - p, v_h)_K + \langle \widehat{\sigma}_h^{\tan} - \widehat{\sigma}^{\tan}, v_h^{\mathrm{nor}} \rangle_{\partial K} \\ &= (p_h - p, v_h)_K + \bigl(\mathrm{d}(\sigma_h - \sigma), v_h \bigr)_K - (\sigma_h - \sigma, \delta v_h)_K \\ &\lesssim \Bigl[\|\sigma - \sigma_h\|_K + h_K \Bigl(\left\| \mathrm{d}(\sigma - \sigma_h) \right\|_K + \|p - p_h\|_K \Bigr) \Bigr] \|\delta v_h\|_K. \end{aligned}$$

In the last step, we have applied Cauchy–Schwarz and the Poincaré inequality with scaling, which says that $\|v_h\|_K \lesssim h_K \|\delta v_h\|_K$. Taking v_h such that $\delta v_h = \delta(\widetilde{u}_h - u_h^*)$ implies the second estimate. Finally, summing over $K \in \mathcal{T}_h$ and applying [5, Theorem 3.11] gives

$$\left\| \delta(\widetilde{u}_h - u_h^*) \right\|_{\mathcal{T}_h} \lesssim \begin{cases} 0, & \text{if } f \in \mathring{\mathfrak{B}}_k^*, \\ h^{s+1} \|f\|_{s,\Omega}, & \text{otherwise, if } \begin{cases} s \leq r+1, & V_h^{k-1} = \mathcal{P}_{r+1} \Lambda^{k-1}(\mathcal{T}_h), \\ s \leq r, & V_h^{k-1} = \mathcal{P}_{r+1}^- \Lambda^{k-1}(\mathcal{T}_h), \end{cases} \end{cases}$$

so the last estimate follows by Assumption A and the triangle inequality. □

Thus far, we have been able to avoid dealing with the error term $\widehat{u}^{\tan} - \widehat{u}_h^{\tan}$, which dominates the postprocessing error, preventing improved convergence of the $\mathfrak{B}_h^*(K)$ components. There is one special exception, however: when $k = n-1$, the space $\mathfrak{B}_h^{*n}(K)$ is trivial, so there is no error in this component of ρ_h^*. In this case, we will see that ρ_h^* is an improved estimate compared to $\mathrm{d}u_h$. Since $\mathfrak{H}_h^{*n}(K) \cong \mathbb{R}$ is nontrivial, though, we need to control the $\overline{\mathfrak{H}}^n$ component of the error, which we will do with the aid of the following lemma.

Lemma 5.5. *If $k = n-1$ and $\eta_h \in \overline{\mathfrak{H}}^n$, then*

$$\langle \widehat{u}^{\tan} - \widehat{u}_h^{\tan}, \eta_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} \lesssim h \Bigl(\left\| \mathrm{d}(\sigma - \sigma_h) \right\|_\Omega + \|p - p_h\|_\Omega \Bigr) \|\eta_h\|_\Omega.$$

In particular, if $f \in \mathring{\mathfrak{B}}_{n-1}^*$, then $\int_{\partial K} \operatorname{tr}(u - u_h) = 0$ for all $K \in \mathcal{T}_h$.

Proof. Since η_h is piecewise constant, $\langle \widehat{u}^{\tan} - \widehat{u}_h^{\tan}, \eta_h^{\mathrm{nor}} \rangle_{\partial\mathcal{T}_h} = \bigl(\mathrm{d}(u - u_h), \eta_h\bigr)_{\mathcal{T}_h}$. Piecewise constants are in $V_h^n = \mathfrak{B}_h^n$, so the estimate follows by Lemma 4.9. In particular, σ and p vanish when $f \in \mathring{\mathfrak{B}}_{n-1}^*$, so in that case the left-hand side is identically zero. ☐

Remark 5.6. This generalizes the well-known property that, when $n = 1$ and $k = 0$, the continuous Galerkin solution equals the exact solution at nodes.

We now show that ρ_h^* approximates $\rho = \mathrm{d}u$ as well as $\mathrm{d}u_h$ does, but no better when $k < n - 1$. However, when $k = n - 1$, we get an improved estimate, and when $f \in \mathring{\mathfrak{B}}_{n-1}^*$, we can obtain an arbitrarily good approximation by taking r^* large enough.

Theorem 5.7. *For each $K \in \mathcal{T}_h$ and $0 \le s \le t_{\max}$,*

$$\|\rho - \widetilde{\rho}_h\|_K \lesssim h_K^{s+1} \|f\|_{s,K}, \quad \text{if } \begin{cases} s \le r^* + 1, & \star W_h^{*k+1}(K) = \mathcal{P}_{r^*+1}\Lambda^{n-k-1}(K), \\ s \le r^*, & \star W_h^{*k+1}(K) = \mathcal{P}_{r^*+1}^-\Lambda^{n-k-1}(K), \end{cases}$$

$$\|\widetilde{\rho}_h - \rho_h^*\|_K \lesssim \bigl\|\mathrm{d}(u - u_h)\bigr\|_K + h_K\Bigl(\bigl\|\mathrm{d}(\sigma - \sigma_h)\bigr\|_K + \|p - p_h\|_K\Bigr).$$

Consequently, if Assumption A holds, then

$$\|\rho - \rho_h^*\|_{\mathcal{T}_h} \lesssim h^{s+1}\|f\|_{s,\Omega}, \quad \text{if } \begin{cases} s \le r + 1, & f \perp \mathring{\mathfrak{B}}_k^*, \\ s \le r, & V_h^k = \mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T}_h) \\ s \le r - 1, & V_h^k = \mathcal{P}_r\Lambda^k(\mathcal{T}_h). \end{cases}$$

In the case $k = n - 1$, this estimate may be improved to

$$\|\rho - \rho_h^*\|_{\mathcal{T}_h} \lesssim h^{s+1}\|f\|_{s,\Omega}, \quad \text{if } \begin{cases} s \le r^* + 1, & f \in \mathring{\mathfrak{B}}_{n-1}^*, \ \star W_h^{*k+1}(K) = \mathcal{P}_{r^*+1}\Lambda^{n-k-1}(K), \\ s \le r^*, & f \in \mathring{\mathfrak{B}}_{n-1}^*, \ \star W_h^{*k+1}(K) = \mathcal{P}_{r^*+1}^-\Lambda^{n-k-1}(K), \\ s \le r + 1, & \textit{otherwise}. \end{cases}$$

Proof. The first estimate is immediate from [5, Theorem 3.11]. Next, subtracting (19a) from (20a) with $\eta_h \in \mathfrak{Z}_h^{*k+1}(K)$ gives

$$(\widetilde{\rho}_h - \rho_h^*, \eta_h)_K = \langle \widehat{u}^{\tan} - \widehat{u}_h^{\tan}, \eta_h^{\mathrm{nor}} \rangle_{\partial K} = \bigl(\mathrm{d}(u - u_h), \eta_h\bigr)_K \le \bigl\|\mathrm{d}(u - u_h)\bigr\|_K \|\eta_h\|_K,$$

which implies

$$\bigl\|P_{\mathfrak{Z}_h^*(K)}(\widetilde{\rho}_h - \rho_h^*)\bigr\|_K \le \bigl\|\mathrm{d}(u - u_h)\bigr\|_K.$$

Furthermore, by the Poincaré inequality and Theorem 5.3,

$$\bigl\|P_{\mathfrak{Z}_h^{*\perp}(K)}(\widetilde{\rho}_h - \rho_h^*)\bigr\|_K \lesssim h_K \bigl\|\delta(\widetilde{\rho}_h - \rho_h^*)\bigr\|_K \le h_K\Bigl(\bigl\|\mathrm{d}(\sigma - \sigma_h)\bigr\|_K + \|p - p_h\|_K\Bigr),$$

so the second estimate follows by the Hodge decomposition and triangle inequality. Summing over $K \in \mathcal{T}_h$ and applying [5, Theorem 3.11] gives

$$\|\widetilde{\rho}_h - \rho_h^*\|_{\mathcal{T}_h} \lesssim h^{s+1}\|f\|_{s,\Omega}, \quad \text{if } \begin{cases} s \le r + 1, & f \perp \mathring{\mathfrak{B}}_k^*, \\ s \le r, & V_h^k = \mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T}_h), \\ s \le r - 1, & V_h^k = \mathcal{P}_r\Lambda^k(\mathcal{T}_h), \end{cases}$$

so the third estimate follows by Assumption A and the triangle inequality.

Finally, consider the special case $k = n - 1$. Taking $\eta_h \in \overline{\mathfrak{H}}^n$ and applying Lemma 5.5 gives

$$(\widetilde{\rho}_h - \rho_h^*, \eta_h)_{\mathcal{T}_h} = \langle \widehat{u}^{\tan} - \widehat{u}_h^{\tan}, \eta_h^{\mathrm{nor}} \rangle_{\partial\mathcal{T}_h} \lesssim h\Bigl(\bigl\|\mathrm{d}(\sigma - \sigma_h)\bigr\|_\Omega + \|p - p_h\|_\Omega\Bigr)\|\eta_h\|_\Omega,$$

and therefore,

$$\bigl\|P_{\overline{\mathfrak{H}}}(\widetilde{\rho}_h - \rho_h^*)\bigr\|_{\mathcal{T}_h} \lesssim h\Bigl(\bigl\|\mathrm{d}(\sigma - \sigma_h)\bigr\|_\Omega + \|p - p_h\|_\Omega\Bigr).$$

Note that this eliminates the $\lVert \mathrm{d}(u-u_h)\rVert_\Omega$ term that appears in the $k<n-1$ case. Hence,

$$\lVert \widetilde{\rho}_h - \rho_h^*\rVert_{\mathcal{T}_h} \lesssim h\Big(\lVert \mathrm{d}(\sigma-\sigma_h)\rVert_\Omega + \lVert p-p_h\rVert_\Omega\Big) \lesssim \begin{cases} 0, & \text{if } f \in \mathring{\mathfrak{B}}^*_{n-1}, \\ h^{s+1}\lVert f\rVert_{s,\Omega}, & \text{if } s \le r+1, \text{ otherwise,}\end{cases}$$

and the improved estimate follows. □

Finally, we show that u_h^* approximates u as well as u_h does, but no better.

Theorem 5.8. *For each $K \in \mathcal{T}_h$ and $0 \le s \le t_{\max}$,*

$$\lVert u - \widetilde{u}_h\rVert_K \lesssim \begin{cases} h_K\lVert f\rVert_K, & \text{if } \star W_h^{*k} = \mathcal{P}_1^-\Lambda^{n-k}(K), \\ h_K^{s+2}\lVert f\rVert_{s,K}, & \text{if } s \le r^*-1, \text{ otherwise,}\end{cases}$$

$$\lVert \widetilde{u}_h - u_h^*\rVert_K \lesssim \lVert u-u_h\rVert_K + h_K\Big(\lVert \mathrm{d}(u-u_h)\rVert_K + \lVert \sigma-\sigma_h\rVert_K\Big) + h_K^2\Big(\lVert \mathrm{d}(\sigma-\sigma_h)\rVert_K + \lVert p-p_h\rVert_K\Big).$$

Consequently, if Assumption A holds, then

$$\lVert u - u_h^*\rVert_K \lesssim \begin{cases} h\lVert f\rVert_\Omega, & \text{if } V_h^k = \mathcal{P}_1^-\Lambda^k(\mathcal{T}_h), \\ h^{s+2}\lVert f\rVert_{s,\Omega}, & \text{if } s \le r-1, \text{ otherwise.}\end{cases}$$

Proof. The first estimate is immediate from [5, Theorem 3.11]. Next, subtracting (19a) from (20a) with $\eta_h \in \mathfrak{Z}_h^{*k+1\perp}(K)$ gives

$$\begin{aligned}(\widetilde{u}_h - u_h^*, \delta\eta_h)_K &= (\widetilde{\rho}_h - \rho_h^*, \eta_h)_K - \langle \widehat{u}^{\tan} - \widehat{u}_h^{\tan}, \eta_h^{\mathrm{nor}}\rangle_{\partial K} \\ &= \big(P_{\mathfrak{Z}_h^{*\perp}(K)}(\widetilde{\rho}_h - \rho_h^*), \eta_h\big)_K - \big(\mathrm{d}(u-u_h), \eta_h\big)_K + (u-u_h, \delta\eta_h)_K \\ &\lesssim \Big(\lVert u-u_h\rVert_K + h_K\lVert \mathrm{d}(u-u_h)\rVert_K + h_K^2\lVert \delta(\widetilde{\rho}_h - \rho_h^*)\rVert_K\Big)\lVert \delta\eta_h\rVert_K,\end{aligned}$$

by Cauchy–Schwarz and the Poincaré inequality. With Theorem 5.3, this implies

$$\lVert P_{\mathfrak{B}_h^*(K)}(\widetilde{u}_h - u_h^*)\rVert_K \lesssim \lVert u-u_h\rVert_K + h_K\lVert \mathrm{d}(u-u_h)\rVert_K + h_K^2\Big(\lVert \mathrm{d}(\sigma-\sigma_h)\rVert_K + \lVert p-p_h\rVert_K\Big).$$

Furthermore, by the Poincaré inequality and Theorem 5.4,

$$\lVert P_{\mathfrak{Z}_h^{*\perp}(K)}(\widetilde{u}_h - u_h^*)\rVert_K \lesssim h_K\lVert \sigma-\sigma_h\rVert_K + h_K^2\Big(\lVert \mathrm{d}(\sigma-\sigma_h)\rVert_K + \lVert p-p_h\rVert_K\Big),$$

so the second estimate follows by the Hodge decomposition and triangle inequality. Finally, summing over $K \in \mathcal{T}_h$ and applying [5, Theorem 3.11] gives

$$\lVert \widetilde{u}_h - u_h^*\rVert_{\mathcal{T}_h} \lesssim \begin{cases} h\lVert f\rVert_\Omega, & \text{if } V_h^k = \mathcal{P}_1^-\Lambda^k(\mathcal{T}_h), \\ h^{s+2}\lVert f\rVert_{s,\Omega}, & \text{if } s \le r-1, \text{ otherwise,}\end{cases}$$

so the last estimate follows by Assumption A and the triangle inequality. □

5.3. **Remarks on the case $k = n$.** Although the case $k=n$ has already been analyzed by Stenberg [40], we now briefly describe this analysis from the FEEC viewpoint, relating it to the techniques developed in this section. In this case, the postprocessing procedure (17) becomes

$$\begin{aligned}(\delta u_h^*, \delta v_h)_K + (\overline{p}_h^*, v_h)_K &= (f, v_h)_K - \langle \widehat{\sigma}_h^{\tan}, v_h^{\mathrm{nor}}\rangle_{\partial K}, && \forall v_h \in W_h^{*n}(K), \\ (u_h^*, \overline{q}_h)_K &= (u_h, \overline{q}_h)_K, && \forall \overline{q}_h \in \mathfrak{H}_h^{*n}(K),\end{aligned}$$

and the intermediate approximation is given by

$$\begin{aligned}(\delta \widetilde{u}_h, \delta v_h)_K + (\widetilde{p}_h, v_h)_K &= (f, v_h)_K - \langle \widehat{\sigma}^{\tan}, v_h^{\mathrm{nor}}\rangle_{\partial K}, && \forall v_h \in W_h^{*n}(K), \\ (\widetilde{u}_h, \overline{q}_h)_K &= (u, \overline{q}_h)_K, && \forall \overline{q}_h \in \mathfrak{H}_h^{*n}(K).\end{aligned}$$

The argument in Theorem 5.4 still works, so applying the Poincaré inequality gives

$$\lVert P_{\mathfrak{Z}_h^{*\perp}(K)}(\widetilde{u}_h - u_h^*)\rVert_K \lesssim h_K\lVert \sigma-\sigma_h\rVert_K + h_K^2\lVert \mathrm{d}(\sigma-\sigma_h)\rVert_K.$$

Furthermore, since $\overline{\mathfrak{H}}^n$ consists of piecewise constants, which are in $V_h^n = \mathfrak{B}_h^n$, we have

$$\|P_{\overline{\mathfrak{H}}}(\widetilde{u}_h - u_h^*)\|_K = \big\|P_{\overline{\mathfrak{H}}}(u - u_h)\big\|_K \le \big\|P_{\mathfrak{B}_h}(u - u_h)\big\|_K.$$

Summing over $K \in \mathcal{T}_h$ and applying [5, Lemma 3.13] implies

$$\|\widetilde{u}_h - u_h^*\|_{\mathcal{T}_h} \lesssim \begin{cases} h^{s+1}\|f\|_{s,\Omega}, & \text{if } s \le 1,\ V_h^n = \mathcal{P}_1^-\Lambda^n(\mathcal{T}_h), \\ h^{s+2}\|f\|_{s,\Omega}, & \text{otherwise, if } \begin{cases} s \le r+1, & V_h^{n-1} = \mathcal{P}_{r+1}\Lambda^{n-1}(\mathcal{T}_h), \\ s \le r, & V_h^{n-1} = \mathcal{P}_{r+1}^-\Lambda^{n-1}(\mathcal{T}_h), \end{cases} \end{cases}$$

so by Assumption A and the triangle inequality, this same estimate holds for $\|u - u_h^*\|_{\mathcal{T}_h}$. This is precisely the improved estimate in Stenberg [40, Theorem 2.2], by essentially the same proof.

6. Illustration of the methods in $n = 3$ dimensions

We now give a concrete illustration of the hybridization and postprocessing schemes in $n = 3$ dimensions, using scalar and vector proxy fields and the familiar operations of vector calculus. Let $\mathcal{T}_h$ be a simplicial triangulation of a bounded, polyhedral domain $\Omega \subset \mathbb{R}^3$. For simplicity, we also assume that Ω is contractible, so that $\mathfrak{H}^0 \cong \mathbb{R}$ and $\mathfrak{H}^k$ is trivial for $k = 1, 2, 3$.

Let V_h be a stable subcomplex of

$$0 \longrightarrow H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl};\Omega) \xrightarrow{\text{curl}} H(\text{div};\Omega) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow 0,$$

containing continuous Lagrange elements, Nédélec edge and face elements, and discontinuous Lagrange elements. Let W_h be the corresponding "broken" complex, with $W_h^k(K) = V_h^k|_K$ for $K \in \mathcal{T}_h$. Using the scalar and vector proxies for tangential traces in Table 1, we have

$$\begin{aligned} \widehat{V}_h^{0,\tan} &= \{v_h|_{\partial\mathcal{T}_h} : v_h \in V_h^0\}, & \widehat{W}_h^{0,\text{nor}} &= \widehat{W}_h^{0,\tan} = \{v_h|_{\partial\mathcal{T}_h} : v_h \in W_h^0\}, \\ \widehat{V}_h^{1,\tan} &= \{v_h|_{\partial\mathcal{T}_h} - (v_h\cdot\mathbf{n})\mathbf{n} : v_h \in V_h^1\}, & \widehat{W}_h^{1,\text{nor}} &= \widehat{W}_h^{1,\tan} = \{v_h|_{\partial\mathcal{T}_h} - (v_h\cdot\mathbf{n})\mathbf{n} : v_h \in W_h^1\}, \\ \widehat{V}_h^{2,\tan} &= \{(v_h\cdot\mathbf{n})\mathbf{n} : v_h \in V_h^2\}, & \widehat{W}_h^{2,\text{nor}} &= \widehat{W}_h^{2,\tan} = \{(v_h\cdot\mathbf{n})\mathbf{n} : v_h \in W_h^2\}, \end{aligned}$$

whose degrees of freedom are just those of V_h^k and W_h^k living on $\partial\mathcal{T}_h$.

For postprocessing on $K \in \mathcal{T}_h$, let $W_h^*(K)$ be a stable subcomplex of

$$0 \longleftarrow L^2(\Omega) \xleftarrow{-\text{div}} H(\text{div};\Omega) \xleftarrow{\text{curl}} H(\text{curl};\Omega) \xleftarrow{-\text{grad}} H^1(\Omega) \longleftarrow 0,$$

whose normal traces have scalar and vector proxies given in Table 1.

6.1. The case $k = 0$.

The hybrid method is

$$\begin{aligned} (\text{grad}\, u_h, \text{grad}\, v_h)_{\mathcal{T}_h} + (p_h, v_h)_{\mathcal{T}_h} - \langle \widehat{\rho}_h^{\text{nor}}, v_h\rangle_{\partial\mathcal{T}_h} &= (f, v_h)_{\mathcal{T}_h}, & \forall v_h &\in W_h^0, \\ \langle \widehat{u}_h^{\tan} - u_h, \widehat{\eta}_h^{\text{nor}}\rangle_{\partial\mathcal{T}_h} &= 0, & \forall \widehat{\eta}_h^{\text{nor}} &\in \widehat{W}_h^{0,\text{nor}}, \\ (u_h, q_h)_{\mathcal{T}_h} &= 0, & \forall q_h &\in \mathbb{R}, \\ \langle \widehat{\rho}_h^{\text{nor}}, \widehat{v}_h^{\tan}\rangle_{\partial\mathcal{T}_h} &= 0, & \forall \widehat{v}_h^{\tan} &\in \widehat{V}_h^{0,\tan}, \end{aligned}$$

which is the hybridized continuous Galerkin method of Cockburn, Gopalakrishnan, and Wang [19] for the Neumann problem. The postprocessing scheme on $K \in \mathcal{T}_h$ is

$$\begin{aligned} (\rho_h^*, \eta_h)_K + (u_h^*, \text{div}\, \eta_h)_K &= \langle \widehat{u}_h^{\tan}, \eta_h\cdot\mathbf{n}\rangle_{\partial K}, & \forall \eta_h &\in W_h^{*1}(K), \\ -(\text{div}\, \rho_h^*, v_h)_K &= (f - p_h, v_h)_K, & \forall v_h &\in W_h^{*0}(K). \end{aligned}$$

6.2. **The case $k = 1$.** The hybrid method is

$$\begin{aligned}
(\sigma_h, \tau_h)_{\mathcal{T}_h} - (u_h, \operatorname{grad} \tau_h)_{\mathcal{T}_h} + \langle \widehat{u}_h^{\mathrm{nor}}, \tau_h \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \tau_h \in W_h^0,\\
(\operatorname{grad} \sigma_h, v_h)_{\mathcal{T}_h} + (\operatorname{curl} u_h, \operatorname{curl} v_h)_{\mathcal{T}_h} - \langle \widehat{\rho}_h^{\mathrm{nor}}, v_h \rangle_{\partial \mathcal{T}_h} &= (f, v_h)_{\mathcal{T}_h}, && \forall v_h \in W_h^1,\\
\langle \widehat{\sigma}_h^{\tan} - \sigma_h, \widehat{v}_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{v}_h^{\mathrm{nor}} \in \widehat{W}_h^{0,\mathrm{nor}},\\
\langle \widehat{u}_h^{\tan} - u_h, \widehat{\eta}_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{\eta}_h^{\mathrm{nor}} \in \widehat{W}_h^{1,\mathrm{nor}},\\
\langle \widehat{u}_h^{\mathrm{nor}}, \widehat{\tau}_h^{\tan} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{\tau}_h^{\tan} \in \widehat{V}_h^{0,\tan},\\
\langle \widehat{\rho}_h^{\mathrm{nor}}, \widehat{v}_h^{\tan} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{v}_h^{\tan} \in \widehat{V}_h^{1,\tan},
\end{aligned}$$

and the postprocessing scheme on $K \in \mathcal{T}_h$ is

$$\begin{aligned}
(\rho_h^*, \eta_h)_K - (u_h^*, \operatorname{curl} \eta_h)_K &= \langle \widehat{u}_h^{\tan}, \eta_h \times \mathbf{n} \rangle_{\partial K}, && \forall \eta_h \in W_h^{*2}(K),\\
(\operatorname{curl} \rho_h^*, v_h)_K + (\operatorname{div} u_h^*, \operatorname{div} v_h)_K &= (f, v_h)_K - \langle \widehat{\sigma}_h^{\tan}, v_h \cdot \mathbf{n} \rangle_{\partial K}, && \forall v_h \in W_h^{*1}(K).
\end{aligned}$$

6.3. **The case $k = 2$.** The hybrid method is

$$\begin{aligned}
(\sigma_h, \tau_h)_{\mathcal{T}_h} - (u_h, \operatorname{curl} \tau_h)_{\mathcal{T}_h} + \langle \widehat{u}_h^{\mathrm{nor}}, \tau_h \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \tau_h \in W_h^1,\\
(\operatorname{curl} \sigma_h, v_h)_{\mathcal{T}_h} + (\operatorname{div} u_h, \operatorname{div} v_h)_{\mathcal{T}_h} - \langle \widehat{\rho}_h^{\mathrm{nor}}, v_h \rangle_{\partial \mathcal{T}_h} &= (f, v_h)_{\mathcal{T}_h}, && \forall v_h \in W_h^2,\\
\langle \widehat{\sigma}_h^{\tan} - \sigma_h, \widehat{v}_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{v}_h^{\mathrm{nor}} \in \widehat{W}_h^{1,\mathrm{nor}},\\
\langle \widehat{u}_h^{\tan} - u_h, \widehat{\eta}_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{\eta}_h^{\mathrm{nor}} \in \widehat{W}_h^{2,\mathrm{nor}},\\
\langle \widehat{u}_h^{\mathrm{nor}}, \widehat{\tau}_h^{\tan} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{\tau}_h^{\tan} \in \widehat{V}_h^{1,\tan},\\
\langle \widehat{\rho}_h^{\mathrm{nor}}, \widehat{v}_h^{\tan} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{v}_h^{\tan} \in \widehat{V}_h^{2,\tan},
\end{aligned}$$

and the postprocessing scheme on $K \in \mathcal{T}_h$ is

$$\begin{aligned}
(\rho_h^*, \eta_h)_K + (u_h^*, \operatorname{grad} \eta_h)_K &= \langle \widehat{u}_h^{\tan}, \eta_h \mathbf{n} \rangle_{\partial K}, && \forall \eta_h \in W_h^{*3}(K),\\
-(\operatorname{grad} \rho_h^*, v_h)_K + (\operatorname{curl} u_h^*, \operatorname{curl} v_h)_K &= (f, v_h)_K - \langle \widehat{\sigma}_h^{\tan}, v_h \times \mathbf{n} \rangle_{\partial K}, && \forall v_h \in W_h^{*2}(K).
\end{aligned}$$

6.4. **The case $k = 3$.** The hybrid method is

$$\begin{aligned}
(\sigma_h, \tau_h)_{\mathcal{T}_h} - (u_h, \operatorname{div} \tau_h)_{\mathcal{T}_h} + \langle \widehat{u}_h^{\mathrm{nor}}, \tau_h \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \tau_h \in W_h^2,\\
(\operatorname{div} \sigma_h, v_h)_{\mathcal{T}_h} + (\overline{p}_h, v_h)_{\mathcal{T}_h} &= (f, v_h)_{\mathcal{T}_h}, && \forall v_h \in W_h^3,\\
(\overline{u}_h - u_h, \overline{q}_h)_{\mathcal{T}_h} &= 0, && \forall \overline{q}_h \in \mathbb{R}^{\mathcal{T}_h},\\
\langle \widehat{\sigma}_h^{\tan} - \sigma_h, \widehat{v}_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{v}_h^{\mathrm{nor}} \in \widehat{W}_h^{2,\mathrm{nor}},\\
(\overline{p}_h, \overline{v}_h)_{\mathcal{T}_h} &= 0, && \forall \overline{v}_h \in \mathbb{R}^{\mathcal{T}_h},\\
\langle \widehat{u}_h^{\mathrm{nor}}, \widehat{\tau}_h^{\tan} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{\tau}_h^{\tan} \in \widehat{V}_h^{2,\tan},
\end{aligned}$$

which is an alternative hybridization of the RT and BDM methods using local Neumann solvers, as in Cockburn [15]. As noted in Section 3.5, this is equivalent to

$$\begin{aligned}
(\sigma_h, \tau_h)_{\mathcal{T}_h} - (u_h, \operatorname{div} \tau_h)_{\mathcal{T}_h} + \langle \widehat{u}_h^{\mathrm{nor}}, \tau_h \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \tau_h \in W_h^2,\\
(\operatorname{div} \sigma_h, v_h)_{\mathcal{T}_h} &= (f, v_h)_{\mathcal{T}_h}, && \forall v_h \in W_h^3,\\
\langle \sigma_h, \widehat{v}_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{v}_h^{\mathrm{nor}} \in \widehat{V}_h^{2,\mathrm{nor}},
\end{aligned}$$

which are the classic hybridized RT and BDM methods of Arnold and Brezzi [3], Brezzi, Douglas, and Marini [8]. The postprocessing scheme on $K \in \mathcal{T}_h$ is exactly that of Stenberg [40],

$$\begin{aligned}(\operatorname{grad} u_h^*, \operatorname{grad} v_h)_K + (\overline{p}_h^*, v_h)_K &= (f, v_h)_K - \langle \widehat{\sigma}_h^{\tan}, v_h \mathbf{n} \rangle_{\partial K}, && \forall v_h \in W_h^{*3}(K),\\ (u_h^*, \overline{q}_h)_K &= (\overline{u}_h, \overline{q}_h)_K, && \forall \overline{q}_h \in \mathbb{R}.\end{aligned}$$

7. Numerical experiments

In this section, we present several numerical experiments in $n = 2$ and $n = 3$ dimensions that illustrate and confirm the theory developed throughout the paper. We omit the cases $k = 0$ and $k = n$, since we have seen that these correspond to known methods for the scalar Poisson equation whose properties are already well understood. The remaining cases correspond to hybridization and postprocessing methods for the vector Poisson equation.

For the sake of brevity, we present only numerical experiments using $\mathcal{P}_{r+1}^- \Lambda$ elements with $\star\mathcal{P}_{r^*+1}^- \Lambda$ postprocessing, where r^* is chosen optimally according to Assumption A, and where f has nonvanishing components in both $\mathfrak{B}^k$ and $\mathring{\mathfrak{B}}_k^*$. We have conducted many additional numerical experiments, which also conform with the theoretical results.

All computations have been carried out using the Firedrake finite element library [37] (version 0.13.0+3719.g8e730839), and a Firedrake component called Slate [23] was used to implement the local solvers for static condensation and postprocessing.

7.1. Numerical experiments in $n = 2$ dimensions.

For these experiments, we take the domain to be the unit square $\Omega = [0, 1]^2$. We identify the $H\Lambda(\Omega)$ complex with

$$0 \longrightarrow H^1(\Omega) \xrightarrow{\operatorname{grad}} H(\operatorname{curl}; \Omega) \xrightarrow{\operatorname{curl}} L^2(\Omega) \longrightarrow 0,$$

so that the $H^*\Lambda(\Omega)$ complex is identified with

$$0 \longleftarrow L^2(\Omega) \xleftarrow{-\operatorname{div}} H(\operatorname{div}; \Omega) \xleftarrow{\operatorname{curl}} H^1(\Omega) \longleftarrow 0.$$

Here, the curl of a scalar field is its rotated gradient. A structured triangle mesh $\mathcal{T}_h$ is formed by partitioning Ω uniformly into $N \times N$ squares, each of which is divided into two triangles.

We apply the "method of manufactured solutions" by choosing a smooth u satisfying the boundary conditions and taking $f = -\Delta u$. Specifically, for $k = 1$, we choose

$$u(x, y) = \begin{bmatrix} \sin(\pi x) \\ \sin(\pi y) \end{bmatrix} + \begin{bmatrix} \sin(\pi x)\cos(\pi y) \\ -\cos(\pi x)\sin(\pi y) \end{bmatrix},$$

where the first term is in $\mathfrak{B}^1$ and the second is in $\mathring{\mathfrak{B}}_1^*$.

Table 2 shows the errors and rates when the hybridized FEEC method with $\mathcal{P}_{r+1}^- \Lambda$ elements is applied to this problem, and Table 3 shows the errors and rates when this numerical solution is postprocessed with $\star\mathcal{P}_{r+2}^- \Lambda$ elements. (Since $\mathcal{P}_{r+1}^- \Lambda^0 \cong \mathcal{P}_{r+1} \Lambda^0$, the minimum degree satisfying Assumption A is $r^* = r + 1$.) These results match the error estimates in Sections 4.3 and 5.2, respectively. In particular, since $k = n - 1$, we see that $\widehat{\rho}_h^{\mathrm{nor}}$ and ρ_h^* superconverge with rate $\mathcal{O}(h^{r+2})$, whereas $\mathrm{d}u_h$ only converges with rate $\mathcal{O}(h^{r+1})$.

For clarity, the captions of Tables 2 and 3 describe the elements used both in FEEC notation and in terms of their classical scalar and vector proxies. Adopting the Unified Form Language (UFL) [1] notation used by Firedrake, we denote Lagrange finite elements by `CG`, rotated Raviart–Thomas $H(\operatorname{curl})$ edge elements by `RTE`, and ordinary Raviart–Thomas $H(\operatorname{div})$ elements by `RT`.

7.2. Numerical experiments in $n = 3$ dimensions.

For the next experiments, the domain is taken to be the unit cube $\Omega = [0, 1]^3$. A structured tetrahedral mesh $\mathcal{T}_h$ is formed by partitioning Ω into $N \times N \times N$ cubes, each of which is divided into six tetrahedra.

r	N	$\lVert\sigma-\sigma_h\rVert_\Omega$		$\lvert\lvert\lvert\sigma^{\tan}-\widehat{\sigma}_h^{\tan}\rvert\rvert\rvert_{\partial\mathcal{T}_h}$		$\lVert u-u_h\rVert_\Omega$		$\lvert\lvert\lvert u^{\tan}-\widehat{u}_h^{\tan}\rvert\rvert\rvert_{\partial\mathcal{T}_h}$		$\lvert\lvert\lvert\widehat{P}_h u^{\mathrm{nor}}-\widehat{u}_h^{\mathrm{nor}}\rvert\rvert\rvert_{\partial\mathcal{T}_h}$		$\lVert\mathrm{d}(u-u_h)\rVert_\Omega$		$\lvert\lvert\lvert\widehat{P}_h\rho^{\mathrm{nor}}-\widehat{\rho}_h^{\mathrm{nor}}\rvert\rvert\rvert_{\partial\mathcal{T}_h}$	
0	1	6.70e-01	—	1.13e+00	—	5.73e-01	—	1.07e+00	—	3.24e-01	—	1.84e+00	—	6.69e-03	—
	2	6.15e-01	0.1	1.57e+00	-0.5	5.35e-01	0.1	1.14e+00	-0.1	4.36e-01	-0.4	1.53e+00	0.3	6.29e-01	-6.6
	4	1.72e-01	1.8	4.49e-01	1.8	2.74e-01	1.0	5.76e-01	1.0	2.19e-01	1.0	8.08e-01	0.9	2.13e-01	1.6
	8	4.52e-02	1.9	1.19e-01	1.9	1.38e-01	1.0	2.85e-01	1.0	1.13e-01	1.0	4.09e-01	1.0	6.10e-02	1.8
	16	1.15e-02	2.0	3.06e-02	2.0	6.93e-02	1.0	1.42e-01	1.0	5.74e-02	1.0	2.05e-01	1.0	1.60e-02	1.9
	32	2.90e-03	2.0	7.71e-03	2.0	3.47e-02	1.0	7.09e-02	1.0	2.89e-02	1.0	1.03e-01	1.0	4.07e-03	2.0
1	1	6.26e-01	—	1.58e+00	—	5.36e-01	—	1.18e+00	—	3.26e-01	—	1.61e+00	—	1.17e+00	—
	2	6.52e-02	3.3	1.40e-01	3.5	1.11e-01	2.3	1.97e-01	2.6	1.61e-01	1.0	4.60e-01	1.8	1.20e-01	3.3
	4	8.46e-03	2.9	1.67e-02	3.1	2.85e-02	2.0	5.07e-02	2.0	5.07e-02	1.7	1.22e-01	1.9	1.58e-02	2.9
	8	1.07e-03	3.0	2.06e-03	3.0	7.21e-03	2.0	1.26e-02	2.0	1.37e-02	1.9	3.11e-02	2.0	2.04e-03	2.9
	16	1.35e-04	3.0	2.58e-04	3.0	1.81e-03	2.0	3.13e-03	2.0	3.55e-03	2.0	7.81e-03	2.0	2.61e-04	3.0
	32	1.70e-05	3.0	3.23e-05	3.0	4.54e-04	2.0	7.77e-04	2.0	8.99e-04	2.0	1.95e-03	2.0	3.30e-05	3.0
2	1	1.95e-02	—	3.29e-02	—	8.78e-02	—	6.88e-02	—	1.18e-01	—	4.18e-01	—	5.64e-03	—
	2	5.59e-03	1.8	1.38e-02	1.3	1.64e-02	2.4	3.34e-02	1.0	3.70e-02	1.7	1.02e-01	2.0	1.46e-02	-1.4
	4	3.65e-04	3.9	8.37e-04	4.0	2.10e-03	3.0	4.16e-03	3.0	5.30e-03	2.8	1.36e-02	2.9	9.98e-04	3.9
	8	2.32e-05	4.0	5.11e-05	4.0	2.66e-04	3.0	5.15e-04	3.0	6.95e-04	2.9	1.73e-03	3.0	6.48e-05	3.9
	16	1.46e-06	4.0	3.15e-06	4.0	3.34e-05	3.0	6.39e-05	3.0	8.88e-05	3.0	2.17e-04	3.0	4.12e-06	4.0
	32	9.13e-08	4.0	1.95e-07	4.0	4.19e-06	3.0	7.96e-06	3.0	1.12e-05	3.0	2.71e-05	3.0	2.60e-07	4.0

TABLE 2. Errors and rates for a manufactured solution with $n = 2$, $k = 1$, using hybridization with $\mathcal{P}_{r+1}^{-}\Lambda^0 \cong \mathtt{CG}_{r+1}$ and $\mathcal{P}_{r+1}^{-}\Lambda^1 \cong \mathtt{RTE}_{r+1}$ elements. Since $k = n-1$, we observe superconvergence of $\widehat{\rho}_h^{\mathrm{nor}}$.

r	N	$\lVert\sigma-\sigma_h\rVert_\Omega$		$\lVert\sigma-\delta u_h^*\rVert_{\partial\mathcal{T}_h}$		$\lVert u-u_h\rVert_\Omega$		$\lVert u-u_h^*\rVert_{\mathcal{T}_h}$		$\lVert\mathrm{d}(u-u_h)\rVert_\Omega$		$\lVert\mathrm{d}u-\rho_h^*\rVert_{\mathcal{T}_h}$		$\lVert\delta\mathrm{d}(u-u_h)\rVert_{\mathcal{T}_h}$		$\lVert\delta(\mathrm{d}u-\rho_h^*)\rVert_{\mathcal{T}_h}$	
0	1	6.70e-01	—	4.38e-01	—	5.73e-01	—	4.85e-01	—	1.84e+00	—	5.24e-01	—	1.40e+01	—	5.70e+00	—
	2	6.15e-01	0.1	4.88e-01	-0.2	5.35e-01	0.1	4.65e-01	0.1	1.53e+00	0.3	3.95e-01	0.4	1.40e+01	-0.0	4.04e+00	0.5
	4	1.72e-01	1.8	1.38e-01	1.8	2.74e-01	1.0	2.46e-01	0.9	8.08e-01	0.9	1.14e-01	1.8	1.40e+01	-0.0	1.94e+00	1.1
	8	4.52e-02	1.9	3.67e-02	1.9	1.38e-01	1.0	1.25e-01	1.0	4.09e-01	1.0	3.07e-02	1.9	1.40e+01	-0.0	9.66e-01	1.0
	16	1.15e-02	2.0	9.40e-03	2.0	6.93e-02	1.0	6.32e-02	1.0	2.05e-01	1.0	7.88e-03	2.0	1.40e+01	0.0	4.84e-01	1.0
	32	2.90e-03	2.0	2.37e-03	2.0	3.47e-02	1.0	3.17e-02	1.0	1.03e-01	1.0	1.99e-03	2.0	1.40e+01	-0.0	2.42e-01	1.0
1	1	6.26e-01	—	4.33e-01	—	5.36e-01	—	4.43e-01	—	1.61e+00	—	5.57e-01	—	1.13e+01	—	4.95e+00	—
	2	6.52e-02	3.3	3.78e-02	3.5	1.11e-01	2.3	6.86e-02	2.7	4.60e-01	1.8	4.06e-02	3.8	7.40e+00	0.6	7.10e-01	2.8
	4	8.46e-03	2.9	4.41e-03	3.1	2.85e-02	2.0	1.51e-02	2.2	1.22e-01	1.9	3.91e-03	3.4	3.92e+00	0.9	1.31e-01	2.4
	8	1.07e-03	3.0	5.36e-04	3.0	7.21e-03	2.0	3.57e-03	2.1	3.11e-02	2.0	4.43e-04	3.1	1.99e+00	1.0	2.86e-02	2.2
	16	1.35e-04	3.0	6.66e-05	3.0	1.81e-03	2.0	8.76e-04	2.0	7.81e-03	2.0	5.45e-05	3.0	9.99e-01	1.0	6.86e-03	2.1
	32	1.70e-05	3.0	8.32e-06	3.0	4.54e-04	2.0	2.17e-04	2.0	1.95e-03	2.0	6.84e-06	3.0	5.00e-01	1.0	1.70e-03	2.0
2	1	1.95e-02	—	1.87e-02	—	8.78e-02	—	3.77e-02	—	4.18e-01	—	4.42e-02	—	6.21e+00	—	6.04e-01	—
	2	5.59e-03	1.8	3.74e-03	2.3	1.64e-02	2.4	8.13e-03	2.2	1.02e-01	2.0	4.59e-03	3.3	2.58e+00	1.3	1.07e-01	2.5
	4	3.65e-04	3.9	2.11e-04	4.1	2.10e-03	3.0	8.38e-04	3.3	1.36e-02	2.9	1.95e-04	4.6	6.89e-01	1.9	8.62e-03	3.6
	8	2.32e-05	4.0	1.27e-05	4.1	2.66e-04	3.0	9.59e-05	3.1	1.73e-03	3.0	9.92e-06	4.3	1.75e-01	2.0	8.25e-04	3.4
	16	1.46e-06	4.0	7.78e-07	4.0	3.34e-05	3.0	1.16e-05	3.0	2.17e-04	3.0	5.76e-07	4.1	4.39e-02	2.0	9.27e-05	3.2
	32	9.13e-08	4.0	4.83e-08	4.0	4.19e-06	3.0	1.43e-06	3.0	2.71e-05	3.0	3.52e-08	4.0	1.10e-02	2.0	1.12e-05	3.0

TABLE 3. Errors and rates for the manufactured solution in Table 2, after local postprocessing with broken $\star\mathcal{P}_{r+2}^{-}\Lambda^0 \cong \mathtt{CG}_{r+2}$ and $\star\mathcal{P}_{r+2}^{-}\Lambda^1 \cong \mathtt{RT}_{r+2}$ elements. Since $k = n-1$, we observe improved convergence of ρ_h^* as compared with $\mathrm{d}u_h$.

7.2.1. *The case $k = 1$.* We apply the method of manufactured solutions with

$$u(x,y,z) = \begin{bmatrix} \sin(\pi x) \\ \sin(\pi y) \\ \sin(\pi z) \end{bmatrix} + \begin{bmatrix} \sin(\pi x)\cos(\pi y) \\ -\cos(\pi x)\sin(\pi y) \\ 0 \end{bmatrix},$$

where the first term is in $\mathfrak{B}^1$ and the second is in $\mathring{\mathfrak{B}}_1^*$.

Table 4 shows the errors and rates when the hybridized FEEC method with $\mathcal{P}_{r+1}^{-}\Lambda$ elements is applied to this problem, and Table 5 shows the errors and rates when this numerical solution is postprocessed with $\star\mathcal{P}_{r+2}^{-}\Lambda$ elements. Again, these results match the error estimates in Sections 4.3 and 5.2, respectively. Since $k < n-1$, we no longer have superconvergence of $\widehat{\rho}_h^{\mathrm{nor}}$ and ρ_h^* as we did

r	N	$\lVert\sigma-\sigma_h\rVert_\Omega$		$\lvert\lvert\lvert\sigma^{\tan}-\widehat{\sigma}_h^{\tan}\rvert\rvert\rvert_{\partial\mathcal{T}_h}$		$\lVert u-u_h\rVert_\Omega$		$\lvert\lvert\lvert u^{\tan}-\widehat{u}_h^{\tan}\rvert\rvert\rvert_{\partial\mathcal{T}_h}$		$\lvert\lvert\lvert\widehat{P}_h u^{\mathrm{nor}}-\widehat{u}_h^{\mathrm{nor}}\rvert\rvert\rvert_{\partial\mathcal{T}_h}$		$\lVert \mathrm{d}(u-u_h)\rVert_\Omega$		$\lvert\lvert\lvert\widehat{P}_h\rho^{\mathrm{nor}}-\widehat{\rho}_h^{\mathrm{nor}}\rvert\rvert\rvert_{\partial\mathcal{T}_h}$	
0	1	7.66e-01	—	4.39e+00	—	6.18e-01	—	2.91e+00	—	2.06e-01	—	1.84e+00	—	7.87e-01	—
	2	6.96e-01	0.1	3.27e+00	0.4	5.75e-01	0.1	2.52e+00	0.2	4.67e-01	-1.2	1.48e+00	0.3	1.56e+00	-1.0
	4	2.12e-01	1.7	9.99e-01	1.7	3.07e-01	0.9	1.29e+00	1.0	3.13e-01	0.6	7.92e-01	0.9	9.38e-01	0.7
	8	5.75e-02	1.9	2.72e-01	1.9	1.58e-01	1.0	6.45e-01	1.0	1.80e-01	0.8	4.04e-01	1.0	5.02e-01	0.9
	16	1.48e-02	2.0	6.96e-02	2.0	7.98e-02	1.0	3.22e-01	1.0	9.42e-02	0.9	2.03e-01	1.0	2.57e-01	1.0
1	1	4.42e-01	—	2.52e+00	—	4.43e-01	—	2.51e+00	—	1.99e-01	—	1.43e+00	—	1.65e+00	—
	2	6.18e-02	2.8	3.36e-01	2.9	1.09e-01	2.0	5.41e-01	2.2	1.76e-01	0.2	4.35e-01	1.7	8.09e-01	1.0
	4	9.61e-03	2.7	5.23e-02	2.7	3.08e-02	1.8	1.41e-01	1.9	5.82e-02	1.6	1.18e-01	1.9	2.44e-01	1.7
	8	1.28e-03	2.9	6.98e-03	2.9	8.06e-03	1.9	3.58e-02	2.0	1.60e-02	1.9	3.04e-02	2.0	6.47e-02	1.9
	16	1.64e-04	3.0	8.94e-04	3.0	2.05e-03	2.0	8.98e-03	2.0	4.14e-03	1.9	7.69e-03	2.0	1.66e-02	2.0
2	1	2.16e-02	—	1.29e-01	—	7.74e-02	—	3.74e-01	—	1.09e-01	—	3.96e-01	—	5.14e-01	—
	2	6.16e-03	1.8	2.90e-02	2.2	1.65e-02	2.2	7.62e-02	2.3	4.61e-02	1.2	9.43e-02	2.1	2.32e-01	1.1
	4	4.12e-04	3.9	1.82e-03	4.0	2.18e-03	2.9	9.66e-03	3.0	7.16e-03	2.7	1.28e-02	2.9	3.52e-02	2.7
	8	2.64e-05	4.0	1.13e-04	4.0	2.79e-04	3.0	1.22e-03	3.0	9.68e-04	2.9	1.65e-03	3.0	4.71e-03	2.9
	16	1.67e-06	4.0	7.01e-06	4.0	3.53e-05	3.0	1.52e-04	3.0	1.25e-04	3.0	2.08e-04	3.0	6.05e-04	3.0

TABLE 4. Errors and rates for a manufactured solution with $n = 3$, $k = 1$, using hybridization with $\mathcal{P}_{r+1}^-\Lambda^0 \cong \mathtt{CG}_{r+1}$ and $\mathcal{P}_{r+1}^-\Lambda^1 \cong \mathtt{N1E}_{r+1}$ elements. Since $k < n-1$, traces converge at the same rate as the corresponding non-hybrid variables, and in particular $\widehat{\rho}_h^{\mathrm{nor}}$ does not superconverge.

r	N	$\lVert\sigma-\sigma_h\rVert_\Omega$		$\lVert\sigma-\delta u_h^*\rVert_{\partial\mathcal{T}_h}$		$\lVert u-u_h\rVert_\Omega$		$\lVert u-u_h^*\rVert_{\mathcal{T}_h}$		$\lVert\mathrm{d}(u-u_h)\rVert_\Omega$		$\lVert\mathrm{d}u-\rho_h^*\rVert_{\mathcal{T}_h}$		$\lVert\delta\mathrm{d}(u-u_h)\rVert_{\mathcal{T}_h}$		$\lVert\delta(\mathrm{d}u-\rho_h^*)\rVert_{\mathcal{T}_h}$	
0	1	7.66e-01	—	5.87e-01	—	6.18e-01	—	5.03e-01	—	1.84e+00	—	1.38e+00	—	1.40e+01	—	6.24e+00	—
	2	6.96e-01	0.1	6.32e-01	-0.1	5.75e-01	0.1	5.27e-01	-0.1	1.48e+00	0.3	1.27e+00	0.1	1.40e+01	-0.0	4.70e+00	0.4
	4	2.12e-01	1.7	1.95e-01	1.7	3.07e-01	0.9	2.91e-01	0.9	7.92e-01	0.9	6.83e-01	0.9	1.40e+01	0.0	2.47e+00	0.9
	8	5.75e-02	1.9	5.35e-02	1.9	1.58e-01	1.0	1.51e-01	0.9	4.04e-01	1.0	3.49e-01	1.0	1.40e+01	-0.0	1.28e+00	1.0
	16	1.48e-02	2.0	1.38e-02	2.0	7.98e-02	1.0	7.66e-02	1.0	2.03e-01	1.0	1.76e-01	1.0	1.40e+01	0.0	6.46e-01	1.0
1	1	4.42e-01	—	3.69e-01	—	4.43e-01	—	3.33e-01	—	1.43e+00	—	1.02e+00	—	1.06e+01	—	3.62e+00	—
	2	6.18e-02	2.8	4.97e-02	2.9	1.09e-01	2.0	8.45e-02	2.0	4.35e-01	1.7	2.81e-01	1.9	7.18e+00	0.6	7.46e-01	2.3
	4	9.61e-03	2.7	7.93e-03	2.6	3.08e-02	1.8	2.56e-02	1.7	1.18e-01	1.9	7.70e-02	1.9	3.83e+00	0.9	2.16e-01	1.8
	8	1.28e-03	2.9	1.06e-03	2.9	8.06e-03	1.9	6.84e-03	1.9	3.04e-02	2.0	1.98e-02	2.0	1.95e+00	1.0	5.71e-02	1.9
	16	1.64e-04	3.0	1.36e-04	3.0	2.05e-03	2.0	1.75e-03	2.0	7.69e-03	2.0	5.01e-03	2.0	9.82e-01	1.0	1.46e-02	2.0
2	1	2.16e-02	—	2.01e-02	—	7.74e-02	—	5.68e-02	—	3.96e-01	—	2.10e-01	—	6.03e+00	—	5.65e-01	—
	2	6.16e-03	1.8	4.46e-03	2.2	1.65e-02	2.2	1.19e-02	2.3	9.43e-02	2.1	5.06e-02	2.1	2.39e+00	1.3	1.09e-01	2.4
	4	4.12e-04	3.9	2.84e-04	4.0	2.18e-03	2.9	1.52e-03	3.0	1.28e-02	2.9	6.67e-03	2.9	6.56e-01	1.9	1.20e-02	3.2
	8	2.64e-05	4.0	1.77e-05	4.0	2.79e-04	3.0	1.92e-04	3.0	1.65e-03	3.0	8.41e-04	3.0	1.69e-01	2.0	1.44e-03	3.1
	16	1.67e-06	4.0	1.11e-06	4.0	3.53e-05	3.0	2.42e-05	3.0	2.08e-04	3.0	1.05e-04	3.0	4.27e-02	2.0	1.78e-04	3.0

TABLE 5. Errors and rates for the manufactured solution in Table 4, after local postprocessing with broken $\star\mathcal{P}_{r+2}^-\Lambda^1 \cong \mathtt{N1E}_{r+2}$ and $\star\mathcal{P}_{r+2}^-\Lambda^2 \cong \mathtt{RT}_{r+2}$ elements. Since $k < n-1$, we observe improved convergence of $\delta\rho_h^*$ but not ρ_h^* itself.

for $n = 2$, $k = 1$, and both now converge with the same rate $\mathcal{O}(h^{r+1})$ as $\mathrm{d}u_h$. However, $\delta\rho_h^*$ still converges with the improved rate $\mathcal{O}(h^{r+1})$, compared to $\mathcal{O}(h^r)$ for $\delta\mathrm{d}u_h$.

As before, we denote Lagrange finite elements by `CG` and Raviart–Thomas $H(\mathrm{div})$ elements by `RT`. We also adopt the UFL notation `N1E` for Nédélec $H(\mathrm{curl})$ edge elements of the first kind.

7.2.2. *The case $k = 2$.* Finally, we apply the method of manufactured solutions with

$$u(x,y,z) = \begin{bmatrix}\sin(\pi y)\sin(\pi z)\\ \sin(\pi x)\sin(\pi z)\\ \sin(\pi x)\sin(\pi y)\end{bmatrix} + \begin{bmatrix}\cos(\pi x)\sin(\pi y)\sin(\pi z)\\ \sin(\pi x)\cos(\pi y)\sin(\pi z)\\ \sin(\pi x)\sin(\pi y)\cos(\pi z)\end{bmatrix},$$

where the first term is in $\mathfrak{B}^2$ and the second is in $\mathring{\mathfrak{B}}_2^*$.

Table 6 shows the errors and rates when the hybridized FEEC method with $\mathcal{P}_{r+1}^-\Lambda$ elements is applied to this problem, and Table 7 shows the errors and rates when this numerical solution is postprocessed with $\star\mathcal{P}_{r+1}^-\Lambda$ elements. (Since $\mathcal{P}_{r+1}^-\Lambda^1$ only contains complete polynomials up to degree r, we need only take $r^* = r$ to satisfy Assumption A.) These results match the error estimates

r	N	$\lVert\sigma-\sigma_h\rVert_\Omega$		$\lvert\lvert\lvert\sigma^{\tan}-\widehat{\sigma}_h^{\tan}\rvert\rvert\rvert_{\partial\mathcal{T}_h}$		$\lVert u-u_h\rVert_\Omega$		$\lvert\lvert\lvert u^{\tan}-\widehat{u}_h^{\tan}\rvert\rvert\rvert_{\partial\mathcal{T}_h}$		$\lvert\lvert\lvert\widehat{P}_h u^{\mathrm{nor}}-\widehat{u}_h^{\mathrm{nor}}\rvert\rvert\rvert_{\partial\mathcal{T}_h}$		$\lVert\mathrm{d}(u-u_h)\rVert_\Omega$		$\lvert\lvert\lvert\widehat{P}_h\rho^{\mathrm{nor}}-\widehat{\rho}_h^{\mathrm{nor}}\rvert\rvert\rvert_{\partial\mathcal{T}_h}$	
0	1	1.83e+00	—	5.57e+00	—	6.42e-01	—	1.35e+00	—	7.19e-01	—	2.28e+00	—	1.28e+00	—
	2	1.62e+00	0.2	6.16e+00	-0.1	5.08e-01	0.3	1.30e+00	0.1	5.98e-01	0.3	1.69e+00	0.4	1.17e+00	0.1
	4	8.47e-01	0.9	3.25e+00	0.9	2.70e-01	0.9	6.85e-01	0.9	3.30e-01	0.9	9.05e-01	0.9	3.98e-01	1.6
	8	4.32e-01	1.0	1.64e+00	1.0	1.38e-01	1.0	3.46e-01	1.0	1.74e-01	0.9	4.60e-01	1.0	1.15e-01	1.8
	16	2.17e-01	1.0	8.24e-01	1.0	6.92e-02	1.0	1.73e-01	1.0	8.84e-02	1.0	2.31e-01	1.0	3.02e-02	1.9
1	1	1.64e+00	—	5.65e+00	—	5.35e-01	—	1.25e+00	—	6.54e-01	—	1.83e+00	—	2.97e+00	—
	2	3.27e-01	2.3	1.25e+00	2.2	1.52e-01	1.8	3.79e-01	1.7	2.83e-01	1.2	5.93e-01	1.6	3.97e-01	2.9
	4	8.45e-02	2.0	3.22e-01	2.0	4.06e-02	1.9	1.01e-01	1.9	8.64e-02	1.7	1.63e-01	1.9	5.31e-02	2.9
	8	2.17e-02	2.0	8.11e-02	2.0	1.04e-02	2.0	2.55e-02	2.0	2.32e-02	1.9	4.16e-02	2.0	6.79e-03	3.0
	16	5.49e-03	2.0	2.03e-02	2.0	2.62e-03	2.0	6.35e-03	2.0	5.97e-03	2.0	1.05e-02	2.0	8.60e-04	3.0
2	1	2.84e-01	—	9.56e-01	—	1.52e-01	—	3.18e-01	—	2.44e-01	—	6.05e-01	—	3.66e-01	—
	2	5.41e-02	2.4	2.04e-01	2.2	3.39e-02	2.2	8.70e-02	1.9	8.47e-02	1.5	1.67e-01	1.9	8.26e-02	2.1
	4	6.81e-03	3.0	2.64e-02	3.0	4.55e-03	2.9	1.15e-02	2.9	1.29e-02	2.7	2.30e-02	2.9	5.51e-03	3.9
	8	8.63e-04	3.0	3.34e-03	3.0	5.82e-04	3.0	1.45e-03	3.0	1.73e-03	2.9	2.95e-03	3.0	3.54e-04	4.0
	16	1.09e-04	3.0	4.20e-04	3.0	7.34e-05	3.0	1.80e-04	3.0	2.22e-04	3.0	3.71e-04	3.0	2.24e-05	4.0

TABLE 6. Errors and rates for a manufactured solution with $n = 3$, $k = 2$, using hybridization with $\mathcal{P}_{r+1}^-\Lambda^1 \cong \mathtt{N1E}_{r+1}$ and $\mathcal{P}_{r+1}^-\Lambda^2 \cong \mathtt{RT}_{r+1}$ elements. Since $k = n-1$, we observe superconvergence of $\widehat{\rho}_h^{\mathrm{nor}}$.

r	N	$\lVert\sigma-\sigma_h\rVert_\Omega$		$\lVert\sigma-\delta u_h^*\rVert_{\partial\mathcal{T}_h}$		$\lVert u-u_h\rVert_\Omega$		$\lVert u-u_h^*\rVert_{\mathcal{T}_h}$		$\lVert\mathrm{d}(u-u_h)\rVert_\Omega$		$\lVert\mathrm{d}u-\rho_h^*\rVert_{\mathcal{T}_h}$		$\lVert\delta\mathrm{d}(u-u_h)\rVert_{\mathcal{T}_h}$		$\lVert\delta(\mathrm{d}u-\rho_h^*)\rVert_{\mathcal{T}_h}$	
0	1	1.83e+00	—	2.14e+00	—	6.42e-01	—	6.97e-01	—	2.28e+00	—	1.97e+00	—	1.81e+01	—	1.31e+01	—
	2	1.62e+00	0.2	1.75e+00	0.3	5.08e-01	0.3	5.07e-01	0.5	1.69e+00	0.4	7.57e-01	1.4	1.81e+01	0.0	9.79e+00	0.4
	4	8.47e-01	0.9	9.51e-01	0.9	2.70e-01	0.9	2.62e-01	0.9	9.05e-01	0.9	2.19e-01	1.8	1.81e+01	-0.0	5.32e+00	0.9
	8	4.32e-01	1.0	4.90e-01	1.0	1.38e-01	1.0	1.33e-01	1.0	4.60e-01	1.0	5.78e-02	1.9	1.81e+01	0.0	2.72e+00	1.0
	16	2.17e-01	1.0	2.47e-01	1.0	6.92e-02	1.0	6.66e-02	1.0	2.31e-01	1.0	1.47e-02	2.0	1.81e+01	0.0	1.37e+00	1.0
1	1	1.64e+00	—	1.72e+00	—	5.35e-01	—	5.55e-01	—	1.83e+00	—	9.23e-01	—	1.36e+01	—	9.40e+00	—
	2	3.27e-01	2.3	4.69e-01	1.9	1.52e-01	1.8	1.59e-01	1.8	5.93e-01	1.6	2.21e-01	2.1	1.02e+01	0.4	3.66e+00	1.4
	4	8.45e-02	2.0	1.24e-01	1.9	4.06e-02	1.9	4.26e-02	1.9	1.63e-01	1.9	3.07e-02	2.8	5.50e+00	0.9	1.01e+00	1.9
	8	2.17e-02	2.0	3.15e-02	2.0	1.04e-02	2.0	1.09e-02	2.0	4.16e-02	2.0	3.94e-03	3.0	2.81e+00	1.0	2.59e-01	2.0
	16	5.49e-03	2.0	7.93e-03	2.0	2.62e-03	2.0	2.74e-03	2.0	1.05e-02	2.0	4.97e-04	3.0	1.41e+00	1.0	6.51e-02	2.0
2	1	2.84e-01	—	6.60e-01	—	1.52e-01	—	2.25e-01	—	6.05e-01	—	6.63e-01	—	9.82e+00	—	6.18e+00	—
	2	5.41e-02	2.4	1.02e-01	2.7	3.39e-02	2.2	4.16e-02	2.4	1.67e-01	1.9	5.40e-02	3.6	4.16e+00	1.2	1.08e+00	2.5
	4	6.81e-03	3.0	1.33e-02	2.9	4.55e-03	2.9	5.71e-03	2.9	2.30e-02	2.9	3.75e-03	3.8	1.15e+00	1.9	1.50e-01	2.8
	8	8.63e-04	3.0	1.68e-03	3.0	5.82e-04	3.0	7.31e-04	3.0	2.95e-03	3.0	2.41e-04	4.0	2.95e-01	2.0	1.92e-02	3.0
	16	1.09e-04	3.0	2.11e-04	3.0	7.34e-05	3.0	9.20e-05	3.0	3.71e-04	3.0	1.51e-05	4.0	7.43e-02	2.0	2.42e-03	3.0

TABLE 7. Errors and rates for the manufactured solution in Table 6, after local postprocessing with broken $\star\mathcal{P}_{r+1}^-\Lambda^0 \cong \mathtt{CG}_{r+1}$ and $\star\mathcal{P}_{r+1}^-\Lambda^1 \cong \mathtt{N1E}_{r+1}$ elements. Since $k = n-1$, we observe improved convergence of ρ_h^* as compared with $\mathrm{d}u_h$.

in Sections 4.3 and 5.2, respectively. Again, since $k = n - 1$, we see that $\widehat{\rho}_h^{\mathrm{nor}}$ and ρ_h^* superconverge with rate $\mathcal{O}(h^{r+2})$, whereas $\mathrm{d}u_h$ only converges with rate $\mathcal{O}(h^{r+1})$.

8. A VIEW TOWARD HDG METHODS FOR FINITE ELEMENT EXTERIOR CALCULUS

In this last section, we briefly present an even more general approach to domain decomposition and hybrid methods for the Hodge–Laplace problem. This includes hybridization of the conforming FEEC methods we have discussed so far, as well as nonconforming and HDG methods. In the cases $k = 0$ and $k = n$, we recover the unified hybridization framework of Cockburn, Gopalakrishnan, and Lazarov [18] for the scalar Poisson equation. When $n = 3$, the cases $k = 1$ and $k = 2$ include some recently proposed HDG methods for the vector Poisson equation and Maxwell's equations. Although we lay out the framework here, we postpone a detailed discussion and analysis of these methods for future work.

8.1. **Variational principle.** To motivate the variational principle for these more general methods, we begin with a new formulation of the exact local solvers for the Hodge–Laplace problem. Given

$\widehat{\sigma}^{\tan}$, $\widehat{u}^{\tan}$ on ∂K, $\overline{u} \in \mathring{\mathfrak{H}}^k(K)$, and $p \in \mathfrak{H}^k$, observe that the exact solution satisfies

$$\begin{aligned}
(\sigma, \tau)_K - (u, \mathrm{d}\tau)_K + \langle u^{\mathrm{nor}}, \tau^{\tan} \rangle_{\partial K} &= 0, && \forall \tau \in H\Lambda^{k-1}(K) \cap H^*\Lambda^{k-1}(K),\\
(\sigma, \delta v)_K + (\rho, \mathrm{d}v)_K + (\overline{p}, v)_K - \langle \rho^{\mathrm{nor}}, v^{\tan} \rangle_{\partial K} &= (f - p, v)_K - \langle \widehat{\sigma}^{\tan}, v^{\mathrm{nor}} \rangle_{\partial K}, &&\\
& && \forall v \in H\Lambda^k(K) \cap H^*\Lambda^k(K),\\
(\rho, \eta)_K - (u, \delta\eta)_K &= \langle \widehat{u}^{\tan}, \eta^{\mathrm{nor}} \rangle_{\partial K}, && \forall \eta \in H\Lambda^{k+1}(K) \cap H^*\Lambda^{k+1}(K),\\
(u, \overline{q})_K &= (\overline{u}, \overline{q})_K, && \forall \overline{q} \in \mathring{\mathfrak{H}}^k(K).
\end{aligned}$$

Here, both d and δ are taken weakly, as they are only applied to test functions.

Now, suppose we choose finite element spaces $W_h^k(K) \subset H\Lambda^k(K) \cap H^*\Lambda^k(K)$ for each $K \in \mathcal{T}_h$, giving the broken space $W_h^k \coloneqq \prod_{K \in \mathcal{T}_h} W_h^k(K)$, and likewise for $W_h^{k\pm 1}$. Suppose we also choose unbroken spaces $\widehat{V}_h^{k-1,\tan} \subset \widehat{V}^{k-1,\tan}$ and $\widehat{V}_h^{k,\tan} \subset \widehat{V}^{k,\tan}$, which do not necessarily correspond to tangential traces of W_h^{k-1} and W_h^k. Then we consider the variational problem: Find

(local variables)	$\sigma_h \in W_h^{k-1}$,	$u_h \in W_h^k$,	$\rho_h \in W_h^{k+1}$,	$\overline{p}_h \in \overline{\mathfrak{H}}_h^k$,
(global variables)	$p_h \in \mathfrak{H}_h^k$,	$\overline{u}_h \in \overline{\mathfrak{H}}_h^k$,	$\widehat{\sigma}_h^{\tan} \in \widehat{V}_h^{k-1,\tan}$,	$\widehat{u}_h^{\tan} \in \widehat{V}_h^{k,\tan}$,

satisfying

$$(\sigma_h, \tau_h)_{\mathcal{T}_h} - (u_h, \mathrm{d}\tau_h)_{\mathcal{T}_h} + \langle \widehat{u}_h^{\mathrm{nor}}, \tau_h^{\tan} \rangle_{\partial\mathcal{T}_h} = 0, \qquad \forall \tau_h \in W_h^{k-1}, \tag{21a}$$

$$\begin{aligned}(\sigma_h, \delta v_h)_{\mathcal{T}_h} &+ (\rho_h, \mathrm{d}v_h)_{\mathcal{T}_h} + (\overline{p}_h + p_h, v_h)_{\mathcal{T}_h}\\ &+ \langle \widehat{\sigma}_h^{\tan}, v_h^{\mathrm{nor}} \rangle_{\partial\mathcal{T}_h} - \langle \widehat{\rho}_h^{\mathrm{nor}}, v_h^{\tan} \rangle_{\partial\mathcal{T}_h} = (f, v_h)_{\mathcal{T}_h}, \qquad \forall v_h \in W_h^k,\end{aligned} \tag{21b}$$

$$(\rho_h, \eta_h)_{\mathcal{T}_h} - (u_h, \delta\eta_h)_{\mathcal{T}_h} - \langle \widehat{u}_h^{\tan}, \eta_h^{\mathrm{nor}} \rangle_{\partial\mathcal{T}_h} = 0, \qquad \forall \eta_h \in W_h^{k+1}, \tag{21c}$$

$$(\overline{u}_h - u_h, \overline{q}_h)_{\mathcal{T}_h} = 0, \qquad \forall \overline{q}_h \in \overline{\mathfrak{H}}_h^k, \tag{21d}$$

$$(u_h, q_h)_{\mathcal{T}_h} = 0, \qquad \forall q_h \in \mathfrak{H}_h^k, \tag{21e}$$

$$(\overline{p}_h, \overline{v}_h)_{\mathcal{T}_h} = 0, \qquad \forall \overline{v}_h \in \overline{\mathfrak{H}}_h^k, \tag{21f}$$

$$\langle \widehat{u}_h^{\mathrm{nor}}, \widehat{\tau}_h^{\tan} \rangle_{\partial\mathcal{T}_h} = 0, \qquad \forall \widehat{\tau}_h^{\tan} \in \widehat{V}_h^{k-1,\tan}, \tag{21g}$$

$$\langle \widehat{\rho}_h^{\mathrm{nor}}, \widehat{v}_h^{\tan} \rangle_{\partial\mathcal{T}_h} = 0, \qquad \forall \widehat{v}_h^{\tan} \in \widehat{V}_h^{k,\tan}. \tag{21h}$$

To complete the specification of the problem, one must define the approximate normal traces $\widehat{u}_h^{\mathrm{nor}}$ and $\widehat{\rho}_h^{\mathrm{nor}}$, which play the same role as the "numerical flux" does in Cockburn, Gopalakrishnan, and Lazarov [18]. The discrete harmonic spaces $\overline{\mathfrak{H}}_h^k$ and $\mathfrak{H}_h^k$ are then defined so that the local and global solvers have unique solutions.

8.2. **The cases $k = 0$ and $k = n$.** We now show that, for the scalar Poisson equation, we recover the unified hybridization framework of Cockburn, Gopalakrishnan, and Lazarov [18]. If $k = 0$, then in terms of scalar and vector proxies, (21) simplifies to

$$\begin{aligned}
(\rho_h, \operatorname{grad} v_h)_{\mathcal{T}_h} + (p_h, v_h)_{\mathcal{T}_h} - \langle \widehat{\rho}_h^{\mathrm{nor}}, v_h \rangle_{\partial\mathcal{T}_h} &= (f, v_h)_{\mathcal{T}_h}, && \forall v_h \in W_h^0,\\
(\rho_h, \eta_h)_{\mathcal{T}_h} + (u_h, \operatorname{div} \eta_h)_{\mathcal{T}_h} - \langle \widehat{u}_h^{\tan}, \eta_h \cdot \mathbf{n} \rangle_{\partial\mathcal{T}_h} &= 0, && \forall \eta_h \in W_h^1,\\
(u_h, q_h)_{\mathcal{T}_h} &= 0, && \forall q_h \in \mathfrak{H}_h^0,\\
\langle \widehat{\rho}_h^{\mathrm{nor}}, \widehat{v}_h^{\tan} \rangle_{\partial\mathcal{T}_h} &= 0, && \forall \widehat{v}_h^{\tan} \in \widehat{V}_h^{0,\tan},
\end{aligned}$$

which gives the methods of [18] for the Neumann problem. As before, essential Dirichlet boundary conditions may be imposed on $\widehat{V}_h^{0,\tan}$, in which case the global harmonic space becomes trivial. Whatever the global boundary conditions on $\partial\Omega$, we have local Dirichlet solvers on each $K \in \mathcal{T}_h$. Different methods are obtained by various choices of W_h^0, W_h^1, $\widehat{V}_h^{0,\tan}$ and the numerical flux $\widehat{\rho}_h^{\mathrm{nor}}$.

Alternatively, if $k = n$, and each $K \in \mathcal{T}_h$ is connected (e.g., simplicial), then (21) becomes

$$\begin{aligned}
(\sigma_h, \tau_h)_{\mathcal{T}_h} - (u_h, \operatorname{div} \tau_h)_{\mathcal{T}_h} + \langle \widehat{u}_h^{\mathrm{nor}}, \tau_h \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \tau_h \in W_h^{n-1}, \\
-(\sigma_h, \operatorname{grad} v_h)_{\mathcal{T}_h} + (\overline{p}_h, v_h)_{\mathcal{T}_h} + \langle \widehat{\sigma}_h^{\tan}, v_h \mathbf{n} \rangle_{\partial \mathcal{T}_h} &= (f, v_h)_{\mathcal{T}_h}, && \forall v_h \in W_h^n, \\
(\overline{u}_h - u_h, \overline{q}_h)_{\mathcal{T}_h} &= 0, && \forall \overline{q}_h \in \mathbb{R}^{\mathcal{T}_h}, \\
(\overline{p}_h, \overline{v}_h)_{\mathcal{T}_h} &= 0, && \forall \overline{v}_h \in \mathbb{R}^{\mathcal{T}_h}, \\
\langle \widehat{u}_h^{\mathrm{nor}}, \widehat{\tau}_h^{\tan} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{\tau}_h^{\tan} \in \widehat{V}_h^{k-1,\tan},
\end{aligned}$$

which is the alternative hybridization of Cockburn [15, Section 5] using local Neumann solvers. In this case, various methods are specified by defining W_h^{n-1}, W_h^n, $\widehat{V}_h^{k-1,\tan}$ and $\widehat{u}_h^{\mathrm{nor}}$.

8.3. **Examples of methods.** Different choices of the finite element spaces and approximate normal traces in (21) yield different families of methods. We now discuss a few specific examples.

8.3.1. *The hybridized FEEC methods.* Suppose we choose the spaces W_h and $\widehat{V}_h$ as in Section 4. We then define $\widehat{u}_h^{\mathrm{nor}} \in \widehat{W}_h^{k-1,\mathrm{nor}}$ and $\widehat{\rho}_h^{\mathrm{nor}} \in \widehat{W}_h^{k,\mathrm{nor}}$ to be new unknown variables, which are determined by augmenting (21) by the equations

$$\langle \widehat{\sigma}_h^{\tan} - \sigma_h^{\tan}, \widehat{v}_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} = 0, \quad \forall \widehat{v}_h^{\mathrm{nor}} \in \widehat{W}_h^{k-1,\mathrm{nor}}, \tag{11d}$$

$$\langle \widehat{u}_h^{\tan} - u_h^{\tan}, \widehat{\eta}_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} = 0, \quad \forall \widehat{\eta}_h^{\mathrm{nor}} \in \widehat{W}_h^{k,\mathrm{nor}}. \tag{11e}$$

Using these, (21b) and (21c) become equivalent to (11b) and $\rho_h = \mathrm{d} u_h$, respectively. Hence, the variational problem is equivalent to (11), so we recover the hybridized FEEC methods of Section 4.

8.3.2. *Mixed and nonconforming hybrid methods.* Suppose we take $\widehat{u}_h^{\mathrm{nor}} = u_h^{\mathrm{nor}}$ and $\widehat{\rho}_h^{\mathrm{nor}} = \rho_h^{\mathrm{nor}}$. Then, using integration by parts, (21) simplifies to

$$\begin{aligned}
(\delta u_h, \delta v_h)_{\mathcal{T}_h} + (\delta \rho_h, v_h)_{\mathcal{T}_h} + (\overline{p}_h + p_h, v_h)_{\mathcal{T}_h} + \langle \widehat{\sigma}_h^{\tan}, v_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} &= (f, v_h)_{\mathcal{T}_h}, && \forall v_h \in W_h^k, \\
(\rho_h, \eta_h)_{\mathcal{T}_h} - (u_h, \delta \eta_h)_{\mathcal{T}_h} - \langle \widehat{u}_h^{\tan}, \eta_h^{\mathrm{nor}} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \eta_h \in W_h^{k+1}, \\
(\overline{u}_h - u_h, \overline{q}_h)_{\mathcal{T}_h} &= 0, && \forall \overline{q}_h \in \overline{\mathfrak{H}}_h^k, \\
(u_h, q_h)_{\mathcal{T}_h} &= 0, && \forall q_h \in \mathfrak{H}_h^k, \\
(\overline{p}_h, \overline{v}_h)_{\mathcal{T}_h} &= 0, && \forall \overline{v}_h \in \overline{\mathfrak{H}}_h^k, \\
\langle u_h^{\mathrm{nor}}, \widehat{\tau}_h^{\tan} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{\tau}_h^{\tan} \in \widehat{V}_h^{k-1,\tan}, \\
\langle \rho_h^{\mathrm{nor}}, \widehat{v}_h^{\tan} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{v}_h^{\tan} \in \widehat{V}_h^{k,\tan},
\end{aligned}$$

and $\sigma_h = \delta u_h$. This is just the Hodge dual of the alternative hybridization approach described in Section 3.5. In particular, when $k = 0$ with Dirichlet boundary conditions on $\partial \Omega$, this becomes

$$\begin{aligned}
-(\operatorname{div} \rho_h, v_h)_{\mathcal{T}_h} &= (f, v_h)_{\mathcal{T}_h}, && \forall v_h \in W_h^0, \\
(\rho_h, \eta_h)_{\mathcal{T}_h} + (u_h, \operatorname{div} \eta_h)_{\mathcal{T}_h} - \langle \widehat{u}_h^{\tan}, \eta_h \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \eta_h \in W_h^1, \\
\langle \rho_h \cdot \mathbf{n}, \widehat{v}_h^{\tan} \rangle_{\partial \mathcal{T}_h} &= 0, && \forall \widehat{v}_h^{\tan} \in \widehat{V}_h^{0,\tan},
\end{aligned}$$

which includes the classic hybridized RT and BDM methods [3, 8] using local Dirichlet solvers. When $k = n$, this becomes

$$\begin{aligned}
(\operatorname{grad} u_h, \operatorname{grad} v_h)_{\mathcal{T}_h} + (\overline{p}_h, v_h)_{\mathcal{T}_h} + \langle \widehat{\sigma}_h^{\tan}, v_h \mathbf{n} \rangle_{\partial \mathcal{T}_h} &= (f, v_h)_{\mathcal{T}_h}, & \forall v_h &\in W_h^n, \\
(\overline{u}_h - u_h, \overline{q}_h)_{\mathcal{T}_h} &= 0, & \forall \overline{q}_h &\in \overline{\mathfrak{H}}_h^n, \\
(\overline{p}_h, \overline{v}_h)_{\mathcal{T}_h} &= 0, & \forall \overline{v}_h &\in \overline{\mathfrak{H}}_h^n, \\
\langle u_h \mathbf{n}, \widehat{\tau}_h^{\tan} \rangle_{\partial \mathcal{T}_h} &= 0, & \forall \widehat{\tau}_h^{\tan} &\in \widehat{V}_h^{n-1,\tan},
\end{aligned}$$

which includes the nonconforming hybrid method of Raviart and Thomas [39].

8.3.3. *Hybridizable discontinuous Galerkin methods.* Suppose we take

$$\widehat{u}_h^{\mathrm{nor}} = u_h^{\mathrm{nor}} - \lambda(\widehat{\sigma}_h^{\tan} - \sigma_h^{\tan}), \qquad \widehat{\rho}_h^{\mathrm{nor}} = \rho_h^{\mathrm{nor}} + \mu(\widehat{u}_h^{\tan} - u_h^{\tan}),$$

where λ and μ are penalty functions on $\partial \mathcal{T}_h$. Section 8.3.2 corresponds to the case $\lambda = \mu = 0$, while the hybridized FEEC methods of Section 4 can be seen as the limiting case $\lambda, \mu \to \infty$.

When $k = 0$, (21) becomes the hybrid local discontinuous Galerkin (LDG-H) method of [18], while $k = n$ gives the alternative implementation of [15, Section 5] using local Neumann solvers. For the vector Poisson equation when $n = 2$ or $n = 3$, (21) corresponds to the recent HDG methods of Nguyen, Peraire, and Cockburn [35], Chen, Qiu, Shi, and Solano [14], which have been applied to Maxwell's equations.

A different family of HDG methods may be constructed by taking

$$\widehat{u}_h^{\mathrm{nor}} = u_h^{\mathrm{nor}} - \lambda\big(\widehat{\sigma}_h^{\tan} - (\delta u_h)^{\tan}\big), \qquad \widehat{\rho}_h^{\mathrm{nor}} = \mathrm{d}u_h^{\mathrm{nor}} + \mu(\widehat{u}_h^{\tan} - u_h^{\tan}),$$

which generalizes the hybrid interior penalty (IP-H) method of [18].

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