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A mathematical model for long waves generated by wavemakers in non-linear dispersive systems

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An initial-boundary-value problem for the equation

$$u_t + u_x + uu_x - u_{xxt} = 0 \qquad (a)$$

is considered for $x, t \geqslant 0$. This system is a model for long water waves of small but finite amplitude, generated in a uniform open channel by a wavemaker at one end. It is shown that, in contrast to an alternative, more familiar model using the Korteweg-deVries equation, the solution of (a) has good mathematical properties: in particular, the problem is well set in Hadamard's classical sense that solutions corresponding to given initial data exist, are unique, and depend continuously on the specified data.

1. *Introduction.* In a recent paper (1), a model equation governing the unidirectional propagation of one-dimensional long waves in non-linear dispersive systems was developed as an alternative to the Korteweg–deVries (KdV) equation. The model considered was the initial-value problem for the equation

$$u_t + u_x + uu_x - u_{xxt} = 0 \qquad (1\cdot 1)$$

with $t \geqslant 0$ and $-\infty < x < \infty$. Equation (1·1) has the same *formal* status as the KdV equation as an approximate model for long waves of small amplitude in an important class of non-linear dispersive systems. It was contended in (1) that (1·1) is in many respects a superior model, having generally good mathematical properties and avoiding certain problematical aspects of the initial-value problem for the KdV equation.

It is the purpose of this paper to consider the initial-boundary-value problem

$$\left.\begin{aligned} u_t + u_x + uu_x - u_{xxt} &= 0,\\ u(x,0) &= g(x),\\ u(0,t) &= h(t),\end{aligned}\right\} \qquad (1\cdot 2)$$

for $t \geqslant 0$ and $x \geqslant 0$, where the boundary value h and initial value g are specified, with $g(0) = h(0)$ for consistency. Thus we consider the propagation of long waves in a semi-infinite medium which is given a specified disturbance at the finite end. One physical system for which (1·2) may serve as a model is an open uniform channel containing water with a wavemaker at one end. The problem (1·2), with $g \equiv 0$ say, is perhaps

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physically more relevant than the pure initial-value problem for (1·1). Certainly for comparing (numerically computed) solutions with actual waves generated in the laboratory, the situation described by (1·2) seems more useful.

We show that (1·2) admits a satisfactory mathematical theory, being well set in Hadamard's sense that solutions corresponding to given initial data g and specified disturbance h exist, are unique, and depend continuously on g and h. The following statement summarizes the main results to be derived, and serves to define the aims of this work.

Main results. Suppose g is twice continuously differentiable and that g and its first two derivatives are bounded for $0 \leqslant x < \infty$. Suppose also that g and g' are square integrable over $0 \leqslant x < \infty$. Let h be continuously differentiable for $0 \leqslant t \leqslant T$. Then there is a unique classical solution $u(x,t)$ of (1·2) for $0 \leqslant t \leqslant T$, such that, for each t, u and u_x are square integrable over $0 \leqslant x < \infty$. Further, this solution depends continuously on perturbations of g and h within these function classes. (The metrics to which this continuity is referred are reviewed in section 2 and in the discussion preceding Theorem 2 in section 4.)

The method employed proceeds in three stages. First, the existence of a solution over a sufficiently small time-interval is proved. This is done by recasting (1·2) in integral-equation form and using the contraction-mapping principle. The extension of the solution to arbitrary time intervals is then made by appeal to an *a priori* estimate for solutions of (1·2). Finally, uniqueness and the continuous dependence of solutions on g and h are established by using *a priori* estimates for the difference of two solutions of (1·2).

The programme is similar to that carried out in sections 3 and 4 of (1) on the initial value problem for (1·1). Emphasis is given, in our proofs, to the points that diverge from those presented in (1).

The present results for (1·2) would not obtain for the same initial-boundary-value problem posed for the KdV equation. In particular, the presence of an extra x-derivative in the KdV equation, (2),

$$u_t + u_x + uu_x + u_{xxx} = 0, \tag{1·3}$$

imposes constraints on the boundary data which are not required in the present model, and which appear unnatural to the physical problem.

A comment on the unidirectionality of the model is deserved. The assumption of unidirectional wave propagation is introduced, along with several other assumptions, in the *derivation* of the model equations (1·1) or (1·3) from the full equations of motion. Out theory is set in function spaces which include, but are not exhausted by, initial data and solutions which come under these assumptions. Hence the term unidirectional is used only to recall an assumption that would have to be made in a derivation of (1·1), and is not to be construed as implying a mathematically established property of the system (1·2) for the general initial data prescribed in our main results.

In section 2 we collect together the function spaces used. Then, in section 3, the existence of a weak solution over a small time-interval is proved under very general conditions. It is shown further that the weak solution provides a classical solution to

(1·2) over the small time-interval when the data are restricted as explained above under the heading Main Results. In section 4 the aforesaid 'solution in the small' is uniquely extended to a classical solution on arbitrary time-intervals. Finally, in section 5, an exact result on the continuous dependence of the solution on g and h is obtained. In contrast with the results formulated in sections 3, 4 and 5, which deal only with an arbitrary finite interval, the Appendix shows how these results may be reinterpreted in a Fréchet-space setting to take account of the entire time axis.

2. *The relevant function spaces.* Before we proceed to prove results leading to our Main Results, it is convenient to collect together the definitions and elementary properties of the function spaces that we shall use. All functions are taken to be real-valued.

Supposing $I = [0, a]$ to be a bounded interval in the real numbers $\mathbb{R}$, we denote by $C^k(I)$ the Banach space of functions on I whose derivatives up to order k are continuous, $k = 0, 1, \ldots$. The norm is

$$\|f\|_{k,I} = \|f\|_k = \sup_{\substack{x\in I\\ 0\leqslant j\leqslant k}} |f^{(j)}(x)|. \tag{2·1}$$

Let $\mathbb{R}^+$ denote the non-negative real numbers. We define $C_b^k(\mathbb{R}^+)$ to be the functions on $\mathbb{R}^+$ whose first k derivatives are bounded and continuous. This is also a Banach space under the norm (2·1) with $\mathbb{R}^+$ in place of I.

By $H^k(\mathbb{R}^+)$, we mean the Sobolev space of (equivalence classes of) measurable square-integrable functions defined on $\mathbb{R}^+$ whose (generalized) derivatives to order k are also square integrable over $\mathbb{R}^+$, $k = 0, 1, \ldots$. This is a Hilbert space with inner product

$$\langle f, g\rangle_k = \sum_{j=0}^{k} \int_0^\infty f^{(j)}(x)\, g^{(j)}(x)\, dx. \tag{2·2}$$

We denote the norm in $H^k(\mathbb{R}^+)$ by

$$\|f\|_{k,2} = \langle f, f\rangle_k^{\frac{1}{2}} \tag{2·3}$$

to distinguish it from the norm (2·1). $H^0(\mathbb{R}^+)$, which is simply $L_2(\mathbb{R}^+)$, and $H^1(\mathbb{R}^+)$ are the only two considered here. We state some useful properties of $H^1(\mathbb{R}^+)$.

DEFINITION. A function $f \in C_b(\mathbb{R}^+)$ is *asymptotically null* if $\lim_{x\to\infty} f(x) = 0$.

PROPOSITION 1. *$H^1(\mathbb{R}^+)$ is continuously embedded in $C_b(\mathbb{R}^+)$. Further, when $H^1(\mathbb{R}^+)$ is identified as a subspace of $C_b(\mathbb{R}^+)$, then $f \in H^1(\mathbb{R}^+)$ implies that f is asymptotically null, and that*

$$\|f\|_0 \leqslant \|f\|_{1,2}. \tag{2·4}$$

Proof. Let g be the extension of f to $\mathbb{R}$ obtained by reflecting f about the origin. Then $g \in H^1(\mathbb{R})$. The usual embedding result ((3), ch. 1, § 9) implies that g is continuous, bounded, and asymptotically null. Hence f is continuous and asymptotically null. It then follows that, for $x \in \mathbb{R}^+$,

$$f^2(x) = -2\int_x^\infty f(\xi) f'(\xi)\, d\xi \leqslant \int_0^\infty [f^2(\xi) + f'^2(\xi)]\, d\xi = \|f\|_{1,2}^2,$$

from which (2·4) follows.

We shall also need to consider spaces of functions of two variables analogous to the one-dimensional spaces just defined. For $T > 0$, let $\mathscr{C}_T$ be the space of bounded continuous functions $u(x,t)$ defined on $\mathbb{R}^+ \times [0,T]$. Similarly, let $\mathscr{C}_T^{l,m}$ be the subspace of elements $u \in \mathscr{C}_T$ such that $\partial_t^i \partial_x^j u \in \mathscr{C}_T$ for $0 \leqslant i \leqslant l$, $0 \leqslant j \leqslant m$. (Thus $\mathscr{C}_T \equiv \mathscr{C}_T^{0,0}$.) We note that $\mathscr{C}_T^{l,m}$ is a Banach space under the norm

$$\|u\|_{\mathscr{C}_T^{l,m}} = \sum_{i=0}^{l} \sum_{j=0}^{m} \|\partial_t^i \partial_x^j u\|_{\mathscr{C}_T}, \tag{2·5}$$

where
$$\|v\|_{\mathscr{C}_T} = \sup_{x \geqslant 0} \sup_{T \geqslant t \geqslant 0} |v(x,t)|. \tag{2·6}$$

We shall often omit the T in this norm symbol when no confusion can thereby result.

Taking $T > 0$ to be finite, we define $\mathscr{W}_T$ to be the functions $f(x,t)$ defined on $\mathbb{R}^+ \times [0,T]$ such that $f(x,t) \in H^1(\mathbb{R}^+)$ for each t and such that the correspondence $t \to f(x,t)$ is continuous from $[0,T]$ to $H^1(\mathbb{R}^+)$. $\mathscr{W}_T$ is a Banach space under the norm

$$\|f\|_{\mathscr{W}_T} = \sup_{0 \leqslant t \leqslant T} \|f(x,t)\|_{1,2}. \tag{2·8}$$

By virtue of Proposition 1, we have:

PROPOSITION 2. *$\mathscr{W}_T$ is embedded in $\mathscr{C}_T$ for each $T > 0$. When $\mathscr{W}_T$ is considered as a subspace of $\mathscr{C}_T$, the classification $u \in \mathscr{W}_T$ implies that u is uniformly continuous in both variables, asymptotically null uniformly in $t \in [0,T]$, and*

$$\|u\|_{\mathscr{C}_T} \leqslant \|u\|_{\mathscr{W}_T}. \tag{2·9}$$

3. *Existence in the small.* By formal operations, we first put (1·2) into integral equation form. The equation may be rewritten

$$(1 - \partial_x^2) u_t = -\partial_x(u + \tfrac{1}{2}u^2),$$

and regarded as an ordinary differential equation for u_t. Formally, the solution is

$$u_t(x,t) = -\tfrac{1}{2}\int_0^\infty e^{-|x-\xi|}\,\partial_\xi(u + \tfrac{1}{2}u^2)\,d\xi + \tfrac{1}{2}\int_0^\infty e^{-(x+\xi)}\,\partial_\xi(u + \tfrac{1}{2}u^2)\,d\xi + h'(t)\,e^{-x}.$$

A formal integration by parts, followed by integration from 0 to t, now gives:

$$u(x,t) = g(x) + (h(t) - h(0))\,e^{-x} + \int_0^t \int_0^\infty K(x,\xi)\,\{u(\xi,\tau) + \tfrac{1}{2}u^2(\xi,\tau)\}\,d\xi\,d\tau, \tag{3·1}$$

where
$$K(x,\xi) = \tfrac{1}{2}\operatorname{sgn}(x-\xi)\,e^{-|x-\xi|} + \tfrac{1}{2}e^{-(x+\xi)}. \tag{3·2}$$

For short, we write this as

$$u = \mathbf{A}u = g(x) + (h(t) - h(0))\,e^{-x} + \mathbf{B}u. \tag{3·3}$$

LEMMA 1. *Let $g \in C_b(\mathbb{R}^+)$ and $h \in C([0,T])$. Then there is an S with $T \geqslant S > 0$, depending only on g and h, such that* (3·3) *has a solution $u \in \mathscr{C}_S$.*

Proof. We view $\mathbf{A}$ as a mapping of $\mathscr{C}_S$, leaving S to be chosen suitably later. Clearly, $\mathbf{A}u \in \mathscr{C}_S$ if $u \in \mathscr{C}_S$. We argue that by choosing R large enough, and S small enough, $\mathbf{A}$ is a contraction mapping of the ball of radius R about the zero element 0 of $\mathscr{C}_S$. The necessary estimates are

$$\|\mathbf{A}u\|_{\mathscr{C}} \leqslant \|g\|_0 + \|h\|_0 + S[\|u\|_{\mathscr{C}}(1+\|u\|_{\mathscr{C}})], \tag{3·4}$$

and
$$\|\mathbf{A}u - \mathbf{A}v\|_{\mathscr{C}} \leqslant S\|u-v\|_{\mathscr{C}}\{1+\tfrac{1}{2}(\|u\|_{\mathscr{C}}+\|v\|_{\mathscr{C}})\}. \tag{3·5}$$

Here we have used the result

$$\sup_{x\geqslant 0}\int_0^\infty |K(x,\xi)|\,d\xi = 1.$$

Note that (3·5) implies $\mathbf{A}$ to be a continuous mapping of $\mathscr{C}_S$ to itself. According to (3·4), $\mathbf{A}$ maps the closed ball of radius R about 0 into itself if

$$R \geqslant r(S) + SR(1+R), \tag{3·6}$$

where
$$r(S) = \|g\|_0 + \|h\|_0 = \|g\|_0 + \|h\|_{0,[0,S]}. \tag{3·7}$$

Hence, by (3·5), $\mathbf{A}$ is a contractive mapping of this ball if

$$S(1+R) = \Theta < 1. \tag{3·8}$$

It is easily verified that a sufficiently small positive value of S ($\leqslant T$) is determined by

$$S(1+2r(S)) \leqslant \tfrac{1}{2}, \tag{3·9}$$

so that we can take
$$R = 2r(S) \tag{3·10}$$

to satisfy (3·6), and then (3·8) is satisfied with $\Theta = \frac{1}{2}$.

Lemma 1 now follows by appeal to the contraction-mapping principle.

Corollary. *Let g and h, R and S be as in Lemma 1, and let*

$$u_0(x,t) = g(x) + e^{-x}(h(t)-h(0)).$$

Then the sequence $u_n(x,t)$ defined by

$$u_n(x,t) = \mathbf{A}u_{n-1}(x,t) = u_0(x,t) + \mathbf{B}u_{n-1}(x,t) \tag{3·11}$$

converges in $\mathscr{C}_S$ to the unique solution u of (3·3) in the ball $\|u\|_{\mathscr{C}} \leqslant R$.

We now want to determine conditions under which the solution u of (3·3) provides, on the time interval $[0,S]$, a classical solution of the problem (1·2).

Lemma 2. *Suppose $g \in C_b^2(\mathbb{R}^+)$ and $h \in C^1([0,T])$. Then any solution $u \in \mathscr{C}_T$ of (3·3) is an element of $\mathscr{C}_T^{1,2}$ and is a classical solution of the initial-boundary value problem (1·2).*

Proof. Since $u = \mathbf{A}u$, the first derivative u_t clearly exists, being given by

$$u_t = (\mathbf{A}u)_t = h'(t)\,e^{-x} + \int_0^\infty K(x,\xi)\{u(\xi,t)+\tfrac{1}{2}u^2(\xi,t)\}\,d\xi, \tag{3·12}$$

which is a bounded continuous function on $\mathbb{R}^+\times[0,T]$ because $h' \in C([0,T])$ and $u \in \mathscr{C}_T$.

By dividing the range of integration at $\xi = x$ we confirm that u_x exists and is equal to

$$u_x = g'(x) - e^{-x}(h(t) - h(0)) + \int_0^t (u + \tfrac{1}{2}u^2)\, dt + \int_0^t \int_0^\infty L(x, \xi)\,(u + \tfrac{1}{2}u^2)\, d\xi\, d\tau, \tag{3·13}$$

where

$$L(x, \xi) = -\tfrac{1}{2}[e^{-|x-\xi|} + e^{-(x+\xi)}]. \tag{3·14}$$

Again, since $g' \in C_b(\mathbb{R}^+)$, and $u \in \mathscr{C}_T$, we have $u_x \in \mathscr{C}_T$. The fact that $u_x \in \mathscr{C}_T$ implies that the right-hand side of (3·13) is also differentiable with respect to x, so that u_{xx} exists, being given by

$$\left.\begin{aligned} u_{xx} &= g''(x) + e^{-x}(h(t) - h(0)) + \int_0^t (u + \tfrac{1}{2}u^2)_x\, d\tau \\ &\qquad\qquad + \int_0^t \int_0^\infty K(x, \xi)\,(u + \tfrac{1}{2}u^2)\, d\xi\, d\tau \\ &= g''(x) + \int_0^t (u + \tfrac{1}{2}u^2)_x\, d\tau + u - g(x), \end{aligned}\right\} \tag{3·15}$$

which is also continuous and bounded. Finally, the above expression for u_{xx} is obviously differentiable with respect to t, and we see that

$$u_{xxt} = (u + \tfrac{1}{2}u^2)_x + u_t,$$

confirming that $u_{xxt} \in \mathscr{C}_T$ and that u provides a solution of the equation in (1·2).

Since $u = \mathbf{A}u$, u automatically satisfies the initial and boundary conditions under the hypotheses about g and h.

Note that if greater regularity of g and h is assumed, any solution $u \in \mathscr{C}_T$ of (3·3) acquires correspondingly greater regularity. This is easily seen by continuing the arguments of (3·12), (3·13), (3·15) and leads to:

Corollary. *Let $g \in C_b^l(\mathbb{R}^+)$ and $h \in C^k([0, T])$, where $k \geqslant 1$, $l \geqslant 2$. Then any solution $u \in \mathscr{C}_T$ of (3·3) lies in $\mathscr{C}_T^{k,\,l}$.*

Account can also be taken of the case in which the initial waveform g, or the forcing term h, are only piecewise continuously differentiable. In particular, we can suppose that h is continuous, and that h' has only discontinuities of the first kind (i.e. jump discontinuities), and that similarly g, g' are bounded and continuous, and g'' is bounded and has only discontinuities of the first kind. By a straightforward modification of the arguments given in Lemma 2, the solution of the integral equation (3·3) corresponding to such data is shown to be piecewise a classical solution of (1·3). The solution responds to discontinuities in a predictable manner, for the arguments of Lemma 2 show that $u - g(x) - e^{-x}(h(t) - h(0)) = \mathbf{B}u$ is still an element of $\mathscr{C}_T^{1,2}$. Thus u is seen to be a classical solution of (1·3), except on the residual set where $g''(x)$ or $h'(t)$ have jump discontinuities, and on this set u has similar discontinuities in u_{xxt} and u_t. Note, however, that a discontinuity in the derivative of the disturbance h is rendered negligible for large x by the factor e^{-x}.

4. *Existence and uniqueness of global solutions*. We extend the solutions obtained in section 3 to arbitrary time intervals by restricting further the initial waveform, and be appealing to an *a priori* estimate for solutions of (1·2).

First we note that if the initial waveform is asymptotically null, then so is the solution determined by Lemmas 1 and 2 for $0 \leqslant t \leqslant S$.

LEMMA 3. *Let $g \in C_b^k(\mathbb{R}^+)$ and $h \in C^l([0,T])$, $k \geqslant 2$, $l \geqslant 1$, and suppose that for some $p \leqslant k$, $g, g', \ldots, g^{(p)}$ are asymptotically null. Then, if $u \in \mathscr{C}_S$ is the solution of* (1·2) *guaranteed by Lemma* 1 *and* 2, *$\partial_t^i \partial_x^j u$ is asymptotically null for $0 \leqslant i \leqslant l$ and $0 \leqslant j \leqslant p$, uniformly for $0 \leqslant t \leqslant S$.*

Proof. The corollary to Lemma 2 assures us that the appropriate derivatives exist. We need only prove that $\partial_t^i \partial_x^j u$ is asymptotically null, uniformly for $0 \leqslant t \leqslant S$, for the special cases $i = 0, 1$ and $j = 0, 1, 2$. For the cases $i \geqslant 1$, $j \geqslant 2$ follow inductively from the differential equation (and the result even holds for the range $0 \leqslant j \leqslant k$ in the case $i \geqslant 1$). The cases $i = 0$ and $p \geqslant j \geqslant 2$ follow inductively by differentiating (3·15) and using the established property of *uniform* asymptotic nullity.

For u itself, the result follows from the corollary to Lemma 1, the fact that $u_0(x,t) = g(x) + e^{-x}[h(t) - h(0)]$ is uniformly asymptotically null for $0 \leqslant t \leqslant S$, and the remark that the integral operator $\mathbf{B}$ of (3·3) preserves uniform asymptotic nullity, as also does the taking of a limit in $\mathscr{C}_S$.

Once u is known to be uniformly asymptotically null, the corresponding fact for u_t, u_x, and u_{xx} are derived immediately from (3·12), (3·13) and (3·15) respectively. Differentiating (3·13) with respect to t, we obtain a similar representation of u_{xt}, whose uniform asymptotic nullity then follows immediately.

The derivation of *a priori* estimates for solutions of (1·2) can now be undertaken.

LEMMA 4. *Suppose $h \in C^1([0,S])$ and $g \in C_b^2(\mathbb{R}^+) \cap H^1(\mathbb{R}^+)$. Then the classical solution u of* (1·2) *guaranteed by Lemmas* 1 *and* 2, *corresponding to g and h, satisfies the estimate*

$$\|u\|_{1,2} \leqslant \|g\|_{1,2}\, e^{BS} + AB^{-1}(e^{BS} - 1), \tag{4·1}$$

where A and B are constants depending only on h and S.

Proof. Multiply (1·2) by u and integrate over $0 \leqslant x \leqslant M$, obtaining

$$\int_0^M u u_t\, dx + \int_0^M u u_x\, dx + \int_0^M u^2 u_x\, dx - \int_0^M u u_{xxt}\, dx = 0.$$

Applying the fundamental theorem of the calculus, and integrating the last term by parts, we have

$$\int_0^M (u u_t + u_x u_{xt})\, dx = -\left[\tfrac{1}{2}u^2 + \tfrac{1}{3}u^3 - u u_{xt}\right]_{x=0}^{x=M}. \tag{4·2}$$

The assumptions on g imply that g and g' are asymptotically null (cf. (1), Appendix 2). Since u and u_{xt} are uniformly asymptotically null by Lemma 3, we conclude that the limit as $M \to \infty$ exists uniformly for $0 \leqslant t \leqslant S$, being given by

$$\int_0^\infty (u u_t + u_x u_{xt})\, dx = \tfrac{1}{2}h^2(t) + \tfrac{1}{3}h^3(t) - h(t)\, u_{xt}(0,t). \tag{4·3}$$

Further, if we integrate (4·2) from 0 to t and apply Fubini's theorem, we have

$$\int_0^M (u^2 + u_x^2)\, dx = \int_0^M (g^2 + g'^2)\, dx - 2\int_0^t \left[\tfrac{1}{2}u^2 + \tfrac{1}{3}u^3 - u_{xt} u\right]_0^M d\tau. \tag{4·4}$$

Also, because $g \in H^1(\mathbb{R}^+)$, we are able to infer the existence of the limit, as $M \to \infty$, of the right-side of (4·4) uniformly for $0 \leqslant t \leqslant S$. Thus u is shown to be in $H^1(\mathbb{R}^+)$ for $0 \leqslant t \leqslant S$. If we let

$$E(u) = \int_0^\infty [u^2(x,t) + u_x^2(x,t)]\,dx, \tag{4·5}$$

the uniform convergence of the integral in (4·3) shows that $E(u)$ is differentiable with respect to t, and that

$$dE/dt = \tfrac{1}{2}h^2(t) + \tfrac{1}{3}h^3(t) - h(t)\,u_{xt}(0,t). \tag{4·6}$$

We now estimate the last term in (4·6). Differentiation of the integral equation (3·13) with respect to t gives

$$u_{xt}(x,t) = -h'(t)\,e^{-x} + u + \tfrac{1}{2}u^2 + \int_0^\infty L(x,\xi)\,(u + \tfrac{1}{2}u^2)\,d\xi,$$

where L is defined in (3·14). By Proposition 1,

$$\sup_{x \geqslant 0} |u(x,t)| \leqslant E(u)^{\frac{1}{2}}; \tag{4·7}$$

hence $|u_{xt}(0,t)|$ can be estimated by

$$\left.\begin{aligned} |u_{xt}(0,t)| &\leqslant |h'(t)| + |h(t)| + \tfrac{1}{2}|h(t)|^2 + \int_0^\infty |L(0,\xi)|\,|u + \tfrac{1}{2}u^2|\,d\xi \\ &\leqslant |h'(t)| + |h(t)| + \tfrac{1}{2}|h(t)|^2 + E(u)^{\frac{1}{2}} + \tfrac{1}{2}E(u), \end{aligned}\right\} \tag{4·8}$$

since $\int |L(0,\xi)|\,d\xi = 1$. Furthermore, by (4·7),

$$|h(t)| = |u(0,t)| \leqslant \sup_{x \geqslant 0} |u(x,t)| \leqslant E(u)^{\frac{1}{2}}. \tag{4·9}$$

Combining (4·8) and (4·9) to estimate dE/dt in (4·6) we obtain

$$\left.\begin{aligned} dE/dt &\leqslant \tfrac{3}{2}E(u) + \tfrac{5}{6}|h(t)|\,E(u) + E(u)^{\frac{1}{2}}\,|h'(t)| + E(u) + \tfrac{1}{2}|h(t)|\,E(u) \\ &\leqslant [|h'(t)|]\,E(u)^{\frac{1}{2}} + [\tfrac{5}{2} + \tfrac{4}{3}|h(t)|]\,E(u) \\ &\leqslant 2AE(u)^{\frac{1}{2}} + 2BE(u), \end{aligned}\right\} \tag{4·10}$$

where constants A and B depend only on h and S. From (4·10) we readily deduce that, for $0 \leqslant t \leqslant S$,

$$E(u)^{\frac{1}{2}} \leqslant E(0)^{\frac{1}{2}}\,e^{Bt} + AB^{-1}(e^{Bt} - 1),$$

which implies

$$\|u\|_{1,2} \leqslant \|g\|_{1,2}\,e^{BS} + AB^{-1}(e^{BS} - 1),$$

where A and B depend only on S and h, and g is the initial waveform.

Remark. From physical considerations, one would expect $\|u\|_{1,2}$ to grow roughly linearly with the energy supplied by the wavemaker (forcing function h), and hence that the estimate (4·1) could be improved. Presumably a better result *does* hold for data which lead to solutions satisfying the hypotheses under which (1·1) is derived as a model equation (see (1), § 2). However, we have been unable to improve this estimate for a general class of initial data.

In any case, Proposition 1 implies that $\|u\|_{1,2}$ is a uniform bound for $u(x, t)$ on $x \geqslant 0$, and that the estimate (4·1) is sufficient to allow us to iterate the arguments of Lemmas 1 and 2, so obtaining our existence theorem.

THEOREM 1. *Let the initial waveform g satisfy the conditions* (i) $g \in H^1(\mathbb{R}^+)$, (ii) $g \in C_b^2(\mathbb{R}^+)$ *and let the boundary forcing function h satisfy* (iii) $h \in C^1([0, T])$. *Then the partial differential equation*

$$u_t + u_x + uu_x - u_{xxt} = 0 \tag{4·11}$$

admits a unique solution $u \in \mathscr{C}_T^{1,2} \cap \mathscr{W}_T$ *which satisfies*

$$u(x, 0) = g(x), \quad u(0, t) = h(t), \tag{4·12}$$

for $x \geqslant 0$ *and* $t \geqslant 0$.

Proof. Uniqueness: First suppose that on the interval $[0, T]$ we have two solutions u and v of (4·11), (4·12) in $\mathscr{C}_T^{1,2} \cap \mathscr{W}_T$. Let $w = u - v$. Then we see that

$$w_t + w_x + uw_x + v_x w - w_{xxt} = 0, \tag{4·13}$$

with

$$w(x, 0) = w(0, t) = 0 \tag{4·14}$$

for $x \geqslant 0$, $T \geqslant t \geqslant 0$. Multiply (4·13) by w and integrate with respect to x over $\mathbb{R}^+$. Taking account of (4·14), we derive

$$\int_0^\infty (ww_t + w_x w_{xt})\, dx = \int_0^\infty ww_x(2v - u)\, dx. \tag{4·15}$$

(It should be shown that the integral on the left-hand side of (4·15) converges uniformly for $0 \leqslant t \leqslant T$. This is easy to derive in the present context, and we omit the details.) If we write

$$m(t) = \|2v - u\|_{\mathscr{C}_t} \quad \text{and} \quad E(t) = \|w(x, t)\|_{1,2}^2,$$

(4·15) implies that

$$dE/dt \leqslant m(t)\, E(t).$$

Since $E \geqslant 0$, this means that

$$E(t) \leqslant E(0) \exp\left(\int_0^t m(\tau)\, d\tau\right).$$

Finally, $E(0) = 0$, so $E(t) \equiv 0$, whence $w \equiv 0$, $u = v$. Thus uniqueness of the solution is established.

Existence. By straightforward iteration of the existence proof of Lemma 1, using the *a priori* estimate (4·1), we infer the existence of an increasing sequence of times $\{T_k\}_{k=0}^\infty$ over which the solution in the small can be extended. Here T_k is determined by (3·9) as

$$T_k = T_{k-1} + \frac{1}{2 + 4r(T_k)}, \tag{4·16}$$

where, according to (3·7),

$$\left.\begin{aligned} r(T_k) &= \sup_{x \geqslant 0} |u(x, T_{k-1})| + \|h\|_{0,[0,T_k]} \\ &\leqslant \|u(x, T_{k-1})\|_{1,2} + \|h\|_{0,[0,T_k]} \\ &\leqslant V(T_{k-1}) + W(T_k). \end{aligned}\right\} \tag{4·17}$$

Here $W(T_k) = \|h\|_{C[0,T_k]}$, and V is obtained from the estimate (4·1) as

$$V(t) = \|g\|_{1,2}\,e^{Bt} + AB^{-1}(e^{Bt}-1),$$

where A and B, defined in (4·10), depend only on t and h, and are uniformly bounded on $[0,T]$. Hence V is uniformly bounded on $[0,T]$. Let M be an upper bound for V on $[0,T]$. Then (4·16) and (4·17) give

$$T_k - T_{k-1} \geqslant \frac{1}{2+4(M+W(T))} > 0$$

for all k. Hence u may be extended over the range $\mathbb{R}^+\times[0,T]$ in a finite number of temporal steps by iterating Lemmas 1 and 2. This method of defining u ensures that $u\in\mathscr{C}_T^{1,2}\cap\mathscr{W}_T$, and that u is a classical solution of the initial-boundary-value problem (4·11), (4·12).

5. *Stability.* In this section, we demonstrate that the unique solution u of (1·2), guaranteed by Theorem 1, depends continuously on the specified data. That is, small perturbations of g and h lead to small perturbations of the corresponding solution u. If we let $\mathbf{U}$ denote the mapping that takes data g and h into the corresponding solutions of (1·2), then by Theorem 1 we have

$$\mathbf{U}:\mathscr{X} = [H^1(\mathbb{R}^+)\cap C_b^2(\mathbb{R}^+)]\times C^1([0,T])\to\mathscr{C}_T^{1,2}\cap\mathscr{W}_T. \tag{5·1}$$

Remark. If A and B are Banach spaces, then $A\cap B$ is a Banach space with norm $\|z\| = \|z\|_A + \|z\|_B$ for $z\in A\cap B$.

Hence $X = H^1(\mathbb{R}^+)\cap C_b^2(\mathbb{R}^+)$ is a Banach space, and we mean $X\times C^1([0,T])$ to have the usual product topology. In fact, because X and $C^1([0,T])$ are both Banach spaces, so is the product. Similarly, $\mathscr{G} = \mathscr{C}_T^{1,2}\cap\mathscr{W}_T$ is a Banach space.

It follows immediately that $\mathbf{U}$ is a continuous mapping in (5·1) if and only if $i\circ\mathbf{U}$ and $j\circ\mathbf{U}$ are both continuous where i and j are the natural inclusions of $\mathscr{G}$ into $\mathscr{C}_T^{1,2}$ and $\mathscr{W}_T$ respectively. The precise result is:

Theorem 2. *$\mathbf{U}$ is continuous.*

Proof. First, by the last remark, it is enough to show that

$$\mathbf{U}:\mathscr{X}\to\mathscr{C}_T^{1,2}\quad\text{and}\quad\mathbf{U}:\mathscr{X}\to\mathscr{W}_T \tag{5·2}$$

are both continuous. Next, since $\mathscr{X}$ is a metric space, and $\mathscr{C}_T^{1,2}$ and $\mathscr{W}_T$ are both metric spaces, it suffices to show that $\mathbf{U}$ is sequentially continuous in both cases.

Let $(g_i,h_i)\in\mathscr{X}$ and $u_i = \mathbf{U}(g_i,h_i)$ be the solution corresponding to the initial waveform g_i and the driving condition h_i, $i = 1, 2$. Define $w = u_1-u_2$. Then w satisfies:

$$w_t + w_x + ww_x + (u_2w)_x - w_{xxt} = 0, \tag{5·3}$$
$$w(x,0) = g(x),\quad w(0,t) = h(t),$$

where $g = g_1-g_2$ and $h = h_1-h_2$. Further, we deduce that w satisfies an integral equation analogous to (4·2):

$$w(x,t) = g(x) + e^{-x}(h(t)-h(0)) + \int_0^t\int_0^\infty K(x,\xi)\{w(\xi,\tau)[1+\tfrac{1}{2}(u_1(\xi,\tau)+u_2(\xi,\tau))]\}\,d\xi\,d\tau. \tag{5·4}$$

Note that the formal operations leading to (4·2) are easily justified because of the regularity of u_1 and u_2 guaranteed by Theorem 1. Suppose now that the distance between (g_1, h_1) and (g_2, h_2) in $\mathscr{X}$ is bounded from above by $\epsilon > 0$, which for convenience, we take to be less than 1. In particular, it follows that

$$\left.\begin{aligned} &\text{(i)} && \|g\|_{1,2} \leqslant \epsilon,\\ &\text{(ii)} && \|g\|_{C_b^2(\mathbb{R}^+)} \leqslant \epsilon,\\ &\text{(iii)} && \|h\|_{C^2([0,T])} \leqslant \epsilon.\end{aligned}\right\} \tag{5·5}$$

We show that $\mathbf{U}\colon \mathscr{X} \to \mathscr{W}_T$ is continuous at (g_2, h_2). If we multiply (5·3) by w and integrate the result over $\mathbb{R}^+$, then after integration by parts and using the fact that u_2, w, and w_{xt} are uniformly asymptotically null for $0 \leqslant t \leqslant T$, we have

$$\int_0^\infty (ww_t + w_x w_{xt})\,dx = \tfrac{1}{2}h(t)^2 + \tfrac{1}{3}h(t)^3 + h(t)^2 h_2(t) + \int_0^\infty u_2 w w_x\,dx - h(t)\,w_{xt}(0,t), \tag{5·6}$$

which is the same as

$$\frac{d}{dt}\int_0^\infty (w^2 + w_x^2)\,dx = h(t)^2[1 + \tfrac{2}{3}h(t) + 2h_2(t)] + 2\int_0^\infty u_2 w w_x\,dx - 2h(t)\,w_{xt}(0,t). \tag{5·7}$$

Here we have omitted the straightforward verification that the first integral on the left of (5·6) converges uniformly for $0 \leqslant t \leqslant T$, so that differentiation under the integral is valid.

We must now estimate $w_{xt}(0,t)$. To this end, differentiate (5·4) with respect to x and t, thus

$$w_{xt}(x,t) = -e^{-x}h'(t) + \int_0^\infty L(x,\xi)\,[w(1 + \tfrac{1}{2}u_1 + \tfrac{1}{2}u_2)]\,d\xi + w[1 + \tfrac{1}{2}u_1 + \tfrac{1}{2}u_2],$$

where L is defined in (3·14). Since

$$\int_0^\infty |L(0,\xi)|\,d\xi \leqslant 1,$$

and $u_1 = w + u_2$, we derive that

$$|w_{xt}(0,t)| \leqslant |h'(t)| + |h(t)|\,|1 + \tfrac{1}{2}h(t) + h_2(t)| + (1 + \|u_2\|_{\mathscr{C}_T})\,\|w\|_{\mathscr{C}_T} + \tfrac{1}{2}\|w\|^2_{\mathscr{C}_T}.$$

If we write

$$F(t) = \int_0^\infty (w^2 + w_x^2)\,dx = \|w\|^2_{1,2}, \tag{5·8}$$

and

$$b = \|u_2\|_{\mathscr{C}_T^{1,2}}, \tag{5·9}$$

and apply Proposition 2 to estimate $\|w\|_{\mathscr{C}}$, then a crude bound for $w_{xt}(0,t)$ is

$$|w_{xt}(0,t)| \leqslant |h'(t)| + |h(t)|\,|1 + \tfrac{1}{2}h(t) + h_2(t)| + (1+b)\,F(t)^{\frac{1}{2}} + \tfrac{1}{2}F(t).$$

Finally, using (5·5,(iii)), we obtain

$$|w_{xt}(0,t)| \leqslant N\epsilon + (1+b)\,F(t)^{\frac{1}{2}} + \tfrac{1}{2}F(t),$$

where N depends only on h_2 and T (since $\epsilon \leqslant 1$). Now certainly $|h(t)| = |w(0,t)| \leqslant F(t)^{\frac{1}{2}}$ by Proposition 2. Hence the last result applied to (5·7) establishes

$$dF/dt \leqslant F(t)\,|1+\tfrac{2}{3}h(t)+2h_2(t)| + bF(t) + 2F(t)^{\frac{1}{2}}[N\epsilon + (1+b)F(t)^{\frac{1}{2}}] + |h(t)|\,F(t).$$

Therefore, since $\epsilon \leqslant 1$,

$$\begin{aligned} dF/dt &\leqslant F(t)[1+\tfrac{2}{3}+2|h_2(t)|+b+2(1+b)+1] + F(t)^{\frac{1}{2}}[2N\epsilon] \\ &\leqslant 2[MF(t)+\epsilon NF(t)^{\frac{1}{2}}], \end{aligned} \tag{5·10}$$

where M and N depend only on T and u_2. Since $F \geqslant 0$, (5·10) implies that

$$F(t)^{\frac{1}{2}} \leqslant F(0)^{\frac{1}{2}} e^{Nt} + \epsilon MN^{-1}(e^{Nt}-1).$$

But $F(0)^{\frac{1}{2}} = \|g\|_{1,2} \leqslant \epsilon$ from (5·5, (i)). Thus for $0 \leqslant t \leqslant T$,

$$\|w\|_{1,2} = F(t)^{\frac{1}{2}} \leqslant \epsilon[e^{NT} + MN^{-1}(e^{NT}-1)]. \tag{5·11}$$

Hence,

$$\|u_1 - u_2\|_{\mathscr{W}_T} = \sup_{0\leqslant t\leqslant T} \|w\|_{1,2} \leqslant \epsilon Q, \tag{5·12}$$

where Q depends only on T and u_2. The continuity of $\mathbf{U}: \mathscr{X} \to \mathscr{W}_T$ at u_2 is now clear.

Since, according to Proposition 2 in section 2, $\mathscr{W}_T$ is continuously embedded in $\mathscr{C}_T$, we may also infer that $\mathbf{U}: \mathscr{X} \to \mathscr{C}_T$ is continuous. In particular, under the assumptions (5·5), we infer from (2·9) that

$$\|u_1 - u_2\|_{\mathscr{C}_T} = \|w\|_{\mathscr{C}_T} \leqslant \epsilon Q. \tag{5·13}$$

This latter fact, and the integral equation (5·4) satisfied by w, is now exploited to show finally that $\mathbf{U}: \mathscr{X} \to \mathscr{C}_T^{1,2}$ is continuous.

Again, g and h are supposed to satisfy (5·5) for a given $1 \geqslant \epsilon \geqslant 0$. Consider

$$\|\partial/\partial t[\mathbf{U}(g_1,h_1) - \mathbf{U}(g_2,h_2)]\|_{\mathscr{C}_T} = \|w_t\|_{\mathscr{C}_T} \leqslant \|h'\|_{C[0,T]} + \left\|\int_0^\infty K(x,\xi)\,[w(1+\tfrac{1}{2}w+u_2)]\,d\xi\right\|_{\mathscr{C}_T},$$

which we obtain by differentiating (5·4) with respect to t. Continuing this estimate,

$$\begin{aligned} &\leqslant \|h\|_{C^1[0,T]} + \|w\|_{\mathscr{C}_T}\left\|\int_0^\infty |K(x,\xi)|\,(1+\tfrac{1}{2}|w|+|u_2|)\,d\xi\right\|_{\mathscr{C}_T} \\ &\leqslant \epsilon + Q\epsilon(1+\tfrac{1}{2}Q\epsilon+b)\sup_{x\geqslant 0}\int_0^\infty |K(x,\xi)|\,d\xi \\ &\leqslant \epsilon(1+Q(1+\tfrac{1}{2}Q+b)) = \epsilon M, \end{aligned}$$

where b is as defined in (5·9) and Q as in (5·13), and hence M is a constant depending only on u_2 and T. This inequality assures that $\mathbf{U}: \mathscr{X} \to \mathscr{C}_T^{1,0}$ is continuous. Similarly, by differentiating (5·4) with respect to x we have

$$\begin{aligned} &\|\partial/\partial x[\mathbf{U}(g_1,h_1) - \mathbf{U}(g_2,h_2)]\|_{\mathscr{C}_T} = \|w_x\|_{\mathscr{C}_T} \\ &\qquad \leqslant \|g'\|_{C_b(\mathbb{R}^+)} + \|h\|_{C[0,T]} + \left\|\int_0^t w(x,\tau)\,[1+\tfrac{1}{2}w(x,\tau)+u_2(x,\tau)]\,d\tau\right\|_{\mathscr{C}_T} \\ &\qquad\qquad + \left\|\int_0^t\int_0^\infty L(x,\xi)\,[w(1+\tfrac{1}{2}w+u_2)]\,d\xi\,d\tau\right\|_{\mathscr{C}_T} \\ &\qquad \leqslant \|g\|_{C_b^2(\mathbb{R}^+)} + \|h\|_{C^1[0,T]} + TQ\epsilon(1+\tfrac{1}{2}Q+b) \\ &\qquad\qquad + TQ\epsilon(1+\tfrac{1}{2}Q+b)\sup_{x\geqslant 0}\int_0^\infty |L(x,\xi)|\,d\xi, \end{aligned}$$

where L is given in (3·14) and b and Q are as before. Applying (5·5, (ii), (iii)), we obtain

$$\|w_x\|_{\mathscr{C}_T} \leqslant \epsilon + \epsilon + \epsilon[2TQ(1+\tfrac{1}{2}Q+b)]$$
$$\leqslant \epsilon M,$$

where again M depends only on T and u_2. Thus we know that $\mathbf{U}: \mathscr{X} \to \mathscr{C}_T^{1,1}$ is continuous.

In a like manner, using (5·13) and differentiating (5·4) further we deduce that $\mathbf{U}: \mathscr{X} \to \mathscr{C}_T^{1,2}$ is continuous. The proof of Theorem 2 is now complete.

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Appendix

Here the results of Theorems 1 and 2 are reinterpreted in a way that allows for the unbounded time interval $0 \leqslant t < \infty$. This is done with the aid of some additional function spaces, which are Fréchet rather than Banach spaces. The new interpretation is tidier in that the rather imprecise time parameter T is eliminated.

Let $C^k(\mathbb{R}^+)$ denote the functions defined on $\mathbb{R}^+$ whose derivatives to order k exist and are continuous but not necessarily bounded on $\mathbb{R}^+$. We give $C^k(\mathbb{R}^+)$ the Fréchet space structure induced by the countable collection of pseudo-norms

$$p_n(f) = \sup_{0 \leqslant x \leqslant n} \sup_{0 \leqslant j \leqslant k} |f^{(j)}(x)| \quad (n = 1, 2, \ldots). \tag{B 1}$$

Thus, if $f, g \in C^k(\mathbb{R}^+)$, then the formula

$$d(f,g) = \sum_{l=1}^{\infty} \frac{1}{2^l} \frac{p_l(f-g)}{1+p_l(f-g)}, \tag{B 2}$$

defines a metric on $C^k(\mathbb{R}^+)$. This space can then be demonstrated to be complete and separable with regard to this metric. We refer the reader to Treves ((4), chapter 8) for an account of the details.

We let $\mathscr{C}_\infty^{l,m}$ denote the space of functions $u(x,t)$ defined on $\mathbb{R}^+ \times \mathbb{R}^+$ whose derivatives $\partial_t^i \partial_x^j u$ exist, are continuous, and are, for each fixed t, bounded as functions of x, for $0 \leqslant i \leqslant l, 0 \leqslant j \leqslant m$. That is, $\mathscr{C}_\infty^{l,m}$ consists of functions u on $\mathbb{R}^+ \times \mathbb{R}^+$ that are in $\mathscr{C}_T^{l,m}$ for all $T > 0$ finite. (Note that there is no restriction on the growth in time.) $\mathscr{C}_\infty^{l,m}$ is given the Fréchet structure induced by the pseudo-norms

$$q_n(u) = \sup_{\substack{x \geqslant 0 \\ n \geqslant t \geqslant 0}} \sup_{\substack{l \geqslant i \geqslant 0 \\ m \geqslant j \geqslant 0}} |\partial_t^i \partial_x^j u(x,t)| \quad (n = 1, 2, \ldots). \tag{B 3}$$

Finally, by analogy with $\mathscr{W}_T$, we define $\mathscr{W}_\infty$ to be the functions $u(x,t)$ on $\mathbb{R}^+ \times \mathbb{R}^+$ such that $u \in \mathscr{W}_T$ for all T finite. The defining pseudo-norms for the Fréchet structure on $\mathscr{W}_\infty$ are

$$r_n(u) = \|u\|_{\mathscr{W}_n} \quad (n = 1, 2, \ldots). \tag{B 4}$$

From Proposition 2 in section 2 there follows:

PROPOSITION 3. *$\mathscr{W}_\infty$ is embedded in $\mathscr{C}_\infty$ $(=\mathscr{C}_\infty^{0,0})$. Considering $\mathscr{W}_\infty$ as a subspace of $\mathscr{C}_\infty$, its elements are continuous functions asymptotically null for each t (and uniformly asymptotically null on bounded intervals) and such that*

$$q_n(u) \leqslant r_n(u) \quad (n = 1, 2, \ldots),$$

where q_n and r_n are the defining pseudo-norms for $\mathscr{C}_\infty$ and $\mathscr{W}_\infty$ respectively.

We can now prove the result anticipated above.

THEOREM 3. *Let the initial waveform g lie in $H^1(\mathbb{R}^+) \cap C_b^2(\mathbb{R}^+)$, and the boundary forcing term h lie in $C^1(\mathbb{R}^+)$. Then there is a one-to-one continuous map*

$$\mathbf{U}: \mathscr{X} = H^1(\mathbb{R}^+) \cap C_b^2(\mathbb{R}^+) \times C^1(\mathbb{R}^+) \to \mathscr{W}_\infty \cap \mathscr{C}_\infty^{1,2}$$

which associates with (g, h) the unique solution u of the system (1·2) in $\mathscr{W}_\infty \cap \mathscr{C}_\infty^{1,2}$.

REMARK. $H^1(\mathbb{R}^+) \cap C_b^2(\mathbb{R}^+)$ is again given its Banach space topology, as in Theorem 2, and $C^1(\mathbb{R}^+)$ its Fréchet topology as defined above. Thus $\mathscr{X}$, being the product of two metric spaces, is itself a metric space. $\mathscr{C}_\infty^{1,2} \cap \mathscr{W}_\infty$ is also a metric space, with metric $d = \sigma + \rho$ where σ and ρ are the metrics on $\mathscr{C}_\infty^{1,2}$ and $\mathscr{W}_\infty$ respectively.

$\mathbf{U}$ is therefore a mapping between metric spaces, and thus it suffices to show that $\mathbf{U}$ is sequentially continuous.

Proof. The hypotheses of Theorem 3 imply those of Theorem 1 for any finite $T > 0$. Hence we may infer the existence of solutions $u^T \in \mathscr{W}_T \cap \mathscr{C}_T^{1,2}$ to the system (1·2). We define a function $u(x,t)$ on $\mathbb{R}^+ \times \mathbb{R}^+$ by $u(x,t) = u^t(x,t)$. This is well defined by the uniqueness assertion of Theorem 1, and it follows from this definition that $u \in \mathscr{W}_T \cap \mathscr{C}_T^{1,2}$ for all finite T, whence $u \in \mathscr{W}_\infty \cap \mathscr{C}_\infty^{1,2}$. Clearly u is a solution, on $\mathbb{R}^+ \times \mathbb{R}^+$, of the system (1·2), since it agrees with a solution on any finite time interval. But u is unique in having these properties, since when restricted to $\mathbb{R}^+ \times [0, T]$ it is unique according to Theorem 1 for each $T > 0$.

Finally, we must show that the correspondence $\mathbf{U}: (g,h) \to u$ is continuous from $\mathscr{X}$ into $\mathscr{W}_\infty \cap \mathscr{C}_\infty^{1,2}$. As we remarked above, it is enough to prove sequential continuity.

Since a sequence in $\mathscr{W}_\infty \cap \mathscr{C}_\infty^{1,2}$ converges if and only if its restriction to $\mathbb{R}^+ \times [0, T]$ converges in $\mathscr{W}_T \cap \mathscr{C}_T^{1,2}$, for each $T > 0$, we may conclude the sequential continuity of $\mathbf{U}$ from the continuity of the restrictions $\mathbf{U}_T$ defined by

$$\mathbf{U}_T(g,h) = \mathbf{U}(g,h)|_{\mathbb{R}^+ \times [0,T]} = u^T.$$

The continuity of $\mathbf{U}_T$ being guaranteed by Theorem 2, the proof of Theorem 3 is now complete.

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