The second boundary value problem for a discrete Monge-Ampère equation

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Outline

Problems in geometric optics lead to the resolution of Monge-Ampère equations (generated Jacobian equations)

$$\det DT_u(x) = \psi(x, u(x), T_u(x)), \quad T_u(x) = T(x, u(x), Du(x))$$
$$T_u(X) = Y.$$

Several of these problems fall in the framework of optimal transport. Model problem for quadratic cost

$$\det D^2 u(x) = \frac{f(x)}{g(Du(x))}$$
$$Du(X) = Y.$$

It is known that the transport map is given by $T(x) = Du(x)$

Solving discretized Monge-Ampère equations leads to the computation of transport maps (for absolutely continuous measures)

That is the point of view taken in semi-discrete optimal transport. The latter is usually said to be a generalization of the Oliker-Prussner method

The Oliker-Prussner method is a discretization of the Dirichlet problem based on the notion of Aleksandrov solution.

Natural generalization of the Oliker-Prussner method for the second boundary condition

As with semi-discrete optimal transport, the computational complexity is $O(N^2)$

Finite difference version, which has linear complexity

May be generalized to other cost functions.

Aleksandrov solutions

Find a convex function $u \in C^2([a,b])$ such that

$$g(u'(x))u''(x) = f(x), x \in (a,b) \quad (1)$$
$$u'((a,b)) = (\alpha, \beta). \quad (2)$$

$x \mapsto u'(x)$ is a surjective mapping from (a,b) onto (α,β). Since u is convex, u' is increasing and hence (2) is equivalent to

$$u'(a) = \alpha \text{ and } u'(b) = \beta.$$

Compatibility : change of variable $x \to \gamma(x) = u'(x) = p$ (gradient mapping), provided that u' is one-to-one, i.e. u is strictly convex

$$\int_a^b f(x)dx = \int_a^b g(u'(x))u''(x)dx = \int_\alpha^\beta g(p)dp.$$

Monge-Ampère measure $M[u](B) = \int_{\gamma(B)} g(p)\, dp$
Replace $\gamma(x)$ by subgradient mapping for non smooth solutions

$$\partial u(x_0) = \{\, p \in \mathbb{R} : u(x) \geq u(x_0) + p(x - x_0), \text{ for all}\, x \in X \,\}.$$

Find u convex such that

$$\int_{\partial u(B)} g(p)\, dp = \int_B f(x)\, dx, \text{ for all Borel sets } B \subset (a, b)$$

Method of supporting paraboloids and Semi-discrete optimal transport

The target density g is approximated by a sum of Dirac masses $\sum_{i=1}^{M} r_i \delta_{P_i}$ for $P_i \in Y$ and $r_i > 0$ for all i.

Energy conservation reads $\sum_{i=1}^{M} r_i = \int_X f(x)\,dx$.

The solution is given by the graph of the convex function

$$u_M(x) = \max_{i=1,\ldots,M} x \cdot P_i - b_i,$$

with rays in the region

$$W_i(b) = \{\, x \in X, x \cdot P_i - b_i \geq x \cdot P_j - b_j \text{ for all } j = 1, \ldots, M \,\},$$

reflected in the direction P_i. We thus need

$$\int_{W_i(b)} f(x)\,dx = r_i, i = 1, \ldots, M.$$

Note that $Du_M \subset \{ P_1, \ldots, P_M \} \subset Y$.

The solution $u_M(x) = \max_{i=1,\dots,M} x \cdot P_i - b_i$ from the method of supporting paraboloids induces a map $T : X \to Y$ defined by

$$T(x) = P_i \text{ for } x \in W_i(b).$$

This map is optimal in the sense that it minimizes the total cost

$$\int_X c(x, T(x))\, dx,$$

where $c(x, y) = |x - y|^2$ is the cost of moving " mass at x to y", among all "measure preserving maps" from X to Y.

The solution $u_M(x) = \max_{i=1,\ldots,M} x \cdot P_i - b_i$ also induces a mapping $\psi : Y \to \mathbb{R}$, the Legendre transform of u_M, such that $\psi(P_i) = b_i$.

The convex envelope $\hat{\psi}$ of ψ, i.e. the largest convex function below ψ, solves in a weak sense (an equation similar to) a Monge-Ampère equation $f(D\hat{\psi}) \det D^2\hat{\psi} = g$.

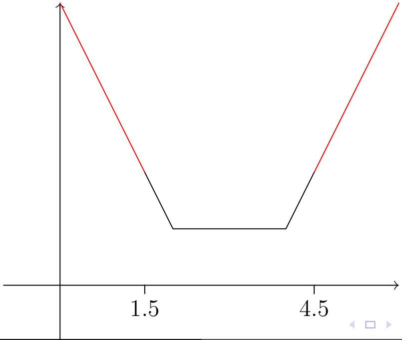
For a direct approximation of a solution of $g(Du) \det D^2 u = f$, one first solves $f(D\psi) \det D^2\psi = g$ for the discrete mapping $\psi : \{P_1, \ldots, P_N\} \to \mathbb{R}$ by rewriting the subdifferential in terms of the so-called Laguerre cells of ψ.

Note the change of point of view : u_M is obtained not by seeking $u_M(x), x \in X$ but by the values b_i at P_i of its Legendre transform.

Second boundary condition in terms of asymptotic cone.
Given $x_0 \in (a, b)$ and $p \in \partial u(x_0)$ the affine function $L(x) = u(x_0) + p(x - x_0)$ is said to be subtangent to u at x_0.
Define

$$\overline{u}(x) = \sup\{\, u(y) + p(x - y), y \in (a, b) \text{ and } p \in \partial u(y)\,\}.$$

It can be shown that $\overline{u}$ is a convex extension of u.



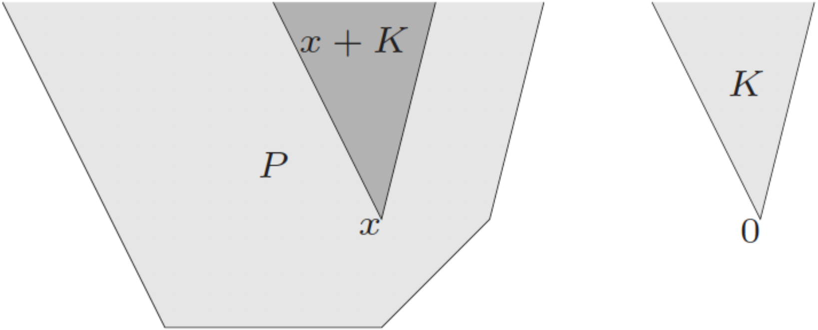
To get the extension $\overline{u}$, one needs to know $\partial u(y)$ for all $y \in (a, b)$. The extension can also be obtained with just the knowledge of $\partial u(a, b) = (\alpha, \beta)$.

Epigraph of $\overline{u}$ is given by

$$\mathrm{epi}\,\overline{u} = \{\, (x, y) \in \mathbb{R}^2, y \geq \overline{u}(x) \,\}.$$

K denotes the epigraph of $k(x) = \max\{\, \alpha x, \beta x \,\}$. For $(r, s) \in \mathbb{R}^2$, $(r, s) + K$ is the epigraph of the function

$$k_{(r,s)}(x) = \max\{\, \alpha(x - r) + s, \beta(x - r) + s \,\}.$$


$x + K$
P
x
K
0

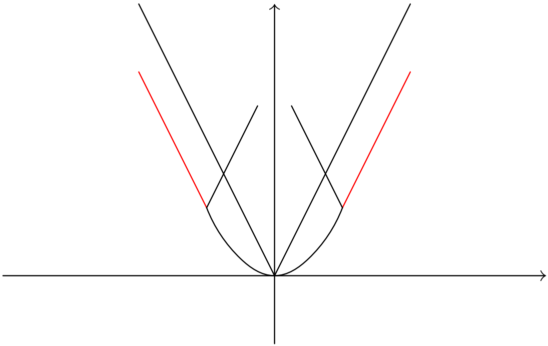


FIGURE – The function $u(x) = x^2$ solves $u'' = 2$ on (-1,1) with $\partial u(-1,1) = (-2,2)$. Its extension to $\mathbb{R}$ is also shown. Here $\alpha = -2$ and $\beta = 2$. The graphs of $k_{0,0}$, $k_{(-1,1)}$ and $k_{(1,1)}$ are shown. Their epigraphs are completely contained in the epigraph of the extension $\overline{u}$ of u.

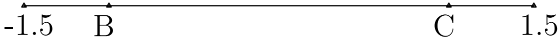
For $x \notin (a, b)$

$$\overline{u}(x) = \inf_{y \in (a,b)} k_{(y,u(y))}(x)$$
$$= \inf_{y \in (a,b)} \max\{\, \alpha(x-y) + u(y), \beta(x-y) + u(y)\,\}$$
$$\overline{u}(x) = \min\{\, k_{(a,\overline{u}(a))}(x), k_{(b,\overline{u}(b))}(x)\,\}.$$

$u \in C(a, b)$ is extended by continuity to $[a, b]$.

Example in 1D of piecewise linear convex function

Assume that $X = (-1.5, 1.5)$ and $Y = (-2, 2)$.



Consider the piecewise linear convex function on X with vertices at $B(-1, 0)$ and $C(1, 0)$.

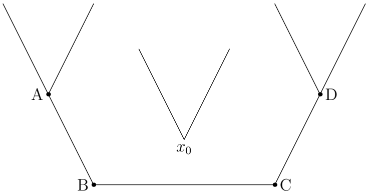
Recall that

$$\partial u(x) = \{\, p \in \mathbb{R}, u(y) \geq u(x) + p(y-x), \forall y \in (a,b) \,\}.$$

For $x \notin \{B, C\}$, $u'(x)$ is constant. At those points $\partial u(x) \in \{\, -2, 0, 2 \,\}$ with $|\partial u(x)| = 0$. We have $\partial u(B) = [-2, 0]$ and $\partial u(C) = [0, 2]$. Thus $|\partial u(B)| = 2$ and $|\partial u(C)| = 2$. Also, $\partial u(X) = \overline{Y}$.

The epigraph of u is a bounded convex set.

The epigraph of the extension of u is an unbounded convex set.



We have $\partial u(\Omega) = \partial u(\mathbb{R}) = \overline{\Omega^*}$.

Monge-Ampère equation reformulated in terms of asymptotic cone.
Find u convex on X which solves equation in X and given outside of X by

$$u(x) = \min_{s \in \partial X} u(s) + k_Y(x - s).$$

g-curvature of convex functions

Let v be a convex function on $\mathbb{R}^d$.

$$\chi_v(E) = \cup_{x \in E} \chi_v(x).$$

g-curvature as the set function

$$\omega(g, v, E) = \int_{\chi_v(E)} g(p) dp.$$

Extend f and g by 0 to $\mathbb{R}^d$ with equation in measures

$$\omega(g, u, E) = \int_E f(x) dx \text{ for all Borel sets } E \subset \overline{X}$$
$$\chi_u(\overline{X}) = \overline{Y}.$$

Approximation by piecewise linear convex functions which have subdifferential a polygon

Sequences of polygons $K^* \subset Y$ $K^* \to Y$. To K^* one associates a cone K which is the epigraph of

$$\max_{j=1,\ldots,N} x \cdot a_j^*,$$

where a_j^* is a vertex of K^*.

Find a piecewise linear convex function u_h with asymptotic cone K such that

$$\omega(g, u_h, x) = \sum_{x \in X_h} c_x \delta_x,$$

where $\sum_{x \in X_h} c_x \delta_x \to \mu_f$.

Subdifferential of the extension

$$\chi_u(y) = \{\, q \in \mathbb{R}^d : \tilde{u}(z) \geq \tilde{u}(x) + q \cdot (z - x), \text{ for all } z \in \mathbb{R}^d \,\}.$$

For X, Y bounded convex $\partial u(X) = Y$ implies $\chi_u(\overline{X}) = \chi_u(\mathbb{R}^d) = \overline{Y}$.

Assume first that Y is polygonal with vertices $a_i^*, i = 1, \ldots, N$. Recall the support function of Y : $p \in \mathbb{R}^d, k_Y(p) = \sup_{y \in Y} p \cdot y$.

Choose a set of vectors V_0 of normals to facets of Y. Then $p \in Y$ iff $p \cdot e \leq k_Y(e)$ for all $e \in V_0$.

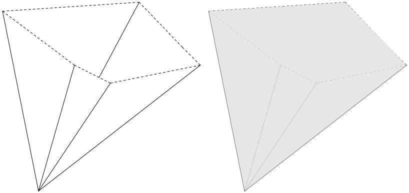


FIGURE – Polyhedral angle in $\mathbb{R}^3$

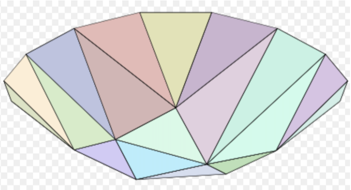


FIGURE – Piecewise linear convex function. Reference : wikipedia.

The unknown are the (finite set of) mesh values $\{ u_h(x), x \in X_h \}$ and the second boundary condition is enforced implicitly using the discrete extension formula

$$u_h(x) = \min_{y \in \partial X_h} \max_{1 \leq j \leq N} (x - y) \cdot a_j^* + u_h(y).$$

(for X polygonal)

The min and the max are over a finite number of points.

This is sometimes called Oliker-Prussner method. Here for the second boundary value problem.

Orthogonal lattice with mesh length $h : \mathbb{Z}_h^d = \{ mh, m \in \mathbb{Z}^d \}$.
Put $X_h = X \cap \mathbb{Z}_h^d$ and

$$\partial X_h = \{ x \in X_h \text{ such that for some } i = 1, \ldots, d, x + hr_i \notin X_h \text{ or } x - hr_i \notin X_h \}.$$

Put $\mathcal{N}_h = X_h \cup \{ x + he, e \in V_0, x \in \partial X_h \}$.

Discrete convexity For a function v_h on $\mathbb{Z}_h^d$, $e \in \mathbb{Z}^d$ and $x \in X_h$

$$\Delta_{he} v_h(x) = v_h(x + he) - 2v_h(x) + v_h(x - he).$$

The unknown in the numerical scheme is a mesh function v_h on X_h which is extended to $\mathbb{Z}^d$ using the extension formula, and which is discrete convex ($\Delta_{he} v_h(x) \geq 0$).

Stencil

Definition

A stencil V is a set valued mapping from X_h to the set of finite subsets of $\mathbb{Z}^d \setminus \{0\}$.

Minimal stencil V_{min} is symmetric with respect to the origin, contains the elements of the canonical basis of $\mathbb{R}^d$ and the set of normals V_0.

Extended mesh $\mathcal{N}_h^2 = X_h \cup \{ x + he, x \in X_h, e \in V_{min} \}$.

Maximal stencil V_{max} such that $e \in V_{max}(x)$ iff $x + he \in \mathcal{N}_h^2$.

$$V_{min} \subset V(x) \subset V_{max}(x).$$

For convergence, and $f \in C(\overline{X})$, choose $V(x)$ to be in addition symmetric with respect to the origin and with vectors with co-prime coordinates.

$$\partial_V v_h(x) = \{ p \in \mathbb{R}^d, p \cdot (he) \geq v_h(y) - v_h(y - he) \forall e \in V(x) \}.$$

$$\omega_V(g, v_h, x) = \int_{\partial_V v_h(x)} g(p) dp, x \in X_h.$$

find $u_h \in \mathcal{C}_h$ with asymptotic cone K such that

$$\omega_V(g, u_h, \{ x \}) = \int_{C_x} \tilde{f}(t) dt, x \in X_h,$$

where C_x with $C_x \cap X_h = \{ x \}$ form a partition of X.

Recall the extension formula

$$u_h(r) = \min_{z \in \partial X_h} u_h(z) + k_Y(r - z).$$

In the case $V = V_{max}$, u_h coincides with its convex envelope and was essentially studied by Bakelman. Existence, uniqueness and convergence of the discretization follows for $f \in L^1(X)$.

$V_{min} \subset V(x) \subsetneq V_{max}(x)$

Existence of solutions follows from the convergence of a damped Newton's method.

Discrete convex mesh functions with asymptotic cone K are Lipschitz continuous with a uniform Lipschitz bound, i.e.

$$|v_h(x) - v_h(y)| \leq C||x - y||_1.$$

If X is a rectangle, which does not require a loss of generality, a subsequence converges to a convex function which is shown to be a viscosity solution for $f \in C(\overline{X})$.

Uniqueness holds under various assumptions for V not necessarily equal to V_{max}

Possibility of having $X_h = X_h^1 \cup X_h^2$ with discrete Monge-Ampère equations on either which do not "interact".

Convergence of the discretization follows from weak convergence of Monge-Ampère measures

Theorem

Let K_m^* be bounded convex polygonal domains increasing to $\overline{Y}$. Then the convex solution u_m of

$$\omega(g, u, E) = \int_E f_{K_m^*}(x)dx \text{ for all Borel sets } E \subset \overline{X}$$

$$\chi_u(\overline{X}) = K_m^*$$

$$u(x^0) = \alpha,$$

for $x^0 \in X$ and $\alpha \in \mathbb{R}$ converges uniformly on compact subsets of X to the solution u with $u(x^0) = \alpha$.

A numerical experiment

Exact solution $u(x, y) = x^2/2 + xy + y^2$ with $X = (0, 1)^2$ and Y is the polygon of area 1 with vertices $(0, 0), (1, 1), (1, 2)$ and $(1, 3)$.

Take $g(x, y) = x + y$ with corresponding right hand side $f(x, y)$.

Initial guess $\partial v_h(X) \subset Y$

Stencil V was taken as $V = -V_1 \cup V_1$ where V_1 consists of the vectors $(1, 0), (0, 1), (1, 1), (1, -1), (2, 1), (-1, 2), (1, 2)$ and $(-2, 1)$.

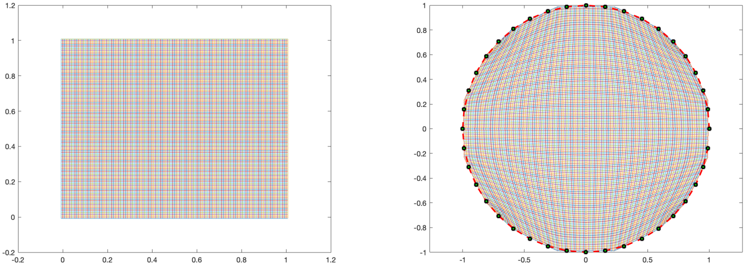
$$g(Du) \det D^2 u = f + u(h, h)$$

Quadrature rules and a damped Newton's method

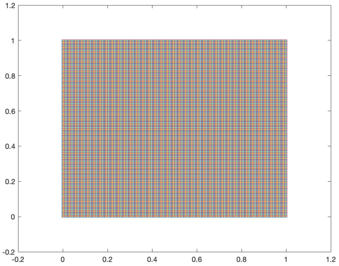
	h				
	$1/2^5$	$1/2^6$	$1/2^7$	$1/2^8$	$1/2^9$
Error for u	$2.72\ 10^{-4}$	$8.01\ 10^{-5}$	$2.31\ 10^{-5}$	$6.52\ 10^{-6}$	$1.82\ 10^{-6}$
Rate		1.76	1.79	1.82	1.84
Error for Du	$6.27\ 10^{-3}$	$3.30\ 10^{-3}$	$1.56\ 10^{-3}$	$8.23\ 10^{-4}$	$3.92\ 10^{-4}$
Rate		0.93	1.07	0.93	1.07

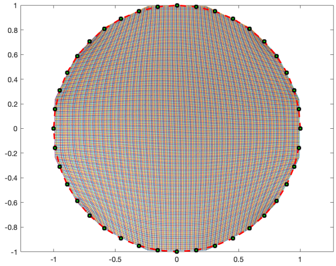
TABLE – Maximum errors for a smooth solution.

Constant density on a square mapped to constant density on the unit disc $h = 1/2^7$



Constant density on a square mapped to Gaussian $e^{-0.5(x^2+y^2)}$ on the unit disc $h = 1/2^8$





Some references

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