STANDARD FINITE ELEMENTS FOR THE NUMERICAL RESOLUTION OF THE ELLIPTIC MONGE-AMPÈRE EQUATION: ALEKSANDROV SOLUTIONS

GERARD AWANOU

Abstract. We prove a convergence result for a natural discretization of the Dirichlet problem of the elliptic Monge-Ampère equation using finite dimensional spaces of piecewise polynomial C^1 functions. Discretizations of the type considered in this paper have been previously analyzed in the case the equation has a smooth solution and numerous numerical evidence of convergence were given in the case of non smooth solutions. Our convergence result is valid for non smooth solutions, is given in the setting of Aleksandrov solutions, and consists in discretizing the equation in a subdomain with the boundary data used as an approximation of the solution in the remaining part of the domain. Our result gives a theoretical validation for the use of a non monotone finite element method for the Monge-Ampère equation.

1. Introduction

Let $\Omega \subset \mathbb{R}^d, d = 2, 3$ be a convex domain with polygonal boundary $\partial\Omega$. In this paper we prove a convergence result for the numerical approximation of solutions to the Dirichlet problem for the Monge-Ampère equation

$$\det D^2 u = f \text{ in } \Omega,\ u = g \text{ on } \partial\Omega, \tag{1.1}$$

by elements of a space V_h of piecewise polynomial functions of some degree $k \geq d$ which are globally C^1. The expression $\det D^2 u$ should be understood in the sense of Aleksandrov c.f. section 2.5. For a smooth function u, $D^2 u = \bigg((\partial^2 u)/(\partial x_i \partial x_j) \bigg)_{i,j=1,\ldots,d}$ is the Hessian of u and f is a given function on Ω satisfying $f \in C(\Omega)$ with $0 < c_0 \leq f \leq c_1$ for constants $c_0, c_1 \in \mathbb{R}$. We assume that $g \in C(\partial\Omega)$ can be extended to a function $\tilde{g} \in C(\overline{\Omega})$ which is convex on Ω.

Let $f_m, g_m \in C^\infty(\overline{\Omega})$ such that $0 < c_2 \leq f_m \leq c_3$, f_m converges uniformly to f on $\overline{\Omega}$ and g_m converges uniformly to $\tilde{g}$ on $\overline{\Omega}$. See for example [3]. Let $u_m \in C(\overline{\Omega})$ denote the Aleksandrov solution of the problem

$$\det D^2 u_m = f_m \text{ in } \Omega,\ u_m = g_m \text{ on } \partial\Omega. \tag{1.2}$$

Finally let $\widetilde{\Omega}$ be a convex polygonal subdomain of Ω. We prove that the problem: find $u_h \in V_h(\widetilde{\Omega})$, $u_h = u_m$ on $\overline{\Omega} \setminus \widetilde{\Omega}$ and

$$\sum_{K \in \mathcal{T}_h(\widetilde{\Omega})} \int_K (\det D^2 u_h - f_m) v_h \, dx = 0, \forall v_h \in V_h(\widetilde{\Omega}) \cap H^1_0(\widetilde{\Omega}), \tag{1.3}$$

has a (locally unique) piecewise strictly convex solution u_h on $\widetilde{\Omega}$ which converges uniformly on compact subsets of $\widetilde{\Omega}$ to the solution $\tilde{u}$ of

$$\det D^2\tilde{u} = f_m \text{ in } \widetilde{\Omega},\ \tilde{u} = u_m \text{ on } \overline{\Omega} \setminus \widetilde{\Omega}, \tag{1.4}$$

which is convex on $\widetilde{\Omega}$ and continuous up to the boundary of $\widetilde{\Omega}$.

Here $\mathcal{T}_h(\widetilde{\Omega})$ denotes a quasi-uniform triangulation of the domain $\widetilde{\Omega}$ and $V_h(\widetilde{\Omega})$ denotes a finite element space on $\widetilde{\Omega}$ of piecewise polynomial C^1 functions of degree $k \geq d$. We make the abuse of notation of writing $u_h = u_m$ to mean that our approximations are discontinuous on the boundary and that u_h coincides with u_m at the Lagrange points on $\partial\widetilde{\Omega}$. For simplicity, we do not indicate the dependence of $\tilde{u}$ on m.

A piecewise strictly convex function which is C^1 is strictly convex as a consequence of [27, Theorem 6, p. 1091] and [41, Lemma 8.32]. Thus our approximations are strictly convex.

1.1. **Relevance of the convergence result for practical computations.** Problems in affine geometry motivated the study of the Dirichlet problem for the Monge-Ampère equation. See for example [7] for a numerical study of the Gauss-curvature equation which is a Monge-Ampère type equation. The Monge-Ampère equation also appears in several applications, e.g. optimal transport and reflector design, but with the so-called second boundary condition, a term used to indicate that this type of boundary condition was studied much later than the Dirichlet problem. Formally, the numerical study of the second boundary condition can be reduced to a sequence of Dirichlet problems using a fixed point algorithm.

Recently, several researchers have used a standard discretization of the type considered in this paper for the numerical study of the reflector design problem [13]. Even if one uses the same type of discretization for the Dirichlet problem (1.1), there is not yet a convergence theory. The convergence result of this paper, as stated above, addresses this issue.

Let $\delta > 0$. It is known, c.f. Theorem 2.15, that the Aleksandrov solution u_m of (1.2) converges uniformly on compact subsets of Ω to the Aleksandrov solution u of (1.1). We choose m such that $|u(x) - u_m(x)| < \delta/2$ for all $x \in \widetilde{\Omega}$. By unicity of the Aleksandrov solution u_m of (1.2), we have $\tilde{u} = u_m$ in $\widetilde{\Omega}$. Thus our results give on each compact subset of $\widetilde{\Omega}$, $|u_h(x) - u_m(x)| < \delta/2$ for h sufficiently small. The solution u of (1.1) can then be approximated within a prescribed accuracy by first choosing m and then h sufficiently small. We emphasize that the solution $\tilde{u}$ of (1.4) is not necessarily smooth.

It remains to chose the data to compute the local solution of (1.3). We may assume that $|f(x) - f_m(x)| < \delta$, $|\tilde{g}(x) - g_m(x)| < \delta$ and since $u_m = g_m$ on $\partial\Omega$ and $u_m \in C(\overline{\Omega})$, we may choose $\widetilde{\Omega}$ such that $|u_m - g_m| < \delta$ on $\overline{\Omega} \setminus \widetilde{\Omega}$. Thus, from a practical point of view, for the implementation, we see that one can take $\widetilde{\Omega} = \Omega$, $f_m = f$ with $u_h = g$ on $\partial\Omega$. A similar situation arises in the routine use in the finite element literature of the approximation of a smooth domain by a polygonal domain. Numerical experiments with the discretization considered in this paper were given in [2] for both smooth and non smooth solutions. For that reason, they are not reproduced here. Another possibility, but with results of less accuracy, is to actually implement the method on a subdomain. This can be easily tested on a code for (1.1) by extending g to a larger domain $\hat{\Omega}$ and using the restriction of g on $\partial\hat{\Omega}$ as boundary value. For the extension of the framework of this paper to the second boundary condition, only the choice of f_m and $\widetilde{\Omega}$ is needed. We wish to address this in a separate work.

1.2. **Methodology.** The purpose of this section is to explain the need for regularization of the data and the need of a subdomain for our convergence result. The methodology of this paper may be applied to other settings where one has numerical evidence of convergence for discretizations of (1.1). The general methodology consists in

1- Prove the convergence and local uniqueness of the solution of the discrete problem (1.3) when (1.1) has a smooth solution. See [5]. Under the assumption that the discrete problem (1.3) has a solution which is piecewise strictly convex, prove local uniqueness using the continuity of the eigenvalues of a matrix as a function of its entries. See section 6.2.

2- Verify that the numerical method is robust enough to handle the standard tests for non smooth solutions. In [2], we prove the convergence of iterative methods which preserve weakly convexity and their effectiveness in capturing a convex solution of (1.3) was illustrated with numerical experiments.

3- Choose m, f_m, g_m and $\widetilde{\Omega}$ as specified in section 1.1.

4- Consider a sequence of smooth uniformly convex domains Ω_s increasing to Ω [9], with the property that $\widetilde{\Omega} \subset \Omega_s$ for all s, and the problems with smooth solutions [43]

$$\det D^2 u_{ms} = f_m \,\text{in}\, \Omega_s, u_{ms} = g_m \,\text{on}\, \partial\Omega_s. \tag{1.5}$$

From Theorem 2.15, u_{ms} converges uniformly on $\widetilde{\Omega}$ to the solution u_m of (1.2) and hence to $\tilde{u}$ as $s \to \infty$.

5- Establish that the discrete approximation $u_{ms,h}$ of the smooth function u_{ms}, on $\widetilde{\Omega}$ and with boundary data u_{ms}, converges uniformly to u_{ms} on $\widetilde{\Omega}$ as $h \to 0$. This takes the form of an error estimate with constants depending on derivatives of u_{ms}.

6- Because $\widetilde{\Omega}$ is an interior domain, interior Schauder estimates allow to get a uniform bound on the derivatives of u_{ms}. In other words, $u_{ms,h}$ converges uniformly to u_{ms} on compact subsets of $\widetilde{\Omega}$ at a rate which depends on $\widetilde{\Omega}$ but is independent of s.

7- The local equicontinuity of convex functions allows to take a subsequence in s. This gives a convex finite element function u_h which solves the finite element problem (1.3). The approximation u_h is shown to converge uniformly on compact subsets of $\widetilde{\Omega}$ to the solution $\tilde{u}$ of (1.4). Local uniqueness of the discrete solution is a consequence of the work done in Step 1.

1.3. **Possible disadvantages of the approach in this paper.** We prove that (1.3) has a strictly convex solution which is locally unique. Even when (1.1) has a smooth solution, global uniqueness of the discrete approximation has not been addressed in previous work. In the standard finite difference context, a variational approach presented in [6] allows to select a special discrete solution. Numerical results reported therein indicate that such an approach is effective when the right hand side of the Monge-Ampère equation is a sum of Dirac masses. The analysis in [6] uses heavily results on the existence of local solutions.

The convergence result in the paper uses results available for the approximation of smooth solutions of (1.1) using standard discretizations. See for example [5]. When (1.1) has a smooth strictly convex solution, these results say that the discrete problem has a solution for $h \leq h_0$ where $h_0 \to 0$ as a high order Sobolev norm of u approaches infinity. Thus for example when $||u||_{C^{k+1}(\Omega)}$ is very big, existence of a discrete solution would hold for h close to machine precision. And this is just for smooth solutions. The interior Schauder estimates give a possibly large upper bound on $||u_{ms}||_{C^{k+1}(\widetilde{\Omega})}$ as the latter depends on the distance of $\widetilde{\Omega}$ to Ω. Thus it is not possible, in this setting, to quantify how small h should be for the existence of $u_{ms,h}$. We recall that u_{ms} and $u_{ms,h}$ were introduced in step 5 of the methodology described in section 1.2.

Results for the numerical approximation of viscosity solutions for (1.1) in the degenerate case $f \geq 0$ are stated in terms of uniform convergence on compact subsets with no quantification of how small h can be. Thus no information is given about how small h should be for a reasonable reduction of the error, although in that setting there is no restriction on the size of h for the existence of a discrete solution.

1.4. **Relation with other work.** A convergence analysis for a discretization of (1.1) starts with a choice of a notion of weak solution. For an analysis based on the notion of viscosity solution, we refer to [24] in the finite difference context, and to [23] in the finite element context for radial solutions with a biharmonic regularization. The discretization proposed in [24] is a monotone scheme and thus enjoys a discrete maximum principle. One of the advantages of a monotone scheme is that one can prove existence of a discrete solution with no restriction on the size of the mesh. Netherthe-less, the reader should be aware that there are many non monotone schemes for problems given in the setting of viscosity solutions e.g. [28]. The lack of a maximum principle for the discretizations analyzed in this paper is related to the difficulty of proving stability of the discretization for smooth solutions without assuming a bound on a high order norm of the solution. For that reason, we introduced the theoretical computational domain $\widetilde{\Omega}$ and fix the parameter m in the regularization of the data.

The weak solution in the viscosity sense is known to be equivalent to the weak solution in the sense of Aleksandrov for $f \in C(\overline{\Omega})$ and $f > 0$ on Ω. The arguments of this paper are based on the notion of Aleksandrov solution. To the best of our knowledge, a proven convergence result for the numerical resolution of (1.1) via the notion of Aleksandrov solution was only considered in [36] for the two dimensional problem. The approach in [36] uses geometric arguments and is different from the one taken here.

When the weak smooth solution of (1.1) is a smooth strictly convex function, Böhmer [10] studied C^1 approximations and his method has been implemented in [18]. See also [17]. Böhmer's method requires a modification of the Argyris space and numerical results in [18] used Newton's method and did not address some of the standard test cases for non smooth solutions. In [13], it is shown that with a standard C^1 approximation based on B-splines, Newton's method coupled with trust region methods is effective for these standard test cases. Newton's method was also used in [22] in the vanishing moment methodology. See also [4]. In [5], we analyzed the discretization (1.3) for C^1 approximations and gave numerical evidence of convergence for non smooth solutions if one uses Lagrange elements and a time marching method. We previously gave the corresponding numerical results with C^1 approximations in [2]. In [34] it is shown that Newton's method is effective if one uses a mixed formulation and implement the resulting method in primal form. See [33] for a description of the method for linear non variational problems. However in all these works, i.e. [10, 18, 13, 22, 4, 34, 5], no proof of convergence is given in the case the solution of (1.1) is not in $H^2(\Omega)$.

In this paper, we present a theory which explains why standard discretizations of the type considered in this paper exhibit numerical convergence for non smooth solutions of the Monge-Ampère equation. The easiest way to get insight into the problem, is through the approach which consists in regularizing the exact solution [3]. The latter approach is less general in the sense that it does not apply to collocation type discretizations such as the standard finite difference method. In fact, it is a standard technique in the analysis of Aleksandrov solutions of the Monge-Ampère equation, e.g. [19, Lemma 3.1], to regularize the data f, g and take a sequence of smooth uniformly convex domains approximating the given domain. It is then natural, following principles of compatible discretization, that a similar approach can be followed for a discretization. Spaces of piecewise polynomials C^1 functions can be constructed using Argyris elements, the spline element method [2] or isogeometric analysis.

Regularization of the data has been used in [26]. If one assumes that the domain Ω is smooth and uniformly convex, we can take $\widetilde{\Omega} = \Omega$ and use global Schauder estimates c.f. [43], and a bootstrapping argument, to implement the compactness argument described in section 1.2. To address the practical issue of dealing with curved boundaries, one should use the approach in [11] which consists in a penalization of the boundary condition and the use of curvilinear coordinates for elements near the boundary. The boundary condition can now be taken as $\tilde{u} = g_m$. The approach of this paper can be easily adapted to explain the numerical results with singular data presented in [1].

Without loss of generality, in subsequent papers on the analysis of schemes for (1.1), one may assume that f and g are smooth. In fact, one can even also assume that the solution is smooth, as the techniques of this paper can be applied to handle the non smooth case.

1.5. **Organization of the paper.** We organize the paper as follows. In the next section, we introduce some notation, recall the main results on the convergence of the discretization (1.3) when (1.1) has a smooth solution and the notion of Aleksandrov solution of (1.1). In section 3 we give preliminary results on smooth and polygonal exhaustions of the domain. In section 4 we give the proof of existence of a convex solution of (1.3). The proof of the convergence of the discretization is given in section

5. In section 6 we prove that our approximations are strictly convex and give a local uniqueness result. The proof of some technical results are given in section 7.

2. Notation and preliminaries

2.1. **General notation.** For two subsets S and T of $\mathbb{R}^d$, we use the usual notation $d(S,T)$ for the distance between them. Moreover, $\operatorname{diam} S$ denotes the diameter of S.

We use the standard notation for the Sobolev spaces $W^{t,p}(\Omega)$ with norms $||.||_{t,p,\Omega}$ and semi-norm $|.|_{t,p,\Omega}$. In particular, $H^t(\Omega) = W^{t,2}(\Omega)$ and in this case, the norm and semi-norms will be denoted respectively by $||.||_{t,\Omega}$ and $|.|_{t,\Omega}$. When there is no confusion about the domain Ω, we will omit the subscript Ω in the notation of the norms and semi-norms. We recall that $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ of elements with vanishing trace on $\partial\Omega$.

We make the usual convention of denoting constants by C but will occasionally index some constants. We assume that the triangulation $\mathcal{T}_h(\Omega)$ of the domain Ω is shape regular in the sense that there is a constant $C > 0$ such that for any element K, $h_K/\rho_K \leq C$, where h_K denotes the diameter of K and ρ_K the radius of the largest ball contained in K. We also require the triangulation to be quasi-uniform in the sense that h/h_{min} is bounded where h and h_{min} are the maximum and minimum respectively of $\{h_K, K \in \mathcal{T}_h\}$.

2.2. **Finite dimensional subspaces.** We will need the broken Sobolev norms and semi-norms

$$||v||_{t,p,h} = \bigg(\sum_{K \in \mathcal{T}_h(\Omega)} ||v||_{t,p,K}^2 \bigg)^{\frac{1}{2}}, 1 \leq p < \infty$$
$$||v||_{t,\infty,h} = \max_{K \in \mathcal{T}_h(\Omega)} ||v||_{t,\infty,K},$$

with a similar notation for $|v|_{t,p,h}$.

We let $V_h(\Omega)$ denote a finite dimensional space of piecewise polynomial $C^1(\Omega)$ functions, of local degree $k \geq d$, i.e., V_h is a subspace of

$$\{v \in C^1(\Omega),\ v|_K \in \mathcal{P}_k,\ \forall K \in \mathcal{T}_h(\Omega)\},$$

and $\mathcal{P}_k$ denotes the space of polynomials of degree less than or equal to k. We make the assumption that the following approximation properties hold:

$$||v - \Pi_h v||_{t,p,h} \leq C_{ap} h^{l+1-t} |v|_{l+1,p}, \tag{2.1}$$

where Π_h is a projection operator mapping the Sobolev space $W^{l+1,p}(\Omega)$ into V_h, $1 \leq p \leq \infty$ and $0 \leq t \leq l \leq k$. We require that the constant C_{ap} does not depend on h and v. We also make the assumption that the following inverse inequality holds

$$||v||_{t,p,h} \leq C_{inv} h^{l-t+\min(0,\frac{d}{p}-\frac{d}{q})} ||v||_{l,q,h}, \forall v \in V_h, \tag{2.2}$$

and for $0 \leq l \leq t, 1 \leq p, q \leq \infty$. We require that the constant C_{inv} be independent of h and v. The approximation property and inverse estimate assumptions are realized for standard finite element spaces [12].

2.3. **Approximations of smooth solutions of the Monge-Ampère equation.** Next, we summarize the results of [5, 2, 10] of estimates for C^1 finite element approximations of smooth solutions of (1.1).

Theorem 2.1. *Let O_s be a convex polygonal subdomain of Ω with a quasi-uniform triangulation $\mathcal{T}_h(O_s)$. Assume that $u_s \in C^\infty(\overline{O}_s)$ is a strictly convex function which solves*

$$\det D^2 u_s = f_s \; in \, O_s, \quad u_s = g_s \; on \, \partial O_s,$$

with $f_s, g_s \in C^\infty(\overline{O}_s)$ and $f_s \geq C > 0$. We consider the problem: find $u_{s,h} \in V_h(O_s)$, $u_{s,h} = g_s$ on ∂O_s and

$$\sum_{K \in \mathcal{T}_h(O_s)} \int_K (\det D^2 u_{s,h} - f_s) v_h \, dx = 0, \forall v_h \in V_h(O_s) \cap H_0^1(O_s). \tag{2.3}$$

Problem (2.3) *has a (locally unique) piecewise convex solution $u_{s,h}$ with*

$$||u_s - u_{s,h}||_{2,h,O_s} \leq C_s h^{l-1}, 2 \leq l \leq k,$$

and the constant C_s is uniformly bounded if $||u_s||_{l+1,\infty,O_s}$ is uniformly bounded.

The result of Theorem 2.1 follows from [10, Theorems 5.1 and 8.7] and an inverse estimate. Equation (2.3) differs from (1.3) in the sense that we assume here that u_s is smooth whereas the solution $\tilde{u}$ of (1.4) is not necessarily smooth.

Corollary 2.2. *Under the assumptions (and notation) of Theorem 2.1, the approximate solution $u_{s,h}$ converges uniformly on compact subsets of O_s to u_s as $h \to 0$.*

Proof. For each element $K \in \mathcal{T}_h(O_s)$, by the embedding of $H^2(K)$ into $L^\infty(K)$, we obtain

$$||u_s - u_{s,h}||_{0,\infty,K} \leq ||u_s - u_{s,h}||_{2,K} \leq C_s h^{l-1} ||u_s||_{l+1,\infty,O_s}.$$

Therefore

$$||u_s - u_{s,h}||_{0,\infty,O_s} \leq C_s h^{l-1} ||u_s||_{l+1,\infty,O_s},$$

and the result follows. $\square$

2.4. **Interior Schauder estimates.** Recall that $\Omega_s \subset \Omega$ is a smooth uniformly convex domain. Recall also that the solution of (1.1) is not in general smooth unless f and $\tilde{g}$ are smooth and Ω is a smooth uniformly convex domain. Thus $||u||_{C^2(\Omega)}$ if defined is not finite in general. We will need estimates which depend on derivatives away from $\partial\Omega_s$ as we assume that Ω is a polygonal domain. This is the main reason for introducing the theoretical computational domain $\widetilde{\Omega}$. Recall that we make the assumption that

$$\tilde{\Omega} \subset \Omega_s, \text{ for all } s,$$

and thus the closure of $\widetilde{\Omega}$ is a compact subset of Ω. The proof of the following lemma is given in section 7.

Lemma 2.3. *Let u_{ms} solve* (1.5). *We have the uniform interior Schauder estimates*

$$||u_{ms}||_{C^{k+1}(\widetilde{\Omega})} \leq C_m,$$

where C_m depends only on $m, d, c_2, ||f_m||_{C^k(\overline{\Omega})}, \widetilde{\Omega}$ and $d(\widetilde{\Omega}, \partial\Omega)$.

2.5. **The Aleksandrov solution.** In this part of the section, we recall the notion of Aleksandrov solution of (1.1) and state several results that will be needed in our analysis. We follow the presentation in [29] to which we refer for further details.

Let Ω be an open subset of $\mathbb{R}^d$. Given a real valued convex function v defined on Ω, the normal mapping of v, or subdifferential of v, is a set-valued mapping N_v from Ω to the set of subsets of $\mathbb{R}^d$ such that for any $x_0 \in \Omega$,

$$N_v(x_0) = \{\, q \in \mathbb{R}^d : v(x) \geq v(x_0) + q \cdot (x - x_0), \text{ for all } x \in \Omega \,\}.$$

Given $E \subset \Omega$, we define $N_v(E) = \cup_{x \in E} N_v(x)$ and denote by $|E|$ the Lebesgue measure of E when the latter is measurable.

If v is a convex continuous function on Ω, the class

$$\mathcal{S} = \{\, E \subset \Omega, N_v(E) \text{ is Lebesgue measurable} \,\},$$

is a Borel σ-algebra and the set function $M[v] : \mathcal{S} \to \overline{\mathbb{R}}$ defined by

$$M[v](E) = |N_v(E)|,$$

is a measure, finite on compact sets, called the Monge-Ampère measure associated with the function v.

We are now in a position to define generalized solutions of the Monge-Ampère equation. Let the domain Ω be open and convex. Given a Borel measure μ on Ω, a convex function $v \in C(\Omega)$, is an Aleksandrov solution of

$$\det D^2 v = \mu,$$

if the associated Monge-Ampère measure $M[v]$ is equal to μ. If μ is absolutely continuous with respect to the Lebesgue measure and with density f, i.e.

$$\mu(B) = \int_B f \, dx, \text{ for any Borel set } B,$$

we identify μ with f. We have

Theorem 2.4 ([31] Theorem 1.1)**.** *Let Ω be a bounded convex domain of $\mathbb{R}^d$. Assume $f \in L^1(\Omega)$ and $g \in C(\partial\Omega)$ can be extended to a function $\tilde{g} \in C(\overline{\Omega})$ which is convex in Ω. Then the Monge-Ampère equation* (1.1) *has a unique convex Aleksandrov solution in $C(\overline{\Omega})$.*

Remark 2.5. *The assumption that $g \in C(\partial\Omega)$ can be extended to a convex function $\tilde{g} \in C(\overline{\Omega})$ can be removed if the domain Ω is uniformly convex,* [29].

We recall that for a convex function v in $C^2(\Omega)$, the Monge-Ampère measure $M[v]$ associated with v is given by

$$M[v](E) = \int_E \det D^2 v(x) \, dx,$$

for all Borel sets $E \subset \Omega$.

Lemma 2.6. *Let $v \in W^{2,d}(\Omega)$ be a piecewise C^2 convex function such that* $\det D^2 v \geq 0$ *a.e. Then $M[v](E) = \int_E \det D^2 v(x) \, dx$ for all Borel sets $E \subset \Omega$.*

The proof of the above lemma is given in section 7.

Definition 2.7. *A sequence μ_m of Borel measures is said to converge weakly to a Borel measure μ if and only if*

$$\int_\Omega p(x)\, d\mu_m \to \int_\Omega p(x)\, d\mu,$$

for every continuous function p with compact support in Ω.

For the special case of absolutely continuous measures μ_m with density a_m with respect to the Lebesgue measure, we have

Definition 2.8. *Let $a_m, a \geq 0$ be given functions. We say that a_m converges weakly to a as measures if and only if*

$$\int_\Omega a_m p\, dx \to \int_\Omega a p\, dx,$$

for all continuous functions p with compact support in Ω.

We have the following weak continuity result of Monge-Ampère measures with respect to local uniform convergence.

Lemma 2.9 (Lemma 1.2.3 [29])**.** *Let u_m be a sequence of convex functions in Ω such that $u_m \to u$ uniformly on compact subsets of Ω. Then the associated Monge-Ampère measures $M[u_m]$ tend to $M[u]$ weakly.*

Remark 2.10. *It follows that if u_m is a sequence of $C^2(\Omega)$ convex functions such that $u_m \to u$ uniformly on compact subsets of Ω, with u solving* (1.1)*, then* $\det D^2 u_m$ *converges weakly to f as measures.*

We will often use the following lemma, the proof of which is given in section 7.

Lemma 2.11. *Let u_j denote a uniformly bounded sequence of convex functions on a convex domain Ω. Then the sequence u_j is locally uniformly equicontinuous and thus has a pointwise convergent subsequence.*

2.6. **Approximations by solutions on subdomains.** For a function b defined on $\partial\Omega$, we denote by b^* its convex envelope, i.e. the supremum of all convex functions below b. If b can be extended to a continuous convex function on $\overline{\Omega}$, then $b^* = b$ on $\partial\Omega$.

Following [39], we define a notion of convergence for functions defined on different subdomains. Recall that $\Omega \subset \mathbb{R}^d$ is bounded and convex. For a function $z : \Omega \to \mathbb{R}$, its upper graph Z is given by

$$Z := \{\, (x, x_{d+1}) \in \Omega \times \mathbb{R}, x_{d+1} \geq v(x) \,\}.$$

For a function $b : \partial\Omega \to \mathbb{R}$, its upper graph is given by

$$B := \{\, (x, x_{d+1}) \in \partial\Omega \times \mathbb{R}, x_{d+1} \geq b(x) \,\}.$$

Definition 2.12. *We say that $z = b$ on $\partial\Omega$ if*

$$B = \overline{Z} \cap (\partial\Omega \times \mathbb{R}).$$

Definition 2.13. *The Hausdorff distance between two nonempty subsets K and H of $\mathbb{R}^d$ is defined as*

$$\max\{\sup[d(x,K), x \in H], \sup[d(x,H), x \in K]\}.$$

Let $\Omega_s \subset \Omega$ be a sequence of convex domains and let $z_s : \Omega_s \to \mathbb{R}$ be a sequence of convex functions on Ω_s. We write $z_s \to z$ if the upper graphs $\overline{Z}_s$ converge in the Hausdorff distance to the upper graph $\overline{Z}$ of z. Similarly, for a sequence $b_s : \partial\Omega_s \to \mathbb{R}$, we say that $b_s \to b$ if the corresponding upper graphs converge in the Hausdorff distance.

Finally, let $a_s : \Omega_s \to \mathbb{R}$ and $a : \Omega \to \mathbb{R}$. We write $a_s \to a$ if the a_s are uniformly bounded and a_s converges to a uniformly on compact subsets of Ω.

To summarize, in Proposition 2.14 below, for a sequence of convex functions on Ω_s or for their restriction to $\partial\Omega_s$, the convergence is convergence of the corresponding upper graphs in the Hausdorff distance whereas for the data a_s we use uniform convergence on compact subsets.

We have

Proposition 2.14 (Proposition 2.4 of [39])**.** *Let $z_s : \Omega_s \to \mathbb{R}$ be convex such that*

$$\det D^2 z_s = a_s \, in \, \Omega_s, z_s = b_s \, on \, \partial\Omega_s.$$

If

$$z_s \to z, a_s \to a, b_s \to b,$$

then

$$\det D^2 z = a \, in \, \Omega, z = b^* \, on \, \partial\Omega,$$

where b^* denotes the convex envelope of b on $\partial\Omega$. In particular if b can be extended to a continuous convex function on $\overline{\Omega}$, $z = b$ on $\partial\Omega$.

We state an approximation result for Monge-Ampère equations which follows from [39, Proposition 2.6]. A detailed proof is given in section 7.

Theorem 2.15. *Let Ω_s be a sequence of convex domains increasing to Ω, i.e. $\Omega_s \subset \Omega_{s+1} \subset \Omega$ and $d(\partial\Omega_s, \partial\Omega) \to 0$ as $s \to \infty$. Assume that $z_s \in C(\overline{\Omega}_s)$ is a sequence of convex functions solving*

$$\det D^2 z_s = a \, in \, \Omega_s, z_s = b \, on \, \partial\Omega_s,$$

with $a \geq 0$, $a \in C(\overline{\Omega})$. Assume that $b \in C(\overline{\Omega})$ and is convex on Ω.

Then z_s converges (up to a subsequence) uniformly on compact subsets of Ω to the unique convex solution z of

$$\det D^2 z = a \, in \, \Omega, z = b \, on \, \partial\Omega,$$

Remark 2.16. *If v_s is a sequence of convex functions which converge on Ω to a convex function v with upper graph V, we can extend v canonically to the boundary by taking the function on $\partial\Omega$ with upper graph $\overline{V} \cap (\partial\Omega \times \mathbb{R})$.*

2.7. **A characterization of weak convergence of measures.** The result we now give is well-known but we give a proof in section 7 for completeness.

Let $C_b(\Omega)$ denote the space of bounded continuous functions on Ω. We have

Lemma 2.17. *Let $a_m, a \in C_b(\Omega), a_m, a \geq 0$ for $m = 0, 1, \ldots$ Assume that the sequence a_m is uniformly bounded on Ω and that a_m converges weakly to a as measures and let $p \in H_0^1(\Omega)$. We have*

$$\int_\Omega a_m p \, dx \to \int_\Omega ap \, dx,$$

as $m \to \infty$.

2.8. **Useful facts about convex functions.** It is known that the pointwise limit of a sequence of convex functions is convex. Also, every pointwise convergent sequence of convex functions converges uniformly on compact subsets. See for example [8, Remark 1 p. 129].

3. Smooth and polygonal exhaustions of the domain

It is known from [9] for example that there exists a sequence of smooth uniformly convex domains Ω_s increasing to Ω, i.e. $\Omega_s \subset \Omega_{s+1} \subset \Omega$ and $d(\partial\Omega_s, \partial\Omega) \to 0$ as $s \to \infty$. An explicit construction of the sequence Ω_s in the special case $\Omega = (0,1)^2$ can be found in [42].

Recall that f_m and g_m are $C^\infty(\overline{\Omega})$ functions such that $0 < c_2 \leq f_m \leq c_3, f_m \to f$ and $g_m \to \tilde{g}$ uniformly on $\overline{\Omega}$. Thus the sequences f_m and g_m are uniformly bounded on Ω. The sequences f_m and g_m may be constructed by extending the given functions to a slightly larger domain preserving the property $f \geq C > 0$ for some constant C and apply a standard mollification. See [3] for a different procedure. By [14], the problem (1.5) has a unique convex solution $u_{ms} \in C^\infty(\overline{\Omega}_s)$. By Theorem 2.15, as $s \to \infty$, the sequence u_{ms} converges uniformly on compact subsets of Ω to the unique convex solution $u_m \in C(\overline{\Omega})$ of Problem (1.2). Moreover, the solution u_m of (1.2) converges uniformly on compact subsets of Ω to the unique convex solution u of (1.1).

Recall that $\widetilde{\Omega}$ is a convex polygonal subdomain of Ω with a quasi-uniform triangulation $\mathcal{T}_h(\widetilde{\Omega})$. We let $\delta > 0$ be a fixed parameter and chose m and $\widetilde{\Omega}$ such that $|f(x) - f_m(x)| < \delta$, $|\tilde{g}(x) - g_m(x)| < \delta$ and $|u(x) - u_m(x)| < \delta$ for all $x \in \widetilde{\Omega}$. Without loss of generality we may assume that $\widetilde{\Omega} \subset \Omega_s$ for all s.

We have

Theorem 3.1. *There exists a convex function $u_h \in V_h(\widetilde{\Omega})$ which is uniformly bounded on compact subsets of $\widetilde{\Omega}$ uniformly in h. The function u_h satisfies $u_h = u_m$ on $\partial\widetilde{\Omega}$ and is obtained as the limit of a subsequence in s of the convex solution $u_{ms,h}$ in $V_h(\widetilde{\Omega})$ of the problem:*

$$\sum_{K \in \mathcal{T}_h} \int_{K \cap \widetilde{\Omega}} (\det D^2 u_{ms,h} - f_m) v_h \, dx = 0, \forall v_h \in V_h(\widetilde{\Omega}) \cap H_0^1(\widetilde{\Omega}), \tag{3.1}$$

with $u_{ms,h} = u_{ms}$ on $\partial\widetilde{\Omega}$.

Proof. Since u_{ms} is smooth on $\overline{\Omega}_s$, Theorem 2.1 yields a solution to Problem (3.1). The latter is convex on $\widetilde{\Omega}$ as a C^1 piecewise convex function, c.f. [16, section 5]. Given a compact subset K of $\widetilde{\Omega}$, we have

$$||u_{ms} - u_{ms,h}||_{0,\infty,K} \leq ||u_{ms} - u_{ms,h}||_{0,\infty,\widetilde{\Omega}} \leq C||u_{ms}||_{k+1,\infty,\widetilde{\Omega}}\ h^{k-1}. \tag{3.2}$$

since $\widetilde{\Omega} \subset \Omega_s$. By the interior Schauder estimates Lemma 2.3, the sequence in s of convex functions $u_{ms,h}$ is uniformly bounded on compact subsets, and hence by Lemma 2.11 has a convergent subsequence also denoted by $u_{ms,h}$ which converges to a function u_h. The function u_h is convex as the pointwise limit of convex functions and the convergence is uniform on compact subsets.

Next, we note that for a fixed h, $u_{ms,h}$ is a piecewise polynomial in the variable x of fixed degree k and convergence of polynomials is equivalent to convergence of their coefficients. Thus u_h is a piecewise polynomial of degree k. Moreover, the continuity conditions on $u_{ms,h}$ are linear equations involving its coefficients. Thus u_h has the same continuity property as $u_{ms,h}$. In other words $u_h \in V_h(\widetilde{\Omega})$.

Finally, since u_{ms} converges uniformly on compact subsets to u_m as $s \to \infty$, we have on $\partial\widetilde{\Omega}$, $u_h = u_m$ as $\partial\widetilde{\Omega}$ is by construction a compact subset of Ω.

As a consequence of the interior Schauder estimates, u_h is uniformly bounded on compact subsets of $\widetilde{\Omega}$ uniformly in h.

□

The goal of the next two sections is to prove that the function u_h given by Theorem 3.1 solves Problem (1.3).

4. Solvability of the discrete problems.

The goal of this section is to prove that (1.3) has a solution. Then Problem (3.1) can be written

$$\int_{\widetilde{\Omega}} (\det D^2 u_{ms,h} - f_m) v_h \, dx = 0, \forall v_h \in V_h(\widetilde{\Omega}) \cap H^1_0(\widetilde{\Omega}). \tag{4.1}$$

To see that the left hand side of the above equation is well defined, we note that $u_{ms,h}$ is a piecewise polynomial C^1 function and is thus in $W^{2,d}(\widetilde{\Omega})$. As a consequence $\det D^2 u_{ms,h} \in L^1(\widetilde{\Omega})$ and since $v_h \in L^\infty(\widetilde{\Omega})$, this gives the result.

Recall that the discrete solution $u_{ms,h}$ being piecewise convex and C^1 is convex on $\widetilde{\Omega}$, c.f. [16, section 5]. We define

$$f_{ms,h} = \det D^2 u_{ms,h}.$$

By Lemma 2.6, we can then view $u_{ms,h} \in W^{2,d}(\widetilde{\Omega})$ as the solution (in the sense of Aleksandrov) of the Monge-Ampère equation

$$\det D^2 u_{ms,h} = f_{ms,h} \,\text{in}\, \widetilde{\Omega}.$$

By Lemma 2.9, $\det D^2 u_{ms_l,h} \to \det D^2 u_h$ weakly as measures for a subsequence $s_l \to \infty$. Then by Lemma 2.17 we get for $v \in V_h(\widetilde{\Omega}) \cap H_0^1(\widetilde{\Omega})$,

$$\int_{\widetilde{\Omega}} (\det D^2 u_{ms_l,h}) v \, dx \to \int_{\widetilde{\Omega}} (\det D^2 u_h) v \, dx. \tag{4.2}$$

It remains to prove that as $l \to \infty$

$$\int_{\widetilde{\Omega}} (\det D^2 u_{ms_l,h}) v \, dx \to \int_{\widetilde{\Omega}} f_m v \, dx.$$

This is essentially what is proved in the next theorem

Theorem 4.1. *Let $V_h(\widetilde{\Omega})$ denote a finite dimensional space of C^1 functions satisfying the assumptions of approximation property and inverse estimates of section 2.2. Then Problem* (1.3) *has a convex solution u_h.*

Proof. Given $v \in V_h(\widetilde{\Omega}) \cap H_0^1(\widetilde{\Omega})$, let v_l be a sequence of infinitely differentiable functions with compact support in $\widetilde{\Omega}$ such that $||v_l - v||_{1,2} \to 0$ as $l \to \infty$. We have by definition of $f_{ms,h}$

$$\int_{\widetilde{\Omega}} (\det D^2 u_{ms_l,h}) v \, dx = \int_{\widetilde{\Omega}} f_{ms_l,h} v \, dx. \tag{4.3}$$

We have

$$\begin{aligned}\int_{\widetilde{\Omega}} f_{ms_l,h} v \, dx = \int_{\widetilde{\Omega}} f_{ms_l,h} (v - v_l) \, dx + \int_{\widetilde{\Omega}} f_{ms_l,h} (v_l - \Pi_h(v_l)) \, dx \\ + \int_{\widetilde{\Omega}} f_{ms_l,h} \Pi_h(v_l) \, dx,\end{aligned}$$

and thus by (4.1)

$$\begin{aligned}\int_{\widetilde{\Omega}} f_{ms_l,h} v \, dx &= \int_{\widetilde{\Omega}} f_{ms_l,h} (v - v_l) \, dx + \int_{\widetilde{\Omega}} f_{ms_l,h} (v_l - \Pi_h(v_l)) \, dx \\ &\quad + \int_{\widetilde{\Omega}} f_m \Pi_h(v_l) \, dx \\ &= \int_{\widetilde{\Omega}} (f_{ms_l,h} - f_m)(v - v_l) \, dx \\ &\quad + \int_{\widetilde{\Omega}} (f_{ms_l,h} - f_m)(v_l - \Pi_h(v_l)) \, dx + \int_{\widetilde{\Omega}} f_m v \, dx.\end{aligned} \tag{4.4}$$

By the inverse estimate (2.2)

$$\begin{aligned}|| \det D^2 u_{ms,h}||_{0,\infty,\widetilde{\Omega}} &\leq C ||u_{ms,h}||^d_{2,\infty,\widetilde{\Omega}} \\ &\leq C h^{-2d} ||u_{ms,h}||^d_{0,\infty,\widetilde{\Omega}}.\end{aligned}$$

Hence by Lemma 2.3

$$|| \det D^2 u_{ms,h}||_{0,\infty,\widetilde{\Omega}} \leq C_h, \tag{4.5}$$

for a constant C_h which depends on h but is independent of s.

Since f_m is uniformly bounded on $\overline{\Omega}$, it follows from (4.5)

$$\left| \int_{\widetilde{\Omega}} (f_{ms_l,h} - f_m)(v - v_l)\, dx \right| \leq C||v - v_l||_{1,2} \to 0 \,\text{as}\, l \to \infty. \tag{4.6}$$

Finally, since $v \in V_h(\widetilde{\Omega})$, we have $\Pi_h(v) = v$ and hence

$$\begin{aligned} \int_{\widetilde{\Omega}} (f_{ms_l,h} - f_m)(v_l - \Pi_h(v_l))\, dx = & \int_{\widetilde{\Omega}} (f_{ms_l,h} - f_m)(v_l - v)\, dx \\ & + \int_{\widetilde{\Omega}} (f_{ms_l,h} - f_m)(\Pi_h(v - v_l))\, dx. \end{aligned}$$

By Schwarz inequality, (4.5) and (2.1)

$$\left| \int_{\widetilde{\Omega}} (f_{ms_l,h} - f_m)(\Pi_h(v - v_l))\, dx \right| \leq C_h||\Pi_h(v - v_l)||_{0,2} \leq C_h||v - v_l||_{1,2} \to 0 \,\text{as}\, l \to \infty.$$

Arguing again as in (4.6), it follows that

$$\int_{\widetilde{\Omega}} (f_{ms_l,h} - f_m)(v_l - \Pi_h(v_l))\, dx \to 0 \,\text{as}\, l \to \infty. \tag{4.7}$$

We conclude by (4.2)–(4.7) that as $l \to \infty$

$$\int_{\widetilde{\Omega}} (\det D^2 u_{ms_l,h}) v\, dx \to \int_{\widetilde{\Omega}} f_m v\, dx.$$

By the unicity of the limit

$$\int_{\widetilde{\Omega}} (\det D^2 u_h) v\, dx = \int_{\widetilde{\Omega}} f_m v\, dx.$$

That is, the limit u_h solves (1.3). The existence of a solution to (1.3) is proved. □

5. Convergence of the discretization

We have

Theorem 5.1. *Under the assumptions set forth in the introduction, the convex solution u_h of Problem 1.3 (given by Theorem 4.1) converges uniformly on compact subsets of $\widetilde{\Omega}$, as $h \to 0$, to the solution $\tilde{u}$ of* (1.4) *which is convex on $\widetilde{\Omega}$ and continuous up to the boundary.*

Proof. We recall from Theorem 3.1 that the function u_h is uniformly bounded on compact subsets of $\widetilde{\Omega}$. It follows from Lemma 2.11 that there exists a subsequence u_{h_l} which converges pointwise to a convex function v. The latter is continuous on $\widetilde{\Omega}$ as it is locally finite. Moreover the convergence is uniform on compact subsets of $\widetilde{\Omega}$.

Recall also from Theorem 3.1 that u_h is obtained as a subsequence in s of the approximations $u_{ms,h}$ of smooth solutions u_{ms} which converge to u_m uniformly on compact subsets of Ω.

Let K be a compact subset of $\widetilde{\Omega}$. There exists a subsequence $u_{m,s_l,h}$ which converges uniformly to u_h on K. By the uniform convergence of u_{ms} to u_m on K, we may assume that u_{m,s_l} converges uniformly to u_m on K.

Let now $\epsilon > 0$. Since u_{h_l} converges uniformly on K to v, $\exists l_0$ such that $\forall l \geq l_0$ $|u_{h_l}(x) - v(x)| < \epsilon/6$ for all $x \in K$.

There exists $l_1 \geq 0$ such that for all $l \geq \max\{ l_0, l_1 \}$, $|u_{ms_l,h_l}(x) - u_{h_l}(x)| < \epsilon/6$ for all $x \in K$.

Moreover, there exists $l_2 \geq 0$ such that for all $l \geq \max\{ l_0, l_1, l_2 \}$, $|u_{ms_l}(x) - u_m(x)| < \epsilon/6$ for all $x \in K$.

Similarly to (3.2), we have on K, $|u_{ms,h_l}(x) - u_{ms}(x)| \leq C_m h_l$ for all $x \in K$. We recall that the constant C_m is independent of s but depends also on $\widetilde{\Omega}$.

We conclude that for $l \geq \max\{ l_0, l_1, l_2 \}$, $|u_m(x) - v(x)| < \epsilon/2 + C h_l$ for all $x \in K$. We therefore have for all $\epsilon > 0$ $|u_m(x) - v(x)| < \epsilon$. We conclude that $u_m = v$ on K.

Since $u_h = u_m$ on $\partial\widetilde{\Omega}$, it follows that $v = u_m$ on $\partial\widetilde{\Omega}$. This proves that $u_m = v$ on $\widetilde{\Omega}$.

The limit u_m being unique, we conclude that u_h converges uniformly on compact subsets of $\widetilde{\Omega}$ to $\tilde{u}$.

□

6. Piecewise strict convexity and local uniqueness

The proof of convergence of the time marching iterative methods for solving (1.3) given in [2, 5] requires the discrete solution to be piecewise strictly convex and locally unique. These results are given in this section. We make the abuse of notation of denoting by $D^2 w_h$ the piecewise Hessian of $w_h \in V_h(\widetilde{\Omega})$. Let $\lambda_1(D^2 w_h)$ denotes the smallest eigenvalue of $D^2 w_h$.

6.1. **Strict convexity of the discrete solution.**

Theorem 6.1. *For $k \geq 2(d+1)$ the C^1 solution u_h of* (1.3) *is piecewise strictly convex and thus strictly convex.*

Proof. Assume that $\det D^2 u_h$ (computed piecewise) is non zero on a set of non zero Lebesgue measure. Then since $\det D^2 u_h$ is a piecewise polynomial, it must vanish identically on an element K_0. Let v denote the unique polynomial of degree $d+1$ which vanishes identically on all faces of K_0 and with average 1 on K_0. We denote as well by v its extension by 0 on all other elements. Then $v > 0$ in K_0 and $v^2 \in V_h(\widetilde{\Omega}) \cap H_0^1(\widetilde{\Omega})$ and thus

$$\int_{\widetilde{\Omega}} f v^2 \, dx = \int_{K_0} f v^2 \, dx > 0.$$

On the other hand

$$\int_{\widetilde{\Omega}} f v^2 \, dx = \sum_{K \in \mathcal{T}_h} \int_{K \cap \widetilde{\Omega}} (\det D^2 u_h) v^2 \, dx = \int_{K_0} (\det D^2 u_h) v^2 \, dx = 0,$$

since $\det D^2 u_h = 0$ on K_0. Contradiction. We therefore have $\det D^2 u_h > 0$ a.e. in $\widetilde{\Omega}$.

□

Let $x_0 \in \widetilde{\Omega}$. If necessary, by identifying u_h with $u_h + \epsilon |x - x_0|^2$, where ϵ is taken to be close to machine precision, we may assume that the solution u_h is strictly convex.

6.2. Uniqueness of the discrete solution.

Theorem 6.2. *Let u_h be a C^1 strictly convex solution of* (1.3)*. Then u_h is locally unique.*

Proof. Define $B_\rho(u_h) = \{ w_h \in V_h, ||w_h - u_h||_{2,\infty} \leq \rho \}$. Then since $\lambda_1(D^2 u_h) \geq c_{00}$, by the continuity of the eigenvalues of a matrix as a function of its entries, w_h is strictly convex for ρ sufficiently small and ρ independent of h.

Let then u_h and v_h be two solutions of (1.3) in $B_\rho(u_h)$. By the mean value theorem, see for example [2], we have for $w_h \in V_h(\widetilde{\Omega}) \cap H_0^1(\widetilde{\Omega})$

$$
\begin{aligned}
0 &= \int_{\widetilde{\Omega}} (\det D^2 u_h - \det D^2 v_h) w_h \, dx \\
&= - \int_0^1 \left\{ \int_{\widetilde{\Omega}} [(\mathrm{cof}(1-t) D^2 v_h + t D^2 u_h)(D u_h - D v_h)] \cdot D w_h \, dx \right\} dt.
\end{aligned}
$$

For each $t \in [0,1]$, $(1-t) v_h + t u_h \in B_\rho(u_h)$ and is therefore strictly convex, that is

$$
[(\mathrm{cof}(1-t) D^2 v_h + t D^2 u_h) D(v_h - u_h)] \cdot D(v_h - u_h) \geq C ||v_h - u_h||^2, C > 0.
$$

Since $u_h = v_h = u_m$ on $\partial\widetilde{\Omega}$, we have $v_h - u_h = 0$ on $\partial\widetilde{\Omega}$ and so integrating both sides, and using $w_h = v_h - u_h$, we obtain $|v_h - u_h|_1 = 0$. Therefore $u_h = v_h$.

□

Remark 6.3. *Our assumption $0 < c_0 \leq f \leq c_1$ is not restrictive. That is, we consider the degenerate case $f \geq 0$ and the case of unbounded f.*

For $M > 0$, if one defines f_M by

$$
f_M(x) = f(x) \text{ for } f(x) \leq M, \text{ and } f_M(x) = 0 \text{ otherwise,}
$$

we showed in [3] *how the Aleksandrov solution of* (1.1) *is a limit of solutions of Monge-Ampère equations with right hand side f_M and boundary data g.*

On the other hand, the constant c_0 may be assumed to be close to machine precision. Moreover, in the case f bounded with $f \geq 0$, for $\epsilon > 0$, it is a simple consequence of [31, Lemma 5.1] *that solutions of Monge-Ampère equations with right hand side $f + \epsilon$ and boundary data g converge uniformly on compact subsets to the Aleksandrov solution of* (1.1) *as $\epsilon \to 0$.*

7. Appendix

We give in this section the proof of some technical results.

Proof of Lemma 2.3. In the homogeneous case, i.e. for $g_m = 0$, the result can be inferred from [15]. See also [20, Theorem 2.16].

In the non homogeneous case, it seems that the only genuine interior Schauder estimates for (1.5), with constant depending only on the diameter of the compact subset $K \subset \Omega_s$ and not on Ω_s is to rely on the corresponding result for the complex Monge-Ampère equation in [21, Theorem 4]. See also the corresponding A.M.S. Mathematical Review. For the convenience of the reader, we finish the proof with a brief introduction to the complex Monge-Ampère equation.

It follows from [21, Theorem 4] that

$$||u_{ms}||_{C^2(\widetilde{\Omega})} \leq C_m,$$

where C_m depends only on $m, d, c_2, ||f_m||_{C^1(\overline{\Omega})}$ and $d(\widetilde{\Omega}, \partial\Omega)$. The estimate for higher order derivatives follows from standard elliptic regularity arguments. For example differentiating the equation one time, and taking into account the smoothness of f_m and the C^2 estimate, one obtains a second order linear equation which, because of the strict convexity of the solution u_{ms}, is uniformly elliptic on compact subsets of Ω and with solution a first derivative of u. The interior Schauder estimates for uniformly elliptic linear equations [25, Theorem 6.2] then applies to give the desired estimate for the third derivatives. Repeating this process is known as a bootstrapping argument.

Let us illustrate the technique with the two dimensional Monge-Ampère equation

$$u_{xx}u_{yy} - u_{xy}^2 = f(x, y),$$

where we use another standard notation for derivatives for simplicity. Put $v = u_x$. Differentiating with respect to x, we get the second order equation

$$u_{yy}v_{xx} + u_{xx}v_{yy} - 2u_{xy}v_{xy} = f_x.$$

By the strict convexity of u, the equation is uniformly elliptic and hence

$$||v||_{C^2(\widetilde{\Omega})} \leq C,$$

with C depending on $\max_\Omega v$, $||f_x||_{C^1(\overline{\Omega})}, \widetilde{\Omega}$, $d(\widetilde{\Omega}, \partial\Omega)$, the smallest eigenvalue of D^2u and a bound on the $C^{2,\alpha}$ norm of u. The latter bound implies an upper bound on the eigenvalues of D^2u, and since $\det D^2u = f \leq C$, we obtain a positive lower bound for the smallest eigenvalue of D^2u. A similar argument applies to u_y and thus $||u||_{C^3(\widetilde{\Omega})} \leq C$, with C depending only on $||f||_{C^2(\overline{\Omega})}, \widetilde{\Omega}$ and $d(\widetilde{\Omega}, \partial\Omega)$.

We finish with a brief introduction to the complex Monge-Ampère equation. First the domain $\Omega \subset \mathbb{R}^d$ is identified with a convex domain of $\mathbb{C}^d$. Let now u be a strictly convex smooth solution and put $z_i = x_i + \sqrt{-1}\, y_i, i = 1, \ldots, d$. We can then view u as a function of z defined by $u(z) = u(x)$. Same for f and g. The complex Monge-Ampère equation is given by

$$\begin{aligned} \det \left(\frac{\partial^2 u}{\partial z_i \partial \overline{z}_j} \right)_{i,j=1,\ldots,d} &= f \text{ in } \Omega \\ u &= g \text{ on } \partial\Omega, \end{aligned} \tag{7.1}$$

where

$$\begin{aligned} \frac{\partial u}{\partial z_i} &= \frac{1}{2}\left(\frac{\partial u}{\partial x_i} - \sqrt{-1}\, \frac{\partial u}{\partial y_i} \right) \\ \frac{\partial u}{\partial \overline{z}_j} &= \frac{1}{2}\left(\frac{\partial u}{\partial x_j} + \sqrt{-1}\, \frac{\partial u}{\partial y_i} \right). \end{aligned}$$

This clearly reduces to (1.1) for real-valued functions defined on a convex domain of $\mathbb{R}^n$. The analogue of convex solution is a plurisubharmonic function, i.e. a function for which the Hessian matrix in (7.1) is positive.

□

Proof of Lemma 2.6. Since $v \in W^{2,d}(\Omega)$, by Hölder's inequality, $\det D^2 v \in L^1(\Omega)$ and thus defines an absolutely continuous finite measure given by

$$\hat{M}[v](E) = \int_E \det D^2 v(x)\, dx,$$

and we recall that v is convex. We will use a comparison principle [37, Theorem 5.1] which permits in general to compare a convex function w with a $W^{2,d}$ function v when $M[w]$ is comparable to $\hat{M}[v]$. It is stated in [37] for a strictly convex domain, but the result also holds for a domain not necessarily strictly convex as a consequence of Theorem 2.4. Let thus w be the Aleksandrov solution of

$$M[w] = \hat{M}[v] \text{ in } \Omega, w = v \text{ on } \partial\Omega.$$

Since $M[w] \geq \hat{M}[v]$ and $w \leq v$ on $\partial\Omega$, we have $w \leq v$ in Ω by [37, Theorem 5.1].

Next, we claim that $\hat{M}[v] \leq M[v]$. For a Borel set $E \subset \Omega$, E is the disjoint union of $\cup_{K \in \mathcal{T}_h} E \cap \mathring{K}$ and $\cup_{K \in \mathcal{T}_h} E \cap \partial K$. Moreover, the number of elements $K \in \mathcal{T}_h$ is countable. Thus

$$\begin{aligned} M[v](E) &= M[v]\bigg(\cup_{K \in \mathcal{T}_h} E \cap \mathring{K} \bigg) + M[v]\bigg(\cup_{K \in \mathcal{T}_h} E \cap \partial K \bigg) \\ &\geq M[v]\bigg(\cup_{K \in \mathcal{T}_h} E \cap \mathring{K} \bigg) = \sum_{K \in \mathcal{T}_h} M[v](E \cap \mathring{K}). \end{aligned}$$

By assumption v is piecewise C^2 and thus

$$\begin{aligned} M[v](E) &\geq \sum_{K \in \mathcal{T}_h} \int_{E \cap \mathring{K}} \det D^2 v(x)\, dx \\ &= \sum_{K \in \mathcal{T}_h} \int_{E \cap K} \det D^2 v(x)\, dx = \int_E \det D^2 v(x)\, dx = \hat{M}[v](E), \end{aligned}$$

since $\det D^2 v \in L^1(\Omega)$.

We have $M[w] \leq \hat{M}[v] \leq M[v]$ and $w \geq v$ on $\partial\Omega$. Thus by the comparison principle [29, Theorem 1.4.6], we have $w \geq v$ in Ω. We conclude that $w = v$ and since $M[w] = \det D^2 v$, i.e. $M[w]$ has density $\det D^2 v$, this proves the result.

□

Proof of Lemma 2.11. For $p_j \in \partial u_j(x)$ and $x \in \Omega$, we have by [29, Lemma 3.2.1]

$$|p_j| \leq \frac{|u_j(x)|}{d(x, \partial\Omega)} \leq \frac{C}{d(x, \partial\Omega)},$$

for a constant C independent of j. Arguing as in the proof of [29, Lemma 1.1.6], it follows that the sequence u_j is uniformly Lipschitz and hence equicontinuous on compact subsets of Ω. By the Arzela-Ascoli theorem, [38, p. 179], we conclude that the result holds.

□

Proof of Theorem 2.15. By convexity of z_s, see [35, Theorem 3.4.7], we have

$$z_s(x) \leq \max_{x\in\Omega_s} b \leq \max_{x\in\Omega} b \leq C, \forall x \in \Omega_s,$$

for a constant $C > 0$.

Let now C denote the minimum of b on $\partial\Omega_s$. We may assume that C is independent of s since by assumption $b \in C(\overline{\Omega})$. Since $z_s = b_s$ on $\partial\Omega_s$, we have $z_s - C \geq 0$ on $\partial\Omega_s$. Either $z_s(x) - C \geq 0$ for $x \in \Omega_s$, or by Aleksandrov's maximum principle, [37, Lemma 3.5] or [30, Proposition 6.15],

$$(-(z_s(x) - C))^n \leq c_n(\operatorname{diam}\Omega_s)^{n-1} d(x, \partial\Omega_s) \int_{\Omega_s} a\, dx,$$

where c_n is a constant which depends only on n. We recall that $a \in C(\overline{\Omega})$. It follows that the sequence z_s is bounded below on Ω_s.

By Lemma 2.11, the sequence z_s being bounded has a pointwise convergent subsequence, also denoted by z_s, to a limit function z. But since z_s is a sequence of convex functions on Ω_s, and Ω_s increases to Ω, the limit function z is a convex function on Ω and the convergence is uniform on compact subsets of Ω. Let us first assume that z_s has a subsequence, also denoted z_s, such that the corresponding upper graphs converge in the Hausdorff distance, i.e. $z_s \to z$. Then by Proposition 2.14, or [39, Proposition 2.4], we have

$$\det D^2 z = a \,\text{in}\, \Omega, z = b \,\text{on}\, \partial\Omega.$$

To complete the proof, it remains to show that z_s has a subsequence such that $z_s \to z$. We define

$$V_s = \{\, (x, x_{d+1}) \in \Omega \times \mathbb{R}, x_{d+1} \geq z_s(x) \,\},$$

the epigraph of the convex function z_s. It is known that $\overline{V}_s$ is a closed convex set. Let $C \geq 0$ such that $|z_s(x)| \leq C$ for all s and $x \in \Omega_s$. Put

$$B = \{\, (x, x_{d+1}) \in \Omega \times \mathbb{R}, |x_{d+1}| \leq C \,\}.$$

Thus $\overline{V}_s \cap B$ is a nonempty compact convex subset of $\mathbb{R}^{d+1}$, i.e. a convex body in the terminology of [40]. By the Blaschke selection theorem [40, Theorem 1.8.7], there exists a subsequence also denoted $\overline{V}_s \cap B$ which converges in the Hausdorff distance to a convex set K.

By [40, Theorem 1.8.7-a], each $(x, r) \in K$ is the limit of a sequence (x_s, r_s) in $\overline{V}_s \cap B$. Since the sequence z_s is bounded from below, we conclude that

$$\{\, r \in \mathbb{R}, (x, r) \in K \,\},$$

has a lower bound for all $x \in \overline{\Omega}$. Note that the sequence x_s converges to an element of $\overline{\Omega}$. Thus the lower bound function of K, i.e. the function $\hat{z}(x)$ defined by

$$\hat{z}(x) = \inf\{\, r \in \mathbb{R}, (x, r) \in K \,\},$$

is well defined on Ω. By [32, Theorem 1.3.1], $\hat{z}$ defines a convex function and if we denote by $\hat{V}$ its epigraph, $K = \overline{\hat{V}} \cap B$. We conclude that $\overline{V}_s$ converges to $\overline{\hat{V}}$ in the Hausdorff distance, i.e. $z_s \to \hat{z}$. We now prove that $\hat{z} = z$ on Ω.

For $x \in \Omega$, let (x, r_s) in $\overline{V}_s \cap B$ such that $(x, r_s) \to (x, r)$. We have $r_s \geq z_s(x)$. If necessary, by taking a subsequence, we get $r \geq z(x)$ and thus by the definition of

$\hat{z}(x)$, we obtain $\hat{z}(x) \geq z(x)$. On the other hand $(x, z_s(x)) \in \overline{V}_s \cap B$ and so by [40, Theorem 1.8.7-b], $(x, z(x)) \in K$. It follows that $\hat{z}(x) \leq z(x)$ and this concludes the proof.

□

Proof of Lemma 2.17. Since $p \in H_0^1(\Omega)$, there exists a sequence p_l of infinitely differentiable functions with compact support in Ω such that $||p_l - p||_{1,2} \to 0$ as $l \to \infty$. We have

$$\int_\Omega (a_m - a)p \, dx = \int_\Omega (a_m - a)(p - p_l) \, dx + \int_\Omega (a_m - a)p_l \, dx.$$

By assumption $\int_\Omega (a_m - a)p_l \, dx \to 0$ as $m \to \infty$. Moreover, since Ω is bounded and $||a_m||_{0,\infty} \leq C$ for all m, we have

$$\begin{aligned} \left| \int_\Omega (a_m - a)(p - p_l) \, dx \right| &\leq ||a_m - a||_{0,\infty} \left| \int_\Omega p - p_l \, dx \right| \\ &\leq C(||a_m||_{0,\infty} + ||a||_{0,\infty}) ||p - p_l||_{0,2} \\ &\leq C ||p - p_l||_{0,2} \to 0 \, \text{as} \, l \to \infty. \end{aligned}$$

This concludes the proof. □

Acknowledgements

The author is grateful to the referees for a careful reading of the manuscript and to Wilfrid Gangbo, Michael Neilan, Ricardo H. Nochetto and Wujun Zhang for many useful discussions. This work began when the author was supported in part by a 2009-2013 Sloan Foundation Fellowship and continued while the author was in residence at the Mathematical Sciences Research Institute (MSRI) in Berkeley, California, Fall 2013. The MSRI receives major funding from the National Science Foundation under Grant No. 0932078 000. The author was partially supported by NSF DMS grant No 1319640.

References

[1] G. Awanou. Isogeometric method for the elliptic Monge-Ampère equation. In *Approximation Theory, XIV (San Antonio, TX, 2013)*, pages 1–13. 2014.
[2] G. Awanou. Pseudo transient continuation and time marching methods for Monge-Ampère type equations. *Adv. Comput. Math.*, 41(4):907–935, 2015.
[3] G. Awanou. Smooth approximations of the Aleksandrov solution of the Monge-Ampère equation. *Commun. Math. Sci.*, 13(2):427–441, 2015.
[4] G. Awanou. Spline element method for Monge–Ampère equations. *BIT*, 55(3):625–646, 2015.
[5] G. Awanou. Standard finite elements for the numerical resolution of the elliptic Monge-Ampère equations: classical solutions. *IMA J. Numer. Anal.*, 35(3):1150–1166, 2015.
[6] G. Awanou and L. Matamba Messi. A variational method for computing numerical solutions of the Monge-Ampère equation. http://arxiv.org/abs/1510.00453, 2015.
[7] F. E. Baginski and N. Whitaker. Numerical solutions of boundary value problems for k-surfaces in $\mathbb{R}^3$. *Numer. Methods Partial Differential Equations*, 12(4):525–546, 1996.
[8] I. J. Bakelman. *Convex analysis and nonlinear geometric elliptic equations.* Springer-Verlag, Berlin, 1994. With an obituary for the author by William Rundell, Edited by Steven D. Taliaferro.

[9] Z. Błocki. Smooth exhaustion functions in convex domains. *Proc. Amer. Math. Soc.*, 125(2):477–484, 1997.

[10] K. Böhmer. On finite element methods for fully nonlinear elliptic equations of second order. *SIAM J. Numer. Anal.*, 46(3):1212–1249, 2008.

[11] S. C. Brenner, T. Gudi, M. Neilan, and L.-Y. Sung. C^0 penalty methods for the fully nonlinear Monge-Ampère equation. *Math. Comp.*, 80(276):1979–1995, 2011.

[12] S. C. Brenner and L. R. Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer-Verlag, New York, second edition, 2002.

[13] K. Brix, Y. Hafizogullari, and A. Platen. Solving the Monge–Ampère equations for the inverse reflector problem. *Math. Models Methods Appl. Sci.*, 25(5):803–837, 2015.

[14] L. Caffarelli, L. Nirenberg, and J. Spruck. The Dirichlet problem for nonlinear second-order elliptic equations. I. Monge-Ampère equation. *Comm. Pure Appl. Math.*, 37(3):369–402, 1984.

[15] L. A. Caffarelli. Interior $W^{2,p}$ estimates for solutions of the Monge-Ampère equation. *Ann. of Math. (2)*, 131(1):135–150, 1990.

[16] W. Dahmen. Convexity and Bernstein-Bézier polynomials. In *Curves and surfaces (Chamonix-Mont-Blanc, 1990)*, pages 107–134. Academic Press, Boston, MA, 1991.

[17] O. Davydov and A. Saeed. Stable splitting of bivariate splines spaces by Bernstein-Bézier methods. In *Curves and surfaces*, volume 6920 of *Lecture Notes in Comput. Sci.*, pages 220–235. Springer, Heidelberg, 2012.

[18] O. Davydov and A. Saeed. Numerical solution of fully nonlinear elliptic equations by Böhmer's method. *J. Comput. Appl. Math.*, 254:43–54, 2013.

[19] G. De Philippis and A. Figalli. Second order stability for the Monge-Ampère equation and strong Sobolev convergence of optimal transport maps. *Anal. PDE*, 6(4):993–1000, 2013.

[20] G. De Philippis and A. Figalli. The Monge-Ampère equation and its link to optimal transportation. *Bull. Amer. Math. Soc. (N.S.)*, 51(4):527–580, 2014.

[21] S. Dinew, X. Zhang, and X. Zhang. The $C^{2,\alpha}$ estimate of complex Monge-Ampère equation. *Indiana Univ. Math. J.*, 60(5):1713–1722, 2011.

[22] X. Feng and M. Neilan. Analysis of Galerkin methods for the fully nonlinear Monge-Ampère equation. *J. Sci. Comput.*, 47(3):303–327, 2011.

[23] X. Feng and M. Neilan. Convergence of a fourth-order singular perturbation of the n-dimensional radially symmetric Monge–Ampère equation. *Appl. Anal.*, 93(8):1626–1646, 2014.

[24] B. Froese and A. Oberman. Convergent finite difference solvers for viscosity solutions of the elliptic Monge-Ampère equation in dimensions two and higher. *SIAM J. Numer. Anal.*, 49(4):1692–1714, 2011.

[25] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[26] R. Glowinski. Numerical methods for fully nonlinear elliptic equations. In *ICIAM 07—6th International Congress on Industrial and Applied Mathematics*, pages 155–192. Eur. Math. Soc., Zürich, 2009.

[27] P. M. Gruber and J. M. Wills, editors. *Handbook of convex geometry. Vol. A, B.* North-Holland Publishing Co., Amsterdam, 1993.

[28] J.-L. Guermond and B. Popov. L^1-approximation of stationary Hamilton-Jacobi equations. *SIAM J. Numer. Anal.*, 47(1):339–362, 2008/09.

[29] C. E. Gutiérrez. *The Monge-Ampère equation*. Progress in Nonlinear Differential Equations and their Applications, 44. Birkhäuser Boston Inc., Boston, MA, 2001.

[30] C. E. Gutiérrez and T. van Nguyen. On Monge-Ampère type equations arising in optimal transportation problems. *Calc. Var. Partial Differential Equations*, 28(3):275–316, 2007.

[31] D. Hartenstine. The Dirichlet problem for the Monge-Ampère equation in convex (but not strictly convex) domains. *Electron. J. Differential Equations*, pages No. 138, 9 pp. (electronic), 2006.

[32] J.-B. Hiriart-Urruty and C. Lemaréchal. *Fundamentals of convex analysis*. Grundlehren Text Editions. Springer-Verlag, Berlin, 2001. Abridged version of ıt Convex analysis and minimization algorithms. I [Springer, Berlin, 1993; MR1261420 (95m:90001)] and ıt II [ibid.; MR1295240 (95m:90002)].

[33] O. Lakkis and T. Pryer. A finite element method for second order nonvariational elliptic problems. *SIAM J. Sci. Comput.*, 33:786–801, 2011.

[34] O. Lakkis and T. Pryer. A finite element method for nonlinear elliptic problems. *SIAM J. Sci. Comput.*, 35(4):A2025–A2045, 2013.

[35] C. P. Niculescu and L.-E. Persson. *Convex functions and their applications*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 23. Springer, New York, 2006. A contemporary approach.

[36] V. I. Oliker and L. D. Prussner. On the numerical solution of the equation $(\partial^2 z/\partial x^2)(\partial^2 z/\partial y^2) - ((\partial^2 z/\partial x \partial y))^2 = f$ and its discretizations. I. *Numer. Math.*, 54(3):271–293, 1988.

[37] J. Rauch and B. A. Taylor. The Dirichlet problem for the multidimensional Monge-Ampère equation. *Rocky Mountain J. Math.*, 7(2):345–364, 1977.

[38] H. L. Royden. *Real analysis*. Macmillan Publishing Company, New York, third edition, 1988.

[39] O. Savin. Pointwise $C^{2,\alpha}$ estimates at the boundary for the Monge-Ampère equation. *J. Amer. Math. Soc.*, 26(1):63–99, 2013.

[40] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 151 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, expanded edition, 2014.

[41] L. Schumaker. *Spline Functions: Computational Methods*. Other titles in applied mathematics. Society for Industrial and Applied Mathematics, 2015.

[42] M. M. Sulman, J. F. Williams, and R. D. Russell. An efficient approach for the numerical solution of the Monge-Ampère equation. *Appl. Numer. Math.*, 61(3):298–307, 2011.

[43] N. S. Trudinger and X.-J. Wang. Boundary regularity for the Monge-Ampère and affine maximal surface equations. *Ann. of Math. (2)*, 167(3):993–1028, 2008.

Department of Mathematics, Statistics, and Computer Science, M/C 249. University of Illinois at Chicago, Chicago, IL 60607-7045, USA

E-mail address: awanou@uic.edu

URL: http://www.math.uic.edu/~awanou