C^1 Quintic Spline Interpolation Over Tetrahedral Partitions

Gerard Awanou and Ming-Jun Lai

Abstract. We discuss the implementation of a C^1 quintic super-spline method for interpolating scattered data in $\mathbb{R}^3$ based on a modification of Alfeld's generalization of the Clough-Tocher scheme described by Lai and LeMéhauté [4]. The method has been implemented in MATLAB, and we test for the accuracy of reproduction on a basis of quintic polynomials. We present numerical evidences that when the partition is refined, the spline interpolant converges to the function to be approximated.

§1. Introduction

There are a few trivariate spline spaces available for interpolation over a tetrahedral partition $\triangle$ of a polygonal domain in $\mathbb{R}^3$. We would like to mention a direct polynomial interpolation by Zenisek in [9]. This scheme requires piecewise polynomials of degree 9 and is globally C^1 over Ω while C^4 around the vertices and C^2 around the edges of $\triangle$. Another scheme is the Alfeld scheme (cf. [1]) which uses polynomials of degree 5 to construct spline functions over a 3D Clough-Tocher refinement of a tetrahedral partition $\triangle$. The scheme produces spline interpolants which are globally C^1 over Ω while C^2 around the vertices and C^1 around the edges of $\triangle$. A further generalization of the Clough-Tocher refinement enables Worsey and Farin in [7] to construct interpolation by C^1 cubic splines. Worsey and Piper constructed C^1 quadratic spline functions based on special tetrahedral partitions in [8].

The present paper is concerned with the implementation of the modification introduced in [4] of Alfeld's Clough-Tocher scheme. It uses polynomials of degree 5 over the Alfeld version of Clough-Tocher refinement

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of $\triangle$ to construct spline interpolants globally C^1 over Ω and locally C^2 around the vertices and C^1 around the edges of $\triangle$. The main difference is that the Alfeld scheme reproduces only polynomials of degree 3, while the new scheme reproduces polynomials of degree 5. Let us mention that the Alfeld scheme was implemented in [2]. Our program like others of the kind involves a great computational complexity.

The paper is organized as follows: First we begin by a review of the B-form of polynomials on tetrahedra. Then we review the construction of [4] in §3. The details on how to compute C^1 quintic spline interpolants are given in §4. We then give the properties of the interpolant. In §6, we present numerical evidence that the scheme reproduces all polynomials of degree 5 and that the interpolation error reduces when the partition is refined. Finally, we will point out our future research topics.

§2. B-form of Polynomials on Tetrahedra

We assume the reader is familiar with the Bernstein representation of polynomials on tetrahedra. An introduction to this topic can be found in [3]. Here, we give only a brief account.

We first recall how to represent trivariate polynomials

$$p(x,y,z) = \sum_{0\le i+j+k\le d} \alpha_{ijk} x^i y^j z^k, \qquad \alpha_{ijk} \in \mathbb{R},$$

of degree d in terms of the barycentric coordinates of the evaluation point (x,y,z) with respect to a given tetrahedron $T = \langle v_1, v_2, v_3, v_4 \rangle$. Recall that any $v \in \mathbb{R}^3$ can be written uniquely in the form

$$v = b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4$$

with

$$b_1 + b_2 + b_3 + b_4 = 1,$$

where b_1, b_2, b_3 and b_4 are the barycentric coordinates of v with respect to T. Let

$$B^d_{ijkl}(v) = \frac{d!}{i!j!k!l!} b_1^i b_2^j b_3^k b_4^l, \qquad i+j+k+l = d$$

be the Bernstein polynomials of degree d. They form a basis of the space of polynomials of degree less than or equal d. As a consequence, any such polynomial can be written uniquely on T in the so-called B-form

$$p = \sum_{i+j+k+l=d} c_{ijkl} B^d_{ijkl}, \qquad c_{ijkl} \in \mathbb{R}.$$

As usual, the c_{ijkl} are associated with the domain points

$$\left\{\frac{iv_1 + jv_2 + kv_3 + lv_4}{d}\right\}_{i+j+k+l=d}.$$

A polynomial of total degree d is uniquely determined by its values on the domain points, i.e. the c_{ijkl}, $i+j+k+l=d$ are completely determined by interpolation on the domain points.

Let $\mathbf{u}= y-x$ be a vector with x and y having barycentric coordinates $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $(\beta_1, \beta_2, \beta_3, \beta_4)$, respectively. One refers to $a = (\beta_1 - \alpha_1, \beta_2 - \alpha_2, \beta_3 - \alpha_3, \beta_4 - \alpha_4) = (a_1, a_2, a_3, a_4)$ as the T-coordinates of $\mathbf{u}$. In terms of a and the $c_{ijkl}, i+j+k+l=d$, the derivative of p in the direction $\mathbf{u}$ can be written in B-form on T as

$$D_{\mathbf{u}}p = d \sum_{i+j+k+l=d-1} \Delta_a(c_{ijkl}) B_{ijkl}^{d-1}$$

with

$$\Delta_a(c_{ijkl}) = a_1 c_{i+1,j,k,l} + a_2 c_{i,j+1,k,l} + a_3 c_{i,j,k+1,l} + a_4 c_{i,j,k,l+1}.$$

Given a spline, *i.e.*,, a piecewise polynomial defined on a collection of tetrahedra, we work with the B-form of each polynomial piece. So it makes sense to look for conditions on the coefficients that will ensure that the spline has global smoothness properties. We explicitly derive the smoothness conditions for a model case.

Let $v_5 = (v_1 + v_2 + v_3 + v_4)/4$, and let

$$p_4 = \sum_{i+j+k+l=d} c_{ijkl}^4 B_{ijkl}^d \qquad \text{on } T_4 = \langle v_1, v_2, v_3, v_5 \rangle$$

and

$$p_1 = \sum_{i+j+k+l=d} c_{ijkl}^1 B_{ijkl}^d \qquad \text{on } T_1 = \langle v_2, v_3, v_4, v_5 \rangle.$$

For p_1 and p_4 to be joined continuously across the common face $\langle v_2, v_3, v_5 \rangle$ they must agree on that face. Since p_4 and p_1 are uniquely determined by their coefficients on that face, the condition of C^0 continuity reads

$$c_{0jkl}^4 = c_{jk0l}^1.$$

To ensure continuity of the first order derivatives, we need only to check continuity of the directional derivatives $D_{v_3-v_2}$, $D_{v_5-v_2}$ and $D_{v_1-v_2}$. We already have

$$D_{v_3-v_2} p_4 = D_{v_3-v_2} p_1; \quad D_{v_5-v_2} p_4 = D_{v_5-v_2} p_1 \text{ on } \langle v_2, v_3, v_5 \rangle$$

since those derivatives depend only on the values of the polynomials on the common face.

Notice that $v_1 - v_2$ has T-coordinates $(1,-1,0,0)$ with respect to T_4 and $(-2,-1,-1,4)$ with respect to T_1. So to ensure C^1 continuity, we need

$$\Delta_{(1,-1,0,0)}(c_{0jkl}) = \Delta_{(-2,-1,-1,4)}(c_{jk0l}), \qquad i+j+k+l = d-1$$

or equivalently

$$c^4_{1jkl} - c^4_{0,j+1,k,l} = -2c^1_{j+1,k,0,l} - c^1_{j,k+1,0,l} - c^1_{jk1l} + 4c^1_{j,k,0,l+1}.$$

Using the C^0 continuity conditions, this gives

$$c^4_{1jkl} = -c^1_{j+1,k,0,l} - c^1_{j,k+1,0,l} - c^1_{jk1l} + 4c^1_{j,k,0,l+1}.$$

Finally we give the following subdivision formulas that give the B-form of p on $\langle v_1, v_2, v_3, w\rangle$ for any point w in $\mathbf{R^3}$. The B-form of p on $\langle v_1, v_2, v_3, w\rangle$ is

$$p = \sum_{i+j+k+l=d} d_{ijkl} B^d_{ijkl}$$

with

$$d_{ijkl} = \sum_{\mu+\nu+\kappa+\delta=l} c_{i+\mu,j+\nu,k+\kappa,\delta} B^l_{\mu,\nu,\kappa,\delta}(w).$$

For example with $w = v_5$,

$$d_{ijkl} = \sum_{\mu+\nu+\kappa+\delta=l} c_{i+\mu,j+\nu,k+\kappa,\delta} \frac{l!}{\mu!\nu!\kappa!\delta!}\left(\frac{1}{4}\right)^l.$$

§3. Description of the Scheme

We describe the new scheme for 3D scattered data interpolation we have implemented.

Let us introduce more notation. For each edge e, let m_e be the midpoint of e and let e_1 and e_2 be two directions which are perpendicular to e and are linearly independent. For each face $f = \langle v_1, v_2, v_3\rangle$, let f_1, f_2, f_3 be the three domain points $\{(iv_1 + jv_2 + kv_3)/5, (i,j,k) = (2,2,1),(1,2,2),(2,1,2)\}$ on f. Let n_f be a unit normal vector to f. For each tetrahedron t, let u_t be the center point of t.

For a tetrahedral partition $\triangle$, for each tetrahedron t, we split t into four subtetrahedra at the center u_t by connecting u_t to any two of four vertices of t. This generates a 3D Clough-Tocher refinement of $\triangle$. For simplicity, we call it the Alfeld refinement and denote it by $A(\triangle)$.

The C^1 quintic spline interpolant on $A(\triangle)$ can then be described as follows. Given a function $g \in C^2(\Omega)$, the interpolant S_g satisfies the following conditions:

1) For each vertex $v \in \Delta$

$$D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3} S_g(v) = D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3} g(v), \qquad \forall\ |\alpha| \le 2\,.$$

2) For each edge e of Δ,

$$D_{e_i} S_g(m_e) = D_{e_i} g(m_e), \qquad i = 1, 2,$$

where D_{e_i} denotes the derivative along the direction e_i and m_e the midpoint of e;

3) For each face f of Δ,

$$D_{n_f} S_g(f_j) = D_{n_f} g(n_f), \qquad j = 1, 2, 3,$$

where D_{n_f} denotes the derivative along the direction n_f;

4) For each tetrahedron t,

$$D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3} S_g(u_t) = D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3} g(u_t), \qquad \forall\ |\alpha| \le 1.$$

This is the scheme which was introduced in [4].

§4. Details of Computation

This is best presented by looking at the case of a single tetrahedron $T = \langle v_1, v_2, v_3, v_4 \rangle$. The faces, edges and vertices of T will be referred to as boundary faces, edges and vertices, respectively, in this case. Let $u_t = v_5 = (v_1 + v_2 + v_3 + v_4)/4$ be the center of T. It subdivides T into 4 subtetrahedra:

$$T_1 = \langle v_2, v_3, v_4, v_5 \rangle, \quad T_2 = \langle v_1, v_3, v_4, v_5 \rangle,$$

$$T_3 = \langle v_1, v_2, v_4, v_5 \rangle, \quad T_4 = \langle v_1, v_2, v_3, v_5 \rangle.$$

The problem then is to determine the approximating polynomial p on each subtetrahedron. Given a polynomial p of degree 5, for each $s = 1, \ldots, 4$, let

$$p(v) = \sum_{i+j+k+l=5} c^s_{ijkl} B^5_{ijkl}(v)$$

be the B-form of p on T_s. For simplicity we show how to compute the c^4_{ijkl}'s. The other coefficients c^s_{ijkl}, $s = 1, 2, 3$, can be computed in a similar fashion.

Clearly $c^4_{5000} = p(v_1)$, and similar relations hold at the other vertices of T. Next we compute coefficients in the first ring around v_1. Because

$$D_{v_2-v_1}p(v_1) = 5(c^4_{4100} - c^4_{5000}),$$

we have

$$c^4_{4100} = \frac{1}{5}D_{v_2-v_1}p(v_1) + c^4_{5000},$$

where $D_{v_2-v_1}p(v_1)$ is computed using the first order derivatives at v_1. Using other directional derivatives, we get

$$\begin{aligned} c^4_{4010} =& \frac{1}{5}D_{v_3-v_1}p(v_1) + c^4_{5000}, \\ c^4_{4001} =& \frac{1}{5}D_{v_4-v_1}p(v_1) + c^4_{5000}. \end{aligned}$$

Proceeding with the coefficients in the ring of radius 2, we need to use second order directional derivatives:

$$\begin{aligned} c^4_{3200} =& \frac{1}{20}D^2_{v_2-v_1}p(v_1) + 2c^4_{4100} - c^4_{5000}, \\ c^4_{3020} =& \frac{1}{20}D^2_{v_3-v_1}p(v_1) + 2c^4_{4010} - c^4_{5000}, \\ c^4_{3002} =& \frac{1}{20}D^2_{v_5-v_1}p(v_1) + 2c^4_{4001} - c^4_{5000}, \\ c^4_{3110} =& \frac{1}{20}D_{v_3-v_1}D_{v_2-v_1}p(v_1) + c^4_{4010} + c^4_{4100} - c^4_{5000}, \\ c^4_{3101} =& \frac{1}{20}D_{v_5-v_1}D_{v_2-v_1}p(v_1) + c^4_{4001} + c^4_{4100} - c^4_{5000}, \\ c^4_{3011} =& \frac{1}{20}D_{v_5-v_1}D_{v_3-v_1}p(v_1) + c^4_{4001} + c^4_{4010} - c^4_{5000}. \end{aligned}$$

It is then clear how to obtain similar formulas in the ring of radius 2 around the other vertices v_3 and v_4.

It is convenient to view the B-net of p over T_1,T_2,T_3 and T_4 as composed of layers. Thus the coefficients on the boundary faces form the first layer. The face $\langle v_1, v_2, v_3\rangle$ of T_4 is also a boundary face of T. On that face only 3 coefficients are to be determined: c^4_{2120}, c^4_{1220} and c^4_{2210}. To compute those coefficients, we use the given data at the midpoints of boundary edges to find directional derivatives along the edges at the midpoint of those edges. The coefficients to be found are then simply expressed in terms of the later derivatives. We show how to compute c^4_{1220} for example.

We consider the edge $\langle v_2, v_3\rangle$, with midpoint $v_{23} := m_{\langle v_2,v_3\rangle}$. At this point a directional derivative along this edge can be computed as

$$D_{v_3-v_2}p(v_{23}) = 5\sum_{j=0}^{4}(c^4_{0,j,5-j,0} - c^4_{0,j+1,4-j,0})B^4_{0,j,4-j,0}(v_{23}).$$

In the equation

$$D_{v_2-v_1}p(v_{23}) = 5\sum_{j=0}^{4}(c^4_{0,j+1,4-j,0} - c^4_{1,j,4-j,0})B^4_{0,j,4-j,0}(v_{23}),$$

there are two unknowns, $D_{v_2-v_1}p(v_{23})$ and c^4_{1220}. We have

$$D_{v_2-v_1}p(v_{23}) = \alpha D_{v_3-v_2}p(v_{23}) + \beta D_{e_1}p(v_{23}) + \gamma D_{e_2}p(v_{23})$$

or equivalently

$$D_{v_2-v_1}p(v_{23}) = \alpha D_{v_3-v_2}p(v_{23}) + \beta D_{e_1}g(v_{23}) + \gamma D_{e_2}g(v_{23})$$

for some constants α, β, and γ. e_1 and e_2 are two directions perpendicular to the edge e= $\langle v_2, v_3\rangle$. Now c^4_{1220} can be computed. Proceeding the same way with the edges $\langle v_1, v_2\rangle$ and $\langle v_1, v_3\rangle$, we get c^4_{2120} and c^4_{2210}. We compute the coefficients on the second layer and get three of them by smoothness conditions

$$\begin{aligned}
c^4_{0221} =& \frac{1}{4}(c^1_{2210} + c^4_{1220} + c^4_{0320} + c^4_{0230}),\\
c^4_{2021} =& \frac{1}{4}(c^2_{2210} + c^4_{2120} + c^4_{3020} + c^4_{2030}),\\
c^4_{2201} =& \frac{1}{4}(c^3_{2210} + c^4_{2210} + c^4_{3200} + c^4_{2300}).
\end{aligned}$$

To get the other coefficients in T_4 on the second layer, namely c^4_{1121}, c^4_{2111} and c^4_{1211}, we use values of the normal derivative to $\langle v_1, v_2, v_3\rangle$ at 3 points on that face: f_1, f_2 and f_3. More precisely, since all coefficients on $f = \langle v_1, v_2, v_3\rangle$ are determined, we can compute the values of $D_{v_1-v_2}p$ and $D_{v_3-v_2}p$ at the points f_1, f_2 and f_3. Since the $D_{\eta_f}p(f_i) := D_{\eta_f}g(f_i)$, $i = 1, \ldots, 3$ are given by interpolation conditions, by expressing $v_5 - v_2$ in terms of η_f, $v_1 - v_2$ and $v_3 - v_2$, we can compute the $D_{v_5-v_2}p(f_i)$, $i = 1, 2, 3$. Notice that

$$\begin{aligned}
D_{v_5-v_2}p(f_1) =& 5\sum_{i+j+k=4}(c^4_{ijk1} - c^4_{i,j+1,k,0})\frac{4!}{5^4 i!j!k!}2^{i+j},\\
D_{v_5-v_2}p(f_2) =& 5\sum_{i+j+k=4}(c^4_{ijk1} - c^4_{i,j+1,k,0})\frac{4!}{5^4 i!j!k!}2^{j+k},\\
D_{v_5-v_2}p(f_3) =& 5\sum_{i+j+k=4}(c^4_{ijk1} - c^4_{i,j+1,k,0})\frac{4!}{5^4 i!j!k!}2^{i+k}.
\end{aligned}$$

These form a system of equations with unknowns c^4_{1121}, c^4_{2111} and c^4_{1211} which can be solved easily. This finishes the computations of all coefficients on the second layer.

On the third layer there are six coefficients in T_4 which are simply computed by using smoothness conditions across interior faces. Those coefficients are $c^4_{2102}, c^4_{1202}, c^4_{0212}, c^4_{0122}, c^4_{2012}, c^4_{1022}$.

At this stage, the only coefficients on this layer which remain to be computed are c^s_{1112}, $s = 1, \ldots, 4$. This layer is viewed as the B-net of a polynomial of degree 3. There are 4 data to be computed and we get them by using the 4 data at the center v_5 of the tetrahedron. More precisely, using the given information at the center, one can compute $p(v_5)$, $D_{v_2-v_1}p(v_5)$, $D_{v_3-v_1}p(v_5)$ and $D_{v_5-v_1}p(v_5)$. It is not difficult to see that they can be expressed in terms of the c^s_{1112}, $s = 1, \ldots, 4$. We therefore have a system of 4 equations in 4 unknowns which had to be solved.

The coefficients in the remaining two layers are obtained by using the subdivision method. In this way, the interpolant will be C^3 at v_5.

§5. Properties of the Interpolant

We prove in this section that the scheme reproduces all quintic polynomials and that the interpolant thus constructed is C^2 around the vertices, C^1 around the edges, C^3 at the center of each tetrahedron, and globally C^1.

Property 1: The scheme reproduces all quintic polynomials. This follows from

Lemma 1. *A polynomial p of degree 5 on $T = \langle v_1, v_2, v_3, v_4\rangle$ with center v_5 is uniquely determined by the following data:*

$$D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3} p(v_i), \quad |\alpha| \le 2, \quad i = 1, \ldots, 4,$$

the values of derivatives in two independent directions perpendicular to each edge of T at the midpoint of the edge and

$$D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3} p(v_5), \quad |\alpha| \le 1.$$

The proof of the lemma is given in the appendix.

Property 2: The interpolant is C^2 around the vertices. If two tetrahedra share the same vertex v, by construction the polynomial pieces share the same values $D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3} g(v)$. So S_g is C^2 at v.

Property 3: The interpolant is C^1 around the edges. Assume for example that two tetrahedra share the common edge $e = \langle v_1, v_2\rangle$. The coefficients of each polynomial piece on e are the same since they are computed using data at the vertices v_1 and v_2. This gives continuity across the edge and also continuity of $D_{v_2-v_1}$ across e. To prove C^1 continuity, we need to show continuity of derivatives in three independent directions. Notice that the interpolation conditions at the vertices determine all derivatives up to order 2 at vertices. Any of the derivatives $D_{e_i} S_g,\ i = 1, 2$ reduce to the same univariate quartic polynomial on e. $D_{e_i} S_g$ is uniquely determined by the 5 pieces of data

$$D_{e_i} S_g(m_e), D_{e_i} S_g(v_1), D_{v_2-v_1} D_{e_i} S_g(v_1), D_{e_i} S_g(v_2), D_{v_2-v_1} D_{e_i} S_g(v_2).$$

This assures continuity of $D_{e_i} S_g, i = 1, 2$ across e.

We show below continuity across interior faces of a tetrahedron which implies continuity across the interior edges.

Property 4: The interpolant is globally C^1. Recall that each tetrahedron is subdivided into 4 subtetrahedra so we study differentiability across tetrahedra in the original partition and differentiability across the subtetrahedra obtained after refinement.

Intertetrahedral continuity: Assume two tetrahedra share a common face f. By construction, coefficients on such a face are determined either by using data at the vertices or data at the midpoints of edges of that face. Hence the continuity of the interpolant S_g across f follows.

Intertetrahedral continuity of derivatives: To prove that the interpolant is C^1 across a face f, it is enough to check continuity of the normal derivative. We show that the restriction of such a derivative to f does not depend on the polynomial pieces. $D_{\eta_f} S_g$ is a polynomial of degree 4 on f which is uniquely determined by the following 15 data:

$$D_{\eta_f} S_g(v_i), \quad i = 1, 2, 3; \qquad D_{v_2-v_1} D_{\eta_f} S_g(v_i), \quad i = 1, 2, 3;$$

$$D_{v_3-v_1} D_{\eta_f} S_g(v_i), \quad i = 1, 2, 3; \qquad D_{\eta_f} g(f_i), \quad i = 1, 2, 3$$

and the values of $D_{\eta_f} S_g$ at the midpoints of the edges of f.

Internal continuity: This is obtained by construction since coefficients on internal faces are computed by using data independent of the faces. So they do not depend on the polynomial piece.

Internal continuity of derivatives: We explicitly show how C^1 smoothness is built across the interior face $\langle v_2, v_3, v_5 \rangle$ which is common to $T_1 = \langle v_2, v_3, v_4, v_5 \rangle$ and $T_4 = \langle v_1, v_2, v_3, v_5 \rangle$. We have

$$c^4_{0jkl} = c^1_{jk0l}$$

and

$$c^4_{1jkl} = -c^1_{j+1,k,0,l} - c^1_{j,k+1,0,l} - c^1_{jk1l} + 4c^1_{j,k,0,l+1}$$

for $j + k + l = d$. We group the c^4_{1jkl} into 6 categories:

(1) c^4_{1400}, c^4_{1310} and c^4_{1301} are determined by data at v_2. They satisfy the conditions since they are entirely computed using these data.

(2) c^4_{1130}, c^4_{1040} and c^4_{1031} are determined by data at v_3. They satisfy the conditions as explained above.

(3) c^4_{1022} and c^4_{1202} are computed by requiring smoothness conditions across $\langle v_2, v_3, v_5\rangle$.

(4) c^4_{1220} is computed by using data on the edge $\langle v_2, v_3\rangle$. It enters the smoothness condition used to set c^4_{0221} above.

(5) c^4_{1121} and c^4_{1211} are determined by using the values of the normal derivative. They enter the smoothness conditions used to set c^4_{0122} and c^4_{0212} respectively. Explicitly

$$c^4_{0122} = \frac{c^1_{1211} + c^4_{1121} + c^4_{0221} + c^4_{0131}}{4}$$

and

$$c^4_{0212} = \frac{c^1_{2111} + c^4_{1211} + c^4_{0221} + c^4_{0311}}{4}.$$

(6) $c^4_{1112}, c^4_{1103}, c^4_{1013}$ and c^4_{1004} are computed by considering a layer as the B-net of a polynomial of degree 3. Such a polynomial is already smooth.

Property 5: The interpolant is C^3 at the center of each tetrahedron. This follows from the construction process.

§6. Numerical Experiments

We have implemented the interpolation scheme in MATLAB. To make sure that our implementation is correct, we have checked that our programs reproduce all polynomials of degree ≤ 5 by testing all 56 basis functions. Starting with a cube subdivided into 12 subtetrahedra by connecting the midpoint of the cube to a diagonal of each face of the cube, the maximum errors of spline interpolants of the 56 basis functions are about $.6661 \times 10^{-15}$. When each of the 12 subtetrahedra is subdivided into 8 subtetrahedra, the maximum errors are around $.7772 \times 10^{-15}$. The slight increase in the maximum errors is probably due to round-off errors.

Next we demonstrate how well this scheme approximates given functions and how the interpolation error evolves when the partition is refined. Starting with a single tetrahedron, we refine this tetrahedral partition 3 times and in a few cases, 4 times. Each time, we subdivide each tetrahedron t into 8 subtetrahedra by using the midpoints of six edges of t and dividing the central octahedron into four subtetrahedra. The central octahedron has three diagonals. The choice of a diagonal determines the kind of refinement one has. A common measure of degeneracy used for a tetrahedron T is

$$\sigma = \frac{h}{\rho},$$

Number of tetrahedra	1	8	64	512
Dimension	68	254	1346	8762

Tab. 1. Numbers of tetrahedra and dimension of spline spaces.

where h is the diameter of T and ρ the diameter of the largest sphere inscribed in T. From the three possible tetrahedral partitions that could arise from the choice of the diagonal of the central octahedron, we choose the diagonal that yields the smallest σ. With the tetrahedron with vertices of coordinates $(0,0,0),(1,0,0),(0,1,0)$ and $(1,0,1)$ this leads to a uniform refinement in the sense that all tetrahedra have the same measure of degeneracy. The choice of this model tetrahedron was suggested by Ong [6]. We first display the dimension of the spline spaces that were used for interpolation. The formula to compute the dimension was given in [4]. If V denotes the number of vertices, E the number of edges, F the number of faces and T the number of tetrahedra in a tetrahedral partition, the dimension of the corresponding spline space is given by

$$10V + 2E + 3F + 4T.$$

The dimensions of the first three refinement levels are given in Table 1. The fourth level of refinement involve 4096 tetrahedra and the dimension of the corresponding spline space is 63338. The limitation of computational power at hand prevents us from displaying additional levels of refinement. We have tested the code on the following functions:

$$f_1(x,y,z) = \exp(x+y+z), \quad f_2(x,y,z) = \sin(x^3+y^3+z^3),$$

$$f_3(x,y,z) = x^6+y^6+z^6, \quad f_4(x,y,z) = 10\exp(-x^2-y^2-z^2).$$

The results are presented in Table 2. We also checked the results of interpolating the homogeneous polynomials of degree 6 and the polynomials $x^7+y^7+z^7$ and $x^8+y^8+z^8$ of degree 7 and 8, respectively, see Table 3.

The maximum errors of the spline interpolants computed by evaluation on each tetrahedron at the domain points

$$\{\psi_{ijkl}\}_{i+j+k+l=10}$$

are displayed as well as the numerical rate of convergence.

These results show that the errors decrease like $\mathrm{O}(h^6)$ when the partition is refined. The convergence rate is specially good for homogeneous polynomials of degree 6.

Number of tetrahedra	$f_1(x,y,z)$	Rate	$f_2(x,y,z)$	Rate
1	3.9822×10^{-3}	0	4.9375×10^{-2}	0
8	9.8569×10^{-5}	40.40	1.5402×10^{-2}	32.05
64	1.9578×10^{-6}	50.34	6.4640×10^{-4}	23.82
512	3.4577×10^{-8}	56.62	1.3401×10^{-5}	48.23
4096	5.7476×10^{-10}	60.16	2.1887×10^{-7}	61.23

Number of tetrahedra	$f_3(x,y,z)$	Rate	$f_4(x,y,z)$	Rate
1	4.6875×10^{-2}	0	1.2769×10^{-1}	0
8	7.3242×10^{-4}	64	6.9864×10^{-3}	18.27
64	1.1444×10^{-5}	64	1.5328×10^{-4}	45.57
512	1.7881×10^{-7}	64	2.6073×10^{-6}	58.78
4096	2.7940×10^{-9}	64	4.1614×10^{-8}	62.65

Tab. 2. Numerical maximum errors of the interpolation scheme.

x^6	Rate	x^5y	Rate	x^5z	Rate
1.5625×10^{-2}	0	1.5625×10^{-2}	0	1.5625×10^{-2}	0
2.4966×10^{-4}	62.59	2.4659×10^{-4}	63.36	2.4414×10^{-4}	64
3.9349×10^{-6}	63.45	3.8618×10^{-6}	63.85	3.8525×10^{-6}	63.37
6.1482×10^{-8}	64	6.0341×10^{-8}	64	6.0341×10^{-8}	63.84

x^4y^2	Rate	x^4yz	Rate	x^4z^2	Rate
1.5625×10^{-2}	0	1.5625×10^{-2}	0	1.5625×10^{-2}	0
2.4529×10^{-4}	63.70	2.4414×10^{-4}	64	2.4414×10^{-4}	64
3.8334×10^{-6}	63.99	3.8147×10^{-6}	64	3.8334×10^{-6}	63.69
5.9897×10^{-8}	64	5.9605×10^{-8}	64	5.9897×10^{-8}	64

x^3y^3	Rate	x^3y^2z	Rate	x^3yz^2	Rate
1.5625×10^{-2}	0	1.5625×10^{-2}	0	1.5625×10^{-2}	0
2.4414×10^{-4}	64	2.4414×10^{-4}	64	2.4414×10^{-4}	64
3.8147×10^{-6}	64	3.8147×10^{-6}	64	3.8147×10^{-6}	64
5.9605×10^{-8}	64	5.9605×10^{-8}	64	5.9605×10^{-8}	64

x^3z^3	Rate	x^2y^4	Rate	x^2y^3z	Rate
1.5625×10^{-2}	0	1.5625×10^{-2}	0	1.5625×10^{-2}	0
2.4414×10^{-4}	64	2.4414×10^{-4}	64	2.4414×10^{-4}	64
3.8147×10^{-6}	64	3.8303×10^{-6}	63.74	3.8147×10^{-6}	64
5.9605×10^{-8}	64	5.9884×10^{-8}	63.97	5.9605×10^{-8}	64

Tab. 3a. Numerical maximum errors of the interpolation scheme.

$x^2y^2z^2$	Rate	x^2yz^3	Rate	x^2z^4	Rate
1.5625×10^{-2}	0	1.5625×10^{-2}	0	1.5625×10^{-2}	0
2.4414×10^{-4}	64	2.4414×10^{-4}	64	2.4414×10^{-4}	64
3.8147×10^{-6}	64	3.8147×10^{-6}	64	3.8147×10^{-6}	64
5.9605×10^{-8}	64	5.9605×10^{-8}	64	5.9897×10^{-8}	63.69

xy^5	Rate	xy^4z	Rate	xy^3z^2	Rate
1.5714×10^{-2}	0	1.5625×10^{-2}	0	1.5625×10^{-2}	0
2.4554×10^{-4}	64	2.4414×10^{-4}	64	2.4414×10^{-4}	64
3.8618×10^{-6}	63.58	3.8147×10^{-6}	64	3.8147×10^{-6}	64
6.0341×10^{-8}	64	5.9605×10^{-8}	64	5.9605×10^{-8}	64

xy^2z^3	Rate	xyz^4	Rate	xz^5	Rate
1.5625×10^{-2}	0	1.5625×10^{-2}	0	1.5625×10^{-2}	0
2.4414×10^{-4}	64	2.4414×10^{-4}	64	2.4414×10^{-4}	64
3.8147×10^{-6}	64	3.8147×10^{-6}	64	3.8365×10^{-6}	63.64
5.9605×10^{-8}	64	5.9605×10^{-8}	64	6.0341×10^{-8}	63.59

y^6	Rate	y^5z	Rate	y^4z^2	Rate
1.6048×10^{-2}	0	1.5625×10^{-2}	0	1.5625×10^{-2}	0
2.5183×10^{-4}	63.72	2.4414×10^{-4}	64	2.4414×10^{-4}	64
3.9349×10^{-6}	64	3.8525×10^{-6}	63.37	3.8334×10^{-6}	63.69
6.1482×10^{-8}	64	6.0341×10^{-8}	63.84	5.9897×10^{-8}	64

y^3z^3	Rate	y^2z^4	Rate	yz^5	Rate
1.5625×10^{-2}	0	1.5625×10^{-2}	0	1.5625×10^{-2}	0
2.4414×10^{-4}	64	2.4414×10^{-4}	64	2.4554×10^{-4}	63.64
3.8147×10^{-6}	64	3.8237×10^{-6}	63.85	3.8484×10^{-6}	63.80
5.9605×10^{-8}	64	5.9897×10^{-8}	63.83	6.0341×10^{-8}	63.77

z^6	Rate	$x^7+y^7+z^7$	Rate	$x^8+y^8+z^8$	Rate
1.5625×10^{-2}	0	1.6406×10^{-1}	0	3.6328×10^{-1}	0
2.5075×10^{-4}	62.31	2.9907×10^{-3}	54.86	8.2966×10^{-3}	43.79
3.9179×10^{-6}	64	5.0068×10^{-5}	59.73	1.6653×10^{-4}	49.82
6.1482×10^{-8}	63.72	8.0839×10^{-7}	61.93	2.9523×10^{-6}	56.41

Tab. 3b. Numerical maximum errors of the interpolation scheme.

§7. Future Research Problems

The authors plan to use the energy minimization method to construct C^1 quintic spline interpolants for given scattered data. This method will not require higher order derivatives information at vertices. Also, the authors plan to apply this interpolation scheme to some real life data sets from oceanography and/or meteorology.

§8. Appendix

We now give the proof of Lemma 1 of Section 5. Let

$$p = \sum_{i+j+k+l=5} c_{ijkl} B^5_{ijkl}$$

be the B-form of a polynomial of degree 5 with respect to T. For a face of T, say $f = \langle v_1, v_2, v_3 \rangle$, from the given data we can determine the following 21 degrees of freedom on f.

1) $p(v_1), p(v_2)$ and $p(v_3)$
2) $D_{v_1-v_2} p(v_i), D_{v_3-v_2} p(v_i), i = 1, 2, 3.$
3) $D^2_{v_1-v_2} p(v_i), D_{v_1-v_2} D_{v_3-v_2} p(v_i)$ and $D^2_{v_3-v_2} p(v_i), i = 1, 2, 3.$
4) Values of the outward normal derivative at the midpoints of the three edges of f.

These data (fifth-degree Argyris element) determine completely p on the given face f. Similarly, p is determined on other faces of T. It remains to determine the coefficients of p which are not associated with domain points on any face of T. These coefficients are $c_{2111}, c_{1211}, c_{1121}$ and c_{1112}. We therefore write

$$p = c_{2111} B^5_{2111} + c_{1211} B^5_{1211} + c_{1121} B^5_{1121} + c_{1112} B^5_{1112} + q$$

where all coefficients in q are determined, i.e. q is known. Now for any point v with barycentric coordinates (b_1, b_2, b_3, b_4),

$$B^5_{ijkl}(v) = \frac{5!}{i!j!k!l!} b_1^i b_2^j b_3^k b_4^l \,,$$

and so

$$p(v) = \frac{5!}{2} b_1 b_2 b_3 b_4 (c_{2111} b_1 + c_{1211} b_2 + c_{1121} b_3 + c_{1112} b_4) + q(v).$$

The data

$$p(v_5), \quad \frac{\partial}{\partial x} p(v_5), \quad \frac{\partial}{\partial y} p(v_5), \quad \frac{\partial}{\partial z} p(v_5),$$

at the center v_5 determine $L(v_5), D_{v_1-v_2}L(v_5), D_{v_1-v_3}L(v_5), D_{v_2-v_3}L(v_5)$, where

$$L(v) = c_{2111}b_1 + c_{1211}b_2 + c_{1121}b_3 + c_{1112}b_4.$$

This is because v_5 is not on any face of T. For example,

$$L(v_5) = \left(\frac{5!}{2}b_1b_2b_3b_4|_{v_5}\right)^{-1}(p(v_5) - q(v_5)) = \frac{4^4 \times 2}{5!}(p(v_5) - q(v_5))$$

and

$$D_{v_1-v_2}p(v_5) = D_{v_1-v_2}(\frac{5!}{2}b_1b_2b_3b_4)|_{v_5}L(v_5) + (\frac{5!}{2}b_1b_2b_3b_4)|_{v_5}D_{v_1-v_2}L(v_5)$$

from which we can compute $D_{v_1-v_2}L(v_5)$ since its coefficient in this last equation is $\frac{5!}{2!}\left(\frac{1}{4}\right)^4$. Using the formula for directional derivatives of Section 2 with $v_5 = (v_1 + v_2 + v_3 + v_4)/4$, we get

$$\begin{aligned} c_{2111} + c_{1211} + c_{1121} + c_{1112} =& 4L(v_5), \\ c_{2111} - c_{1211} =& \frac{1}{5}D_{v_1-v_2}L(v_5), \\ c_{2111} - c_{1121} =& \frac{1}{5}D_{v_1-v_3}L(v_5), \\ c_{1211} - c_{1121} =& \frac{1}{5}D_{v_2-v_3}L(v_5). \end{aligned}$$

These equations form a system of equations which can be readily solved. This determines all the coefficients of p and concludes the proof of the lemma. □

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Gerard Awanou
University of Georgia
Math Department
Athens, Ga 30602
`gawanou@math.uga.edu`

Ming-Jun Lai
University of Georgia
Math Department
Athens, Ga 30602
`mjlai@math.uga.edu`