Decisions in Economics and Finance manuscript No.
(will be inserted by the editor)

Shortfall Risk Minimization in a Discrete Regime Switching Model

Gerard Awanou

Northern Illinois University, Department of Mathematical Sciences, Dekalb, IL, 60115 *

The date of receipt and acceptance will be inserted by the editor

Mathematics Subject Classification (2000): 49L20, 90C39, 91B28, 91B70, 93C55

Journal of Economic Literature Classification Number : D81, G11, G13

1. Introduction

There have been profound ideas on how to measure risk which have influenced the financial market. Shortfall risk minimization is one of these methods which has attracted considerable attention. This problem has been studied for the binomial model in Runggaldier et al (2000) and Runggaldier et al (2002) and for the trinomial model in Scagnelatto and Vargiolu (2002). In this paper, we investigate the shortfall risk minimization in a discrete regime switching model. In the model, we have two possible regimes, which are both binomial. To fix the ideas, we can think of the second regime as being the consequence of the presence of inside information, but this can also be due to other factors. Explicit solutions for one-period models are given.

The binomial model has as limiting case the Black-Scholes model, c.f. Musiela and Rutkowski (1997), while the trinomial model has limiting case a stochastic volatility model, c.f. Avallaneda et al (1995). The discrete model used in this paper was introduced in Guo (1999) and was shown to have as limiting case, the Black-Scholes model with Markov-modulated volatility, i.e. the Black-Scholes model in which the mean and volatility are assumed to depend on a discrete-time finite-state Markov chain. The Markov chain may be taken to model the presence of inside information or to capture market trends as well as various economic factors.

The paper is organized as follows: In the first section, we introduce the discrete model in the context of inside information and recall the concept of shortfall risk minimization. An alternative is to locally minimize the expected shortfall by adopting myopic strategies, i.e. minmize the expected shortfall over the following period. In the second section, we give explicit formulas for the local expected shortfall and a numerical example in the multiperiod case of the difference between the shortfall risk when the market is normal and the shortfall risk in the presence of inside information.

* Phone 1-815-753-6749, fax 1-815-753-1112, e-mail: `awanou@math.niu.edu`

2. Model and Shortfall risk

The fluctuations of stock prices are modeled in a discrete time economy with trading dates $n = 0, 1, 2, \ldots, N$ and with two primary traded securities: the stock and a risk-free asset, a bond. A tree diagram can be found in [2], p. 9.

We begin by assuming that the distribution of information among investors is modeled as a discrete time homogeneous Markov chain $\epsilon = \epsilon_t$ which moves among two states; ϵ_t may model more complex information structures if it is assumed to move among more states. We let $\epsilon_t = 0$ at those times when people believe that they are well informed and $\epsilon_t = 1$, when there is information asymmetry and that a group of people may have inside information. Put

$$a_1 = P(\epsilon_{t+1} = 1|\epsilon_t = 1) \quad \text{and}$$
$$a_0 = P(\epsilon_{t+1} = 0|\epsilon_t = 0).$$

We also assume that the rate of return of the risky asset is $u_i - 1$ with probability p_i and $d_i - 1$ with probability $1 - p_i$ when the market is normal, $i = 0$, and when there is inside information, $i = 1$. It follows that if we denote by $\eta_k^{(\epsilon_k,\epsilon_{k-1})}$ the appreciation rates of the stock price, and X_k the stock price at time k, we have

$$X_k^{\epsilon_k} = \eta_k^{(\epsilon_k,\epsilon_{k-1})} X_{k-1}^{\epsilon_{k-1}},$$

where $\eta_k^{i,j}$ are i.i.d. random variables taking values u_i with probability

$$p_i(\delta_{i,1-j} + (-1)^{\delta_{i,1-j}} a_j)$$

and d_i with probability

$$(1 - p_i)(\delta_{i,1-j} + (-1)^{\delta_{i,1-j}} a_j).$$

Here $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$. We will use the notation

$$\Phi(i, j) = \delta_{i,1-j} + (-1)^{\delta_{i,1-j}} a_j.$$

Throughout the paper, we will assume that

$$0 < d_0 < d_1 < 1 < u_1 < u_0, \tag{1}$$

which implies that the rate of return is smaller when there is inside information.

In addition to the stock with price $X_k^{\epsilon_k}$, we are also interested in a risk-free asset with price B_k. We will assume zero interest rate and $B_0 = 1$ so that $B_k = 1$ for all k.

The underlying probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is discrete with a filtration $\mathbb{F} = (\mathcal{F}_n)_{n=0,\ldots,N}$. We let $\mathcal{F}_N = \mathcal{F}$ and $\mathcal{F}_n$ be the σ-algebra generated by $X_k^{\epsilon_k}, k = 0, \ldots, n$. A portfolio strategy is an adapted pair $\phi = (\eta_n, \psi_n)_{n=0,\ldots,N}$, where η_n is the number of units of bonds, ψ_n the number of units of the stock, the investor carries in the period $[n, n+1)$. We will assume that ϕ is self-financing, c.f. Pliska (1997). Under the self-financing assumption, $(\psi_n)_n$, is enough to characterize the trading strategy ϕ. Denoting by V_n the value of the portfolio in the period $[n, n+1)$

$$V_{n+1} = \eta_{n+1} + \psi_{n+1} X_{n+1}^{\epsilon_{n+1}} = V_n + \psi_n (X_{n+1}^{\epsilon_{n+1}} - X_n^{\epsilon_n}),$$

c.f. Runggaldier et al (2002).

Let H be a liability to be hedged at some fixed future time N. The model considered in this paper will be shown below to be incomplete. Hence it might not be possible to hedge exactly the claim. On the other hand superhedging might require an initial capital $\overline{V_0} = V_0(\psi)$ too high since $V_N =$

$V_0 + \sum_{k=0}^{N-1} \psi_k (X_{k+1}^{\epsilon_{k+1}} - X_k^{\epsilon_k})$. An alternative is to minimize the expected shortfall for an initial capital $V_0 < \overline{V_0}$ that is

$$J(\epsilon_0, X_0, V_0) = \inf_{\psi \in \mathcal{A}} E^{\mathbf{P}}_{X_0^{\epsilon_0}, V_0} \{ [H_N - V_N(\psi)]^+ \}, \tag{2}$$

for a given initial distribution of information ϵ_0, an initial price $X_0^{\epsilon_0}$ of the stock and a given initial capital V_0.

In particular we will be interested in the difference

$$J(1, X_0, V_0) - J(0, X_0, V_0)$$

between the shortfall risk when the market is normal and the shortfall risk in the presence of inside information.

3. Local expected shortfall risk minimization

A classical approach to solve the optimization problem (2) is the use of dynamic programming. In the binomial and trinomial models (which are special cases of the model considered here) explicit solutions of the dynamic programming algorithm can be given under more or less restrictive assumptions. However this is not possible for the regime switching model in the multiperiod case. We have identified, with the help of numerical experiments, conditions under which explicit formulas can be given in a one-period model. A numerical example in the multiperiod case is given at the end of this section. As mentionned in the introduction, one can also adopt myopic strategies by minimizing the local expected shortfall, that is the expected shortfall over the following period

$$J(\epsilon_{n-1}, X_{n-1}, V_{n-1}) = \inf_{\psi_{n-1}} E^{\mathbf{P}}_{X_{n-1}^{\epsilon_{n-1}}, V_{n-1}} \{ [H(X_n) - V_n(\psi)]^+ \},$$

We first give explicit formulas for one-period models, or the local expected shortfall. The analysis reveals a number of measures which can be shown to be on the boundary of the set of equivalent martingale measures. Define $q_0^* = \frac{1-d_0}{u_0-d_0}$, $q_1^* = \frac{1-d_1}{u_1-d_1}$, $r_0^* = \frac{1-d_1}{u_0-d_1}$, $r_1^* = \frac{1-d_0}{u_1-d_0}$, $s_0^* = \frac{1-u_1}{u_0-u_1}$ and $s_1^* = \frac{1-d_0}{d_1-d_0}$, and set

$$E^{Q_0}[H(X_N)|\mathcal{F}_{N-1}] = q_0^* H(X_{N-1}u_0) + (1-q_0^*) H(X_{N-1}d_0)$$

and similar formulas for $E^{Q_1}[H(X_N)|\mathcal{F}_{N-1}], E^{R_0}[H(X_N)|\mathcal{F}_{N-1}], E^{R_1}[H(X_N)|\mathcal{F}_{N-1}]$, $E^{S_0}[H(X_N)|\mathcal{F}_{N-1}]$ and $E^{S_1}[H(X_N)|\mathcal{F}_{N-1}]$ with q_0^* replaced respectively by q_1^*, r_0^*, r_1^*, s_0^* and s_1^*.

3.1. Explicit formulas in One-Period Models

Explicitly $J(0, X_{n-1}, V_{n-1})$ is the minimizer over admissible strategies ψ of

$$\begin{aligned} j(0, X_{n-1}, V_{n-1}, \psi) &= p_0 \varPhi(0,0)[H(S_{n-1}u_0) - V_{n-1} - \psi S_{n-1}(u_0 - 1)]^+ \\ &+ (1-p_0)\varPhi(0,0)[H(S_{n-1}d_0) - V_{n-1} - \psi S_{n-1}(d_0 - 1)]^+ \\ &+ p_1 \varPhi(1,0)[H(S_{n-1}u_1) - V_{n-1} - \psi S_{n-1}(u_1 - 1)]^+ \\ &+ (1-p_1)\varPhi(1,0)[H(S_{n-1}d_1) - V_{n-1} - \psi S_{n-1}(d_1 - 1)]^+. \end{aligned}$$

This functional of ψ is the sum of piecewise linear functions so the minimum is reached at the zeros of these functions. They are

$$\psi_{n-1,1} = \frac{H(S_{n-1}u_0) - V_{n-1}}{S_{n-1}(u_0-1)}, \quad \psi_{n-1,2} = \frac{H(S_{n-1}d_0) - V_{n-1}}{S_{n-1}(d_0-1)},$$
$$\psi_{n-1,3} = \frac{H(S_{n-1}u_1) - V_{n-1}}{S_{n-1}(u_1-1)}, \quad \psi_{n-1,4} = \frac{H(S_{n-1}d_1) - V_{n-1}}{S_{n-1}(d_1-1)}.$$

We simply need to compute the value of j at those points. Recalling that $u_0 > u_1 > 1 > d_1 > d_0$, we have (omitting lengthy computational details):

$$\begin{aligned} j(0, X_{n-1}, V_{n-1}, \psi_{n-1,1}) &= \Phi(0,0)\frac{1-p_0}{1-q_0^*}[E^{Q_0}[H(S_n)|\mathcal{F}_{n-1}] - V_{n-1}]^+ \\ &+ \Phi(1,0)\frac{p_1}{1-s_0^*}[E^{S_0}[H(S_n)|\mathcal{F}_{n-1}] - V_{n-1}]^+ \\ &+ \Phi(1,0)\frac{1-p_1}{1-r_0^*}[E^{R_0}[H(S_n)|\mathcal{F}_{n-1}] - V_{n-1}]^+, \end{aligned}$$

$$\begin{aligned} j(0, X_{n-1}, V_{n-1}, \psi_{n-1,2}) &= \Phi(0,0)\frac{p_0}{q_0^*}[E^{Q_0}[H(S_n)|\mathcal{F}_{n-1}] - V_{n-1}]^+ \\ &+ \Phi(1,0)\frac{p_1}{r_1^*}[E^{R_1}[H(S_n)|\mathcal{F}_{n-1}] - V_{n-1}]^+ \\ &+ \Phi(1,0)\frac{1-p_1}{s_1^*}[E^{S_1}[H(S_n)|\mathcal{F}_{n-1}] - V_{n-1}]^+, \end{aligned}$$

$$\begin{aligned} j(0, X_{n-1}, V_{n-1}, \psi_{n-1,3}) &= -\Phi(0,0)\frac{p_0}{s_0^*}[V_{n-1} - E^{S_0}[H(S_n)|\mathcal{F}_{n-1}]]^+ \\ &+ \Phi(0,0)\frac{1-p_0}{1-r_1^*}[E^{R_1}[H(S_n)|\mathcal{F}_{n-1}] - V_{n-1}]^+ \\ &+ \Phi(1,0)\frac{1-p_1}{1-q_1^*}[E^{Q_1}[H(S_n)|\mathcal{F}_{n-1}] - V_{n-1}]^+ \text{ and} \end{aligned}$$

$$\begin{aligned} j(0, X_{n-1}, V_{n-1}, \psi_{n-1,4}) &= \Phi(0,0)\frac{p_0}{r_0^*}[E^{R_0}[H(S_n)|\mathcal{F}_{n-1}] - V_{n-1}]^+ \\ &- \Phi(0,0)\frac{1-p_0}{1-s_1^*}[V_{n-1} - E^{S_1}[H(S_n)|\mathcal{F}_{n-1}]]^+ \\ &+ \Phi(1,0)\frac{p_1}{q_1^*}[E^{Q_1}[H(S_n)|\mathcal{F}_{n-1}] - V_{n-1}]^+. \end{aligned}$$

Remark 1. In the above formulas, if the term in front of the minus sign is nonzero, the expression inside the brackets turns out to be positive.

Recalling that $d_0 < d_1 < 1 < u_1 < u_0$, we have the following theorem

Theorem 1. *Let H be a convex function. For $i, j = 0, 1$ we have*

$$\begin{aligned} E^{S_i}[H(S_n)|\mathcal{F}_{n-1}] \le E^{Q_1}[H(S_n)|\mathcal{F}_{n-1}] &\le E^{R_j}[H(S_n)|\mathcal{F}_{n-1}] \\ &\le E^{Q_0}[H(S_n)|\mathcal{F}_{n-1}]. \end{aligned}$$

In general $E^{S_i}[H(S_n)|\mathcal{F}_{n-1}],\ i = 0, 1$ and $E^{R_j}[H(S_n)|\mathcal{F}_{n-1}],\ j = 0, 1$ are not ordered.

Proof. Let $f : I \to \mathbb{R}$ be a convex function and let $a, b, c \in I$ with $a < b < c$. Then

$$\frac{f(b) - f(a)}{b - a} \leq \frac{f(c) - f(a)}{c - a} \leq \frac{f(c) - f(b)}{c - b}.$$

This implies that $f(a) \geq \frac{a-c}{b-c} f(b) + \frac{b-a}{b-c} f(c)$, $f(c) \geq \frac{b-c}{b-a} f(a) + \frac{c-a}{b-a} f(b)$ and $f(b) \leq \frac{c-b}{c-a} f(a) + \frac{b-a}{c-a} f(c)$. Using $1 < u_1 < u_0$ and $d_0 < d_1 < 1$, we have $H(S_{n-1}) \geq E^{S_0}[H(S_n)|\mathcal{F}_{n-1}]$ and $H(S_{n-1}) \geq E^{S_1}[H(S_n)|\mathcal{F}_{n-1}]$. Next using $d_1 < 1 < u_1$ we get $H(S_{n-1}) \leq E^{Q_1}[H(S_n)|\mathcal{F}_{n-1}]$. It follows that $E^{S_i}[H(S_n)|\mathcal{F}_{n-1}] \leq E^{Q_1}[H(S_n)|\mathcal{F}_{n-1}],\ i = 1, 2$. Next it can be easily checked that for $a, b, c, d \in I$ with $a < b < c < d$ we have

$$\frac{c-b}{d-b} f(d) + \frac{d-c}{d-b} f(b) \leq \frac{c-a}{d-a} f(d) + \frac{d-c}{d-a} f(a)$$
$$\frac{b-a}{c-a} f(c) + \frac{c-b}{c-a} f(a) \leq \frac{b-a}{d-a} f(d) + \frac{d-b}{d-a} f(a).$$

Using the first inequality with $d_0 < d_1 < 1 < u_0$ and $d_0 < d_1 < 1 < u_1$, we obtain $E^{R_0}[H(S_n)|\mathcal{F}_{n-1}] \leq E^{Q_0}[H(S_n)|\mathcal{F}_{n-1}]$ and $E^{Q_1}[H(S_n)|\mathcal{F}_{n-1}] \leq E^{R_1}[H(S_n)|\mathcal{F}_{n-1}]$. Finally using $d_1 < 1 < u_1 < u_0$ and $d_0 < 1 < u_1 < u_0$ we get $E^{R_1}[H(S_n)|\mathcal{F}_{n-1}] \leq E^{Q_0}[H(S_n)|\mathcal{F}_{n-1}]$ and $E^{Q_1}[H(S_n)|\mathcal{F}_{n-1}] \leq E^{R_0}[H(S_n)|\mathcal{F}_{n-1}]$. $\square$

A consequence of the previous theorem is that if $V_{n-1} \geq E^{Q_0}[H(S_n)|\mathcal{F}_{n-1}]$, then the local expected shortfall risk is zero.

We can now give an explicit formula for the local expected shortfall assuming that the initial capital is not too small.

Theorem 2. *Assume that*

$$\max\left(E^{R_1}[H(S_n)|\mathcal{F}_{n-1}], E^{R_0}[H(S_n)|\mathcal{F}_{n-1}]\right) < V_{n-1} < E^{Q_0}[H(S_n)|\mathcal{F}_{n-1}],$$

then the shortfall risk in the last stage of the dynamic programming is given by

$$J(0, X_{n-1}, V_{n-1}) = \Phi(0,0) \min\left(\frac{1-p_0}{1-q_0^*}, \frac{p_0}{q_0^*}\right)[E^{Q_0}[H(S_n)|\mathcal{F}_{n-1}] - V_{n-1}].$$

Proof. We only need to compare J_1, J_2, J_3 and J_4 where

$$J_1 = \Phi(0,0)(1-p_0)\frac{u_0 - d_0}{u_0 - 1}[E^{Q_0}[H(S_n)|\mathcal{F}_{n-1}] - V_{n-1}]$$
$$J_2 = \Phi(0,0)p_0\frac{u_0 - d_0}{1 - d_0}[E^{Q_0}[H(S_n)|\mathcal{F}_{n-1}] - V_{n-1}]$$
$$J_3 = \Phi(0,0)p_0\frac{u_0 - u_1}{u_1 - 1}[V_{n-1} - E^{S_0}[H(S_n)|\mathcal{F}_{n-1}]]$$
$$J_4 = \Phi(0,0)(1-p_0)\frac{d_1 - d_0}{1 - d_1}[V_{n-1} - E^{S_1}[H(S_n)|\mathcal{F}_{n-1}]]$$

Next it can be checked that

$$J_3 \geq J_2 \iff V_{n-1} \geq E^{R_1}[H(S_n)|\mathcal{F}_{n-1}] \text{ and } J_4 \geq J_1 \iff V_{n-1} \geq E^{R_0}[H(S_n)|\mathcal{F}_{n-1}].$$ $\square$

The optimal strategy is given by $\Psi_{n-1,1}$ or $\Psi_{n-1,2}$ depending on the mimimum of $\frac{1-p_0}{1-q_0^*}$ and $\frac{p_0}{q_0^*}$. It is immediate that in general under the assumptions of the theorem,

$$J(\epsilon_{N-1}, X_{n-1}, V_{n-1}) = \Phi(0, \epsilon_{N-1}) \min \left(\frac{1-p_0}{1-q_0^*}, \frac{p_0}{q_0^*} \right) [E^{Q_0}[H(S_n)|\mathcal{F}_{n-1}] - V_{n-1}].$$

4. Numerical example for the shortfall in the multiperiod case

Let $N = 2$ and consider an European call option with strike price $K = \$45$ on a stock whose value at time 0 is $X_0 = \$50$ independent of the value of ϵ_0. With an initial capital $V_0 = 6$, take

$$a_0 = 0.9, a_1 = 0.3, p_0 = 0.7, p_1 = 0.3, u_0 = 1.1, d_0 = 0.8, u_1 = 1.05, \text{ and } d_1 = 0.9.$$

When the inital state is normal, the shortfall is found to be 0.7472 and in the first period one should hold 0.8 units of the stock and in the second period depending on the values of the stock price X_0u_0, X_0d_0, X_0u_1 or X_0d_1, the holdings are respectively 1, 0.5, 0.9048 and 0.4444. In the presence of inside information, the shortfall is 0.6619 and the corresponding holdings are 0.6001 in the first period and 1.1817, 0.0002, 1 and 0.3334 in the second. The difference between $J(1, 45, 6)$ and $J(0, 45, 6)$ gives an indication of the value of the information ϵ_0, thus giving an edge if one can determine that a group of people have inside information. We chose the initial capital high enough to have $J(1, 45, 6) < J(0, 45, 6)$.

5. Conclusion

We have investigated the problem of shortfall risk minimization in the presence of inside information. It appears that the investor can take different positions which correspond to different amounts of risk. It seems more practical due to the complexity of the problem to use instead local risk minimization. We believe that the discrete model investigated here should receive more attention since regime switching models are gaining popularity in finance.

References

1. Avellaneda M., A. Levy, A. Paras, 1995, Pricing and hedging derivative securities in markets with uncertain volatilities, *Applied Mathematical Finance* **2**, 73-88.
2. Guo X., 1999, Inside Information and Stock Fluctuations, Ph.D. Dissertation, Mathematics Department, Rutgers University.
3. Musiela, M. and Rutkowski M., 1997, Martingale Methods in Financial Modeling, Springer, New-York.
4. Pliska, S.R. 1997, Introduction to Mathematical Finance: Discrete Time Models, Blackwell Publishers.
5. Runggaldier R, Trivellato B. and Vargiolu T., 2002, A Bayesian Adaptive Control Approach to Risk Management in a Binomial Model, *Seminar on Stochastic analysis, Random Fields, and Applications* (R.C. Dalang, M. Dozzi and F. Russo. eds.). Progress in Probability, Vol 52, Birkhauser Verlag, 243-258.
6. Runggaldier R. and Zaccaria A., 2000, A Stochastic Control Approach to Risk Management under Restricted Information, *Mathematical Finance*, **10**, N° 2, 277-288.
7. Scagnelatto C. and Vargiolu T., 2002, Explicit Solutions for Shortfall Risk Minimization in Multinomial Models, *Decisions in Economics and Finance*, **25**, 145-155.
8. Schulmerich, M., S. Trautmann, 2003, Local Expected Shortfall-Hedging in Discrete Time, *European Finance Review* **7**, 75-102.