Isogeometric method for the elliptic Monge-Ampère equation

Gerard Awanou

Abstract We discuss the application of isogeometric analysis to the fully non linear elliptic Monge-Ampère equation, an equation nonlinear in the highest order derivatives. The construction of smooth discrete spaces renders isogeometric analysis a natural choice for the discretization of the equation.

1 Introduction

We are interested in the numerical resolution of the nonlinear elliptic Monge-Ampère equation

$$\begin{aligned} \det D^2 u &= f \, \text{in} \, \Omega \\ u &= 0 \, \text{on} \, \partial\Omega, \end{aligned} \tag{1}$$

where $D^2 v$ denotes the Hessian of a smooth function v, i.e. $D^2 v$ is the matrix with (i,j)-th entry $\partial^2 v/(\partial x_i \partial x_j)$. Here Ω is a smooth uniformly convex bounded domain of $\mathbb{R}^2$ which is at least $C^{1,1}$and $f \in C(\overline{\Omega})$ with $f \geq c_0 > 0$ for a constant c_0. If $f \in C^{0,\alpha}, 0 < \alpha < 1$, (1) has a classical convex solution in $C^2(\Omega) \cap C(\overline{\Omega})$ and its numerical resolution assuming more regularity on u is understood e.g. [6, 11, 7]. In the non smooth case, various approaches have been proposed e.g. [17, 16]. For various reasons, it is desirable to use standard discretization techniques, which are valid for both the smooth and the non smooth cases. We propose to solve numerically (1) by the discrete version of the sequence of iterates

Gerard Awanou
University of Illinois at Chicago, Department of Mathematics, Statistics, and Computer Science, M/C 249, Chicago, IL 60607-7045, USA e-mail: awanou@uic.edu

$$
\begin{aligned}
(\mathrm{cof}(D^2 u_\varepsilon^k + \varepsilon I)) : D^2 u_\varepsilon^{k+1} &= \det D^2 u_\varepsilon^k + f \,\mathrm{in}\, \Omega \\
u_\varepsilon^{k+1} &= 0 \,\mathrm{on}\, \partial\Omega,
\end{aligned}
\tag{2}
$$

where $\varepsilon > 0$, I is the 2×2 identity matrix and we use the notation $\mathrm{cof} A$ to denote the matrix of cofactors of A, i.e. for all i, j, $(-1)^{i+j}(\mathrm{cof} A)_{ij}$ is the determinant of the matrix obtained from A by deleting its i-th row and its j-th column. For two $n\times n$ matrices A, B, we recall the Frobenius inner product $A : B = \sum_{i,j=1}^n A_{ij} B_{ij}$, where A_{ij} and B_{ij} refer to the entries of the corresponding matrices.

Our recent results [1] indicate that an appropriate space to study a natural variational formulation of (1) is a finite dimensional space of piecewise smooth C^1 functions. For the numerical experiments we will let V_h be a finite dimensional space of piecewise smooth C^1 functions constructed with the isogeometric analysis paradigm. Numerical results indicate that the proposed iterative regularization (2) is effective for non smooth solutions. Formally the sequence defined by (2) converges to a limit u_ε and u_ε converges uniformly on compact subsets of Ω to the solution u of (1) as $\varepsilon \to 0$.

For $\varepsilon = 0$, (2) gives the sequence of Newton's method iterates applied to (1). Surprisingly, for the two dimensional problem, the formal limit u_ε of the sequence u_ε^{k+1} solves the vanishing viscosity approximation of (1)

$$
\begin{aligned}
\varepsilon \Delta u_\varepsilon + \det D^2 u_\varepsilon - f &= 0 \,\mathrm{in}\, \Omega \\
u_\varepsilon &= 0 \,\mathrm{on}\, \partial\Omega.
\end{aligned}
\tag{3}
$$

However discrete versions of Newton's method applied to (3) do not in general perform well for non smooth solutions. This led to the development of alternative methods, e.g. the vanishing moment methodology [11]. The key feature in (2) is that the perturbation εI is included to prevent the matrix $D^2 u_\varepsilon^k + \varepsilon I$ from being singular.

The difficulty of constructing piecewise polynomials C^1 functions is often cited as a motivation to seek alternative approaches to C^1 conforming approximations of the Monge-Ampère equation. In [1] Lagrange multipliers are used to enforce the C^1 continuity, but the extent to which this constraint is enforced in the computations is comparable to the accuracy of the discretization. With the isogeometric method, the basis functions are also C^1 at the computational level. On the other hand another advantage of the isogeometric method is the exact representation of a wide range of geometries which we believe would prove useful in applications of the Monge-Ampère equation to geometric optics. Finally the isogeometric method is widely reported to have better convergence properties than the standard finite element method.

The main difficulty of the numerical resolution of (1) is that Newton's method fails to capture the correct numerical solution when the solution of (1) is not smooth. We proposed in [1] to use a time marching method for solving the discrete equations resulting from a discretization of (1). Moreover in [3] we argued that the correct solution is approximated if one first regularizes the data. However numerical experiments reported in [1] and in this paper indicate that regularization of the data may not be necessary.

It is known that the convex solution u of (1) is the unique minimizer of a certain functional J in a set of convex functions S. It is reasonable to expect, although not very easy to make rigorous, that the set S can be approximated by a set of smooth convex functions S_m and minimizers of J in S_m would approximate the minimizer of J in S. We prove that the functional J has a unique minimizer in a ball of C^1 functions centered at a natural interpolant of a smooth solution u. With a sufficiently close initial guess, a minimization algorithm can be used for the computation of the numerical solution. The difficulty of choosing a suitable initial guess may be circumvented by using a global minimization strategy as in [14]. Netherveless our result can be considered a first step towards clarifying whether regularization of the data is necessary for a proven convergence theory of C^1 approximations of (1) in the non smooth case.

In this paper the numerical solution u_h is computed as the limit of the sequence $u^k_{\varepsilon,h}$ which solve the discrete variational problem associated with (2). For the case of smooth solutions we use $\varepsilon = 0$ in the resulting discrete problem. See Remark 3.3. Since (1) is not approximated directly there is a loss of accuracy. Netherveless our algorithm can be considered a step towards the development of fast iterative methods capable of retrieving the correct numerical approximation to (1) in the context of C^1 conforming approximations.

Let $u_{\varepsilon,h}$ denote the solution of the discrete problem associated to (3). The existence of $u_{\varepsilon,h}$ and $u^k_{\varepsilon,h}$, the convergence of the sequence $(u^k_{\varepsilon,h})_k$ as $k \to \infty$ as well as the behavior of $u_{\varepsilon,h}$ as $\varepsilon \to 0$ will be addressed in a subsequent paper. These results parallel our recent proof of the convergence of the discrete vanishing moment methodology [2].

This paper falls in the category of papers which do not prove convergence of the discretization of (1) to weak solutions but give numerical evidence of convergence as well results in the smooth case and/or in particular cases e.g. [13, 12, 10]. We organize the paper as follows: in the next section we describe the notation used and some preliminaries. In section 3 we prove minimization results at the discrete level. We also derive in section 3 the vanishing viscosity approximation (3) from (2) as well as the discrete variational formulation used in the numerical experiments. In section 4 we recall the isogeometric concept and give numerical results in section 5.

2 Notation and preliminaries

We denote by $C^k(\Omega)$ the set of all functions having all derivatives of order $\leq k$ continuous on Ω where k is a nonnegative integer or infinity and by $C^0(\overline{\Omega})$, the set of all functions continuous on $\overline{\Omega}$. A function u is said to be uniformly Hölder continuous with exponent $\alpha, 0 < \alpha \leq 1$ in Ω if the quantity

$$\sup_{x \neq y} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}},$$

is finite. The space $C^{k,\alpha}(\Omega)$ consists of functions whose k-th order derivatives are uniformly Hölder continuous with exponent α in Ω.

We use the standard notation of Sobolev spaces $W^{k,p}(\Omega)$ with norms $||.||_{k,p}$ and semi-norm $|.|_{k,p}$. In particular, $H^k(\Omega)=W^{k,2}(\Omega)$ and in this case, the norm and semi-norms will be denoted respectively by $||.||_k$ and semi-norm $|.|_k$.

For a function u, we denote by Du its gradient vector and recall that D^2u denotes its Hessian. For a matrix field A, we denote by $\operatorname{div} A$ the vector obtained by taking the divergence of each row.

Using the product rule one obtains for sufficiently smooth vector fields v and matrix fields A

$$\operatorname{div}(Av)=(\operatorname{div} A^T)\cdot v+A:(Dv)^T. \tag{4}$$

Moreover by [8, p. 440]

$$\operatorname{div}\operatorname{cof} D^2 v=0. \tag{5}$$

For computation with determinants, the following results are needed.

Lemma 1. *We have*

$$\det D^2 v=\frac{1}{2}(\operatorname{cof} D^2 v):D^2 v=\frac{1}{2}\operatorname{div}\big((\operatorname{cof} D^2 v)Dv\big). \tag{6}$$

And for $F(v)=\det D^2 v$ we have

$$F'(v)(w)=(\operatorname{cof} D^2 v):D^2 w=\operatorname{div}\big((\operatorname{cof} D^2 v)Dw\big),$$

for v,w sufficiently smooth.

Proof. For a 2×2 matrix A, one easily verifies that $2\det A=(\operatorname{cof} A):A$. It follows that $\det D^2 v=1/2(\operatorname{cof} D^2 v):D^2 v$. Using (4) and (5) we obtain $(\operatorname{cof} D^2 v):D^2 v=\operatorname{div}\big((\operatorname{cof} D^2 v)Dv\big)$ and $(\operatorname{cof} D^2 v):D^2 w=\operatorname{div}\big((\operatorname{cof} D^2 v)\,Dw\big)$. Finally the expression of the Fréchet derivative is obtained from the definition of Fréchet derivative and the expression $\det D^2 v=1/2(\operatorname{cof} D^2 v):D^2 v$. □

Lemma 2. *Let $v,w\in W^{2,\infty}(\Omega)$ and $\psi\in H^2(\Omega)\cap H_0^1(\Omega)$, then*

$$\left|\int_\Omega(\det D^2 v-\det D^2 w)\psi\,dx\right|\leq C(|v|_{2,\infty}+|w|_{2,\infty})|v-w|_1|\psi|_1. \tag{7}$$

The above lemma is a simple consequence of the mean value theorem and Cauchy-Schwarz inequalities. For additional details we refer to [1].

We require our approximation spaces V_h to satisfy the following properties: There exists an interpolation operator Q_h mapping $W^{l+1,p}(\Omega)$ into the space V_h for $1\leq p\leq\infty, 0\leq l\leq d$ with d a constant that depends on V_h and such that

$$||v-Q_h v||_{k,p}\leq C_{ap}h^{l+1-k}||v||_{l+1,p}, \tag{8}$$

for $0\leq k\leq l$ and

$$||v||_{s,p}\leq C_{inv}h^{l-s+\min(0,\frac{n}{p}-\frac{n}{q})}||v||_{l,q}, \forall v\in V_h, \tag{9}$$

for $0 \leq l \leq s, 1 \leq p,q \leq \infty$.

The discussion in [1] is for a space V_h of piecewise polynomials. However, the results quoted here are valid for spaces of piecewise smooth C^1 functions.

We consider the following discretization of (1): find $u_h \in V_h \cap H^1_0(\Omega)$ such that

$$\int_\Omega (\det D^2 u_h) v \, dx = \int_\Omega f v \, dx, \forall v \in V_h \cap H^1_0(\Omega). \tag{10}$$

It can be shown that for $u_h \in H^2(\Omega)$, the left hand side of the above equation is well defined [1]. We recall from [1] that under the assumption that $u \in C^4(\overline{\Omega})$ is a strictly convex function, there exists $\delta > 0$ such that if we define

$$X_h = \{ v_h \in V_h, v_h = 0 \, \text{on} \, \partial\Omega, ||v_h - Q_h u||_1 < \frac{\delta h^2}{4} \},$$

then for h sufficiently small and $v_h \in X_h, ||v_h - Q_h u||_1 < \delta h^2/2$, v_h is convex with smallest eigenvalue bounded a.e. below by $m'/2$ and above by $3M'/2$. Here m' and M' are respectively lower and upper bounds of the smallest and largest eigenvalues of $D^2 u$ in Ω. The idea of the proof is to use the continuity of the eigenvalues of a matrix as a function of its entries. Thus using (8) with $k = 2, p = \infty$ and $l = d$ one obtains that $D^2 Q_h u(x)$ is also positive definite element by element for h sufficiently small. The same argument shows that a C^1 function close to $D^2 Q_h u$ is also piecewise convex and hence convex due to the C^1 continuity. The power of h which appears in the definition of X_h arises from the use of the inverse estimate (9).

We note that by an inverse estimate, for $v_h \in X_h$,

$$||v_h - Q_h u||_{2,\infty} \leq C_{inv} h^{-2} ||v_h - Q_h u||_1 \leq C_{inv} \delta.$$

3 Minimization results

We first note

Lemma 3. *Let v_n, v, w_n and $w \in W^{2,\infty}(\Omega) \cap H^1_0(\Omega)$ such that $||v_n - v||_{2,\infty} \to 0$ and $||w_n - w||_{2,\infty} \to 0$. Then*

$$\int_\Omega (\det D^2 v_n) w_n \, dx \to \int_\Omega (\det D^2 v) w \, dx \tag{11}$$

$$\int_\Omega f v_n \, dx \to \int_\Omega f v \, dx. \tag{12}$$

Proof. Put $\alpha = \int_\Omega (\det D^2 v_n) w_n \, dx - \int_\Omega (\det D^2 v) w \, dx$. We have

$$\alpha = \int_\Omega (\det D^2 v_n - \det D^2 v) w_n \, dx + \int_\Omega (\det D^2 v)(w_n - w) \, dx.$$

Using (7) we obtain

$$|\alpha| \leq C(|v_n|_{2,\infty} + |v|_{2,\infty})|v_n - v|_1 |w_n|_1 + C|v|_{2,\infty}|v|_1 |w_n - w|_1.$$

Since $|v_n - v|_1 \leq C||v_n - v||_{2,\infty}$ and convergent sequences are bounded, (11) follows. We have

$$\left| 3 \int_\Omega f(v_n - v)\, dx \right| \leq C||f||_0 ||v_n - v||_0,$$

and so (12) holds.

We consider the functional J defined by

$$J(v) = -\int_\Omega v \det D^2 v\, dx + 3 \int_\Omega f v\, dx.$$

We have

Lemma 4. *For $v, w \in W^{2,\infty}(\Omega) \cap H_0^1(\Omega)$*

$$J'(v)(w) = 3 \int_\Omega (f - \det D^2 v) w\, dx.$$

Proof. Note that for v, w smooth, vanishing on $\partial\Omega$ and by Lemma 1

$$J'(v)(w) = 3 \int_\Omega f w\, dx - \int_\Omega w \det D^2 v\, dx - \int_\Omega v \operatorname{div}[(\operatorname{cof} D^2 v) Dw]\, dx.$$

But by integration by parts, the symmetry of $D^2 v$ and Lemma 1

$$\int_\Omega v \operatorname{div}[(\operatorname{cof} D^2 v) Dw]\, dx = -\int_\Omega [(\operatorname{cof} D^2 v) Dw] \cdot Dv\, dx = -\int_\Omega [(\operatorname{cof} D^2 v) Dv] \cdot Dw\, dx$$
$$= \int_\Omega w \operatorname{div}[(\operatorname{cof} D^2 v) Dv]\, dx = 2 \int_\Omega w \det D^2 v\, dx.$$

Thus

$$J'(v)(w) = 3 \int_\Omega (f - \det D^2 v) w\, dx.$$

We have proved that for v, w smooth, vanishing on $\partial\Omega$

$$J(v + w) - J(v) = 3 \int_\Omega (f - \det D^2 v) w\, dx + O(|w|_1^2).$$

Since the space of infinitely differentiable functions with compact support is dense in $W^{2,\infty}(\Omega) \cap H_0^1(\Omega)$, the result holds for $v, w \in W^{2,\infty}(\Omega) \cap H_0^1(\Omega)$ by a density argument and using Lemma 3. □

The Euler-Lagrange equation for J is therefore (10).

Remark 3.1 *It has been shown in [4, 19] that a generalized solution of* (1) *is the unique minimizer of the functional J on the set of convex functions vanishing on the boundary.*

Theorem 3.2 *Let $u \in C^4(\overline{\Omega})$ be the unique strictly convex solution of* (1). *Then for h sufficiently small, the functional J has a unique minimizer $\hat{u}_h$ in X_h. Moreover $||u-\hat{u}_h||_1 \to 0$ as $h \to 0$.*

Proof. We first note that by (7), the functional J is sequentially continuous in $W^{2,\infty}(\Omega) \cap H_0^1(\Omega)$. For $v_n, v \in W^{2,\infty}(\Omega) \cap H_0^1(\Omega)$ we have

$$J(v_n) - J(v) = 3\int_\Omega f(v_n - v)\,dx + \int_\Omega (v \det D^2 v - v_n \det D^2 v_n)\,dx.$$

We conclude from Lemma 3 that $J(v_n) \to J(v)$ as $||v_n - v||_{2,\infty} \to 0$. Moreover using the expression of $J'(v)(w)$ given in Lemma 4, we obtain

$$J''(v)(w)(z) = -3\int_\Omega w \operatorname{div}[(\operatorname{cof} D^2 v) Dz]\,dx = 3\int_\Omega [(\operatorname{cof} D^2 v)Dz] \cdot Dw\,dx.$$

We conclude that

$$J''(v)(w)(w) = 3\int_\Omega [(\operatorname{cof} D^2 v)Dw] \cdot Dw\,dx.$$

That is, J is strictly convex in X_h by definition of X_h. A minimizer, if it exists, is therefore unique.

The argument to prove that J has a minimizer follows the lines of Theorem 5.1 in [9]. We have for some $\theta \in [0,1]$

$$\begin{aligned} J(v) &= J(0) + J'(0)(v) + \frac{1}{2} J''(\theta v)(v)(v) \\ &= 0 + 3\int_\Omega f v\,dx + \frac{3}{2}\theta \int_\Omega [(\operatorname{cof} D^2 v)Dv] \cdot Dv\,dx. \end{aligned} \tag{13}$$

We claim that for $v \in X_h, v \neq 0$, we have $\theta \neq 0$. Assume otherwise. Then

$$\begin{aligned} 0 &= -\int_\Omega v \det D^2 v\,dx = -\frac{1}{2}\int_\Omega v \operatorname{div}(\operatorname{cof} D^2 v) Dv\,dx \\ &= \frac{1}{2}\int_\Omega [(\operatorname{cof} D^2 v)Dv] \cdot Dv\,dx \geq \frac{m}{2}|v|_1^2, \end{aligned} \tag{14}$$

where m is a lower bound on the smallest eigenvalue of $\operatorname{cof} D^2 v$. By the assumption on $v \in X_h$ we have $m > 0$. We obtain the contradiction $v = 0$ and conclude that $\theta \in (0,1]$.

Next, note that

$$\left| \int_\Omega f v\,dx \right| \leq ||f||_0 ||v||_0 \leq ||f||_0 ||v||_1. \text{Thus} \int_\Omega f v\,dx \geq -||f||_0 ||v||_1.$$

By (13), we obtain using Poincare's inequality

$$J(v) \geq -3||f||_0||v||_1 + \frac{3}{2}\theta m|v|_1^2 \geq -3||f||_0||v||_1 + C||v||_1^2 \\ \geq ||v||_1(-3||f||_0 + C||v||_1), \tag{15}$$

for a constant $C > 0$. Let now $R > 0$ such that

$$X_h \cap \{ v \in V_h \cap H_0^1(\Omega), ||v||_1 \leq R \} \neq \emptyset.$$

Since J is continuous, J is bounded below on the above set. Moreover for $||v||_1 \geq R$, we have

$$J(v) \geq R(-3||f||_0 + CR).$$

We conclude that the functional J is bounded below. We show that its infimum is given by some $\hat{u}_h$ in X_h. Let $v_n \in X_h$ such that $\lim_{n\to\infty} J(v_n) = \inf_{v\in X_h} J(v)$ which has just been proved to be finite. Then the sequence $J(v_n)$ is bounded and by (15), the sequence v_n is also necessary bounded. Let v_{k_n} be a weakly convergent subsequence with limit $\hat{u}_h$. We have

$$\lim_{n\to\infty} J'(\hat{u}_h)(v_{k_n}) = J'(\hat{u}_h)(u_h).$$

Since J is strictly convex in X_h,

$$J(v_{k_n}) \geq J(\hat{u}_h) + J'(\hat{u}_h)(v_{k_n} - \hat{u}_h),$$

and so at the limit $\inf_{v\in X_h} J(v) \geq J(\hat{u}_h)$. This proves that $\hat{u}_h$ minimizes J in X_h.

We now prove that $||u - \hat{u}_h||_1 \to 0$ as $h \to 0$. Note that since $u_h \in X_h$, $||\hat{u}_h - Q_h u||_1 \leq \delta h^2/4$. By (8) and triangle inequality, we obtain the result. □

Remark 3.3 *From the approach taken in [1] we may conclude that* (10) *has a unique convex solution u_h in X_h which therefore solves the Euler-Lagrange equation for the functional J. Since X_h is open and convex and J convex on X_h, by Theorem 3.9.1 of [15] we have*

$$J(v) \geq J(u_h) + J'(u_h)(v - u_h), \forall v \in X_h.$$

Since u_h is a critical point of J in X_h, we get

$$J(v) \geq J(u_h), \forall v \in X_h.$$

We conclude that both u_h and $\hat{u}_h$ are minimizers of J in X_h. By the strict convexity of J in X_h, they are equal. Therefore the unique minimizer of J in X_h solves (10)*.*

We now turn to the regularized problems (2) and (3). The formal limit of u_ε^k as $k \to \infty$ solves

$$(\mathrm{cof}(D^2 u_\varepsilon + \varepsilon I)) : D^2 u_\varepsilon = \det D^2 u_\varepsilon + f \text{ in } \Omega \\ u_\varepsilon = 0 \text{ on } \partial\Omega.$$

But since I and $D^2 u_\varepsilon$ are 2×2 matrices, we have $\operatorname{cof}(D^2 u_\varepsilon + \varepsilon I) = \operatorname{cof} D^2 u_\varepsilon + \operatorname{cof} \varepsilon I = \operatorname{cof} D^2 u_\varepsilon + \varepsilon I$ and we obtain

$$(\operatorname{cof} D^2 u_\varepsilon) : D^2 u_\varepsilon + \varepsilon I : D^2 u_\varepsilon = \det D^2 u_\varepsilon + f.$$

Since $\varepsilon I : D^2 u_\varepsilon = \varepsilon \Delta u_\varepsilon$ and by (6) we have $(\operatorname{cof} D^2 u_\varepsilon) : D^2 u_\varepsilon = 2 \det D^2 u_\varepsilon$, we obtain (3).

Next we present the discrete variational formulation used in the numerical experiments. To avoid large errors, we used a damped version of (2). Let $\nu > 0$. We consider the problem

$$\begin{aligned}(\operatorname{cof}(D^2 u_\varepsilon^k + \varepsilon I)) : D^2 u_\varepsilon^{k+1} &= 2\det D^2 u_\varepsilon^k + \frac{1}{\nu}(-\det D^2 u_\varepsilon^k + f) \operatorname{in} \Omega \\ u_\varepsilon^{k+1} &= 0 \operatorname{on} \partial\Omega.\end{aligned} \tag{16}$$

We note that for $\nu = 1$, (16) reduces to (2). Also the formal limit, as $\varepsilon \to 0$ and $k \to \infty$, of u_ε^k solving (16) is a solution of $1/\nu(f - \det D^2 u) = 0$.

Let $|x|$ denote the Euclidean norm of $x \in \mathbb{R}^2$. Note that that $D^2(|x|^2/2) = I$ and thus for u_ε^k smooth, $\operatorname{cof}(D^2 u_\varepsilon^k + \varepsilon I) = \operatorname{cof} D^2(u_\varepsilon^k + \varepsilon/2|x|^2)$ and thus using (4) and (5) we obtain

$$\begin{aligned}\operatorname{div}\left((\operatorname{cof}(D^2 u_\varepsilon^k + \varepsilon I)) D u_\varepsilon^{k+1}\right) &= 2\det D^2 u_\varepsilon^k + \frac{1}{\nu}(-\det D^2 u_\varepsilon^k + f) \operatorname{in} \Omega \\ u_\varepsilon^{k+1} &= 0 \operatorname{on} \partial\Omega.\end{aligned}$$

This leads to the following discretization: find $u_{\varepsilon,h}^{k+1} \in V_h \cap H_0^1(\Omega)$ such that $\forall v \in V_h \cap H_0^1(\Omega)$

$$\begin{aligned}-\int_\Omega \left((\operatorname{cof}(D^2 u_{\varepsilon,h}^k + \varepsilon I)) D u_{\varepsilon,h}^{k+1}\right) \cdot Dv \, dx = \int_\Omega \Big(& 2\det D^2 u_{\varepsilon,h}^k \\ & + \frac{1}{\nu}(-\det D^2 u_{\varepsilon,h}^k + f)\Big) v \, dx.\end{aligned} \tag{17}$$

For the initial guess $u_{\varepsilon,h}^0$ when $\varepsilon \geq 0$, we take the discrete approximation of the solution of the problem

$$\begin{aligned}\Delta u_\varepsilon^0 &= 2\sqrt{f} \operatorname{in} \Omega \\ u_\varepsilon^0 &= 0 \operatorname{on} \partial\Omega.\end{aligned}$$

While this does not assure that $u_{\varepsilon,h}^0 \in X_h$ the above choice appears to work in all our numerical experiments.

Remark 3.4 *For a possible extension of the minimization result in Theorem 3.2 to the case of non smooth solutions, the homogeneous boundary condition is necessary.*

4 Isogeometric analysis

We refer to [20] for a short introduction to isogeometric analysis. Here we give a shorter overview suitable for our needs. Precisely, we are interested in the ability of this approach to generate finite dimensional spaces of piecewise smooth C^1 functions which can be used in the Galerkin method for approximating partial differential equations.

A univariate NURBS of degree p is given by

$$\frac{w_i N_{i,p}(u)}{\sum_{j\in\mathscr{J}} w_j N_{j,p}(u)}, u \in [0,1],$$

with B-splines $N_{i,p}$, weights w_i and an index set $\mathscr{J}$ which encodes its smoothness. The parameter h refers to the maximum distance between the knots $u_i, i \in \mathscr{J}$.

A bivariate NURBS is given by

$$R_{kl}(u,v) = \frac{w_{kl} N_k(u) N_l(v)}{\sum_{i\in\mathscr{I}} \sum_{j\in\mathscr{J}} w_j N_i(u) N_j(v)}, u, v \in [0,1],$$

with index sets $\mathscr{I}$ and $\mathscr{J}$. In the above expression, we omit the degrees pU and pV of the NURBS R_{kl} in the u and v directions.

The domain Ω is described parametrically by a mapping $F : [0,1]^2 \to \Omega, F(u,v) = \sum_{i\in\mathscr{I}} \sum_{j\in\mathscr{J}} R_{ij}(u,v) c_{ij}$ with NURBS R_{ij} and control points c_{ij}. We take equally spaced knots u_i, v_j and hence h refers to the size of an element in the parametric domain.

We say that a NURBS R_{kl} has degree p if the univariate NURBS N_k and N_l all have degree p. The NURBS considered in this paper are all of a fixed degree p and C^1.

The basis functions R_{ij} used in the description of the domain are also used in the definition of the finite dimensional space $V_h \subset \text{span}\{R_{ij} \circ F^{-1}\}$. Thus the numerical solution takes the form

$$T_h(x,y) = \sum_{i\in\mathscr{I}} \sum_{j\in\mathscr{J}} R_{ij}(F^{-1}(x,y)) q_{ij},$$

with unknowns q_{ij}.

It can be shown [18] that there exists an interpolation operator Q_h mapping $H^r(\Omega), r \geq p+1$ into V_h such that if $0 \leq l \leq p+1, 0 \leq l \leq r \leq p+1$, we have

$$|u - Q_h u|_l \leq C h^{r-l} ||u||_r,$$

with C independent of h. Thus the approximation property (8) holds for spaces constructed with the isogeometric analysis concept. For the inverse estimates (9) we refer to [5].

5 Numerical results

The implementation was done by modifying the companion code to [20]. The computational domain is taken as the unit circle: $x^2+y^2-1=0$ with an initial triangulation depicted in Figure 1. The numerical solutions are obtained by computing $u^k_{\varepsilon,h}$ defined by (17).

We consider the following test cases.

Test 1: (smooth solution) $u(x,y)=(x^2+y^2-1)e^{x^2+y^2}$ with $f(x,y)=4e^{2(x^2+y^2)}(x^2+y^2)^2(2x^2+3+2y^2)$. Numerical results are given in Table 1. Since $pU=2, pV=2$, the approximation space in the parametric domain contains piecewise polynomials of degree $p=2$. The analysis in [1] suggests that the rate of convergence for smooth solutions is $O(h^p)$ in the H^1 norm, $O(h^{p+1})$ and $O(h^{p-1})$ in the L^2 and H^2 norms respectively. No regularization or damping was necessary for this case.

Test 2 (No known exact solution) $f=e^{x^2+y^2}, g=0$. As expected the numerical solution displayed in Figure 2 appears to be a convex function.

Test 3 (solution not in $H^1(\Omega)$) $u(x,y)=-\sqrt{1-x^2-y^2}$ with $f(x,y)=1/(x^2+y^2-1)^2$. With regularization and damping, we were able to avoid the divergence of the discretization. These results should be compared with the ones in [1] where iterative methods with only a linear convergence rate were proposed for non smooth solutions of (1). Note that u in this case is highly singular as f vanishes on $\partial\Omega$.

In the tables n_{it} refers to the number of iterations for Newton's method.

Table 1 Smooth solution $u(x,y)=(x^2+y^2-1)e^{x^2+y^2}$

h	n_{it}	L^2 norm	rate	H^1 norm	rate	H^2 norm	rate
$1/2^6$	3	$4.5620\ 10^{-1}$		$1.5565\ 10^{-0}$		$1.1877\ 10^{+1}$	
$1/2^7$	6	$8.4903\ 10^{-3}$	5.75	$1.6442\ 10^{-1}$	3.24	$5.0963\ 10^{-0}$	1.2
$1/2^8$	4	$7.7160\ 10^{-4}$	3.46	$3.9573\ 10^{-2}$	2.05	$2.5880\ 10^{-0}$	0.97
$1/2^9$	4	$9.0321\ 10^{-5}$	3.09	$9.8122\ 10^{-3}$	2.01	$1.3019\ 10^{-0}$	0.99
$1/2^{10}$	4	$1.1077\ 10^{-5}$	3.03	$2.4462\ 10^{-3}$	2.00	$6.5184\ 10^{-1}$	0.99

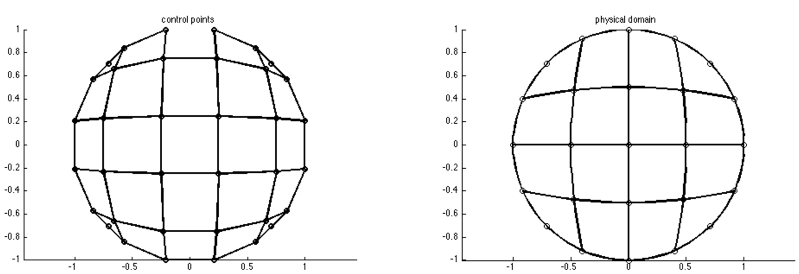

Fig. 1 Circle represented exactly. $pU=2, pV=2$

Table 2 Solution not in $H^1(\Omega)$ $u(x,y) = -\sqrt{1-x^2-y^2}$ with $\nu = 2.5, \varepsilon = 0.01$

h	n_{it}	L^2 norm	rate
$1/2^5$	42	4.0261 10^{-1}	
$1/2^6$	2	1.7529 10^{-1}	1.20
$1/2^7$	5	1.3612 10^{-1}	0.36
$1/2^8$	3	1.0609 10^{-1}	0.36
$1/2^9$	2	9.6321 10^{-2}	0.14
$1/2^{10}$	4	7.8179 10^{-2}	0.30

Acknowledgements The author thanks the two referees for their suggestions and their careful reading of the manuscript. This work began when the author was supported in part by a 2009-2013 Sloan Foundation Fellowship and NSF grant DMS-1319640. Part of this work was completed when the author was in residence at the Mathematical Sciences Research Institute (MSRI) in Berkeley, California, Fall 2013. The MSRI receives major funding from the National Science Foundation under Grant No. 0932078 000.

References

1. Awanou, G.: Pseudo transient continuation and time marching methods for Monge-Ampère type equations (2013). http://arxiv.org/pdf/1301.5891.pdf

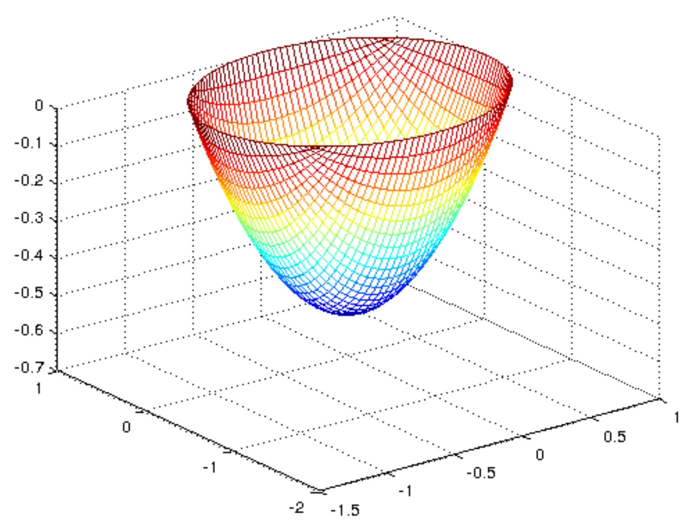

Fig. 2 Convex solution with data $f = e^{x^2+y^2}, g = 0$ with $\nu = 2.5, \varepsilon = 0.01, h = 1/32$. No known analytical formula

2. Awanou, G.: Spline element method for the Monge-Ampère equation (2013). http://arxiv.org/abs/1012.1775
3. Awanou, G.: Standard finite elements for the numerical resolution of the elliptic Monge-Ampère equation: Aleksandrov solutions (2013). http://arxiv.org/pdf/1310.4568v1.pdf
4. Bakelman, I.J.: Variational problems and elliptic Monge-Ampère equations. J. Differential Geom. **18**(4), 669–699 (1984) (1983)
5. Bazilevs, Y., Beirão da Veiga, L., Cottrell, J.A., Hughes, T.J.R., Sangalli, G.: Isogeometric analysis: approximation, stability and error estimates for h-refined meshes. Math. Models Methods Appl. Sci. **16**(7), 1031–1090 (2006)
6. Böhmer, K.: On finite element methods for fully nonlinear elliptic equations of second order. SIAM J. Numer. Anal. **46**(3), 1212–1249 (2008)
7. Brenner, S.C., Gudi, T., Neilan, M., Sung, L.Y.: C^0 penalty methods for the fully nonlinear Monge-Ampère equation. Math. Comp. **80**(276), 1979–1995 (2011)
8. Evans, L.C.: Partial differential equations, *Graduate Studies in Mathematics*, vol. 19. American Mathematical Society, Providence, RI (1998)
9. Faragó, I., Karátson, J.: Numerical solution of nonlinear elliptic problems via preconditioning operators: theory and applications, *Advances in Computation: Theory and Practice*, vol. 11. Nova Science Publishers Inc., Hauppauge, NY (2002)
10. Feng, X., Neilan, M.: Convergence of a fourth order singular perturbation of the n-dimensional radially symmetric Monge-Ampere equation. Submitted
11. Feng, X., Neilan, M.: Analysis of Galerkin methods for the fully nonlinear Monge-Ampère equation. J. Sci. Comput. **47**(3), 303–327 (2011)
12. Froese, B.D., Oberman, A.M.: Fast finite difference solvers for singular solutions of the elliptic Monge-Ampère equation. J. Comput. Phys. **230**(3), 818–834 (2011)
13. Glowinski, R.: Numerical methods for fully nonlinear elliptic equations. In: ICIAM 07—6th International Congress on Industrial and Applied Mathematics, pp. 155–192. Eur. Math. Soc., Zürich (2009)
14. Mohammadi, B.: Optimal transport, shape optimization and global minimization. C. R. Math. Acad. Sci. Paris **344**(9), 591–596 (2007)
15. Niculescu, C.P., Persson, L.E.: Convex functions and their applications. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 23. Springer, New York (2006). A contemporary approach
16. Oberman, A.M.: Wide stencil finite difference schemes for the elliptic Monge-Ampère equation and functions of the eigenvalues of the Hessian. Discrete Contin. Dyn. Syst. Ser. B **10**(1), 221–238 (2008)
17. Oliker, V.I., Prussner, L.D.: On the numerical solution of the equation $(\partial^2 z/\partial x^2)(\partial^2 z/\partial y^2) - ((\partial^2 z/\partial x\partial y))^2 = f$ and its discretizations. I. Numer. Math. **54**(3), 271–293 (1988)
18. Tagliabue, A., Dede, L., Quarteroni, A.: Isogeometric analysis and error estimates for high order partial differential equations in fluid dynamics. Preprint
19. Tso, K.: On a real Monge-Ampère functional. Invent. Math. **101**(2), 425–448 (1990)
20. Vuong, A.V., Heinrich, C., Simeon, B.: ISOGAT: a 2D tutorial MATLAB code for isogeometric analysis. Comput. Aided Geom. Design **27**(8), 644–655 (2010)