RECTANGULAR MIXED FINITE ELEMENTS FOR ELASTICITY

DOUGLAS N. ARNOLD AND GERARD AWANOU

Abstract. We present a family of stable rectangular mixed finite elements for plane elasticity. Each member of the family consists of a space of piecewise polynomials discretizing the space of symmetric tensors in which the stress field is sought, and another to discretize the space of vector fields in which the displacement is sought. These may be viewed as analogues in the case of rectangular meshes of mixed finite elements recently proposed for triangular meshes. As for the triangular case the elements are closely related to a discrete version of the elasticity differential complex.

1. Introduction

Let Ω be a simply connected polygonal domain of $\mathbb{R}^2$, occupied by a linearly elastic body which is clamped on $\partial\Omega$ and let $H(\operatorname{div}, \Omega, \mathbb{S})$ be the space of square-integrable fields taking values in $\mathbb{S}$, the space of symmetric tensors, and which have square integrable divergence. We denote as usual by $L^2(\Omega, \mathbb{R}^2)$ the space of square integrable vector fields with values in $\mathbb{R}^2$ and $H^k(R, X)$ the space of functions with domain $R \subset \mathbb{R}^2$, taking values in the finite dimensional space X, and with all derivatives of order at most k square integrable. For our purposes, X will be either $\mathbb{S}$, $\mathbb{R}^2$, or $\mathbb{R}$, and in the later case, we will simply write $H^k(R)$. The norms in $H^k(R, X)$ and $H^k(R)$ are denoted respectively $\|\cdot\|_{H^k}$ and $\|\cdot\|_k$. We also denote by $\mathcal{P}_{k_1,k_2}(R, X)$ the space of polynomials on R of degree at most k_1 in x and of degree at most k_2 in y. We write $\mathcal{P}_{k_1,k_2}$ for $\mathcal{P}_{k_1,k_2}(R, \mathbb{R})$. For a vector field $v : \Omega \to \mathbb{R}^2$, $\operatorname{grad} v$ is the matrix field with rows the gradient of each component, and $\epsilon(v) = [(\operatorname{grad} v) + (\operatorname{grad} v)^T]/2$. For a matrix field τ, div τ is the vector obtained by applying the divergence operator row-wise and

$$\sigma : \tau = \sum_{i,j=1}^{2} \sigma_{ij}\tau_{ij}.$$

The solution $(\sigma, u) \in H(\operatorname{div}, \Omega, \mathbb{S}) \times L^2(\Omega, \mathbb{R}^2)$ of the elasticity problem can be characterized as the unique critical point of the Hellinger-Reissner functional

$$\mathcal{J}(\sigma, v) = \int_\Omega \big(\frac{1}{2} A\tau : \tau + \operatorname{div} \tau \cdot v - f \cdot v\big)\, dx.$$

The compliance tensor $A = A(x) : \mathbb{S} \to \mathbb{S}$ is given, bounded, and symmetric positive definite uniformly with respect to $x \in \Omega$, and the body force f is also given. The unknowns, σ and u, represent the the stress and displacement fields, respectively.

A mixed finite element method determines approximate stress field and approximate velocity field (σ_h, u_h) as the unique critical point of the Hellinger–Reissner functional in a finite

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element space $\Sigma_h \times V_h \subset H(\operatorname{div}, \Omega, \mathbb{S}) \times L^2(\Omega, \mathbb{R}^2)$, where h denotes the mesh size. It can be shown that for a stable approximation, the following two conditions are sufficient:[2]

- $\operatorname{div} \Sigma_h \subset V_h$.
- There exists a linear operator $\Pi_h : H^1(\Omega, \mathbb{S}) \to \Sigma_h$, bounded in $\mathcal{L}(H^1, L^2)$ uniformly with respect to h, and such that $\operatorname{div} \Pi_h \sigma = P_h \operatorname{div} \sigma$ for all $\sigma \in H^1(\Omega, \mathbb{S})$, where $P_h : L^2(\Omega, \mathbb{R}^2) \to V_h$ denotes the L^2-projection.

In recent work of Arnold and Winther,[2] a family of finite element spaces based on triangular meshes was proposed and shown to satisfy these two conditions. In this paper, we will derive an analogous family of finite element spaces based on rectangular meshes and show that they satisfy these two stability conditions.

A key ingredient in the design of mixed finite element methods is the use of differential complexes.[1,2] In two dimensions, the elasticity complex is

$$0 \longrightarrow \mathcal{P}_1(\Omega) \xrightarrow{\subset} C^\infty(\Omega) \xrightarrow{J} C^\infty(\Omega, \mathbb{S}) \xrightarrow{\operatorname{div}} C^\infty(\Omega, \mathbb{R}^2) \longrightarrow 0,$$

where $\mathcal{P}_k(\Omega)$ is the space of polynomials in two variables of total degree k and J, the Airy stress operator defined by

$$Jq := \begin{pmatrix} \frac{\partial^2 q}{\partial y^2} & -\frac{\partial^2 q}{\partial x \partial y} \\ -\frac{\partial^2 q}{\partial x \partial y} & \frac{\partial^2 q}{\partial x^2} \end{pmatrix}$$

An analoguous sequence with less smoothness is

$$0 \longrightarrow \mathcal{P}_1(\Omega) \xrightarrow{\subset} H^2(\Omega) \xrightarrow{J} H(\operatorname{div}, \Omega, \mathbb{S}) \xrightarrow{\operatorname{div}} L^2(\Omega, \mathbb{R}^2) \longrightarrow 0. \tag{1.1}$$

We denote by

$$\begin{pmatrix} \mathcal{P}_{k_1,k_2} & \mathcal{P}_{k_3,k_4} \\ \mathcal{P}_{k_3,k_4} & \mathcal{P}_{k_5,k_6} \end{pmatrix}_{\mathbb{S}}$$

the space of symmetric matrix fields $\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$ such that $\sigma_{11} \in \mathcal{P}_{k_1,k_2}$, $\sigma_{12} = \sigma_{2,1} \in \mathcal{P}_{k_3,k_4}$ and $\sigma_{22} \in \mathcal{P}_{k_5,k_6}$. Similarly, we sometimes write $\begin{pmatrix} \mathcal{P}_{k_1,k_2} \\ \mathcal{P}_{k_3,k_4} \end{pmatrix}$ at the place of $\mathcal{P}_{k_1,k_2} \times \mathcal{P}_{k_3,k_4}$. Let $k \geq -2$ be an integer. The following sequence is exact

$$0 \longrightarrow \mathcal{P}_1(\Omega) \xrightarrow{\subset} \mathcal{P}_{k+4,k+4}(\Omega) \xrightarrow{J} \begin{pmatrix} \mathcal{P}_{k+4,k+2} & \mathcal{P}_{k+3,k+3} \\ \mathcal{P}_{k+3,k+3} & \mathcal{P}_{k+2,k+4} \end{pmatrix}_{\mathbb{S}} \xrightarrow{\operatorname{div}} \begin{pmatrix} \mathcal{P}_{k+3,k+2} \\ \mathcal{P}_{k+2,k+3} \end{pmatrix} \longrightarrow 0.$$

To verify the surjectivity of the last map, one can notice that the alternating sum of the dimensions of the spaces in the sequence is zero.

The paper is organized as follows. In the next section, we present our rectangular element in the lowest order case and then show the relation to the elasticity differential complex in the third section. The error estimates are given in the fourth section. In Section 5, we present the higher order elements. Finally we describe a simplified element of low order in the last section.

2. The Elements in the Lowest Order Case

Let $\mathcal{T}_h$ denote a conforming partition of Ω into rectangles of diameter bounded by h, which is quasi-uniform in the sense that the aspect ratio of the rectangles is bounded by a fixed constant.

In this section, we define our finite element spaces $\Sigma_h \subset H(\operatorname{div}, \mathbb{S})$ and $V_h \subset L^2(\Omega, \mathbb{R})$ and a bounded operator $\Pi_h : H^1(\Omega, \mathbb{S}) \to \Sigma_h$ satisfying the commutativity condition

$$\operatorname{div} \Pi_h \tau = P_h \operatorname{div} \tau, \qquad \tau \in H^1(\Omega, \mathbb{S}) \tag{2.1}$$

and the bound

$$\|\Pi_h \tau\|_0 \le C \|\tau\|_1, \qquad \tau \in H^1(\Omega, \mathbb{S}), \tag{2.2}$$

with c independent of h. It follows easily that our pair of elements is stable.[2]

We first describe our elements on a single rectangle R. We will denote by n the outward normal to an edge of R and define

$$V_R = \begin{pmatrix} \mathcal{P}_{2,1} \\ \mathcal{P}_{1,2} \end{pmatrix}, \quad \Sigma_R = \left\{ \tau \in \begin{pmatrix} \mathcal{P}_{5,3} & \mathcal{P}_{4,4} \\ \mathcal{P}_{4,4} & \mathcal{P}_{3,5} \end{pmatrix}_{\mathbb{S}} \mid \operatorname{div} \tau \in V_R \right\}$$

The dimension of V_R is 12 and the degrees of freedom are given by the values of each component at 6 interior nodes of R. The dimension of Σ_R is at least 45. To see this, notice that the dimension of the space of matrices with values in $\begin{pmatrix} \mathcal{P}_{5,3} & \mathcal{P}_{4,4} \\ \mathcal{P}_{4,4} & \mathcal{P}_{3,5} \end{pmatrix}_{\mathbb{S}}$ is 73 and for those matrix fields, $\operatorname{div} \tau \in \mathcal{P}_{4,3} \times \mathcal{P}_{3,4}$. Since $\dim \mathcal{P}_{4,3} = \dim \mathcal{P}_{4,3} = 20$ and $\dim \mathcal{P}_{1,2} = \dim \mathcal{P}_{2,1} = 6$, the condition $\operatorname{div} \tau \in V_R$ imposes 28 constraints. We will exhibit 45 degrees of freedom after a few preliminaries. This will establish that the dimension of Σ_R is 45. (This could also be done with a dimension counting argument.)

The kernel of ϵ is the space of infinitesimal rigid motions so

$$\dim \epsilon(V_R) = 12 - 3 = 9.$$

Define

$$M_1(R) := \left\{ \tau \in \begin{pmatrix} \mathcal{P}_{5,3} & \mathcal{P}_{4,4} \\ \mathcal{P}_{4,4} & \mathcal{P}_{3,5} \end{pmatrix}_{\mathbb{S}} \mid \operatorname{div} \tau = 0 \text{ and } \tau n = 0 \text{ on } \partial R \right\}.$$

The following lemma characterizes $M_1(R)$.

Lemma 2.1. *Let L_i, $i = 1, \dots, 4$ be linear functions which define the edges e_i, $i = 1, \dots, 4$, of R and define $b_R = L_1 L_2 L_3 L_4$. We have $J(b_R^2 \mathcal{P}_{1,1}) = M_1(R)$, and so the dimension of $M_1(R)$ is 4.*

Proof. First we note that $J(b_R^2 \mathcal{P}_{1,1}) \subset M_1$. Indeed for $q \in \mathcal{P}_{1,1}$, $b_R^2 q \in \mathcal{P}_{5,5}$ and $\tau = J(b_R^2 q) \in \begin{pmatrix} \mathcal{P}_{5,3} & \mathcal{P}_{4,4} \\ \mathcal{P}_{4,4} & \mathcal{P}_{3,5} \end{pmatrix}_{\mathbb{S}}$ with $\operatorname{div} \tau = 0$. It remains to show that $\tau n = 0$ on ∂R. This follows from the identities

$$J(q) n \cdot n = \frac{\partial^2 q}{\partial s^2}, \quad J(q) n \cdot t = \frac{\partial^2 q}{\partial s \partial n}. \tag{2.3}$$

Next, let v_i, $i = 1, \dots, 4$, be the vertices of R so that the edge e_i is the segment from v_i to v_{i+1} with $v_5 \equiv v_1$. Let also $\tau \in M_1$. Since $\operatorname{div} \tau = 0$, there exists $q \in \mathcal{P}_{5,5}(\Omega)$ such that

$\tau = J(q)$. Using the identities (2.3) and the vanishing of τn, we infer that implies that q is linear on each edge and $\partial q/\partial n$ is constant on each edge.

By adding a linear function to q, we may assume that $q \equiv 0$ on the edges e_1 and e_4 meeting at v_1. It follows that q and ∇q vanish at v_1 and so q and $\partial q/\partial n$ vanish on e_1. This implies that L_1^2 divides q. Similarly L_2^2, L_3^2, and L_4^2 divide q. Thus $q = b_R^2 \tilde{q}$ for some $\tilde{q}$ in $\mathcal{P}_{1,1}$. □

We can now give the degrees of freedom for Σ_R.

Lemma 2.2. *A matrix field $\tau \in \Sigma_R$ is uniquely determined by the following degrees of freedom*

(i) *the values of each component of $\tau(x)$ at the vertices of R (12 degrees of freedom)*
(ii) *the first two moments of $(\tau n) \cdot n$ on each edge (8 degrees of freedom)*
(iii) *the first three moments of $(\tau n) \cdot t$ on each edge (12 degrees of freedom)*
(iv) *the values of $\int_R \tau : \phi$ for all ϕ in $\epsilon(V_R)$ (9 degrees of freedom)*
(v) *the values of $\int_R \tau : \phi$ for all ϕ in $M_1(R)$ (4 degrees of freedom)*

Proof. We assume that all listed degrees of freedom in the lemma vanish and show that $\tau = 0$. Notice that $\tau n \cdot n \in \mathcal{P}_3$ and $\tau n \cdot t \in \mathcal{P}_4$ on each edge. Since $\tau = 0$ at the vertices, the second and third set of degrees of freedom imply that $\tau n = 0$ on ∂R. Recall that for $v \in H^1(\Omega, \mathbb{R}^2)$ and $\tau \in H(\mathrm{div}, \Omega, \mathbb{S})$,

$$\int_R (\mathrm{div}\, \tau) v \, dx = -\int_R \tau : \epsilon(v) \, dx + \int_{\partial R} \tau n \cdot v \, ds. \tag{2.4}$$

Since for $\tau \in \Sigma_R$, $\mathrm{div}\, \tau \in V_R$, using (2.4) for $v \in V_R$, we get $\mathrm{div}\, \tau = 0$, i.e., $\tau \in M_1(R)$. Finally, using the last set of degrees of freedom, we get $\tau = 0$. □

We now describe the finite elements on the triangulation $\mathcal{T}_h$. We denote by V_h the space of vector fields which belong to V_R for each $R \in \mathcal{T}_h$ and Σ_h the space of matrix fields which belong piecewise to Σ_R, are continuous at mesh vertices, and have normal components which are continuous across mesh edges.

It remains to define the interpolation operator described at the beginning of this section and to verify the properties (2.1) and (2.2). Because of vertex degrees of freedom, the canonical interpolation operator for Σ_h is not bounded on $H^1(\Omega, \mathbb{S})$, so as in the triangular case,[2] we consider a family of bounded linear operators $E_h^x : H^1(\Omega, \mathbb{S}) \to \mathbb{S}$ for each vertex x of the triangulation, and define the interpolation operator $\Pi_h : H^1(\Omega, \mathbb{S}) \to \Sigma_h$ by

$$\Pi_h \tau(x) = E_h^x \tau \quad \text{for all vertices } x, \tag{2.5}$$

$$\int_e (\tau - \Pi_h \tau) n \cdot n \, v \, ds = 0 \quad \text{for all edges } e \text{ and all } v \in \mathcal{P}_1(e), \tag{2.6}$$

$$\int_e (\tau - \Pi_h \tau) n \cdot t \, v \, ds = 0 \quad \text{for all edges } e \text{ and all } v \in \mathcal{P}_2(e), \tag{2.7}$$

$$\int_R (\tau - \Pi_h \tau) : \phi \, dx = 0 \quad \text{for all rectangles } R \text{ and } \phi \in \epsilon(V_R), \tag{2.8}$$

$$\int_R (\tau - \Pi_h \tau) : \phi \, dx = 0 \quad \text{for all rectangles } R \text{ and } \phi \in M_1(R). \tag{2.9}$$

For $\tau \in H^1(\Omega, \mathbb{S})$, $R \in \mathcal{T}_h$ and $v \in V_R$, we have

$$\int_R (\operatorname{div} \Pi_h \tau - \operatorname{div} \tau) \cdot v \, dx = -\int_R (\Pi_h \tau - \tau) : \epsilon(v) \, dx + \int_{\partial R} (\Pi_h \tau - \tau) n \cdot v \, ds.$$

The first term on the right vanishes because of (2.8). On the other hand, for $v \in V_R$, $v \cdot t \in \mathcal{P}_2$ and $v \cdot n \in \mathcal{P}_1$ and so the second term also vanishes by (2.6) and (2.7). We conclude that the commutativity condition (2.1) holds. (Note that the proof of (2.1) depended only on the properties (2.6)–(2.8). We need to verify the boundedness condition (2.2). For this, let $\hat{R} = [0,1] \times [0,1]$ be the reference rectangle and let $F : \hat{R} \to R$ be an affine mapping onto R, $F(\hat{x}) = B\hat{x} + b$, with $b = (b_1, b_2)$ and

$$B = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}.$$

Although B is symmetric we will write B^T so as not to disguise the analogy to more general situations. Given a matrix field $\hat{\tau} : \hat{R} \to \mathbb{S}$, define $\tau : R \to \mathbb{S}$ by the matrix Piola transform

$$\tau(x) = B\hat{\tau}(\hat{x})B^T.$$

We first define an interpolation operator Π_h^0 which is local with respect to the triangulation so that its norm can be estimated by scaling arguments. We let $\Pi_{\hat{R}}^0 : H^1(\hat{R}, \mathbb{S}) \to \Sigma_{\hat{R}}$ by

$$\begin{aligned}
\Pi_{\hat{R}}^0 \hat{\tau}(\hat{x}) &= 0 \quad \text{for all vertices } \hat{x} \text{ of } \hat{R} \\
\int_{\hat{e}} (\hat{\tau} - \Pi_{\hat{R}}^0 \hat{\tau}) n \cdot n \, v \, ds &= 0 \quad \text{for all edges } \hat{e} \text{ of } \hat{R} \text{ and all } v \in \mathcal{P}_1(\hat{e}) \\
\int_{\hat{e}} (\hat{\tau} - \Pi_{\hat{R}}^0 \hat{\tau}) n \cdot t \, v \, ds &= 0 \quad \text{for all edges } \hat{e} \text{ of } \hat{R} \text{ and all } v \in \mathcal{P}_2(\hat{e}) \\
\int_{\hat{R}} (\hat{\tau} - \Pi_{\hat{R}}^0 \hat{\tau}) : \phi \, dx &= 0 \quad \text{for all } \phi \in \epsilon \begin{pmatrix} \mathcal{P}_{2,1} \\ \mathcal{P}_{1,2} \end{pmatrix} \\
\int_{\hat{R}} (\hat{\tau} - \Pi_{\hat{R}}^0 \hat{\tau}) : \phi \, dx &= 0 \quad \text{for all } \phi \in M_1(\hat{R}).
\end{aligned}$$

In view of Lemma 2.2, $\Pi_{\hat{R}}^0$ is well defined and is easily seen to be bounded. Next define $\Pi_R^0 : H^1(R, \mathbb{S}) \to \Sigma_R$ by

$$\Pi_R^0 \tau(x) = B \Pi_{\hat{R}}^0 \hat{\tau}(\hat{x}) B^T,$$

for each rectangle R of $\mathcal{T}_h$ and define $\Pi_h^0 : H^1(\Omega, \mathbb{S}) \to \Sigma_h$ by

$$(\Pi_h^0 \tau)|_R = \Pi_R^0 \tau.$$

Note that Π_h^0 is bounded on $H^1(\Omega, \mathbb{S})$. At the end of this section, we will verify the commutativity property (2.1) with Π_h replaced by Π_h^0. Moreover a standard scaling argument gives

$$\|\Pi_h^0 \tau\|_0 \le c(\|\tau\|_0 + h\|\tau\|_1) \tag{2.10}$$

where c does not depend on h.

Next, let R_h be a Clement interpolation operator,[3,4] which maps $L^2(\Omega, \mathbb{R})$ into

$$\{\, \theta_h \in C^0(\bar{\Omega}) \mid \theta_{h|R} \in \mathcal{P}_{1,1}, \forall R \in \mathcal{T}_h \,\}$$

and denote as well by R_h the corresponding operator which maps $L^2(\Omega, \mathbb{S})$ into the subspace of Σ_h of continuous matrix fields whose components are piecewise in $P_{1,1}$. We have

$$\|R_h\tau - \tau\|_j \leq ch^{m-j}\|\tau\|_m, \quad 0 \leq j \leq 1, \quad j \leq m \leq 2, \tag{2.11}$$

with c independent of h. We define our interpolation operator Π_h by

$$\Pi_h = \Pi_h^0(I - R_h) + R_h. \tag{2.12}$$

Then using (2.10) and (2.11), we obtain

$$\|\Pi_h^0(I - R_h)\tau\|_0 \leq c(\|(I - R_h)\tau\|_0 + h\|(I - R_h)\tau\|_1) \leq ch^m\|\tau\|_m, \tag{2.13}$$

for $1 \leq m \leq 2$. It follows that

$$\|\Pi_h\tau\|_0 \leq \|\Pi_h^0(I - R_h)\tau\|_0 + \|R_h\tau\|_0 \leq c\|\tau\|_1,$$

i.e., (2.2) holds. Finally, we check the commutativity property (2.1). We have

$$\begin{aligned} \operatorname{div}\Pi_h &= \operatorname{div}\Pi_h^0(I - R_h)\tau + \operatorname{div} R_h \\ &= \operatorname{div}\Pi_h^0 + \operatorname{div} R_h - \operatorname{div}\Pi_h^0 R_h \\ &= P_h \operatorname{div} + (I - P_h)\operatorname{div} R_h \\ &= P_h \operatorname{div}, \end{aligned}$$

where we have used the commutativity property for Π_h^0 (which we are about to prove) and $R_h\tau \in \Sigma_h$.

It remains to establish the commutativity property of Π_h^0. For this it is sufficient to verify (2.6)–(2.8) with Π_h replaced by Π_h^0. Notice that for an edge $\hat{e}$ of $\hat{R}$, and the corresponding edge e of R, $n_{\hat{e}}$ is equal to n_e. Similarly, $\hat{t} = t$. Also $B^T n_e = \|B^T n_e\| n_{\hat{e}}$ as $B\hat{t} \perp n_e$. Let $v \in \mathcal{P}_1(e)$ and let $\hat{v}(\hat{x}) = v(x)$, so $\hat{v} \in \mathcal{P}_1(\hat{e})$. Since, $|\hat{e}| = 1$, we have

$$\begin{aligned} \int_e \Pi_R^0\tau(x) n_e \cdot n_e v(x)\, ds &= |e| \int_{\hat{e}} B\Pi_{\hat{R}}^0\hat{\tau}(\hat{x}) B^T n_{\hat{e}} \cdot n_{\hat{e}}\hat{v}(\hat{x})\, d\hat{s}, \\ &= |e| B \int_{\hat{e}} \Pi_{\hat{R}}^0\hat{\tau}(\hat{x}) B^T n_{\hat{e}} \cdot n_{\hat{e}}\hat{v}(\hat{x})\, d\hat{s} \\ &= |e|\|B^T n_e\| B \int_{\hat{e}} \Pi_{\hat{R}}^0\hat{\tau}(\hat{x}) n_{\hat{e}} \cdot n_{\hat{e}}\hat{v}(\hat{x})\, d\hat{s} \\ &= |e|\|B^T n_e\| B \int_{\hat{e}} \hat{\tau}(\hat{x}) n_{\hat{e}} \cdot n_{\hat{e}}\hat{v}(\hat{x})\, d\hat{s} \\ &= |e| \int_{\hat{e}} B\hat{\tau}(\hat{x}) B^T n_{\hat{e}} \cdot n_{\hat{e}}\hat{v}(\hat{x})\, d\hat{s} \\ &= \int_e \tau(x) n_e \cdot n_e v(x)\, ds, \end{aligned}$$

verifying (2.6). Similarly, for $v \in \mathcal{P}_2(e)$,

$$\int_e \Pi_R^0\tau(x) n_e \cdot t_e v(x)\, ds = \int_e \tau(x) n_e \cdot t_e v(x)\, ds,$$

verifying (2.7). To verify (2.8), we infer from (2.6) and (2.7) that

$$\int_{\partial R} \Pi_R^0 \tau(x) n \cdot v(x)\, ds = \int_{\partial R} \tau(x) n \cdot v(x)\, ds \tag{2.14}$$

for all $v \in \mathcal{P}_{2,1} \times \mathcal{P}_{1,2}$. Next notice that since $\Pi^0_{\hat{R}}$ satisfies (2.6)–(2.8), we have for $\hat{v} \in P_{2,1} \times P_{1,2}$,

$$\begin{aligned} \int_{\hat{R}} \operatorname{div} \hat{\tau} \hat{v}(\hat{x})\, d\hat{x} &= -\int_{\hat{R}} \hat{\tau} : \epsilon(\hat{v})\, d\hat{x} + \int_{\partial \hat{R}} \hat{\tau} \hat{n} \cdot \hat{v}\, d\hat{s} \\ &= -\int_{\hat{R}} \Pi^0_{\hat{R}} \hat{\tau} : \epsilon(\hat{v})\, d\hat{x} + \int_{\partial \hat{R}} \Pi^0_{\hat{R}} \hat{\tau} \hat{n} \cdot \hat{v}\, d\hat{s} \\ &= \int_{\hat{R}} \operatorname{div} \Pi^0_{\hat{R}} \hat{\tau} \hat{v}(\hat{x})\, d\hat{x}. \end{aligned} \tag{2.15}$$

Next, a direct computation shows that for $\tau(x) = B\hat{\tau}(\hat{x})B^T$,

$$\operatorname{div} \tau = B \operatorname{div} \hat{\tau}.$$

We therefore have

$$\begin{aligned} \int_R (\operatorname{div} \tau - \operatorname{div} \Pi_h^0 \tau) \cdot v\, dx &= (\det B) B \int_{\hat{R}} \operatorname{div}(\hat{\tau} - \Pi^0_{\hat{R}} \hat{\tau}) \cdot \hat{v}\, d\hat{x} \\ &= (\det B) \int_{\hat{R}} \operatorname{div}(\hat{\tau} - \Pi^0_{\hat{R}} \hat{\tau}) \cdot (B^T \hat{v})\, d\hat{x} = 0, \end{aligned} \tag{2.16}$$

where we have noted that $B^T \hat{v} \in P_{2,1} \times P_{1,2}$ and applied (2.15) at the last step. Now, for $v \in P_{2,1} \times P_{1,2}$,

$$\begin{aligned} \int_R \Pi_h^0 \tau(x) &: \epsilon(v)(x)\, dx = -\int_R \operatorname{div} \Pi_h^0 \tau(x) \cdot v(x)\, dx + \int_{\partial R} \Pi_h^0 \tau(x) n \cdot v(x)\, ds \\ &= -\int_R \operatorname{div} \Pi_h^0 \tau(x) \cdot v(x)\, dx + \int_{\partial R} \tau(x) n \cdot v(x)\, ds, \\ &= \int_R \tau(x) : \epsilon(v(x))\, dx + \int_R \operatorname{div} \tau(x) \cdot v(x)\, dx - \int_R \operatorname{div} \Pi_h^0 \tau(x) \cdot v(x)\, dx \\ &= \int_R \tau(x) : \epsilon(v)(x)\, dx, \end{aligned}$$

where we have used (2.14) in the second step and (2.16) in the last. This completes the proof.

3. Discrete Version of the Elasticity Sequence

The element described in this paper was discovered by looking for a discrete version of (1.1). The starting point is the construction of an H^2 element which we now describe first on a single rectangle.

We take $Q_R = \mathcal{P}_{5,5}(R)$ with the following 36 degrees of freedom

(i) derivatives up to order 2 at each vertex ($6 \times 4 = 24$ degrees of freedom)
(ii) moments of degree 0 and 1 of $\partial q / \partial n$ on each edge ($2 \times 4 = 8$ degrees of freedom)
(iii) $\int_R J(q) : \phi\, dx$ for all $\phi \in M_1$ (4 degrees of freedom)

We next show that we have described a unisolvent set of degrees of freedom. For this we assume that all degrees of freedom of Q_R vanish. On each edge, q is a polynomial of degree 5 with triple roots at each vertex. Therefore $q \equiv 0$ on each edge and we can write $q = \bar{q} L_1 L_2 L_3 L_4$, with $\bar{q} \in \mathcal{P}_{3,3}$. We have

$$\frac{\partial q}{\partial n_1}|_{e_1} = \bar{q} \frac{\partial L_1}{\partial n_1} L_2 L_3 L_4$$

so that $q_1 := \partial q / \partial n_1 \in \mathcal{P}_5(e_1)$ on the edge e_1; (this can also be seen easily on the reference rectangle). Clearly, q_1 and $\partial q_1/\partial s$ are zero at the vertices of e_1, so using the second set of degrees of freedom

$$0 = \int_{e_1} q_1 \frac{\partial^4 q_1}{\partial s^4} = -\int_{e_1} \frac{\partial q_1}{\partial s} \frac{\partial^3 q_1}{\partial s^3} = \int_{e_1} \frac{\partial^2 q_1}{\partial s^2} \frac{\partial^2 q_1}{\partial s^2}.$$

We conclude that $\partial^2 q_1 / \partial s^2 = 0$ on e_1, so that $q_1 = 0$ on e_1 and therefore $\bar{q} = 0$ on e_1. Since a similar argument show that $\bar{q}$ vanishes on each edge, we have $q = \tilde{q} b_R^2$ with $\tilde{q} \in \mathcal{P}_{1,1}$. Taking $\phi = J(b_R^2 \tilde{q})$ in the last set of degrees of freedom, we get $q = 0$, since the kernel of J is $\mathcal{P}_1(R)$.

The finite element $Q_h \subset H^2(\Omega)$ is assembled the usual way and we define an interpolation operator $I_h : C^\infty(\Omega) \to Q_h$ by requiring at each vertex x

$$I_h q(x) = q(x), \tag{3.1}$$

$$(\nabla I_h q)(x) = (\nabla q)(x), \tag{3.2}$$

$$(J I_h q)(x) = E_h^x (J q)(x), \tag{3.3}$$

and

$$\int_e \frac{\partial I_h q}{\partial n}(s) f \, ds = \int_e \frac{\partial q}{\partial n}(s) f \, ds \quad \text{for all edges and all } f \in \mathcal{P}_1(e) \tag{3.4}$$

$$\int_R J(I_h q) : \phi \, dx = \int_R J(q) : \phi \, dx \quad \text{for all } \phi \in M_1 \tag{3.5}$$

We can now describe our discrete version of the elasticity differential complex:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathcal{P}_1(\Omega) & \xrightarrow{\subset} & C^\infty(\Omega) & \xrightarrow{J} & C^\infty(\Omega, \mathbb{S}) & \xrightarrow{\text{div}} & C^\infty(\Omega, \mathbb{R}^N) & \longrightarrow 0 \\
& \Big\downarrow id & & \Big\downarrow I_h & & \Big\downarrow \pi_h & & \Big\downarrow P_h & \\
0 \longrightarrow & \mathcal{P}_1(\Omega) & \xrightarrow{\subset} & Q_h & \xrightarrow{J} & \Sigma_h & \xrightarrow{\text{div}} & V_h & \longrightarrow 0
\end{array}$$

We only need to check the commutativity property $\Pi_h J q = J I_h q$, i.e. $\Pi_h \tau = \sigma$ with $\tau = J(q)$ and $\sigma = J I_h q$.

We have by definition of I_h, $(J I_h q)(x) = (E_h^x J q)(x)$. Recall that for $\tau = Jq$, $\partial^2 q / \partial s^2 = \tau n \cdot n$ and $\partial^2 q / (\partial s \partial n) = \tau n \cdot t$. Now, for $v \in \mathcal{P}_1(e)$, using (3.1) and (3.2) after a double integration by parts

$$\int_e \big(J I_h q - J q \big) n \cdot n \, v \, ds = \int_e \frac{\partial^2 (I_h q - q)}{\partial s^2} v \, ds = 0.$$

By (3.2) and (3.4), we also have for $v \in \mathcal{P}_2(e)$

$$\int_e \big(J I_h q - J q \big) n \cdot t \, v \, ds = \int_e \frac{\partial^2 (I_h q - q)}{\partial s \partial n} v \, ds = -\int_e \frac{\partial (I_h q - q)}{\partial n} \frac{\partial v}{\partial s} \, ds = 0.$$

Next, for $\phi = \epsilon(v) \in \epsilon(V_R)$, using (2.4)

$$\begin{aligned}\int_R \big(JI_h q - Jq\big) : \phi &= \int_{\partial R} \big(JI_h q - Jq\big) n \cdot v \, ds \\ &= 0\end{aligned}$$

by the above identities since for $v \in V_R$, $v \cdot t \in \mathcal{P}_2(e)$ and $v \cdot n \in \mathcal{P}_1(e)$ for each edge e.

Finally, for $\phi \in M_1(R)$,

$$\int_R \big(JI_h q - Jq\big) : \phi = 0$$

by (3.5).

4. Error Analysis

Since our pair of elements is stable, we have from the theory of mixed methods the following quasioptimal estimate

$$\|\sigma - \sigma_h\|_{H(\mathrm{div})} + \|u - u_h\|_{L^2} \le c \inf_{\tau \in \Sigma_h, v \in V_h} (\|\sigma - \tau\|_{H(\mathrm{div})} + \|u - v\|_{L^2}).$$

This gives an $\mathrm{O}(h^2)$ error estimate for smooth solutions. It can be refined in various ways. For the interpolation operator Π_h, c.f. (2.12), we have

$$I - \Pi_h = (I - R_h) - \Pi_h^0 (I - R_h),$$

so using (2.11) and (2.13), we get

$$\|\Pi_h \tau - \tau\|_0 \le c h^m \|\tau\|_m, \quad 1 \le m \le 2.$$

We also recall that the projection operator P_h satisfies the error estimate

$$\|P_h v - v\|_0 \le c h^m \|v\|_m, \quad 0 \le m \le 2,$$

since V_R contains the vector fields with components in $P_{1,1}$. We then have the following result, whose proof is similar to the one in Arnold–Winther.[2]

Theorem 4.1. *Let (σ, u) denote the unique critical point of the Hellinger-Reissner functional over $H(\mathrm{div}, \Omega, \mathbb{S}) \times L^2(\Omega, \mathbb{R}^2)$ and let (σ_h, u_h) be the unique critical point over $\Sigma_h \times V_h$. Then*

$$\begin{aligned}\|\sigma - \sigma_h\|_0 &\le c h^m \|\sigma\|_m, & 1 \le m \le 2,\\ \|\operatorname{div} \sigma - \operatorname{div} \sigma_h\|_0 &\le c h^m \|\operatorname{div} \sigma\|_m, & 0 \le m \le 2,\\ \|u - u_h\|_0 &\le c h^m \|u\|_{m+1}, & 1 \le m \le 2.\end{aligned}$$

5. Higher Order Elements

In this section we describe a family of stable element pairs, one for each degree $k \ge 1$. The case $k = 1$ is the one treated above. We first describe the elements on a single rectangle:

$$V_R = \begin{pmatrix} \mathcal{P}_{k+1,k} \\ \mathcal{P}_{k,k+1} \end{pmatrix}, \quad \Sigma_R = \left\{ \tau \in \begin{pmatrix} \mathcal{P}_{k+4,k+2} & \mathcal{P}_{k+3,k+3} \\ \mathcal{P}_{k+3,k+3} & \mathcal{P}_{k+2,k+4} \end{pmatrix}_{\mathbb{S}} \mid \operatorname{div} \tau \in V_R \right\}.$$

We have dim $V_R = 2(k+1)(k+2)$ and the degrees of freedom are given by the values of each component at $(k+1)(k+2)$ interior nodes of R.

$$\begin{aligned}\dim \Sigma_R \geq d_k &:= 2 \ \dim \mathcal{P}_{k+4,k+2} + \dim \mathcal{P}_{k+3,k+3} - 2(\dim \mathcal{P}_{k+3,k+2} - \dim \mathcal{P}_{k+1,k}) \\ &= 2(k+3)(k+5) + (k+4)^2 - 2[(k+3)(k+4) - (k+1)(k+2)] \\ &= 3k^2 + 16k + 26.\end{aligned}$$

As in the lowest order case, let us define

$$M_k(R) := \left\{ \tau \in \begin{pmatrix} \mathcal{P}_{k+4,k+2} & \mathcal{P}_{k+3,k+3} \\ \mathcal{P}_{k+3,k+3} & \mathcal{P}_{k+2,k+4} \end{pmatrix}_{\mathbb{S}} \mid \operatorname{div} \tau = 0, \quad \tau n = 0 \text{ on } \partial R \right\}.$$

Similar arguments to Lemma 2.1 show that for τ in $M_k(R)$, $\tau = J(b_R^2 q)$ for some $q \in \mathcal{P}_{k,k}$ with $b_R = L_1 L_2 L_3 L_4$, hence dim $M_k(R) = (k+1)^2$. Next, the dimension of $\epsilon(V_R)$ is $2(k+1)(k+2) - 3 = 2k^2 + 6k + 1$ and $\epsilon(V_R)$ is orthogonal to $M_k(R)$ by (2.4). We conclude that the dimension of

$$N_k(R) = \epsilon(V_R) + M_k(R)$$

is $3k^2+8k+2$. Notice also that for $\tau \in \begin{pmatrix} \mathcal{P}_{k+4,k+2} & \mathcal{P}_{k+3,k+3} \\ \mathcal{P}_{k+3,k+3} & \mathcal{P}_{k+2,k+4} \end{pmatrix}_{\mathbb{S}}$, $\tau n \cdot n \in \mathcal{P}_{k+2}$ and $\tau n \cdot t \in \mathcal{P}_{k+3}$. We can now give the degrees of freedom of Σ_R:

(i) the values of each component of τ at each vertex of R, ($3 \times 4 = 12$ degrees of freedom)
(ii) the moments of degree at most k of $\tau n \cdot n$ on each edge ($4(k+1)$ degrees of freedom)
(iii) the moments of degree at most $k+1$ of $\tau n \cdot t$ on each edge ($4(k+2)$ degrees of freedom)
(iv) the values of $\int_R \tau : \phi$ for all $\phi \in N_k(R)$.

The proof of unisolvency is identical to the lowest order case. The corresponding error estimates are similar to those in the triangular case.[2] We recall that the Clement interpolant which takes values into the space of continuous matrix fields with components in $\mathcal{P}_{k,k}$ satisfies the error estimate

$$\|R_h v - v\|_j \leq c h^{m-j} \|v\|_m, \quad 0 \leq j \leq 1, \ j \leq m \leq k+1,$$

and the projection operator into V_R satisfies

$$\|P_h^k v - v\|_0 \leq c h^m \|v\|_m, \quad 0 \leq m \leq k+1.$$

We have

Theorem 5.1. *Let (σ, u) denote the unique critical point of the Hellinger-Reissner funtional over $H(\operatorname{div}, \Omega, \mathbb{S}) \times L^2(\Omega, \mathbb{R}^2)$ and let (σ_h, u_h) be the unique critical point over $\Sigma_h \times V_h$. Then*

$$\begin{aligned}\|\sigma - \sigma_h\|_0 &\leq c h^m \|\sigma\|_m, & 1 \leq m \leq k+1, \\ \|\operatorname{div} \sigma - \operatorname{div} \sigma_h\|_0 &\leq c h^m \|\operatorname{div} \sigma\|_m, & 0 \leq m \leq k+1, \\ \|u - u_h\|_0 &\leq c h^m \|u\|_{m+1}, & 1 \leq m \leq k+1.\end{aligned}$$

6. A Simplified Element of Low Order

Let $RM(R)$ be the space of infinitesimal rigid motions on R, i.e., vector fields of the form $(a - cx_2, b - cx_1)$. We take $V_R = RM(R)$ and

$$\Sigma_R = \left\{ \tau \in \begin{pmatrix} \mathcal{P}_{5,3} & \mathcal{P}_{4,4} \\ \mathcal{P}_{4,4} & \mathcal{P}_{3,5} \end{pmatrix}_{\mathbb{S}} \text{ and } \operatorname{div} \tau \in RM(R) \right\}.$$

Then, dim $\Sigma_R = 36$ since the condition div $\tau \in RM(R)$ imposes $40 - 3 = 37$ conditions on τ. We take as degrees of freedom

(i) the values of each component of $\tau(x)$ at the vertices of R (12 degrees of freedom)
(ii) the first two moments of $(\tau n) \cdot n$ on each edge (8 degrees of freedom)
(iii) the first three moments of $(\tau n) \cdot t$ on each edge (12 degrees of freedom)
(iv) the values of $\int_R \tau : \phi$ for all ϕ in

$$M_1(R) := \left\{ \tau \in \begin{pmatrix} \mathcal{P}_{5,3} & \mathcal{P}_{4,4} \\ \mathcal{P}_{4,4} & \mathcal{P}_{3,5} \end{pmatrix}_{\mathbb{S}} \mid \operatorname{div} \tau = 0, \quad \tau n = 0 \text{ on } \partial R \right\}$$

(4 degrees of freedom)

The proof of unisolvency is as in the lowest order case, since for $v \in RM(R), \epsilon(v) = 0$ and again the error estimates are similar to those of the triangular case since Σ_R contains the matrix fields with components in $\mathcal{P}_{0,0}$ and V_R contains the vector fields with components in $\mathcal{P}_{0,0}$.

$$\begin{aligned} \|\sigma - \sigma_h\|_0 &\leq ch\|\sigma\|_1, \\ \|\operatorname{div} \sigma - \operatorname{div} \sigma_h\|_0 &\leq ch^m \|\operatorname{div} \sigma\|_m, \qquad 0 \leq m \leq 1, \\ \|u - u_h\|_0 &\leq ch\|u\|_2. \end{aligned}$$

References

[1] D. Arnold, Differential complexes and numerical stability, in *Proceedings of the International Congress of Mathematicians, Vol. I: Plenary Lectures and Ceremonies*, L. Tatsien, ed. (Higher Education Press, Beijing, 2002), pp. 137–157.

[2] D. Arnold and R. Winther, Mixed finite elements for elasticity, *Numer. Math.* **92** (2002) 401–419.

[3] P. Clément, Approximation by finite element functions using local regularization, *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge* **9** (1975), 77–84.

[4] G. Bernardi and V. Girault, A local regularization operator for triangular and quadrilateral finite elements, *SIAM J. Numer. Anal.* **35** (1998) 1893–1916.

Institute for Mathematics and its Applications, University of Minnesota, 400 Lind Hall, 207 Church Street S.E., Minneapolis, MN, 55455, USA

E-mail address: arnold@ima.umn.edu

URL: http://www.ima.umn.edu/~arnold

E-mail address: awanou@ima.umn.edu

URL: http://www.ima.umn.edu/~awanou