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STATISTICALLY WELL-SET CAUCHY PROBLEMS

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I. Introduction

A. Physical Background

This paper is concerned with *random solutions* of initial value problems for partial differential equations, or so-called *Cauchy problems*. This is a new and mathematically formidable subject which is best understood through its analogy to the well-established and simpler (but already quite difficult!) field of classical statistical mechanics.

Classical statistical mechanics is concerned with systems having a *finite* number of degrees of freedom, whose evolution in time is governed by *ordinary* differential equations of a very special kind, associated with holonomic Lagrangian dynamical systems. Since we shall be concerned with problems which involve continuous media but are analogous to those of classical statistical mechanics, our systems will be governed by *partial* differential equations, and will have a countably infinite number of degrees of freedom. We will also consider systems whose evolution is not determined by the laws of mechanics, such as the evolution of temperature in some continuous medium.

More specifically, our paper is intended to provide a rigorous general mathematical foundation for treating simplified mathematical models of physical problems arising naturally in connection with (a) vibrating strings, (b) gravity waves in the ocean, (c) sound in a box (e.g., noise in a room), (d) turbulence in an incompressible fluid and the Burgers–Hopf one-dimensional model for it, (e) the heat conduction equation (which also describes diffusion and shear flows in a viscous liquid), etc.

A careful review of what was then known about many specific problems, with bibliographical references, was given 10 years ago by one of us [34], in a book [56] which also contains several other relevant articles on the statistics of gravity waves (for which cf. also [36]), turbulence, etc.[1] We shall concentrate here on recent developments.

Classical statistical mechanics, which has the kinetic theory of gases as an important application, as formulated by Maxwell (1859, 1866), Boltzmann (1895), and J. W. Gibbs (1902), consists esssentially in taking the following steps[2]:

(Ia) Definition of the phase-space Ω (a $2k$-dimensional variety, if the system has k degrees of freedom), whose points represent the states of the system (position and velocity). Thus for a pendulum, Ω is a 2-dimensional cylinder in $\mathbf{R}^3$; for a system of m molecules (points) in Euclidean space, Ω is $\mathbf{R}^{6m}$; and so on.

(Ib) Proof of an existence, uniqueness and continuity theorem for the solutions of the associated system of ordinary differential equations (Hamilton–Jacobi equations). To each $\omega_0 \in \Omega$ corresponds one and only one solution $\omega_t = T_t(\omega_0)$, which represents the state (motion) for $-\infty < t < +\infty$ of the system whose state at $t = 0$ is represented by ω_0; this solution defines the orbit $\Gamma(\omega_0) = \{\omega_t\}_{-\infty < t < \infty}$. Corresponding to the evolution of the system, the set of solutions defines a steady flow $\omega_t = T_t \omega_0$ in Ω; the set of transformations $\{T_t\}$ has the group property (Huyghens principle) $T_{t+s} = T_t T_s = T_s T_t$.

(Ic) Choice of a probability measure μ. The initial states of the system occur with a specified probability distribution μ, so that, for any Borel set A,

$$\operatorname{prob}[\omega_0 \in A] = \mu(A) \qquad \text{is defined.}$$

[1] See also the article by one of us in A. Blanc-Lapierre and R. Fortet, "Théorie des Fonctions Aléatoires," Masson, 1953, and the articles in Chapters 8 and 9 of M. Rosenblatt (ed.), "Time Series Analysis," Wiley, New York, 1963.

[2] See Khinchine [38] for a more complete discussion.

Owing to the special form of the *DE*, one can choose in classical statistical mechanics a probability measure μ for which the flow in Ω is measure preserving (Liouville theorem):

$$\mu(T_t A) = \mu(A) \qquad \text{for any Borel set} \quad A.$$

The continuous media of classical physics we shall consider have the following properties instead:

(a) Each continuous medium fills a domain (connected open set) D in a Euclidean space $\mathbf{R}^n$ with a boundary ∂D (as a rule $n = 3$, but this is immaterial in most cases); let $\mathbf{x} = (x_1, \ldots, x_n)$ be the general point of D.

(b) The state of the medium is defined by a set of q real-valued functions of $\mathbf{x}$ on D:

$$\mathbf{u}(\mathbf{x}) = [u_1(\mathbf{x}), \ldots, u_q(\mathbf{x})]. \tag{1.1a}$$

(c) During an open interval of time the function from D to $\mathbf{R}^q$

$$\mathbf{u}(\mathbf{x}, t) = [u_1(\mathbf{x}, t), \ldots, u_q(\mathbf{x}, t)], \tag{1.1b}$$

which defines the state of the system at each time t, satisfies the partial differential equations

$$\partial^m u_j/\partial t^m = B_j(u_1, \ldots, u_q) \qquad (j = 1, 2, \ldots, q), \tag{1.2a}$$

possibly subject to constraints such as incompressibility ($\nabla \cdot \mathbf{u} = 0$), of the form

$$A_k(u_1, \ldots, u_q) = 0 \qquad (k = 1, 2, \ldots, r), \tag{1.2b}$$

where A_k and B_j are time-independent differential operators, typically polynomials in the linear partial derivatives $(\partial/\partial x_1), \ldots, (\partial/\partial x_n)$ which do not depend on time t.

Let us clarify these generalities by a few examples.

Example 1. *Vibrating string.* Here the transverse displacement $u(x, t)$ satisfies

$$u_{tt} = u_{xx}, \tag{1.3}$$

with $n = q = 1$ and $m = 2$.

Example 2. Three-dimensional sound waves are governed by the differential equations[3]

[3] Other mathematical formulations are possible; we here adopt that of H. Lamb, "Hydrodynamics," Section 285. Cambridge Univ. Press, London, 1932.

$$\partial^2 p/\partial t^2 = c^2 \nabla^2 p, \qquad \nabla^2 = \sum_{k=1}^{3} \frac{\partial^2}{\partial x_k{}^2}, \tag{1.4a}$$

$$\partial\phi/\partial t = (p - p_0)/\rho_0. \tag{1.4b}$$

Here ϕ is the velocity potential, so that $\mathbf{u} = -\nabla\phi$ is the velocity.

For sound waves in a rectangular box, the functions p, ϕ can be continued analytically by reflection so as to fill all space (all of $\mathbf{R}^3$).

Example 3. For the *heat equation* in a homogeneous medium, we have

$$v_t = \alpha \nabla^2 v. \tag{1.5}$$

We can have $n = 1, 2$, or 3, depending on whether we are dealing with a rod, a slab or plate, or a solid. In the case $n = 1$, the same differential equation (DE) also governs shear viscous flow in a homogeneous incompressible fluid.

The preceding DE's were *linear*. The DE of shear viscous flow is also a special case of the following example.

Example 4. The *Navier–Stokes* equations for incompressible viscous flow are

$$\frac{\partial u_j}{\partial t} = -\sum_k u_k \frac{\partial u_j}{\partial x_k} + \nu \nabla^2 u_j - \frac{1}{\rho}\frac{\partial p}{\partial x_j} \tag{1.6a}$$

$$\sum \frac{\partial u_k}{\partial x_k} = \nabla \cdot \mathbf{u} = 0. \tag{1.6b}$$

Formally, one can always reduce normal systems of DE's such as those above to systems which are *first-order in time* (i.e., for which $m = 1$). For instance the vibrating string equation of Example 1 can be reduced to

$$\partial u/\partial t = v, \qquad \partial v/\partial t = \partial^2 u/\partial x^2. \tag{1.7}$$

Here the auxiliary function $v(x, t)$ represents the transverse velocity of the point x of the string at time t; $m = n = 1$ and $q = 2$.

In order to reduce the mathematical problem to its simplest form, we will always suppose that the domain has no boundary (is the whole of $\mathbf{R}^n$). Thus the solutions will depend only on the initial conditions $\mathbf{v}(\mathbf{x})$. Hopefully, our conclusions will be applicable to physical problems in which the boundaries have little effect on the evolution; homogeneous turbulence in a wind tunnel is an important example of these circumstances.

These examples suggest that any discussion of the statistical mechanics of continuous media should include the following steps, analogous to those of Ia–Ic above:

(a) specifying a class of "admissible vector fields" on a physical domain which describe the possible "states" of the medium;

(a′) constructing a function space $\mathscr{U}$ whose points are these states, with an appropriate topology (i.e., notion of continuity);

(b) specifying a system of partial DE's governing (in the mathematical model) the evolution of the medium in time;

(b′) proof of existence, uniqueness, and continuity theorems for this system of partial DE's, thereby realizing a flow which represents the evolution of the system [26];

(c) specification of a probability measure μ on $\mathscr{U}$;

(c′) for *conservative* systems such as those of Examples 1–3, one has a *group* and can hope to find a probability measure *invariant* under the action of the group $\{T_t\}$ defining the flow of (*b*′). In such cases one can hope to have a full analogy to classical statistical mechanics, with such deeper properties as maximum entropy, ergodic theory, etc. However, one cannot hope to have an invariant probability measure for *dissipative* systems such as those associated with the heat equation, viscous flow without forcing terms, homogeneous turbulence, etc.

Because they are exceptional, we shall defer our review of these deeper aspects of classical statistical mechanics until Section V.F below. We make here only the general observation that, whereas conservative systems in continuum physics typically correspond to hyperbolic systems of partial DE's and are still associated with *groups* of transformations T_t on $\mathscr{U}$, whose domain is $-\infty < t < \infty$, dissipative systems usually correspond to *parabolic* systems of DE's which give rise only to semigroups $\{T_t\}$ defined for $t \geq 0$.

We emphasize that, to avoid becoming overwhelmed by technical difficulties, we will *not* deal with many fascinating problems. Thus random motions can be produced by random forces (e.g., a vibrating string in a liquid submitted to random impacts), but we will not consider DE's with random coefficients. Furthermore, as already said, we will not consider the effects of boundary conditions, but will stick to *pure* Cauchy problems: as in [34], we shall consider only randomness introduced by the choice of the initial conditions. Again as in [34], our existence, uniqueness, and continuity theorems refer exclusively to linear partial DE's; indeed, the theory of nonlinear partial DE's (and of nonlinear semigroups) is very fragmentary and itself very special. Also, many of our deeper results (e.g., those involving metric transitivity in Section V.F) are valid only for *normal* probability distributions.

Finally, we shall consider almost exclusively problems having *spatial*

homogeneity; it will be recalled that classical statistical mechanics suffers from a similar limitation. Mathematically, this will limit us to linear partial DE's with *constant coefficients*, a limitation that has the great advantage of enabling us to use the techniques of Fourier analysis (classical harmonic analysis).

B. Mathematical Background

Our paper is devoted to the analysis of mathematical models for physical systems such as those we have described. The evolution of such a system having been defined by DE's (1.2a)–(1.2b), the first step usually consists in choosing a function space in which the passage of time corresponds to a continuous flow. That is, one usually tries to establish existence, uniqueness, and continuity theorems; when this has been done, one says that the Cauchy problem is *well set* (in the function space selected).

We introduce a new point of view by relaxing the usual condition of *sure continuity*. The existence and uniqueness property of the solutions still being true, it may happen that continuous dependence on the initial values fails. Thus it may happen that for initial values $\mathbf{u}_0(\mathbf{x})$ very near to $\mathbf{v}_0(\mathbf{x})$, the value of $\mathbf{u}(\mathbf{x}, t)$ may be very far from $\mathbf{v}(\mathbf{x}, t)$ with a small probability. We shall call a Cauchy problem *statistically well set* in a given function space when, for the probability measure μ appropriate to the problem considered, orbits initiating in a small enough neighborhood V of $\mathbf{v}_0(\mathbf{x})$ are, at the time t, with *probability very near one* in any given neighborhood of $\mathbf{v}(\mathbf{x}, t)$. We shall see in Section III that this gives a good deal more freedom in the choice of a function space.

There has been an abundant literature concerned with the selection of function spaces in which Cauchy problems are well set. Our main purpose is more to clarify and to introduce unifying principles than to add new ideas in this field.

In broad outline, the literature on function spaces has passed through four stages.

In the *classical* stage [24], mathematicians used the "space" of all functions $\mathbf{u}(\mathbf{x})$ continuous and having all the derivatives explicitly contained in the DE, while "continuity" meant the pointwise convergence of all functions and derivatives referred to.

Next there developed in the 1930's an emphasis on the *Hilbert space* $L_2(\mathbf{R}^n)$ of all Lebesgue square-integrable functions, with convergence signifying mean square convergence. The appropriateness of this space for many purposes had become apparent earlier from work of D. Hilbert, E. Schmidt, and F. Riesz.[4] But it was its successful technical application by von Neumann

[4] See the classical textbook, R. Courant and D. Hilbert, "Methoden der Mathematische Physik," Springer, New York, 1924.

and Stone[5] to the Schroedinger equation, which generates a unitary group in $L_2(\mathbf{R}^3)$, which led to its widespread adoption. Hilbert space has the special attraction that it provides the most perfect infinite-dimensional analogue of Euclidean space; moreover it corresponds in many cases to problems in which the *total energy* of the system is finite, especially if one supplements $L_2(\mathbf{R}^n)$ by the Sobolev spaces $W_s(\mathbf{R}^n)$, which are also Hilbert spaces but whose functions have square-integrable derivatives.

Third came Banach spaces, in which distance has all the usual properties except that of satisfying the Pythagorean theorem (orthogonality is missing). It was primarily in this setting that Hille, Yosida, and Phillips developed their abstract theory of C_0-semigroups in the years 1945–1955 [26, 27]. Their theory was based on a highly technical use of resolvents and Laplace transforms ([27, Part 3], [55, Chapter IX]); a more direct method for constructing solutions of deterministically well-set Cauchy problems (1.2a)–(1.2b) in Banach spaces was developed a decade later by one of us [7].

Some parts of the theory of C_0-semigroups apply more generally to *Fréchet* spaces, and physical problems involving infinite total energy (e.g., turbulence in $\mathbf{R}^n$) have physically natural formulations in particular Fréchet spaces Γ and Λ_p $(1 \le p \le +\infty)$ introduced in [8]. These consist of all functions which are continuous, respectively, whose pth powers are Lebesgue integrable in compact sets. These are *Polish* spaces (metrizable, separable, and complete); moreover their topologies are defined by a countable family of seminorms; they therefore have relatively nice topological properties.

Last came spaces of (Laurent) Schwartz *distributions*, which have the nice property of being closed under differentiation. Since 1955, there has been a general tendency to shift the emphasis in the theory of partial DE's to generalized solutions given by distributions. In the works of Friedman [20], Gel'fand and Shilov [21], L. Schwartz, Schultz [47], and Hörmander [29], the spaces D' of distributions and S' of tempered distributions have become the natural frame of the theory of DE's. Thus, one of our aims is to show, using the results of one of us [10], how one can develop a unified *statistical* theory of well-set Cauchy problems in the space S' of tempered distributions. Unlike the spaces Γ and Λ_p, S' is not a Polish space.

Not only continuity theorems but also statistical properties of random solutions depend on the topological structure of the function space $\mathscr{U}$ used to represent a given problem. Thus, in order for the *spectrum* of a random vector field to uniquely determine its probability measure, one must require that this measure be *regular* in a suitable sense. After a careful review (given in Appendix B and summarized in Section II) of the different definitions of regularity proposed by Carathéodory [14], Halmos [25], Gnedenko and Kolmogoroff [23], Blackwell [9], and Bourbaki [13], we adopted the following one already given in [8]: "A measure μ on a topological space E is regular when it is defined on all Borel sets of E, and is the Lebesgue completion of its restriction to these Borel sets."

Thus great attention must be paid to the Borel structure of the function spaces; using the basic ideas of Mackey [40] and the results of Parthasarathy [43] on Borel structures in Polish spaces, our main result is as follows:

The spaces Λ_p, Γ, S', and D' all have standard Borel structures (a countably generated Borel space is called standard if it has the same Borel structure as the real line). In each of the topological linear spaces Λ_p, Γ, S' the Borel sets are just the Borel sets of D' which happen to be contained in these subspaces; moreover they are the same in the weak and in the strong topologies.

This result gives a unifying background for the development of the theory of random solutions of the (well-set) Cauchy problems we consider.

[5] J. von Neumann, "Grundlagen der Quantenmechanik," Springer, New York, 1932; M. H. Stone, "Linear Transformations in Hilbert Space," Amer. Math. Soc., Providence, Rhode Island, 1933.

Fourier transform

In order to give a coherent discussion of the *spectrum* of random solutions of a Cauchy problem, one not only needs to restrict attention to suitably "regular" measures, one also needs a decent Fourier transform theory. This incidentally also gives increased depth to the concepts of autocorrelation and spectrum.

It is therefore fortunate that such a theory exists in the spaces S', D' as well as in the spaces $L^2(\mathbf{K}^r)$ of square-integrable functions on tori (and other locally compact Abelian groups). Unfortunately, the theory is restricted to linear partial DE's with constant coefficients in domains without boundaries: in physical language, to problems having *spatial homogeneity*.

The Fourier transform also gives valuable clues concerning which Cauchy problems should be considered as well set. Thus, reducing any given linear partial DE or system to the standard first-order form

$$\partial u_j/\partial t = \sum p_{jl}(D_1, \ldots, D_n)u_l \qquad (1 \le j, l \le q), \tag{1.8}$$

we see that the Fourier component $c_l(\mathbf{k}, t)e^{i\mathbf{k}\cdot\mathbf{x}}$ with wave-vector $\mathbf{k}$ should evolve according to the system

$$dc/dt = Pc, \qquad P = P(\mathbf{k}) = \|p_{jl}(ik_1, \ldots, ik_n)\|. \tag{1.9}$$

Hence, if $\lambda_1, \ldots, \lambda_q$ are the eigenvalues of $P(\mathbf{k})$, the rate of growth of the Fourier component with wave-vector $\mathbf{k}$ should be $\max \operatorname{Re}\{\lambda_l(\mathbf{k})\}$. Hence a natural condition for the system (1.9) to be deterministically well set is that

$$\Lambda = \sup_{\mathbf{k},\, l} \operatorname{Re}\{\lambda_l(\mathbf{k})\} < +\infty; \tag{1.10}$$

this is the Hadamard–Petrowsky condition for the Cauchy problem (1.8) to be well set.

Physical considerations

The reasons for preferring one function space $\mathscr{U}$ or probability measure μ to another need not be strictly mathematical. For example, consider again the diffusion equation $u_t = u_{xx}$ (Example 3 in one space dimension). Here $u(x, t)$ can be interpreted as the absolute temperature at time t of the point x on an infinite rod, which forces one to set $u(x, t) \geq 0$. Or, alternatively, $u(x, t)$ can also be interpreted as the velocity of the sheet x in a shear flow, in which case the condition of positivity has physically no sense; the natural assumption is that the energy of the portion of the fluid contained in a finite box is finite:

$$\int_a^b u(x, t)^2 \, dx < +\infty \qquad \text{for any} \quad a, b \in \mathbf{R}.$$

Physical considerations may also dictate the choice of function space. Thus, for Cauchy problems (1.8) in which the natural physical condition is finite total energy and having finite stability index Λ, one can construct a Hilbert space $\mathscr{U}$ relative to which the problem is deterministically well set, but the norm in $\mathscr{U}$ depends on the energy functional [7].

It is therefore remarkable that *all* Cauchy problems (1.8) having finite stability index should be well set in the space S' of tempered distributions (whereas $u_t = u_{xx}$ is not well set in D', for example). This was essentially proved by Gel'fand and Shilov [21] and Friedman [20]; Schultz [47] has given this basic result an especially simple formulation and proof.

C. Summary

The main body of this paper is quite technical in tone; to make it more readable, we give here an informal summary of its contents. It is divided into Sections II–VII and Appendixes A and B, most of which also have individual introductions.

Section II and Appendixes A to D are preparatory; they deal with *regular probability measures* on a number of particular *function spaces* which seem to provide especially appropriate settings for the consideration of Cauchy problems from a statistical standpoint. These spaces are locally convex, sequentially complete topological linear spaces. In such spaces, a probability measure is called *regular* when it is definable (up to sets of probability zero) from the probabilities of being in specified open (or closed) sets by repeated use of "not" and countable "and", "or" combinations. *Borel sets* are just sets which can be constructed from open (or closed) sets by such combinations. It is shown in Section II and Appendixes A and B that all the function spaces of interest for this paper have a *standard Borel structure*, just like that of the Borel sets of the real continuum. This means that, even though these function spaces may be quite sophisticated, regular probability measures on them have most of the nice properties of ordinary measure on [0, 1].

Section III is concerned with the question of when an initial value or Cauchy problem for a linear partial differential equation (DE) or system with constant coefficients should be considered as *well set*. We first review the classical *deterministic* viewpoint of Hadamard: that a Cauchy problem is well set when solutions for $t > 0$ *exist* and are *unique* for any reasonable initial data, and depend *continuously* on those data. Results of Hadamard, Petrowsky, one of us [7], and Martin Schultz [47] are then recalled which give a satisfactory criterion for this. Namely, for any system of DE's of the form

$$\partial u_j/\partial t = \sum_{l=1}^{q} p_{jl}(D_1, \ldots, D_n)u_l \qquad (j = 1, \ldots, q). \tag{1.11}$$

and wave-vector $\mathbf{k}$, the Cauchy problem for the initial value $\mathbf{u}(\mathbf{x}, 0) = e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{f}$ is easily seen to have the solution

$$\mathbf{u}(\mathbf{x}, t) = \exp[tP(i\mathbf{k})]\mathbf{f}e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{1.12}$$

The criterion is that the spectral radii of the matrices $\exp[tP(i\mathbf{k})]$ be uniformly bounded [see (3.6)]: that the stability index $\Lambda = \sup\{\mathrm{Re}\, \lambda_j(P(i\mathbf{k}))\}$ be finite. Systems (1.11) with this property are called "regular."

Next, we adopt the point of view of classical statistical mechanics already described in Section I.A: we assume that a Cauchy problem is deterministically well set, but that its initial conditions are chosen at random. The properties of the solutions are then only known statistically. This point of view has been developed by one of us for many years (see [34–36]); the resulting theory is greatly extended and generalized here.

Then a new *probabilistic* viewpoint is adopted: that one can obtain approximate values of the solutions for $t > 0$ with arbitrarily high *probability*, by approximating the initial data sufficiently closely. This is shown to have a major technical advantage for Cauchy problems with "random" (i.e., not precisely known) initial data: it is less sensitive to the selection of the function space E chosen to represent the problem. It also seems to offer a new philosophical perspective, which could have interesting implications for physics. Cauchy problems with this property are called *statistically* well set. Every deterministically well set problem is statistically well set; Appendix C presents a simple example of a Cauchy problem with random initial values which is statistically well set without being deterministically well set.

Section IV calls attention to various particular classes of Cauchy problems with random initial data having special properties, in which one can either give simpler proofs, prove stronger results, or use weaker hypotheses. These are the cases of a compact (i.e., toroidal) domain, of a hyperbolic DE or system, and of normal (i.e., Gaussian) and/or time-independent probability measures. In the first case, one can avoid the technical theory of distributions altogether and work in Hilbert space; in the second, one can use ordinary distributions and avoid tempered distributions. Hence one can hope to avoid, in these two cases, the restriction to linear DE's with constant coefficients, i.e., the assumption of a "translation group" structure on the underlying space X, which limits the applicability of our results.

The assumptions that the probability measure μ is normal and/or time independent are fulfilled in various applications. Thus the classical theory of holonomic systems in statistical equilibrium depends essentially on the existence of a time-invariant measure (Liouville's theorem). Analogous theories can only hold for continuous media when their governing equations permit such a measure. This is the case for the wave equation, but not for the diffusion (heat) equation, which is dissipative. We hope that, by considering the implications of the above assumptions and those of compactness of X and the hyperbolicity of $P(D)$, readers will develop perspective on and appreciation for the technical problems solved in the rest of our paper.

In Section V, we study properties associated with the correlation and spectrum of a homogeneous random vector field (HRVF)—i.e., one whose probability distribution is invariant under translations $\mathbf{x} \mapsto \mathbf{x} + \mathbf{c}$ of the underlying space X. Here the first four sections (Sections V.B–E) recall the known [8] general properties of these quantities in spaces of ordinary HRVF, and extend them to homogeneous random tempered distributions (HRTD) using a result from Appendix D. Then some important special properties of the *metrically transitive* case[6] are derived. Finally, in Sections V.F–G, sufficient conditions (on the smallness of spectral energy at large wave numbers) are derived for generalized ("weak") solutions of a Cauchy problem to be classical ("smooth") solutions with probability one.

Section VI determines the evolution in time of the spectral matrix measure ν_t of admissible normal, homogeneous RVF under the action of a (partial) differential operator associated with a regular Cauchy problem. This is done in the context of tempered distributions. That is, we consider the RVF as a HRTD in $S'(X)$. We show (Theorem 6.2) that ν_t evolves through the action of the differential equation

$$\partial \nu / \partial t = P(i\mathbf{k})\nu + \nu P(i\mathbf{k})^H, \tag{1.13}$$

where $P(i\mathbf{k})$ is the matrix $\|P_{jl}(ik_1, \ldots, ik_n)\|$ of (1.12), and $A^H = A^{*T}$ denotes the Hermite conjugate (transpose complex conjugate) of a matrix A.

[6] It is generally believed that homogeneous turbulence is metrically transitive; see Batchelor [2, Chapter 2].

This result was conjectured by one of us (Birkhoff) several years ago; it was first proved in Bona's thesis [10], and we present his proof. We regard the precise formulation and rigorous proof of this result as the high point of our paper: it gives a unified theoretical treatment (in the space of tempered distributions) of the evolution in time of normal homogeneous RVF, for all linear Cauchy problems with constant coefficients which can reasonably be considered as well set. When the spectrum is continuous, the complete statistical behavior is then determined because this implies metric transitivity.

Finally, in Section VII, we apply *classical* methods to parabolic problems. We show how the existence of a highly smoothing Green function for $t > 0$ gives sharp results about the special properties of solutions of the Cauchy problem in this case, which do not follow directly from the general theory of Section VI. We have included this section largely to illustrate the need for supplementing the general methods of modern functional analysis by classical arguments, when "best possible" results are wanted.

II. Probability in Function Spaces

A. Fréchet Spaces; the Function Space Λ_p

We begin by presenting some ideas and facts from modern functional analysis, which will enable us to carry out steps (a) and (a′) of the program outlined in Section I.A. As was explained in Section I.B (see also Section III.A), the basic problem is that of choosing the right *function space* $\mathscr{U}$ whose elements are a suitable class of vector fields, suitably topologized. Each vector field in $\mathscr{U}$ is considered to specify a possible "state" of the system under consideration.

In this and the next section, we shall describe function spaces which seem to us appropriate for treating "pure" Cauchy problems in domains without boundaries. Specifically, we shall consider throughout only domains $X = \mathbf{K}^s\mathbf{R}^{n-s}$ which are *products* of s copies of the unit circle $\mathbf{K}$ and $n - s$ copies of the real line $\mathbf{R}$. (It is known that any connected locally Euclidean Abelian group manifold can be presented in this way.)

We shall also restrict attention to a few sequentially complete, locally convex topological linear function spaces with separable duals. Indeed, our primary concern will be with the spaces Γ and Λ_p taken as basic in [8], and certain spaces S' and D' of distributions to be defined in Section II.B. We shall pay special attention to Λ_2 and S'. Although we shall define these spaces below, for convenient reference, we will seldom use their definitions. Instead, we shall be concerned with *general* properties of topological linear spaces defined by pseudonorms; we shall review these in Appendix A.

In general, a topological linear space (TLS) is a linear space over the real or complex numbers with a topology in which the operations of addition and scalar multiplication are continuous. A TLS is said to be *locally convex* if it has a basis for neighborhoods of zero consisting of convex sets. The condition of locally convexity insures that the space has a nontrivial collection of continuous linear functionals [55, p. 107].

The particular space Λ_2, which was defined in [8] and will be redefined in this section, arises naturally in many physical situations since the elements of Λ_2 are typically those which have finite energy on bounded subsets of X. The space S' of q-vector-valued tempered distributions is technically convenient, since systems (1.11) can be handled there in a uniform and satisfactory way (see Section III.B). Both spaces are subspaces of D', the space of all q-vector-valued Schwartz distributions on X. In D', the intersection $\Lambda_p \cap S'$ is important since it enjoys many properties of both the spaces Λ_p and S'.

Definition 2.1. We let $\Lambda_p = \Lambda_p(X, q)$ be the space of all real (or complex if we allow complex values) q-vector fields defined on X such that $|\mathbf{u}(\mathbf{x})|^p$ is absolutely Lebesgue integrable on compact subsets of X. [This space, with $p = 2$, was used extensively in [8], where Λ_2 was simply denoted Λ and only the case $X = \mathbf{R}^n$ was considered. Evidently, when $X = \mathbf{K}^s$ is a (compact) torus, $\Lambda_p(X, q)$ is just the familiar Banach space $L_p(X, q)$.]

The topology on Λ_p is most easily defined by pseudonorms. A *pseudo*norm p on a linear space F is a mapping $p\colon F \to [0, \infty)$ such that if $x, y \in F$, then

$$\begin{aligned} p(0) &= 0 \\ p(\beta x) &= |\beta| p(x) \qquad \text{if} \quad \beta \text{ is a scalar} \\ p(x+y) &\le p(x) + p(y). \end{aligned} \tag{2.1}$$

A pseudonorm is a norm if $p(x) = 0$ implies $x = 0$ in F.

One way that a linear space F can be made locally convex is by specifying a collection of pseudonorms which define convergence in F. Thus suppose that $\{p_\alpha\}$, $\alpha \in A$, is a collection of pseudonorms on F. Then it is easy to see [55, p. 24] that sets of the form

$$\{x \in F \mid p_{\alpha_i}(x) \le \varepsilon_i,\ 1 \le i \le n\} \tag{2.2}$$

can be taken as a basis for neighborhoods of zero in F. The topology obtained by taking as a basis for neighborhoods of an arbitrary point y the translations of the basis at zero by y is called the *topology induced by the pseudonorms* $\{p_\alpha\}$. With this topology, F is a locally convex TLS [55, p. 26]. When the set of pseudonorms is countable, F is metrizable with metric

$$\rho(x, y) = \sum_{\alpha=1}^{\infty} 2^{-\alpha} \min[p_\alpha(x-y), 1]. \tag{2.2'}$$

A locally convex TLS whose topology is metrizable, and which is sequentially complete in this metric, is called a Fréchet space. The spaces $\Lambda_p(X, q)$ are all Fréchet spaces. We refer the reader to [30, 52, 55] and to [1] for expositions of the theory of a locally convex TLS and Fréchet spaces.

The topology on Λ_p is induced by the countable family of pseudonorms

$$\|\mathbf{u}\|_n = \left\{\int_{K_n} \sum |u_j(\mathbf{x})|^p dm(\mathbf{x})\right\}^{1/p} \qquad (n = 1, 2, \dots), \tag{2.3}$$

where $K_n = \{\mathbf{x} \in X \mid |x_j| \le n, j = s+1, \dots, n\}$, and m is Lebesgue measure on X. The topology induced by these pseudonorms turns Λ_p into a Fréchet space. We can define $\Lambda_\infty = \Lambda_\infty(X, q)$ similarly, as the space of all (real or) complex q-vector functions on X which are essentially bounded on every compact subset of X (with respect to Lebesgue measure on X). The topology on Λ_∞ is defined by the countable family of pseudonorms

$$\|\mathbf{u}\|_n = \operatorname*{ess\,sup}_{\mathbf{x} \in K_n} \left\{\sum_{j=1}^{q} |u_j(\mathbf{x})|\right\} \qquad (n = 1, 2, \dots). \tag{2.4}$$

Λ_∞ is also a Fréchet space.

If we let $L_p(K, q)$ be the Banach space of all q-vector-valued functions $\mathbf{u}$ defined on a compact set K with values in ($\mathbf{R}^q$ or) $\mathbf{C}^q$ for which each $|u_j|^p$ is integrable (on K) (or, if $p = \infty$, for which u_j is essentially bounded on K), then the topology on Λ_p defined above coincides with the projective limit topology on Λ_p induced by the spaces $L_p(K, q)$ with respect to the filter of all compact sets $K \subset X$ and the restriction maps $\rho_{K\tilde{K}}\colon L_p(K, q) \to L_p(\tilde{K} q)$ for $K \subset \tilde{K}$ given by restricting $f \in L_p(K, q)$ to $\tilde{K}$.

See [52, p. 514] for the definitions of these topologies.

Definition 2.2. Letting X and q be as above, we define $\Gamma = \Gamma(X, q)$ to be the space of all continuous q-vector fields on X.

Here the topology is defined by the pseudonorms

$$\|\mathbf{u}\|_n = \sup_{\mathbf{x} \in K_n} \{|\mathbf{u}(\mathbf{x})|\} \qquad (n = 1, 2, \dots).$$

This topology, the topology of uniform convergence on compact subsets of X, turns Γ into a Fréchet space. Again, the topology coincides with that obtained by making Γ the projective (or inductive) limit of the spaces $C(K)$ with respect to the filter of all compact sets $K \subset X$. Clearly, Γ is a Borel subspace of $\Lambda_p \subset \Lambda_{p'}$, when $p \ge p'$.

For applications to the theory of homogeneous turbulence of an incompressible fluid, which was the primary concern of [8], Λ_2 is the most natural space since it contains precisely those velocity fields which have finite kinetic energy on compact sets. The Fréchet spaces Γ were also considered in [8] because the classic notion of a Khinchine window set applies most naturally to spaces of continuous functions.

B. Spaces of Distributions

In treating general Cauchy problems, it is convenient to consider also spaces of distributions, since a powerful theory of well-set Cauchy problems in such spaces is available [20, 21, 29, 47, 55].

Definition 2.3. Letting $X = \mathbf{R}^n$, we denote by $\mathscr{S}(X)$ the space of all complex-valued infinitely differentiable functions f defined on X such that f and all its derivatives tend to zero at infinity faster than any power of $1/|x|$, where $|x|^2 = x_1^2 + \cdots + x_n^2$.

The space $\mathscr{S}(X)$ is normed as follows (see [30, 52]). Let m and k be nonnegative integers and $f \in \mathscr{S}(X)$. Then we define the collection of pseudonorms.

$$\|f\|_{m,k} = \sup_{|\alpha| \le m} \sup_{x \in X} (1 + |x|)^k |D^\alpha f(x)|, \tag{2.5}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index of nonnegative integers, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and

$$D^\alpha f = \partial^{|\alpha|} f / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}.$$

The topology induced on $\mathscr{S}(X)$ by these pseudonorms is the locally convex translation invariant topology defined by the system of neighborhoods of zero given in (2.2). In this topology, $\mathscr{S}(X)$ is a Fréchet space [52, p. 94].

Definition 2.4. We will denote by $\mathscr{S}'(X)$ the dual space [52, p. 35] of all continuous linear functionals from $\mathscr{S}(X)$ to the complex numbers. The elements of $\mathscr{S}'(X)$ are called *tempered distributions.*

Tempered distributions have a simple (global) representation in terms of the concepts of classical mathematical physics. Namely, any tempered distribution can be interpreted as a superposition of Borel distributions of charges, dipoles, quadrupoles, . . . , 2^n-poles, each of whose densities has at most a polynomial order of growth. In mathematical language [52, p. 272], *any tempered distribution is a finite sum of (distributional) derivatives of continuous functions, each growing at infinity more slowly than some polynomial.* We have not used this result below, but think it could be applied with profit.

We distinguish two topologies on $\mathscr{S}'(X)$; the weak and the strong topology [52, Chapter 19]. A basis for neighborhoods of zero for the weak topology are the sets

$$\{T \in \mathscr{S}'(X) \mid |T(\phi_j)| \le 1, 1 \le j \le r\}, \tag{2.6}$$

where r is a positive integer and $\phi_1, \ldots, \phi_r$ are elements of $\mathscr{S}(X)$. Actually, this topology is generally called the weak-star topology; but since $\mathscr{S}(X)$ is *reflexive* [52, p. 376], which means that the second dual of $\mathscr{S}(X)$ [the dual of $\mathscr{S}'(X)$] is naturally isomorphic to $\mathscr{S}(X)$, the weak-star topology coincides with the weak topology on $\mathscr{S}'(X)$ induced by $\mathscr{S}(X)$ as above.

A basis for neighborhoods of zero in the strong topology are the sets

$$\left\{T \in \mathscr{S}'(X) \mid \sup_{A \in \phi} |T(\phi)| \le 1\right\}, \tag{2.7}$$

where A is allowed to be any bounded subset of $\mathscr{S}(X)$. The latter topology is the usual strong topology put on the dual of a locally convex topological linear space. It is not a metrizable topology in this case.

We shall consider more generally the space of vector-valued tempered distributions. We let $S(X) = [\mathscr{S}(X)]^q$ be the product of q copies of $\mathscr{S}(X)$, and endow $S(X)$ with the product topology. Then $S(X)$ is a Fréchet space, called the space of infinitely differentiable vector fields rapidly decreasing at infinity.

The space of *vector-valued tempered distributions* is denoted by $S'(X)$ and is equal to $[\mathscr{S}'(X)]^q$, the product of q copies of $\mathscr{S}'(X)$. We give $S'(X)$ the product topologies induced by the weak and the strong topology on $\mathscr{S}'(X)$. These are called the weak and the strong topologies on $S'(X)$, respectively.

It is useful to have alternative characterizations of $S'(X)$. One such characterization is obtained by thinking of $S'(X)$ as the collection of all continuous linear maps from $\mathscr{S}(X)$ to q-dimensional complex space $\mathbf{C}^q$. When we think of $S'(X)$ in this way, if $T \in S'(X)$ and $\phi \in \mathscr{S}(X)$, then $T(\phi)$ is a complex q-vector, and if $T = (T_1, \ldots, T_q)$, where $T_i \in \mathscr{S}'(X)$, then the ith component of $T(\phi)$ is $T_i(\phi)$. That is,

$$T(\phi) = (T_1(\phi), \ldots, T_q(\phi)). \tag{2.8}$$

Finally, $S'(X)$ can also be defined as the dual space of $S(X)$; the continuous linear functions on $S(X)$. When we think of $S'(X)$ in this way, we will denote the value of the functional $T \in S'(X)$ at $f \in S(X)$ by $\langle T \cdot f \rangle$. If $T = (T_1, \ldots, T_q)$ and $f = (f_1, \ldots, f_q)$, then

$$\langle T \cdot f \rangle = \sum_{j=1}^{q} T_j(f_j). \tag{2.9}$$

The spaces $\mathscr{S}(X)$, $\mathscr{S}'(X)$, $S(X)$, and $S'(X)$ are all amenable to Fourier analysis. When $X = \mathbf{R}^n$, the Fourier transform of an element $\phi(x) \in \mathscr{S}(\mathbf{R}^n)$ is defined by

$$\hat{\phi}(\xi) = \mathscr{F}\phi(\xi) = \int_{\mathbf{R}^n} e^{-i\mathbf{x}\cdot\xi}\,\phi(\mathbf{x})\,dm(\mathbf{x}),$$

where $\mathbf{x}\cdot\xi = \sum x_i\xi_i$ is the usual inner product on $\mathbf{R}^n$. It is known [1, p. 71] that the mapping $\mathscr{F}: \mathscr{S}(\mathbf{R}^n) \to \mathscr{S}(\mathbf{R}^n)$ above is an isomorphism of the topological linear space $\mathscr{S}(\mathbf{R}^n)$ in (2.5) whose inverse is the mapping

$$\overline{\mathscr{F}}\psi(\mathbf{x}) = (2\pi)^{-n}\int_{\mathbf{R}^n} e^{i\xi\cdot\mathbf{x}}\,\psi(\xi)\,dm(\xi).$$

This isomorphism extends to a unique continuous automorphism of $\mathscr{S}'(\mathbf{R}^n)$ by transposition; the extended mappings are denoted $\mathscr{F}$ and $\overline{\mathscr{F}}$ again for simplicity.

For the vector case of $S(\mathbf{R}^n)$ and $S'(\mathbf{R}^n)$, we define $\mathscr{F}$ and $\overline{\mathscr{F}}$ componentwise and the same results hold. In particular, all the usual formulas for the transform of derivatives, etc., hold. For details, the reader may consult Arsac [1], Friedman [20], Horvath [30], Treves [52], or Yosida [55].

The space $D'(X)$ of all q-vector-valued distributions on X is defined similarly. We let $\mathscr{D}(X)$ be the collection of infinitely differentiable functions on X with compact support. We give $\mathscr{D}(X)$ its usual topology [30, p. 170], defined by the uncountable collection of pseudonorms

$$p_\Theta(f) = \sup_\alpha \sup_{x\in X} |\theta_\alpha(x)\,D^\alpha f(x)| \tag{2.10}$$

for $f \in \mathscr{D}(X)$, where α is any multi-index of nonnegative integers and $\Theta = \{\theta_\alpha\}$ is any collection of continuous functions defined on X whose supports are "locally finite" [e.g., for each $x \in X$, $\theta_\alpha(x) = 0$ except for a finite number of multi-indices α]. Its dual $\mathscr{D}'(X)$ is the space of all distributions on X. As above, the space $D'(X)$ of all continuous linear maps from $\mathscr{D}(X)$ to $\mathbf{C}^q$ will be called the space of all vector-values distributions over X. The comments regarding weak and strong topologies on $S'(X)$ and $\mathscr{S}'(X)$ apply to $D'(X)$ and $\mathscr{D}'(X)$.

These spaces are closely related as the following fairly obvious results indicate; see also Appendix A.

Lemma 2.1. *There are sequences of natural continuous linear one–one mappings (continuous "monomorphisms"):*

$$\Gamma \subset \Lambda_\infty \subset \Lambda_p \subset \Lambda_{p'} \subset \Lambda_1 \subset D' \qquad (p > p' > 1), \tag{2.11}$$

$$\Gamma \cap S' \subset \Lambda_\infty \cap S' \subset \Lambda_p \cap S' \subset \Lambda_{p'} \cap S' \subset S' \subset D', \tag{2.12}$$

where the topologies of $\Gamma \cap S' \subset \Lambda_\infty \cap S'$ and $\Lambda_p \cap S'$ are those induced by Γ, Λ_∞, and Λ_p, respectively.

The preceding statements refer to the strong topologies that we defined earlier in this section on the various spaces. The result can be rephased in terms of the topologies themselves; for example, Lemma 2.1 says that the topology on Λ_p that we have defined is stronger than the topology that Λ_p inherits as a subset of D'. Each of the spaces in Lemma 2.1 is dense in all the containing spaces for the topology of the containing space. Concerning the weak topologies on these spaces, we have a similar result.

Lemma 2.2. *The sequences of natural continuous linear one–one mappings in* (2.11) *and* (2.12) *are continuous for the weak topologies on these spaces also.*

Proof. This follows directly from Lemma 2.1 and Proposition 3 on [30, p. 256].

C. Borel Sets

As usual, we define a subset B of a topological space E to be a *Borel set* when it is a member of the σ-field $\mathscr{B}(E)$ generated by the closed subsets of E. It is a remarkable fact that, although the function spaces discussed in Sections II.A and II.B have very different topologies, they all have essentially the same Borel sets—moreover these are the same whether the strong or the weak topology is used (cf. Theorem 2.1).

Even more remarkable, these function spaces all have isomorphic σ-fields of subsets; they are all "standard T_1-spaces" which have "standard Borel structures" in the terminology of Mackey [40]. We shall now make these notions precise.

Definition 2.5. A *Borel mapping* (of topological spaces) is a mapping such that the inverse image of any Borel set is a Borel set. A topological space E will be called a *standard* T_1-space when it is a T_1-space[7] and there exists a Borel bijection $\beta: E \to [0, 1]$ (with Borel inverse $\beta^{-1}: [0, 1] \to E$) from E onto the interval [0, 1].

Evidently, any continuous mapping is Borel. It is a remarkable fact that, in the real interval [0, 1], and hence by definition in any standard T_1-space E, the Borel algebra of all Borel sets is the *free* Borel algebra (Boolean σ-algebra) $\mathscr{B}_\omega$ with countably many generators.[8]

[7] A topological space is a T_1-space when every point is a closed set.

[8] G. Birkhoff, "Lattice Theory," 3rd ed., Chapter XI, §3, Amer. Math. Soc., Providence, Rhode Island, 1967.

As we shall see in Section II.D, standard T_1-spaces furnish an excellent setting in which to apply probability theory. Moreover, fortunately, all the function spaces of greatest interest to us are standard T_1-spaces. More precisely, we have the following theorem.

Theorem 2.1. *The spaces Λ_p, Γ, S', and D' are all standard T_1-spaces. In each of the topological linear spaces Λ_p, Γ, S', the Borel sets are just those Borel sets of $D'(X)$ which happen to be also contained in these subspaces*[9]*; moreover, they are the same in the weak as in the strong topology.*

Theorem 2.1. will be proved in Appendix A as Theorem A1. It is a corollary, as we shall also show there, that if E and F denote one of the function spaces Λ_p, Γ, and S', or D' or any intersection of these spaces, and $E \subset F$, then the σ-field of Borel sets in E forms a closed ideal in the Borel algebra of Borel sets in F. For example, a set in $\Lambda_p \cap S'$ is Borel as a subset of S' if and only if it is a Borel subset of Λ_p. In particular, Γ, Λ_p, and S' are all Borel subsets of D'.

In proving Theorem 2.1, extensive use will be made of the following concept.

Definition 2.6 (Bourbaki [13]). A complete separable metrizible topological space is called a *Polish space*.

It is a basic theorem that any Polish space with a continuum of points (e.g., which contains a straight line) is a standard T_1-space.

More generally, from any topological space S one can construct the *Borel structure* $(S, \mathscr{B})$ consisting of S and the σ-field of its Borel subsets. Such Borel structures have been thoroughly investigated by Mackey [40], Parthasarathy [43], and others. These authors define a Borel structure to be *standard* when it is isomorphic with the Borel structure of a Polish space (e.g., of [0, 1]). Thus a T_1-space with a continuum of points is "standard" in our sense if and only if it has a standard Borel structure.

We next show the equivalence, in the above spaces, of the usual notion of Borel set as defined above with the different notion of a "Borel set" proposed by Gel'fand and Vilenkin [22] and Mourier [41]. These authors first define a *cylinder set* in a topological linear space E as a subset $C \subset E$ such that, for some finite set of continuous linear functionals $\phi_1, \ldots, \phi_n$ on E (i.e., elements of the dual space E' of E), and some Borel set $B \subset \mathbf{R}^n$,

$$f \in C \quad \text{if and only if} \quad (\phi_1(f), \ldots, \phi_n(f)) \in B. \tag{2.13}$$

[9] See Lemma 2.1 of Section II.B.

These authors then define a "Borel set" in E as a member of the σ-field $\mathscr{C}$ generated by all the cylinder sets of the form $C = \{x \in E \mid a_i < \phi_i(x) < b_i \ (1 \le i \le n)\}$ in the real case and, in the complex case, those of the form $C_1 = \{x \in E \mid a_{2i-1} < \operatorname{Re} \phi_i(x) < b_{2i-1}\}$ and $C_2 = \{x \in E \mid a_{2i} < \operatorname{Im} \phi_i(x) < b_{2i}\}$, $i = 1, \ldots, n$. These cylinder sets are analogues of the classical open Khinchine window sets, which provide the most natural way to reconstruct a function from experiments in physical applications.

Since the Borel sets in $\mathbf{R}^n$ are generated by the open intervals or "slices" $a_i < x_i < b_i$, the σ-field $\mathscr{C}$ contains all the cylinder sets. Actually, $\mathscr{C}$ consists precisely of the *weak Borel sets* of E (i.e., the Borel sets in its weak topology). Although the weak Borel sets of a general topological linear space are not all Borel sets in the strong topology, we shall prove the following result in Theorem A2.

Theorem 2.2. *Let E be a locally convex topological linear space whose strong Borel structure is standard, and suppose that its dual E' is separable in the weak-star topology. Then the weak Borel sets of E are precisely the strong Borel sets generated by the strong topology on E.*

Theorems 2.1 and 2.2 yield the following corollary.

Theorem 2.3. *In the function spaces Λ_p, Γ, S', and D', the weak Borel sets are the same as the strong Borel sets.*

It also follows that, in the preceding spaces, the definitions of a "Borel set" proposed by Mourier [41] and by Gel'fand and Vilenkin [25] are equivalent to the standard definition (in *either* the strong *or* weak topology).

D. Regular Probability Measures

In many applications [34] one is interested in the *statistical* behavior of solutions of Cauchy problems (1.8) having random initial values—e.g., whose initial data are given by homogeneous random vector fields (HRVF). In treating such *statistical Cauchy problems*, we will rely heavily on the concept of a *regular* (probability) measure introduced in [8].

Definition 2.7. A measure μ on a topological space E is *regular* when it is defined on all Borel sets of E, and is the Lebesgue completion [25, p. 55] of its restriction to these Borel sets.

This concept was introduced in [8] in order to provide a rigorous mathematical formulation for the following generally accepted[10] principle. A

[10] At least, in the theory of homogeneous turbulence!

separable stationary Gaussian random function is *uniquely* determined (i.e., its probability measure μ is uniquely determined) by its autocorrelation function—and hence by its "energy spectrum" (if continuous). To make this literally true, one must identify all μ which assign the same measure $\mu(B)$ to all Borel sets B, and the simplest and most natural way to do this seems to be to consider only probability measures which are "regular" in the sense just defined. Our definition is closely related to other notions of "regular" measures in the literature.

The theory of regular measures in standard T_1-spaces is especially nice, because there is a natural bijection between regular measures in any such space and completely additive set-functions on the Borel algebra $\mathscr{B}_\omega$ of all Borel sets in [0, 1] which, as we noted in Section II.C, is simply the free Borel algebra with countably many generators. It follows that any regular *probability* measure on a standard T_1-space is isomorphic (with respect to measure and set-theoretic operations) to Lebesgue measure on [0, 1]; see [40] and [43]. This avoids various pathological possibilities which might otherwise occur; see Blackwell [9].

Our definition of a "regular" measure is closely related to other notions of "regular" (probability) measures in the literature, due to Halmos [25, p. 224], Carathéodory [14, pp. 238–9], Berberian [6, Section 59], Mourier [41, p. 162], and Gnedenko and Kolmogoroff [23, p. 18]. The following theorem summarizes some of the more important results in this vein.

Theorem 2.4. *Let μ be a regular probability measure on Γ, Λ_p, S', or D'. Then μ is perfect in the sense of Gnedenko and Kolmogoroff and is both inner and outer regular in the sense of Halmos.*

Using the results on the Borel structure of the relevant function spaces stated above, we can derive some useful facts concerning regular measure on these spaces.

First, since the Borel sets of these spaces are generated by the cylinder sets, which in turn are generated by the Khinchine window sets, a regular measure μ is just the measure constructable from the values of μ on the window sets by the classic constructions of Borel and Lebesgue. Thus to specify a regular measure on the spaces of interest to us, it is enough to define it consistently on the window sets.

Second, since the Borel structures of Γ, Λ_p, S', and D' are so nicely related we see that, if E and F are among these spaces or intersections of two of them and $E \subset F$, then the regular measures on F whose support lies in E are (ignoring sets of measure zero) the regular measures on E. For example, the regular probability measures μ on Λ_p with $\mu(\Lambda_p \cap S') = 1$ coincide with the regular probability measures on $\Lambda_p \cap S'$. This last proposition generalizes Theorem 4 of [8], where the same result was obtained for $\Gamma \subset \Lambda_p$.

Finally, since all the Borel structures in question are standard, they are countably generated. Therefore any regular measure μ on these spaces is strictly separable, in the sense that the σ-algebra of μ-measurable sets modulo μ-null sets (i.e., the Borel sets in this case) is countably generated; see [8, p. 669] and [25, p. 168]. It follows from Theorem B of [25, p. 168] that, if μ is σ-finite (e.g., if μ is a probability measure), then μ is metrically separable. That is, the space of Borel sets is a complete separable metric space (a Polish space) under the stochastic distance $d(S, T) = \mu(S \cup T) - \mu(S \cap T)$.

Theorem 2.5. *The space of Borel sets in any of the spaces Γ, Λ_p, S', or D is a Polish space under the stochastic metric induced by any regular probability measure μ.*

E. Admissible and Homogeneous Probability Measures

In this section, E will denote one of the function spaces described in Sections II.A–B, and μ will denote a regular probability measure on the Borel sets $\mathscr{B}(E)$ of E. We now consider a further limitation of the measure μ which will be necessary in some of what follows. The condition we will require will be called admissibility. It is a generalization of a notion defined in [8] for the space Λ_2 (which was denoted Λ in [8]).

In [8, Part A], the term admissible was used for probability measures on the Borel sets of Λ_2 which gave finite energy expectation to bounded (or, equivalently, compact) subsets of X. That is, if $\mathbf{u}(\mathbf{x}, \omega)$ is the random vector field associated with a measure μ on Λ_2, then if D is a compact (or bounded) subset of X, the "energy in D" is defined as

$$E_D(\omega) = \int_D |\mathbf{u}(\mathbf{x}, \omega)|^2 \, dx, \tag{2.14}$$

and the "energy expectation" as

$$E(D) = \int_{\Lambda_2} E_D(\omega) \, d\mu(\omega). \tag{2.15}$$

The measure μ was called *admissible* in [8] when $E(D)$ was finite for all compact sets $D \subset X$.

This definition is equivalent to another definition. The dual space of all continuous linear functionals on Λ_2 is denoted Λ_2'. It is not difficult to determine that Λ_2' consists of all q-vector fields g which are square integrable on X and which have compact support. The duality between Λ_2 and Λ_2' is given by the inner product

$$(\mathbf{u}, \mathbf{g}) = \int_X \mathbf{u}(\mathbf{x})\, \mathbf{g}(\mathbf{x})\, dx, \tag{2.16}$$

where $\mathbf{u} \in \Lambda_2$ and $\mathbf{g} \in \Lambda_2'$. Then it is easy to check that $E(D) < \infty$ for all compact D is equivalent to

$$\int_{\Lambda_2} |(\mathbf{u}(\mathbf{x}, \omega), \mathbf{g}(\mathbf{x}))|^2\, d\mu(\omega) < \infty. \tag{2.17}$$

for all $\mathbf{g} \in \Lambda_2'$. Thus admissibility in the sense of [8] is the same as requiring (2.17) to hold for all $\mathbf{g} \in \Lambda_2'$.

Stated in the form (2.17), the appropriate generalization is straightforward. If E' denotes the dual space of E, and (T, f) denotes the value of $T \in E'$ applied to $f \in E$, then we proceed as follows. Let $L^2(\mu) = L^2(E, \mathscr{B}(E), \mu)$ be the collection of μ-measurable functions $h: E \to \mathbf{C}$ that are square integrable with respect to μ.

Definition 2.8. The measure μ is called *admissible* if $E' \subset L^2(\mu)$ and if this inclusion is continuous for the topology on E'.

That is, if $T \in E'$, then we first require that

$$\int_E |(T, f)|^2\, d\mu(f) < \infty. \tag{2.18}$$

Second, we require the mapping

$$T \to \int_E |T(f)|^2\, d\mu(f) \tag{2.18'}$$

to be continuous. For the spaces discussed in Sections II.A–B, the continuity of (2.18′) follows from (2.18) by a category argument given by Dudley in [19, p. 776].

In particular, a regular measure μ on Λ_p is admissible if and only if, for any compact set $D \subset \mathbf{R}^n$, we have

$$\int_{\Lambda_p} \int_D |\mathbf{u}(\mathbf{x}, \omega)|^p\, dm\,(\mathbf{x})\, d\mu(\omega) < +\infty. \tag{2.19}$$

A final notion that can apply to regular measures on E is the following notion of (statistical) homogeneity.[11]

[11] This notion was first introduced by G. I. Taylor in the context of turbulence. Taylor also designed an apparatus for producing (nearly) homogeneous turbulence experimentally in a wind tunnel.

Definition 2.9. A regular measure μ on E, a function space over X, is called *homogeneous* when μ is invariant under any transformation of E induced by a translation of X.

More precisely, we define the transformations of E as follows. If $f \in E$ is a function on X and $\mathbf{x} \to \mathbf{x} + \mathbf{c}$ is any translation of X, we define $\tau_c f = g$ by $g(\mathbf{x}) = f(\mathbf{x} - \mathbf{c})$. (We assume that the topology of E is invariant under translations of X, so that τ_c automatically carries Borel sets of E into Borel sets, and regular measures into regular measures.)

If $A \in \mathscr{B}(E)$ is a Borel set in E, then $\tau_c A$ means $\{\tau_c g \mid g \in A\}$; clearly μ is homogeneous if and only if

$$\mu(\tau_c\, A) = \mu(A) \tag{2.20}$$

for all $A \in \mathscr{B}(E)$ and all $\mathbf{c} \in X$.

F. Product Measure Theorem

Many authors, including Wiener and Doob,[12] consider random functions and random vector fields (RVF) as defined by a measurable function $\mathbf{u}(\mathbf{x}, \omega)$ on the *product* $X \times \Omega$ of the domain X with an abstract probability space Ω (the "sample space"). We will now show how to obtain such a function from our definition, according to which a RVF is defined by a regular probability measure μ on an appropriate function space E, whose points are vector-valued functions on X.

In many cases, as suggested by Doob [18, Chapter 2, §2], it suffices to set $\Omega = E$. If $E = \Gamma(X)$, this can be done since for each given $\mathbf{x}$ and $\omega = \mathbf{u}(\mathbf{x}) \in \Omega = \Gamma(X)$ the value of $\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{x}, \omega)$ is unambiguous. For random distributions, one can set $\Omega = D'(X)$; while for tempered distributions, one can set $\Omega = S'(X)$. If one does this, the derivation of a satisfactory $\mathbf{u}(\mathbf{x}, \omega)$ is again trivial. Thus, if μ is a regular probability measure on $S'(X)$, then the function $\mathbf{u} = \mathbf{u}(\phi, \omega)$ on $\mathscr{S}(X) \times S'(X)$ to $\mathbf{C}^q$ given by $u(\phi, \omega) = \omega(\phi)$ is continuous, and hence measurable on the Borel sets of $\mathscr{S}(X)$ and the μ-measurable sets of $S'(X)$.

For $E = \Lambda_p(X)$, the following product measure theorem interprets each $\omega \in \Lambda_p(X)$ as a sample function $\mathbf{u}(\mathbf{x}, \omega)$ in a way which is consistent with our formulation in terms of probability measures on the Borel sets of E. This result was essentially established in [8, p. 670, Lemma 4] and [8, p. 674, Theorem 5]; we shall simplify and generalize the proof below. Our result

[12] In [18, Chapter 2, §2], Doob considers a definition of stochastic processes which, like ours, makes Ω be a function space. He refers to stochastic processes so defined as "of function space type."

avoids a serious difficulty in the usual theory, mentioned by Halmos [25, p. 142, Ex. 2] and discussed carefully by Blackwell [9]; see Appendix B.

Theorem 2.6. *Let μ be any regular probability measure on $\Lambda_p = \Lambda_p(X, q)$, $1 \leq p < \infty$. Then there exists a Borel function $\mathbf{v}(\mathbf{x}, \omega)$ from $X \times \Lambda_p$ to $\mathbf{R}^q$ such that $\mathbf{v}(\mathbf{x}, \omega)$ is represented for any fixed $\omega \in \Lambda_p$ by itself.*

Proof. We construct such a $\mathbf{v}(\mathbf{x}, \omega)$ as follows: Let ω stand for one equivalence class of functions of Λ_p. For each positive integer j, let $h = 2^{-j}$, and define for any $\mathbf{u}(\mathbf{x}) \in \omega$ of $\Lambda_p(X, q)$:

$$\mathbf{v}_j(\mathbf{x}, \omega) = (2h)^{-n} \int_{-h}^{h} \cdots \int_{-h}^{h} \mathbf{u}(x_1 + c_1, \ldots, x_n + c_n)\, dc_1 \cdots dc_n. \tag{2.21}$$

This is a continuous function of $\mathbf{x}$ for each $h = h(j)$, and the mappings $\alpha_j : \omega \to \mathbf{v}_j(\mathbf{x}, \omega)$, $\beta_j : (\mathbf{x}. \omega) \to \mathbf{v}_j(\mathbf{x}, \omega)$ are continuous from Λ_p to $\Gamma \subset \Lambda_p$ and from $X \times \Lambda_p$ to $\mathbf{R}^q$, respectively.

Consider the function $\mathbf{v}(\mathbf{x}, \omega)$ defined as follows:

$$\mathbf{v}(\mathbf{x}, \omega) = \lim_{j \to \infty} \mathbf{v}_j(\mathbf{x}, \omega) \qquad \text{wherever this limit exists,} \tag{2.22}$$

$$\mathbf{v}(\mathbf{x}, \omega) = 0 \qquad \text{elsewhere.} \tag{2.23}$$

The hypotheses imply those of the Lebesgue density theorem [25, p. 268], which implies that, for any ω, $\mathbf{v}(\mathbf{x}, \omega)$ belongs to the equivalence class of ω. Moreover, since the mappings β_j are continuous, the set defined in (2.22) is Borel in $X \times \Lambda_p$ and the function $\mathbf{v}(\mathbf{x}, \omega)$ is a Borel function fron $X \times \Lambda_p$ to $\mathbf{R}^q$.

Corollary 2.1. *The function $\mathbf{v}(\mathbf{x}, \omega)$ is $(m \times \mu)$-measurable on the product $X \times \Lambda_p$.*

Proof. It is measurable since it is Borel, and the measure on Λ_p is regular.

Again, we note that by [8, p. 699, Lemma 1], the functional E_D defined in (2.14) is μ-measurable.

Random functions $\mathbf{v}(\mathbf{x}, \omega)$ which are $(m \times \mu)$-measurable have especially nice properties. Thus one can apply to them the Fubini and Tonelli theorems, a fact which has many important implications such as the following.

Corollary 2.2. *If μ is admissible in Theorem 2.6, then the coordinate functions $v_j(\mathbf{x}, \omega)$ of $\mathbf{v}$ are pth power $m \times \mu$ integrable over $D \times \Lambda_p$, for each compact domain $D \subset X$.*

Proof. Since the functions are jointly measurable by Corollary 2.1, it suffices by the Fubini and Tonelli theorems to prove that the iterated integral of $|v_j(\mathbf{x}, \omega|^p$ is finite. However, as we have already remarked above (2.19), this is a direct consequence of the admissibility of μ.

III. Well-Set Cauchy Problems

A. Well-Set Cauchy Problems

By definition, *Cauchy problems* are concerned with the evolution in time of solutions $u(\mathbf{x}, t)$ of specified partial differential equations (DE's) or systems of partial differential equations, given the *initial* values $u(\mathbf{x}, 0) = v(\mathbf{x})$. For this reason, they are also called "initial value" problems. Loosely speaking, a Cauchy problem is called "well set" when this solution exists, is unique, and depends continuously on the "initial data" v. It is classic, for example, that the Cauchy problem is well set for the heat equation $u_t = u_{xx}$ and the wave equation $u_{tt} = u_{xx}$; it is not well set for the Laplace equation $u_{tt} = -u_{xx}$. The same is true of their n-dimensional generalizations to $u_t = \nabla^2 u$, $u_{tt} = \nabla^2 u$, and $u_{tt} = -\nabla^2 u$.

We shall deal here primarily with a special class of Cauchy problems: namely, those involving systems of *linear* partial DE's with *constant coefficients*. These include Maxwell's electromagnetic equations, the equations of homogeneous elastic solids, and many other systems arising in classical mathematical physics.

Technically, then, we shall consider a $(q \times q)$-matrix differential operator $P(D) = \|P_{ij}(D_1, \ldots, D_n)\|$, where $P_{ij}(x_1, \ldots, x_n)$ is an element of the polynomial ring $\mathbf{C}[x_1, \ldots, x_n]$ over the complex field $\mathbf{C}$, and $D_j = \partial/\partial x_j$. Thus we shall be studying initial value or Cauchy problems defined by systems of partial DE's ("evolution" equations) of the form

$$\partial u_j/\partial t = \sum_{k=1}^{q} P_{jk}(D_1, \ldots, D_n) u_k \qquad (j = 1, \ldots, q). \tag{3.1}$$

We shall consider them both classically (see Section I.B) and as defining "flows" in the function spaces described in Sections II.A–B.

Classically, essentially following Hadamard, the *deterministic* Cauchy problem for a given system (3.1) is said to be *well set* (bien posé) in a given function space E (e.g., a Fréchet space) of q-vector-valued functions, when there exists a *dense* subset $D \subset E$ of initial values

$$u_j(\mathbf{x}, 0) = v_j(\mathbf{x}) \qquad (j = 1, \ldots, q) \tag{3.2}$$

such that (i) for each $v = \mathbf{v}(\mathbf{x}) \in D$, the DE (3.1) has one and only one smooth (i.e., classical) solution $\mathbf{u}(\mathbf{x}, t)$ for $t \geq 0$ with $\mathbf{u}(\mathbf{x}, 0) = \mathbf{v}(\mathbf{x})$, and (ii) $\mathbf{u}(\mathbf{x}, t) = u(t) \in E$ depends continuously in E on $v = u(0) \in D$. Since the set D of initial values is dense in E, there is at most one way to *extend* the classical solutions by continuity so as to define a *flow* in all of E, whose streamlines ("trajectories" or "orbits") are the paths $u(t)$ in $E \times [0, \infty)$ which corresponds to solutions $\mathbf{u}(\mathbf{x}, t)$ of the system (3.1) or to their limits.

When such an extension exists, we shall say that the Cauchy problem for the DE (3.1) is (deterministically) *well set in E*. This idea has strong physical and intuitive appeal. An excellent informal discussion of it was given by Hille already in 1948 [26, Chapter XX]. In particular, Hille emphasized the fact that, whereas (linear) hyperbolic problems gave rise to *groups*, parabolic problems led only to *semigroups*.

Hille's approach was developed further, and the notion of an "infinitesimal transformation" made more rigorous some years later by Hille and Phillips; see [27, § 23.6–23.8] for their comments on the connection with the Cauchy problem. Yosida[13] working largely independently, developed similar ideas in the more general context of Fréchet spaces. However, his applications to concrete Cauchy problems were limited to a few special equations in Banach spaces [55, Chapter XIV].

Concurrently, the powerful theory of distributions was being developed; it extended the idea of a flow in phase-space (and corresponding theorems of existence, uniqueness, and continuity) to a class of orbits consisting of a still more general class of "solutions" of (3.1).

The following definition applies to all of the preceding phase spaces.

Definition 3.1. A *C_0-semigroup* in a topological linear space E is a one-parameter family of continuous linear operators $T_t : E \to E$ $(t \geq 0)$ such that

(a) $T_s T_t = T_{s+t}$, $\quad T_0 = I$,

(b) $\lim_{t \to s} T_t x = T_s x \quad$ for all $s \geq 0$ and $x \in E$.

Every C_0-semigroup $\{T_t\}$ on E has an *infinitesimal generator* L, defined by

$$Lv = \lim_{h \downarrow 0} h^{-1}(T_h v - v); \tag{3.3}$$

the domain of L need not be all of E.

Conversely, if L is any linear operator on a dense subspace of E, then the *abstract Cauchy problem* for L consists in finding a C_0-semigroup whose infinitesimal operator is an extension of L. For instance, if $L = P(D)$ is a

[13] See [55, p. 231] and the references given there.

linear differential operator, a C_0-semigroup on E whose infinitesimal operator is an extension of L may be called a *solution semigroup* (in E) of the abstract Cauchy problem

$$\partial u/\partial t = P(D)u \qquad [u(0) = v], \tag{3.4}$$

which is just (3.1) in an abbreviated notation.

For a C_0-semigroup to be a solution semigroup of (3.1) in E is essentially equivalent to the assertion that the Cauchy problem is well set in the classical deterministic sense of Hadamard and Hille; and the preceding conditions express the idea of physical *determinism* for individual solutions very naturally. But unfortunately, they are ambiguous: their fulfillment is sensitive to the particular choice of E. For example, when $X = \mathbf{R}^n$, the heat equation defines a "well-set" Cauchy problem in the Hilbert space $L^2(X)$ and in the space $S'(X)$ of tempered distributions, but not in the space Λ_2 of [8], nor in the space $\Gamma(X)$ of all continuous functions, nor in the space $D'(X)$ of Schwartz distributions. Again [7, Example 4], consider the Cauchy problem for the system $u_t = w$, $w_t = u_{xx}$ obtained from the wave equation $u_{tt} = u_{xx}$ by setting $u_t = w$. This problem is not (deterministically) well set in $E = [L^2(X)]^2$, because $(u(0), w(0)) \in [L_2(X)]^2$ does not necessarily imply $(u(t), w(t)) \in [L_2(X)]^2$. However, it is well set in the Hilbert space having the norm $[\int(w^2 + u_x^2)\,dx]^{1/2}$ associated with the square root of the "wave energy" $\int(w^2 + u_x^2)\,dx$.

Regular Cauchy problems. Fortunately, there exists another purely algebraic criterion for the Cauchy problem to be (deterministically) well set for the constant-coefficient system (3.1), which is independent of the choice of E. This can be easily motivated in terms of the concepts of Fourier analysis.

For each wave-vector $\mathbf{k}$, the system (3.1) is equivalent to the following system of *ordinary* DE's acting on the Fourier component $\mathbf{f}(t)e^{i\mathbf{k}\cdot\mathbf{x}}$ associated with this particular wave vector:

$$\mathbf{f}'(t) = P(i\mathbf{k})\mathbf{f}(t),$$

where $P(i\mathbf{k})$ is the $q \times q$ *amplification matrix* $\| p_{jl}(ik_1, \ldots, ik_n) \|$. As t increases, the asymptotic *growth rate* of $\mathbf{f}(t)$ is

$$\Lambda(P(i\mathbf{k})) = \sup_{1 \leq j \leq q} \mathrm{Re}\{\lambda_j(P(i\mathbf{k}))\}, \tag{3.5}$$

where the $\lambda_j(P(i\mathbf{k}))$ are the eigenvalues of $P(i\mathbf{k})$.

Definition 3.2. The *stability index* of the constant-coefficient system (3.1) is the number

$$\Lambda(P) = \sup_{\mathbf{k} \in X'} \Lambda(P(i\mathbf{k})). \tag{3.6}$$

The Cauchy problem defined by (3.1) is *regular* when the system has a finite stability index—in other words, when the asymptotic growth rate of all Fourier components is uniformly bounded.

Notice that this definition, which is essentially due to Hadamard and Petrowsky, does not depend on the function space E. This contrasts with the ideas of an abstract Cauchy problem formulated above.

Most mathematicians today [20, 29, 47, 55] agree with Hadamard and Petrowsky that the Cauchy problem should be considered as *well set* for (3.1) if and only if it is regular. Moreover it is known [7] that, given a regular system (3.1), one can always construct a Hilbert space in which the deterministic Cauchy problem is well set. The relevant norm is the square root of physical energy in many classical examples; this observation is in line with the dictum of Hadamard and Poincaré: that physics is the best guide for deciding which initial or boundary value problems are well set.

From a technical standpoint, the analysis of [7] is much simpler than that of Hille, Phillips, and Yosida; thus it avoids altogether the problem of defining abstract "infinitesimal operators" with dense domains, and the "abstract Cauchy problem" of reconstructing C_0-semigroups from their infinitesimal generators. Moreover, as will be explained in Section IV.B, it is adequate for treating problems on *compact* domains $X = \mathbf{K}^n$.

Unfortunately, in *unbounded* domains such as $\mathbf{R}^n$, Banach spaces are not adequate. Since the total "energy" of a HRVF is infinite with probability one, it seems to be impossible to construct reasonable Hilbert or Banach spaces for such functions in which the Cauchy problem is well set. Hence the results of [7], which refer to C_0-semigroups on Banach spaces, are inapplicable.

Instead, one must use Fréchet spaces like $\Lambda_p(X)$ which are not Banach spaces, or more general spaces of distributions like $S'(X)$ and $D'(X)$ which are not even metrizable. Moreover, the theory of C_0-semigroups on Fréchet spaces other than Banach spaces has not been developed to the point of being applicable to concrete Cauchy problems; see [55]. Hence, when X is unbounded, it is necessary to appeal to the theory of distributions.

B. The Theorem of Schultz

Fortunately, for linear constant-coefficient systems (3.1), one can use a beautiful recent theorem of Martin Schultz [47], of which we now give a slightly simplified proof. This theorem shows that, in the space $S'(X)$, a Cauchy problem (3.1) is well set if and only if $P(D)$ is regular in the sense of Hadamard and Petrowski as defined in Section III.A.

Schultz's result will be important for us in Section VI, for it is precisely the class of *regular* systems (3.1) for which we can characterize the evolution of random solutions in terms of the evolution of the spectral matrix. His result is analogous to a theorem of Birkhoff [7, Section IV]; it is technically closely related to results of Friedman [20, Chapter 7, §3], and Gel'fand and Shilov [21, Chapter 2].

Theorem 3.1 (M. H. Schultz). *The abstract Cauchy problem* (3.1) *in* $S'(X)$ *is regular if and only if the matrix functions of* $tP(i\mathbf{k})$, $\{\exp[tP(i\mathbf{k})]\}$, *form a one-parameter* C_0*-semigroup in* $S'(X)$ *and, in this case, there is a unique* C_0*-semigroup that solves* (3.1) *given by* $T_t v = \bar{\mathscr{F}}(\exp[tP(i\mathbf{k})]) * v$, *where* $\bar{\mathscr{F}}$ *is the inverse Fourier transform and* $*$ *denotes convolution.*

Proof. First suppose the Cauchy problem is regular. Using the fact that the Fourier transform $\mathscr{F}$ is an isomorphism of $S'(X)$ (see Section II.B), we see that a C_0-semigroup solving (3.1) in $S'(X)$ is equivalent to a C_0-semigroup solving

$$\partial\hat{u}/\partial t = P(i\mathbf{k})\hat{u}, \qquad \hat{u}(0) = \hat{v}, \tag{3.7}$$

in $S'(X)$, where the overcaret signifies "Fourier transform of." A matrix $Q(x)$ of polynomials is regular in our sense if and only if $\exp[tQ(i\mathbf{x})]$ is a multiplier in $S'(X)$, as Schultz proves or as one can see from [20, (2.13), p. 171]. Thus $\exp[tP(i\mathbf{k})]$ is a multiplier in $S'(X)$. Let M_t be multiplication by $\exp[tP(i\mathbf{k})]$ in $S'(X)$. Then $\{M_t\}_{t\geq 0}$ defines a C_0-semigroup in $S'(X)$. Indeed, the semigroup property is trivial and the continuity property is a consequence of the continuity in t of $\exp[tP(i\mathbf{k})]$ viewed as an operator on $S'(X)$. The latter can be seen again from [20, (2.13), p. 171].

It is a straightforward calculation that the semigroup $\{M_t\}$ solves the initial value problem (3.7) in $S'(X)$. Hence, the C_0-semigroup $\{T_t\}$ defined by

$$T_t v = \bar{\mathscr{F}} M_t * v \tag{3.8}$$

solves the Cauchy problem (3.1) in $S'(X)$.

The uniqueness of this semigroup derives from an abstract uniqueness theorem in this context. For if $\{T_t\}$ is any solution semigroup as defined in Section III.A, then a semigroup $\{M_t\}$ acting on $S'(X)$ which "solves" (3.7) is defined by

$$M_t \hat{v} = \mathscr{F} T_t \bar{\mathscr{F}} \hat{v}, \tag{3.9}$$

Applying [20, Theorem 6, p. 177] to (3.7), and recalling from Section II.B that the dual of $S'(X)$ is $S(X) = [\mathscr{S}(X)]^q$, if there exists a solution to the modified adjoint problem (3.10)

$$\partial\phi/\partial t = -P(i\mathbf{k})^T\phi \quad \text{for} \quad 0 \leq t \leq t_0, \quad \phi(t_0) = \phi_0, \tag{3.10}$$

in $S(X)$, then there is at most one solution to the Cauchy problem (3.7) on the interval $0 \leq t \leq t_0$. Since t_0 is arbitrary, this will yield the uniqueness of $\{M_t\}$ and thus of $\{T_t\}$.

But (3.10) can be solved explicitly. For P is regular and hence so is its transpose P^T. Therefore, $\exp[-(t-t_0)P(i\mathbf{k})^T]$ is a multiplier in $S(X)$ for $t \leq t_0$. Given ϕ_0 in $S(X)$, we define

$$\phi(\mathbf{k}, t) = \exp[-(t-t_0)P(i\mathbf{k})^T]\,\phi_0(\mathbf{k}). \tag{3.11}$$

One checks without difficulty that this solves the modified adjoint problem (3.10) in $S(X)$.

This shows that if P is regular in our sense, then $\exp[tP(i\mathbf{k})]$ is a C_0-semigroup in $S'(X)$ and that $T_t v = \mathscr{F}\exp[tP(i\mathbf{k})] * v$ defines the only C_0-semigroup in $S'(X)$ which solves (3.1).

Now suppose that $\{\exp[tP(i\mathbf{k})]\}_{t \geq 0}$ is a C_0-semigroup in $S'(X)$. Then, following Schultz [47, Theorem 5.4], we note that this means that $\exp[tP(i\mathbf{k})]$ is a multiplier in $S'(X)$ for each t. Now since $\exp[tP(i\mathbf{k})]$ is continuous in t as an operator on $S'(X)$, each entry of the matrix $\exp[tP(i\mathbf{k})]$ is bounded by a polynomial in $|\mathbf{k}|$ on any compact interval $[0, T]$. Let $\rho(A)$ denote the spectral radius of a $(q \times q)$-matrix A. Then

$$\rho(A)^2 \leq \|A\|^2 \leq \sum_{j,k} |a_{jk}|^2. \tag{3.12}$$

Hence

$$\begin{aligned}\rho(\exp[tP(i\mathbf{k})])^2 &\leq \sum_{j,k} \exp[tP(i\mathbf{k})]^2_{j,k} \\ &\leq C(1+|\mathbf{k}|)^R \quad \text{for} \quad 0 \leq t \leq T,\end{aligned} \tag{3.13}$$

where C is a constant. But

$$\rho(\exp[tA]) = \exp[t \sup_j \operatorname{Re} \lambda_j(A)] \tag{3.14}$$

for any $(q \times q)$-matrix A with complex entries where $\lambda_j(A)$, $1 \leq j \leq q$, are the eigenvalues of A. Using this and taking the logarithm of (3.13), we obtain

$$2t \sup_{1 \leq j \leq q} \operatorname{Re} \lambda_j(P(i\mathbf{k})) \leq \log C + R \log(1+|\mathbf{k}|). \tag{3.15}$$

An application of the result of [20, Corollary, p. 219] yields

$$\sup_{1 \leq j \leq q} \operatorname{Re} \lambda_j(P(i\mathbf{k})) \leq \text{const} \tag{3.16}$$

and this holds for all $\mathbf{k}$. Therefore,

$$\Lambda(P) = \sup_{\mathbf{k} \in \mathbf{R}^n} \sup_{1 \leq j \leq q} \operatorname{Re} \lambda_j(P(i\mathbf{k})) < \infty \tag{3.17}$$

and so P is regular. This completes the proof of Theorem 3.1.

The condition that the linear system (3.1) be *strictly stable* in the usual sense is, of course [7, Section III], just the condition that $\Lambda(P) < 0$. Any regular system (3.1) can be reduced to the strictly stable case by introducing a new variable $\mathbf{w}(\mathbf{x}, t) = e^{-(\Lambda(P)+1)t}\,\mathbf{u}(\mathbf{x}\ t)$. Yosida has proved that if $\Lambda(P) \leq 0$, then the semigroup $\{M(t)\}$ is *equicontinuous* in the sense [55] that:

For any continuous seminorm p on E, there exists a continuous seminorm $\tilde{p}$ on E such that $p(T_t u) \leq \tilde{p}(u)$ for all $t \geq 0$ and $u \in F$.

C. Statistically Determinate Problems

One of us has already studied many special Cauchy problems of physical interest from the point of view of statistical mechanics in a series of publications; see [34] for a survey of this work. These problems were investigated in special function spaces, chosen to make them "deterministically well set" in the sense defined in Section III.A; they were almost all of the form (3.1).

We are here trying to initiate a more general and systematic statistical theory which will be applicable not only to deterministically well-set problems of the form (3.1) but to many others. To emphasize the broad scope of the ideas, we begin in a much more general setting than was considered in the last two sections.

Let L_j be a time-independent partial differential operator $(j = 1, \ldots, q)$. Let E be a real (or complex) space of q-vector-valued (possibly generalized) functions defined on our underlying domain $X = \mathbf{K}^s\mathbf{R}^{n-s}$. We shall assume that the operator $L = [L_j]$ is defined on a dense subset of E and consider the Cauchy problem

$$\partial u/\partial t = L[u], \qquad u(0) = v \in E, \tag{3.18}$$

where v is a given initial value in the function space E.

Essentially, as in [34], we define a *Cauchy problem with random initial values* to be a space E and a system (3.18), as above, together with a regular probability measure μ on E. We refer the reader to Section II.D or [34, §16] and Appendix B for discussions of regular measures; the measure μ is to be thought of as assigning the statistical distribution of initial values in E.

Much as in (3.4), we define a *solution* of the Cauchy problem (3.18) in E as a function $u(t)$ from $[0, \infty)$ to E such that

$$\lim_{h \to 0}\ [u(t+h) - u(t)]/h = L[u(t)] \qquad (\text{all } t \geq 0).$$

As in Section III.A, we shall call $u(t)$ the *orbit* (or trajectory) of the solution, and $u(0) = v$ its initial value.

However, we shall propose in this section and the next some notions of a "well-set" Cauchy problem that are weaker than Hadamard's. We shall not require that the mappings e^{tL} defined by (3.18) be everywhere defined and continuous, but only that they be *almost* everywhere defined and Borel. More precisely, we require that, for all v in some subset M of E of μ-probability one,[14] there exists a unique solution of (3.18) with $u(0) = v$, thus defining a "flow" along the orbits $u(t) = T_t v$ with initial values in M. These flows need not be linear or continuous, but must be Borel mappings. Moreover for the *systematic theory* to be given in Sections IV.B–C and Sections VI and VII, we will specialize to the regular linear constant-coefficient systems described in Sections III.A–B.

Definition 3.3. A Cauchy problem with random initial values is called *statistically determinate* when there is a dense Borel set $M \subset E$ with the following properties:

(i) $\mu(M) = 1$.

(ii) For all $v \in M$, the problem (3.18) with initial value v has a unique solution with orbit $\{u(t)\}$ in E.

(iii) The mapping $T_t : M \to E$ given by $T_t(v) = u(t)$ is a Borel mapping for all $t > 0$.

We call the set M the *subset of initial values for orbits.*

When condition (iii) is replaced by the stronger condition

(iv) the mapping $T_t : M \to E$ of (iii) is continuous for each $t > 0$,

then we call the problem *statistically continuous.*

The following result relates this notion to the concept of a deterministically well-set problem defined in Section III.A, it is a simple consequence of the definitions involved.

Theorem 3.2. *A statistically determinate Cauchy problem is statistically continuous if it is deterministically well set in E, and the subset M of initial values for orbits has probability one.*

Induced measures

When the solutions of a Cauchy problem (3.18) evolve in time under the action of L, it is natural to ask how their probability distribution changes. Provided that the mappings T_t are all Borel mappings, the following concept allows us to answer this question.

[14] In Sections V.G–H, we shall show that this condition is implied by a wide class of conditions on the (energy) spectrum.

Definition 3.4. Let $T: E \to F$ be a Borel mapping between topological spaces, and let μ be any regular probability measure on E. Then the measure *induced*[15] *by T on F* is the regular probability measure $\nu = T(\mu)$ defined by

$$\nu(S) = \mu(T^{-1}(S)) \qquad \text{for any Borel set} \quad S \subset F. \tag{3.19}$$

Trivially, (3.19) defines a Borel probability measure ν_0 on F, with $\nu_0(T(E)) = 1$ and, since $[T^{-1}(F)]' = \varnothing$ (the void set), $\nu_0[T(E)'] = 0$. The Lebesgue completion of ν_0 is the regular probability measure induced on F.

Clearly also, for any statistically determinate Cauchy problem with initial probability measure μ_0, since the T_t are Borel mappings, at any time $t > 0$ the random functions $\mathbf{u}(\mathbf{x}, t, \omega)$ can be considered as distributed with the regular probability measure μ_t induced by T_t from μ on E.

The preceding ideas can be generalized to other linear boundary value problems (e.g., to elliptic problems), by letting E be a (function) space of boundary value functions, and F that of interior values.

D. Statistically Well-Set Problems

The definitions of Section III.C weaken the classic concept of a (deterministically) "well-set" Cauchy problem adopted by Hadamard, Petrowsky, and Hille in two simple ways. They weaken the existence and uniqueness requirements on solutions from "all" to "almost all" initial values, and they weaken the requirement that the T_t be continuous to the requirement that they be Borel mappings.

The weakening of the concept of a well-set problem which we propose in this section is much more subtle; instead of requiring that each T_t carry all (sufficiently) nearby points into nearby points, it requires that this be true with a very high degree of probability.

Definition 3.5. Consider a statistically determinate Cauchy problem (3.18) with random initial values given by a regular probability measure μ in a metric space with metric d. Let M be the specified Borel set of initial values for orbits with $\mu(M) = 1$. Then the problem (3.18) is *statistically well set* for μ when, for any $v(0) \in M$, $\varepsilon > 0$, $\eta > 0$, and $t > 0$, there exists $\delta > 0$ such that the conditional probability that $d(u(t), v(t)) < \eta$, given that $d(u(0), v(0)) < \delta$, exceeds $1 - \varepsilon$. [In symbols, we must be able to choose $\delta > 0$ so that

$$\mu(d(u(t), v(t)) < \eta \mid d(u(0), v(0)) < \delta) > 1 - \varepsilon, \tag{3.20}$$

where $\mu(A \mid B)$ means the conditional probability of A given B.]

[15] In the terminology of [25, p. 162], the mapping T is "μ-measurable.

This definition asserts that, given the probability distribution of initial states of a system, if we know the initial state sufficiently accurately, then we know its current state at any later time t very accurately with a high degree of probability.

The definition above is made in a metrizable linear space setting, and so is valid for the Fréchet spaces Γ and Λ_p. For more general nonmetrizable spaces such as S' or D', we must substitute a definition based on neighborhoods. In such spaces, we propose the following definition (which is equivalent in Fréchet spaces to the one indicated above).

Definition 3.6. A statistically determinate initial value problem (3.18) [such as (3.1)] will be called *statistically well set* when, given a solution $v(t)$ of (3.18) with $v(0) \in M$, $\varepsilon > 0$, and $t > 0$, and a neighborhood V of $v(t)$, there exists a convex circled neighborhood U of $v(0)$ such that the conditional probability

$$\mu(u(t) \in V \mid u(0) \in U) > 1 - \varepsilon. \tag{3.21}$$

In other words, we require that

$$\mu[(u(t) \in V) \cap (u(0) \in U)] > (1 - \varepsilon)\mu[u(0) \in U]. \tag{3.22}$$

We now digress briefly to discuss the technical question of the existence of the conditional probability in (3.20). We define

$$N_v^{\delta} = \{u \in E \mid d(u, v) < \delta\}. \tag{3.23}$$

The conditional probabilities of (3.20) will all exist provided $\mu(N_v^{\delta}) > 0$ for all $\delta > 0$ and $v \in M$, where M [with $\mu(M) = 1$] is the specified set of initial values for orbits. If we define the set W by

$$W = \{u \in E \mid \exists \delta > 0 \text{ with } \mu(N_u^{\delta}) = 0\}, \tag{3.24}$$

then it suffices for the proof to show that $\mu(W) = 0$. For then we can make all the conditional probabilities in (3.20) exist by subtracting W from M. This alteration does not change the statistical determinacy of the problem.

Now, as will be shown in Appendix B, any regular probability measure μ defined on any of the function spaces of primary importance in this paper (see Sections II.A,B) is inner regular in the sense of [25]. Thus, in these spaces,

$$\mu(W) = \sup\{\mu(K) \mid K \subset W, K \text{ compact}\}. \tag{3.25}$$

But if $K \subset W$ is compact, then K can be covered by a finite number of spheres of the form $B = \{u \mid d(u, v) < \delta\}$, with $v \in W$ and $\mu(B) = \mu(N_v^{\delta}) = 0$. It follows that any such compact set K has probability measure $\mu(K) = 0$. Hence, from (3.25), $\mu(W) = 0$ as desired. Throughout the above calculations, we have assumed that we are working in a linear metric space. By substituting convex circled neighborhoods for the metric spheres used in (3.23), etc., the same proof works in the more general situations of interest here. We have proved

Theorem 3.3. *Let μ be a regular probability measure on Γ, Λ_p, S', or D', and let W be defined for a Borel set M with $\mu(M) = 1$ as in* (3.24) *above. Then $\mu(W) = 0$, and hence the conditional probabilities of* (3.20) *and* (3.21) *exist.*

We turn to the relation between being statistically well set and the notions of Section III.C.

Theorem 3.4. *A Cauchy problem with random initial values that is statistically continuous is statistically well set.*

Proof. Both problems are statistically determinate, and so we need only establish (3.21). Given v and a neighborhood V of $v(t)$, we choose a convex circled neighborhood of U of v so that

$$T_t(U) \subset V. \tag{3.26}$$

We can arrange this by the continuity of the mapping T_t. Then (3.21) will be satisfied for any $\varepsilon > 0$.

Combining Theorem 3.4 with Theorem 3.2, we see that a deterministically well-set problem is statistically well set if the initial values are sufficiently smooth with probability one.

Corollary 3.1. *A Cauchy problem with random initial values which is deterministically well set and for which the dense subset M of initial values for orbits has probability one is statistically well set.*

Perhaps more interesting is the fact that the converse of this corollary does not hold; there are problems that are statistically well set but not deterministically well set. In Appendix C, some examples of this phenomenon are given. These examples establish our claim that our notion of a statistically well-set Cauchy problem is indeed a weaker concept than the Hille–Phillips–Yosida notion of a (deterministically) well-set problem. They show that even if a problem is not well set in a particular function space, it may be statistically well set when one takes into account the *a priori* distribution of initial values.

IV. Some Special Conditions

A. Introduction

Our most complete and systematic general results, to be presented in Sections V and VI, rely basically on Schultz's theorem. They apply to Cauchy problems for regular systems (3.1) of linear DE's with constant coefficients in Euclidean spaces $\mathbf{R}^n$ and their Cartesian products with tori, $X = \mathbf{K}^s\mathbf{R}^{n-s}$. In such spaces, one can use the theory of tempered distributions.

Unfortunately, no analogue of this theory is known for other manifolds. Consequently, extensions of our results to well-set Cauchy problems involving linear partial DE's with *variable* coefficients on general manifolds may prove difficult.

However, there are two important special cases in which our results can be obtained without appealing to the theory of tempered distributions. These are the cases of a compact manifold and of a hyperbolic system (3.1). We shall treat these cases in Section IV.B and in Section IV.C, respectively.

Specifically, in the *compact* case one seems not to need the theory of distributions at all; the theory of C_0-semigroups acting on *Banach spaces* seems to be adequate. Moreover, in the special case $X = \mathbf{K}^n$ and a system (3.1), the spectrum is always *discrete* and so one can expand solutions in multiple Fourier series with random coefficients. These facts are established and some of their implications exploited in Section IV.B.

With *hyperbolic* systems (3.1) by contrast, one can always use the space D' of Schwartz distributions, and analogous spaces can be defined for vector fields on general manifolds. A few obvious implications of this fact will be derived in Section IV.C.

The preceding special conditions (of compact X and of a hyperbolic system) were considered primarily because of their mathematical implications. In Section IV.D, we turn our attention to the consequences of assuming the physically interesting condition of "homogeneity," especially as regards the "order of growth" of functions.

Next, in Section IV.E, we derive some consequences of assuming that the distribution of the random initial vector field is *normal* (Gaussian), and show that the property of normality is preserved in time under the action of any statistically determinate *linear* system (3.1), and hence in the space of tempered distributions for any Cauchy problem which is regular.

Finally, in Section IV.F, we discuss the conditions under which a nontrivial *time-independent* "statistical mechanics" can be constructed, so that one has an analogue of the ergodic theorem in classical mechanics.

B. Compact Domains

We shall now apply the preceding results to Cauchy problems (3.1) in *compact* spatial domains $X = \mathbf{K}^n$ (with duals $X' = \mathbf{Z}^n$). In this case, the expected total "energy" is usually finite, and $\Lambda_2 = \Lambda_2(X) = [L^2(X)]^q$ is ordinary Hilbert space. Since $X = \mathbf{K}^n$ and the expected energy is finite, one can express almost every RVF as the sum of a suitable multiple Fourier series; thus

$$\mathbf{v}(\mathbf{x}, \omega) = \sum_{\mathbf{k} \in X'} \mathbf{f}(\mathbf{k}, \omega) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{4.1}$$

where $\mathbf{k} = (k_1, \ldots, k_n)$ with $k_i = 0, \pm 1, \pm 2, \ldots$. Using the results of Section II.C and Appendix A, the Borel sets in Λ_2 are easily described, as follows.

Theorem 4.1. *The Borel sets in Λ_2 are generated by the "window sets" of Fourier coefficients in* (4.1) *such that, for some $\mathbf{k} \in \mathbf{Z}^n$ and a_i, b_i, c_i, d_i $(i = 1, \ldots, q)$,*

$$a_i \le \operatorname{Re}\{f_i(\mathbf{k})\} \le b_i, \qquad c_i \le \operatorname{Im}\{f_i(\mathbf{k})\} \le d_i. \tag{4.2}$$

Hence the *regular* probability measures are those defined by compatible measures on the window sets (4.2); we omit the proofs, which are straightforward.

The series (4.1) represents a *normal homogeneous* RVF if the vectors $\mathbf{f}(\mathbf{k}, \omega)$ and $\mathbf{f}(\mathbf{l}, \omega)$ are normally distributed and independent for any $\mathbf{k} \neq \mathbf{l}$.

When X is compact, moreover, $S' = D'$, and so Schultz's theorem (Section III.B) ensures that every regular Cauchy problem (3.1) is deterministically and hence also statistically well set in D' by the corollary to Theorem 3.4. However, even for compact X, the usual norms need not be satisfactory. Relying on [7] for various technical facts, we now investigate this situation in detail.

By [7, Section V], any regular system (3.1) with $q = 1$ does define a C_0-semigroup in $\Lambda_2 = L_2(X)$, hence a deterministically well-set problem there. Moreover it was proved in [7] that one can always construct a "direct integral" norm on the Fourier transform space of the functions of finite "energy," for any regular Cauchy problem. This makes the Cauchy problem well set on the resulting Hilbert space of functions of finite norm = $(\text{energy})^{1/2}$, in the sense that (3.1) acts on this Hilbert space as a C_0-semigroup. Since the initial values for orbits are the whole space, we can combine this result with Theorem 3.2, to obtain

Corollary 4.1. *Any regular system* (3.1) *defines a statistically determinate problem on a suitable Hilbert space (defined by a suitable direct integral*[16] *norm).*

[16] Since $X = \mathbf{K}^n$ is compact, this direct integral is actually a direct sum over $X' = \mathbf{Z}^n$.

But when $q \geq 2$, unfortunately, a regular system (3.1) need *not* define a *deterministically* well-set problem even when X is compact. As we observed in Section III.A, the Cauchy problem for $u_t = v$, $v_t = u_{xx}$ is not well set in the Hilbert space $[L^2(\mathbf{K})]^2$, $\mathbf{K}$ the unit circle, because $(u(0), v(0)) \in [L^2(\mathbf{K})]^2$ need not imply $u(t) \in [L^2(\mathbf{K})]^2$ for $t > 0$. [This problem is, however, *statistically* well set in the preceding space for many typical probabilities and associated energy spectra; see Appendix C.]

We shall now sharpen the result stated in the corollary above for a general class of regular Cauchy problems (3.1) and compact X. As in [7, Section II], the Cauchy problem (3.1) for random initial data $\mathbf{u}(\mathbf{x}, 0, \omega) = \mathbf{v}(\mathbf{x}, \omega)$ given by (4.1) is solved formally by writing

$$\mathbf{u}(\mathbf{x}, t, \omega) = \sum e^{tP(i\mathbf{k})} \mathbf{f}(\mathbf{k}, \omega)\, e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{4.3}$$

Moreover, as was shown in [7], one can obtain a C_0-semigroup on another Hilbert space H by using a *distorted basis* of vectors $\mathbf{b}_1(\mathbf{k}), \ldots, \mathbf{b}_q(\mathbf{k})$ for each $\mathbf{k}$ obtained by a linear transformation $B = B(\mathbf{k})$, relative to which $B^{-1}P(i\mathbf{k})B$ assumes a Jordan canonical form. If

$$\beta(\mathbf{k}) = \sup_{|\mathbf{f}| = |\mathbf{g}| = 1} (|B\mathbf{f}| / |B\mathbf{g}|) \tag{4.4}$$

is the associated *distortion factor*, then the inequality

$$\sum |\mathbf{f}(\mathbf{k}, \omega)|^2 \beta^2(\mathbf{k}) < +\infty \tag{4.5}$$

for almost all ω guarantees that $\mathbf{u} \in [L^2(X)]^q$ almost surely in (4.3) for all $t > 0$. Moreover, by Theorem 4.1 (4.3) defines for each $t > 0$ a Borel mapping of the subset of $\mathbf{v}$ whose Fourier coefficients satisfy (4.5) into $[L^2(X)]^q = H$.

We introduce the (discrete) *energy spectrum* E of the HRVF $\mathbf{u}$ of (4.3) by

$$E(\mathbf{k}) = \int_H |e^{tP(i\mathbf{k})}\, \mathbf{f}(\mathbf{k}, \omega)|^2\, d\mu(\omega), \tag{4.6}$$

where μ is the probability measure associated with $\mathbf{u}$ [8, Part B] and $\mathbf{k}$ is any wave vector in X'. The energy spectrum is the trace of the Fourier coefficients ("transform") of the $q \times q$ *covariance matrix* $\Gamma = \|\Gamma_{ij}\|$ given by

$$\Gamma_{ij}(\mathbf{h}) = \int_H u_i(\mathbf{x} + \mathbf{h}, t, \omega)\, u_j(\mathbf{x}, t, \omega)^*\, d\mu(\omega) \tag{4.7}$$

or, using (4.3) [we assume $\mathbf{f}(\mathbf{k}, \omega)$ and $\mathbf{f}(\mathbf{l}, \omega)$ are independent for $\mathbf{k} \neq \mathbf{l}$],

$$\Gamma_{ij}(\mathbf{h}) = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{h}} \int_H [e^{tP(i\mathbf{k})}\mathbf{f}(\mathbf{k}, \omega)]_i [e^{tP(i\mathbf{k})}\mathbf{f}(\mathbf{k}, \omega)]_j^*\, d\mu(\omega).$$

These notions are only defined in the special case of discrete spectrum (X a torus) considered in this section. In Sections V and VI, we shall say more

about covariance and spectrum in the general case. Here we remark only the following result.

Theorem 4.2. *The Cauchy problem for any regular equation* (3.1) *is statistically determinate in* $\Lambda_2 = [L^2(X)]^q$ *for any torus X and initial HRVF whose energy spectrum satisfies*

$$\sum_{\mathbf{k}} \beta^2(\mathbf{k})\, E(\mathbf{k}) < +\infty, \tag{4.8}$$

where $\beta(\mathbf{k})$ is the distortion factor required to reduce $P(i\mathbf{k})$ to Jordan canonical form.

Proof. By hypothesis, the initial spectrum [(4.6) with $t = 0$] satisfies (4.8). Hence the function

$$\sum_{\mathbf{k}} \beta^2(\mathbf{k}) |f(\mathbf{k}, \omega)|^2 \tag{4.9}$$

is integrable with respect to μ. Hence it must be finite almost everywhere with respect to μ and the desired result now follows from (4.5) and the definition of statistically determinate in Section III.C.

C. Hyperbolic Systems

We next consider *hyperbolic* systems. As defined in Friedman [20, p. 196] (see also Hörmander [29]), these are systems (3.1) which are "regular" in the sense that the roots $\lambda_i(\mathbf{k})$ of the characteristic equation $|P(i\mathbf{k}) - \lambda I| = 0$ have real parts uniformly bounded above and below:

$$A \leq \mathrm{Re}\{\lambda_i(\mathbf{k})\} \leq B, \qquad \text{where} \quad -\infty < A \leq B < +\infty. \tag{4.10}$$

Certainly a hyperbolic system is regular in the sense of (3.6).

For $q = n = 1$, the most general hyperbolic system (3.1) has the form

$$\partial u/\partial t = i \sum_{j=0}^{n} i^j a_j\, \partial^j u/\partial x^j \qquad (\text{all } a_j \text{ real}, \quad i = \sqrt{-1}). \tag{4.11}$$

Another thoroughly understood class of hyperbolic systems with $n = 1$ is defined, for real a_{ij} and b_j, by

$$\sum_{j=0}^{q} a_{ij}(\partial u_i/\partial t + b_j\, \partial u_i/\partial x) = 0, \tag{4.12}$$

where a_{ij} is any nonsingular (real) square matrix.[17]

[17] See Courant–Hilbert, Vol. 2, for a complete analysis of systems of the form (4.12).

Every hyperbolic system (3.1) defines a deterministically well-set Cauchy problem in $D'(X)$. In more detail [20, pp. 197–8] any linear hyperbolic system (3.1) has a *Green's function* $G(\mathbf{x}, t)$ for any real t, which is a Schwartz distribution with compact support. There is thus for any fixed $\mathbf{x}$ and t a finite domain of dependence, which is a *cone* in $(\mathbf{x}, t)$-space for systems having constant coefficients like (3.1).

If $\mathbf{v}(\mathbf{x})$ is of class $C^{(m)}$ with $m = q + n + 2$, then the *convolution* $\mathbf{v}(\mathbf{x}) * G(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t)$ is a classical solution of (3.1). Furthermore, the mapping $v \to u$ defined above is *continuous* on $D'(X)$ [20, p. 78, Theorem 40]. Hence $\mathbf{u}(\mathbf{x}, t)$ is a solution, which is unique in $D'(X)$. These facts have the following immediate corollary.

Theorem 4.3. *For any hyperbolic system* (3.1), *the Cauchy problem is deterministically well set in the space $D'(X)$ of Schwartz distributions.*

To prove even statistical determinacy, we need information about the initial values. Here the facts stated above imply the following result, in which $\Gamma^{(m)}$ denotes the subset of those functions on X whose derivatives of order m exist and are continuous. Thus, the Γ of our previous notation is $\Gamma^{(0)}$.

Theorem 4.4. *The Cauchy problem for any hyperbolic system* (3.1) *is statistically determinate in Γ and hence in any Λ_p, provided that the initial data correspond to $\mathbf{v}(\mathbf{x}) \in \Gamma^{(m)}$ with probability one, where $m = q + n + 2$.*

In the following sections, we shall try to determine sufficient conditions to make $\mathbf{v}(\mathbf{x}) \in S'(X)$ and $\mathbf{v}(\mathbf{x}) \in C^{(m)}$ with probability one. Such conditions, taken in combination with theorems like those just above, imply that weak solutions to random Cauchy problems are actually classical solutions with probability one.

D. Spatial Homogeneity; Wiener's Theorem

As we have mentioned already, the Cauchy problem is not (deterministically) well set in D' or in Λ_p for (regular) parabolic DE's such as the heat equation: uniqueness fails. Hence the approach of Section IV.B also fails for such parabolic DE's in D' and Λ_p.

We shall adopt a different approach in the present section, exploring the consequences of assuming *spatial homogeneity*, an assumption which was made in [8]. Random vector fields $\mathbf{u}(\mathbf{x}, \omega)$, whose measure μ is invariant under translations of X, are also called (strictly) "stationary" [18, p. 94], and the notions of correlation, spectrum, and harmonic analysis have been

developed for such (spatially[18]) homogeneous RVF. We recall the definition of homogeneity from Section II.E.

Definition 4.1. A (regular) measure μ on a function space $E = E(X)$, whose topology is invariant under every translation $\mathbf{x} \to \mathbf{x} + \mathbf{a}$ of X, will be called *homogeneous* when, for any Borel set S in E and every $\mathbf{a} \in X$, if $U_{\mathbf{a}}(S)$ is the set of all $\mathbf{u}(\mathbf{x} + \mathbf{a})$ for $\mathbf{u}(\mathbf{x}) \in S$, $\mu[U_{\mathbf{a}}(S)] = \mu[S]$.

The assumption of homogeneity will also be important in Parts V and VI, while in Part VII, it will yield some new results about random classical solutions of parabolic problems.

Cauchy problems whose initial data are homogeneous in the above sense have a number of fairly obvious properties such as the following.

Theorem 4.5. *If μ is a homogeneous regular probability measure on E, and the well-set Cauchy problem in question is invariant under space translations, then so is the induced measure* (*Section* III.C) *μ_t for any $t > 0$.*

Indeed, one easily verifies

$$\begin{aligned} \mu_t[U_{\mathbf{a}}(S)] &= \mu[T_t^{-1}(U_{\mathbf{a}}(S))] = \mu[U_{\mathbf{a}}(T_t^{-1}(S))] \\ &= \mu[T_t^{-1}(S)] = \mu_t[S]. \end{aligned} \tag{4.13}$$

Theorem 4.5 applies to any system of partial differential equations with constant coefficients (e.g., the Navier–Stokes equations), *provided* that the system defines a statistically well-set Cauchy problem.

Theorem 4.6. *If $\mathscr{G}$ is a semigroup of linear transformations $\{T_t\}$ of E, and if the restriction of each T_t to some fixed Borel subset M of E with $\mu(M) = 1$ is a Borel function, then $\mathscr{G}$ is μ-measurable.*

Proof. For any Borel subset B of E, let $S = B \cap T_t(M)$. Then $T_t^{-1}(S)$ will be a Borel subset of M, while $T_t^{-1}(B - S)$ will be a subset of $E - M$, and hence of μ-measure zero. It follows that $T_t^{-1}(B)$ will be μ-measurable, with $\mu(T_t^{-1}(B)) = \mu(T_t^{-1}(S))$.

It is a corollary that T_t induces from the μ-measure on E a μ_t-measure on every Borel subset of $T_t(M)$. This can be uniquely extended to a perfect probability measure on E.

The next result, whose proof depends on a generalization of a classic theorem of Wiener, shows that homogeneous measures on Λ_p give RVF in

[18] We shall discuss temporal homogeneity in Section IV.F.

S' with probability one. This result allows us, for example, to reduce questions about homogeneous measures on Λ_p to questions in $\Lambda_p \cap S'$, where Fourier analysis can be applied. Actually, we prove more than is stated in Theorem 4.7; see Corollary 4.3.

Theorem 4.7. *Let μ be an admissible homogeneous regular probability measure on Λ_p, $1 \le p < \infty$. Then $\mu(\Lambda_p \cap S') = 1$; that is, almost every sample function is a tempered distribution.*

Proof. Define $w(\mathbf{x})$ by

$$w(\mathbf{x}) = \int_{\Lambda_p} |\mathbf{u}(\mathbf{x}, \omega)|^p \, d\mu(\omega).$$

This is finite for almost every $\mathbf{x}$ since, for any compact set $D \subset \mathbf{R}^n$,

$$\int_D w(\mathbf{x}) \, d\mathbf{x} < \infty$$

by Corollary 2.2. By homogeneity, w must be constant almost everywhere, say $w \equiv L$ a.e. Hence,

$$\int_D w(\mathbf{x}) \, d\mathbf{x} = m(D)L,$$

where $m(D)$ is the volume of D, and so, by Fubini's theorem, since $\mathbf{u}$ is jointly measurable by Theorem 2.6,

$$\int_{\Lambda_p} \left[\frac{1}{m(D)} \int_D |\mathbf{u}(\mathbf{x}, \omega)|^p \, d\mathbf{x}\right] d\mu(\omega) = L. \tag{4.14}$$

Letting $D = S(a) = \{|\mathbf{x}| \le a\}$ be the ball of radius a centered at the origin of $\mathbf{R}^n$, and defining

$$f_a(\omega) = a^{-n} \int_{S(a)} |\mathbf{u}(\mathbf{x}, \omega)|^p \, d\mathbf{x}, \tag{4.15}$$

we see that (4.14) yields

$$\int_{\Lambda_p} f_a(\omega) \, d\mu(\omega) = K \tag{4.16}$$

for all $a > 0$ and in particular for $a = 1, 2, \ldots$. Here, $K = \pi_n L$ where π_n is the volume of the unit ball in $\mathbf{R}^n$.

We claim that, for any fixed $\alpha > 1$, $f_a(\omega) \le a^\alpha$ for all sufficiently large a almost surely. To prove this, let

$$A_a = \{\omega \,|\, f_a(\omega) > a^\alpha\} \tag{4.17}$$

and

$$A = \bigcap_{l=1}^{\infty} \bigcup_{a \ge l} A_a.$$

Then $\mu(A) = 0$, for

$$\mu(A) = \lim_{l \to \infty} \mu\left(\bigcup_{a \ge l} A_a\right) \le \overline{\lim_{l \to \infty}} \sum_{a \ge l} \mu(A_a). \tag{4.18}$$

But (4.16) and the definition (4.17) together with the fact that $f_a \ge 0$ imply

$$\mu(A_a) \le K/a^\alpha \qquad \text{for all} \quad a. \tag{4.19}$$

Since $\sum_a a^{-\alpha}$ converges for $\alpha > 1$, the right-hand side of (4.18) is zero. Thus, almost surely, ω is not in A, whence $f_a(\omega) \le a^\alpha$ for sufficiently large a. Hence,

$$\overline{\lim_{a \to \infty}} \, (f_a(\omega)/a^\alpha) < \infty,$$

almost surely. That is,

$$\overline{\lim_{a \to \infty}} \, (1/a)^{n+\alpha} \int_{S(a)} |\mathbf{u}(\mathbf{x}, \omega)|^p d\mathbf{x} < \infty, \tag{4.20}$$

almost surely.

The desired result now follows from a generalization of one of Norbert Wiener's basic tools [54, p. 138, Theorem 20], which we state and prove separately. We let Σ be the unit sphere $|\mathbf{x}| = 1$ in $\mathbf{R}^n$, $d\sigma$ surface Lebesgue measure on Σ, $\rho = |\mathbf{x}|$, and $\boldsymbol{\xi} = \mathbf{x}/\rho \in \Sigma$.

Lemma 4.1. *If $\mathbf{u}(\mathbf{x}) \in \Lambda_p(\mathbf{R}^n)$ satisfies, for some $r > 0$,*

$$\int_{S(a)} |\mathbf{u}(\mathbf{x})|^p \, dm(\mathbf{x}) \le Ma^r, \tag{4.21}$$

for all $a > 0$ sufficiently large, where $S(a)$ is the ball $|\mathbf{x}| \le a$, then

$$\mathbf{u}(\mathbf{x})/(1 + |\mathbf{x}|)^q \in L^p(\mathbf{R}^n) \qquad \text{for all} \quad q > r/p. \tag{4.22}$$

Proof. By the Fubini theorem, in the above notation

$$g(\rho) = \int_\Sigma |\mathbf{u}(\rho\boldsymbol{\xi})|^p \, d\sigma(\boldsymbol{\xi})$$

exists for almost all $\rho > 0$. Moreover,

$$\int_{S(a)} |\mathbf{u}(\mathbf{x})|^p \, dm(\mathbf{x}) = \int_0^a g(\rho)\rho^{n-1} \, d\rho = G(a)$$

and

$$\int_{S(a)} |\mathbf{u}(\mathbf{x})|^p (1 + |\mathbf{x}|)^{-pq} \, dm(\mathbf{x}) = \int_0^a \frac{g(\rho)\rho^{n-1}}{(1 + \rho)^{pq}} \, d\rho = J(a). \tag{4.23}$$

Integrating (4.23) by parts,

$$J(a) = \frac{G(a)}{(1+a)^{pq}} + pq \int_0^a \frac{G(\rho)}{(1+\rho)^{pq+1}} \, d\rho.$$

Hence by (4.21) we have, for all large enough a,

$$J(a) \leq \frac{Ma^r}{(1+a)^{pq}} + Mpq \int_0^a \frac{\rho^r}{(1+\rho)^{pq+1}} \, d\rho.$$

Since $J(a)$ is an increasing function of a and bounded if $pq > r$, the conclusion follows.

Corollary 4.2. *If, for some* $K(\xi) \in L(\Sigma)$, *and* $r > 0$,

$$\int_0^a |\mathbf{u}(\rho\xi)|^p \, d\rho \leq (1+a)\, K(\xi), \tag{4.24}$$

then (4.21) *holds with* $r = n$.

Proof. From (4.24), we deduce

$$\int_0^a |\mathbf{u}(\rho\xi)|^p \rho^{n-1} \, d\rho \leq K(\xi)\,(1+a)^n.$$

Hence, by an application of Fubini's theorem,

$$\int_{S(a)} |\mathbf{u}(\mathbf{x})|^p \, dm(\mathbf{x}) = \int_\Sigma \int_0^a |\mathbf{u}(\rho\xi)|^p \rho^{n-1} \, d\rho \, d\sigma(\xi)$$

$$\leq (1+a)^n \int_\Sigma K(\xi)\, d\sigma \leq M(1+a)^n \leq La^n$$

for a sufficiently large, the inequalities (4.23), etc., can now be repeated. [N.B. The assertion is that the factor a^r in (4.21) can be replaced by $(1+a)^r$, which is more convenient for some applications.]

To complete the proof of Theorem 4.7, we need only remark that, by combining the last lemma with (4.20), we have, almost surely,

$$\mathbf{v}(\mathbf{x}, \omega) = \frac{\mathbf{u}(\mathbf{x}, \omega)}{(1 + |\mathbf{x}|^2)^{q/2}} \in L_p(\mathbf{R}^n), \tag{4.25}$$

whenever $pq > (n + \alpha)$. Thus $\mathbf{v} \in L_p(\mathbf{R}^n) \subset S'(\mathbf{R}^n)$ almost surely, and thus $\mathbf{u} = (1 + |\mathbf{x}|^2)^{q/2}\mathbf{v} \in S'(\mathbf{R}^n)$ almost surely.

We note the following corollary to the above proof. This will be useful in Section VII.C. It states that, with probability one, a homogeneous RVF on Λ_p cannot grow very rapidly at infinity.

Corollary 4.3. *Let μ be a regular admissible homogeneous probability on $\Lambda_p(\mathbf{R}^n)$. Then, almost surely*

$$\frac{\mathbf{u}(\mathbf{x}, \omega)}{(1 + |\mathbf{x}|^2)^s} \in L_p(\mathbf{R}^n),$$

whenever $s > (n+1)/2p$.

Proof. Fix $s > (n+1)/2p$, and choose $\alpha > 1$ so small that $s > (n+\alpha)/2p$. Lemma 4.1 then combines with (4.20) to show directly that $\mathbf{u}(\mathbf{x}, \omega)/(1 + |\mathbf{x}|^2)^s \in L_p(\mathbf{R}^n)$.

E. Normal Probability Measures

For *linear* Cauchy problems with random initial values, one can say more. Following Fréchet, we define a probability measure μ on a linear topological space E to be *normal* when, for every continuous linear functional α on E, the regular probability measure μ_α induced on the real or complex field (as in Section III.C) by the map α is a normal (Gaussian) measure.

For various reasons (see [8] and Section II.C), one is often especially interested in normal homogeneous random vector fields (normal HRVF). [This is not always the case; thus real temperatures cannot be normally distributed except in the trivial deterministic case of zero variance, because $u(\mathbf{x}, \omega) \geq u_0$ (absolute zero temperature) otherwise holds with probability zero.]

Let μ be a normal probability measure on a topological vector space E, let $T\colon E \to F$ be a continuous linear transformation, and let ν be the probability measure induced as in Section III.C on F by T from μ. If α is any continuous linear functional on F, then the composite $\beta = \alpha \circ T$ is a continuous linear functional on E. Moreover the measure on the real or complex numbers induced by α from ν is the same as the measure induced by β from μ: in symbols, $\alpha[\nu] = \alpha[T[\mu]] = \beta[\mu]$. Since $\beta[\mu]$ is normal on $\mathbf{R}$ or $\mathbf{C}$ by hypothesis, it follows that so is $\alpha[\nu]$; hence ν is normal on F.

It will be essential in Section VI to know that normality is preserved under the action of a system (3.1). This we now prove.

Theorem 4.8. *Suppose the Cauchy problem* (3.1) *is statistically determinate in the function space E for a normal probability measure μ. Then μ_t as defined in Section* III.C *is normal for all $t > 0$.*

Proof. By the definition of statistical determinacy, there is a subset $M \subset E$ with $\mu(M) = 1$ such that (3.1) has a unique solution for any initial value in M. Further, the mapping $T_t : M \to E$ given by $T_t v = u(t)$ where $u(t)$ is the unique solution of (3.1) for initial value v, is Borel measurable. Hence the measure $\mu_t = \mu \circ T_t^{-1}$ is a Borel probability measure on E.

Since (3.1) is *linear*, the mappings T_t are linear transformations. By the remarks preceding the theorem, then, μ_t is also normal.

Note that the preceding discussion does *not* imply that a given normal measure μ on E is determined by the associated μ_α. However, this is true in certain cases (see Section V).

Again, Fréchet's idea and the preceding discussion apply just as well to linear *Borel* mappings (including functionals) as they apply to *continuous* linear mappings. Hence we have the following result, which has implications in many important physical problems.

Corollary 4.4. *Let $\mathcal{G}$ be a semigroup of linear transformations T_t of a function space E associated with a statistically determinate Cauchy problem* (3.1). *Then every regular normal probability measure μ for initial values defines a regular normal probability measure μ_t on the* $\mathbf{u}(\mathbf{x}, t, \omega)$, *for any $t > 0$.*

F. Time-Independent Measures

An interesting question concerns the determination of those nontrivial Cauchy problems with random initial values which have time-independent measures. Such measures occur typically with *conservative* systems of continuum mechanics—e.g., for $u_t = u_x$ (convection equation); for $u_t = v_x$, $v_t = u_x$ (wave equation); and for $u_t = v_{xx}$, $v_t = -u_{xx}$ (vibrating beam equations). When they occur, one can apply many concepts of ergodic theory, as one of us has pointed out elsewhere [34]. We now show that no asymptotically stable system (3.1) can admit a nontrivial time-independent measure.

Accordingly, let E be any locally convex topological linear space, and let T be any *one-one*[19] linear operator acting on E. Let μ be any regular probability measure on E which is invariant under T. We call T *asymptotically stable* when, for all $v \in E$, $\lim_{n\to\infty} T^n(v) \to 0$. A special case arises when we have a system (3.1) which is *strictly stable* (as defined in Section III.B, $\Lambda(P) < 0$), and which defines a C_0-semigroup $\{T_t\}_{t\geq 0}$ on E. (This is always the case if $E \equiv S'$, by Theorem 3.1.) Setting $T = T_1 = e^{P(i\mathbf{k})}$, it follows from the semigroup property that $T^n = T_n$, and it follows from strict stability that $T_n v \to 0$ as $n \to \infty$, for all $v \in E$.

[19] This hypothesis plays an essential role in the proof of Poincaré's recurrence theorem, *Acta Math.* **13** (1890), 67; C. Caratheodory, *Berliner Ber.* (1919), 580–4. For a more recent version, see M. Kac, *Bull. Amer. Math. Soc.* **33** (1947), 1002–10, where the condition $\mu A > 0$ is omitted at the foot of p. 1005. Note that the hypothesis of one-oneness is always fulfilled in Cauchy problems (3.1).

Lemma 4.2. *If μ is a time-independent probability measure on E which is invariant under an asymptotically stable linear transformation T, then $\mu(V) = 1$ for any neighborhood V of* 0.

Proof. Since μ is regular, V and its complement V' are both measurable. By Poincaré's recurrence theorem, if $\mu(V') > 0$, then almost every $v \in V'$ would recur to V' infinitely often, contradicting the hypothesis $T^n(v) \to 0$.

Theorem 4.9. *If $0 \in E$ has a countable set $\{V_k\}$ of neighborhoods with $\bigcap V_k = \{0\}$, and the regular probability measure μ is invariant under T, any asymptotically stable linear operator acting on E, then $\mu(0) = 1$.*

This result applies to the heat equation $u_t = \nabla^2 u$ in $E = L^2(-\infty, \infty)$, even though this DE is not strictly stable.

Proof. Let $W_n = \bigcap_{k=1}^n V_k$; then $\mu(W_n) = 1$ by Lemma 4.2. Hence $\mu(0) = \mu(\bigcap W_n) = 1$. [Note that the hypothesis of Theorem 4.9 is fulfilled in the spaces Λ_p, Γ, S', D'.]

Heuristically, a *conservative* system (3.1) is a system such that all matrices $P(i\mathbf{k})$ are diagonalizable and have pure imaginary eigenvalues. This is the condition that all the components of formal Fourier transforms $\mathbf{f}(\mathbf{k}, t)$ of solutions $\mathbf{u}(\mathbf{x}, t)$ of (3.1) should be bounded away from 0 and ∞ as $t \to \infty$. Hence, physical intuition suggests that this condition [essentially $\Lambda(P) = \Lambda(-P) = 0$] should be necessary and sufficient for "energy" conservation, and hence (perhaps) for the existence of a nontrivial, time-invariant measure.

V. Correlation and Spectrum

A. The Correlation Matrix

Two of the most important and characteristic properties of a homogeneous random function $u(x, \omega)$ are its correlation function and its spectrum. These were originally defined by Wiener [54, p. 150] for individual sample functions, essentially as follows. The *correlation function* of $u(x)$ is

$$R(h) = \lim_{X\to\infty} \frac{1}{2X} \int_{-X}^{X} u(x+h)\, u^*(x)\, dx, \qquad (5.1)$$

where $u^*(x)$ is the complex conjugate of $u(x)$; the *spectrum* of $u(x)$ has for (spectral) energy density at the wave-number k

$$\sigma(k) = \frac{1}{2\pi} \lim_{H\to\infty} \int_{-H}^{H} R(h)e^{-ihk}\, dh, \tag{5.2}$$

where we have assumed a continuous spectrum for simplicity.

Remark. Here and in Part VI, many of the proofs are restricted to the case $X = \mathbf{R}^n$. This is mainly because the theory of distributions has been developed primarily for this case, but also (for metric transitivity) spectral matrix measures can be continuous only in this case.

In the metrically transitive case to be considered in Section V.F, this definition [by means of the space-average (5.1) or, in the case of a time series, the corresponding time-average] is equivalent to an *average with respect to probability measure.* This latter average has the advantage of always *existing* for any random vector-valued function (RVF) $\mathbf{u}(\mathbf{x}, \omega)$ which is *admissible in* $\Lambda = \Lambda_2(X, q)$ in the sense defined in (2.15). The condition for this is that, for any bounded set D,

$$\int_\Lambda \int_D |\mathbf{u}(\mathbf{x}, \omega)|^2\, dm(\mathbf{x})\, d\mu(\omega) < +\infty. \tag{5.3}$$

A quite satisfactory theory of correlation and spectrum was developed in [8] for RVF which are admissible in Λ. We shall summarize this theory next.

Definition 5.1. For any admissible RVF $\mathbf{u}(\mathbf{x}, \omega)$, we define (i) its average or *mean* as

$$\mathbf{f}(\mathbf{x}) = \int_\Lambda \mathbf{u}(\mathbf{x}, \omega)\, d\mu(\omega), \tag{5.4}$$

and (ii) its *covariance matrix* as $\Gamma = \|\Gamma_{jk}\|$, where

$$\Gamma_{jk}(\mathbf{x}, \mathbf{y}) = \int_\Lambda u_j(\mathbf{x}, \omega)\, u_k^*(\mathbf{y}, \omega)\, d\mu(\omega). \tag{5.5}$$

Lemma 5.1. *The mean and covariance of any admissible RVF are well defined.*

Proof. First notice that, since μ is admissible, the energy expectation (5.3) is finite for any bounded set D. This shows that $\mathbf{u}(\mathbf{x}, \omega) \in [L^2(D \times \Lambda_2)]^q$, and hence, by the Fubini theorem, the integral in (5.5) converges for almost all $\mathbf{x}$ and $\mathbf{y}$. Further, by the Schwarz inequality (using the fact that μ is a finite measure), the integral in (5.4) converges a.e. by the same reasoning.

The covariance matrix has some obvious properties. For example, it is Hermitian symmetric:

$$\Gamma_{jk}(\mathbf{x}, \mathbf{y}) = \Gamma_{kj}^*(\mathbf{y}, \mathbf{x}). \tag{5.6}$$

Furthermore, the matrix $\Gamma(\mathbf{x}, \mathbf{y})$ is positive semidefinite for each choice of $\mathbf{x}$ and $\mathbf{y}$ in X. That is,

$$\sum_{j,k=1}^{q} \Gamma_{jk}(\mathbf{x}, \mathbf{y})\, \xi_j \xi_k^* \geq 0 \tag{5.7}$$

for all elements $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_q) \in \mathbf{C}^q$. $\Gamma(\mathbf{x}, \mathbf{y})$ is also of *positive type.* That is,

$$\sum_{jk} \sum_{\alpha\beta} \Gamma_{jk}(\mathbf{x}^{(\alpha)}, \mathbf{x}^{(\beta)})\, \xi_j^{(\alpha)} \xi_k^{(\beta)*} \geq 0, \tag{5.8}$$

whatever the values of $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)} \in X$ and $\boldsymbol{\xi} = (\xi_1^{(1)}, \ldots, \xi_1^{(m)}, \ldots, \xi_q^{(m)}) \in \mathbf{C}^{mq}$. Notice that (5.8) implies (5.7) by specializing to $m = 1$. These facts are established in [8, Part C, Section 1].

Lemma 5.2. (See [8, Part C, Section 8].) *If the probability measure μ is homogeneous as defined in Section* II.E, *then the mean is constant and* $\Gamma(\mathbf{x}, \mathbf{y}) = R(\mathbf{x} - \mathbf{y})$ *depends only on the difference of its arguments.*

Proof. Let $\mathbf{x} - \mathbf{y} = \mathbf{h}$ so that $\mathbf{x} = \mathbf{y} + \mathbf{h}$. Then

$$\begin{aligned}\Gamma_{jk}(\mathbf{y} + \mathbf{h}, \mathbf{y}) &= \int_\Lambda u_j(\mathbf{y} + \mathbf{h}, \omega)\, u_k^*(\mathbf{y}, \omega)\, d\mu(\omega)\\ &= \int_\Lambda \tau_\mathbf{y} u_j(\mathbf{h}, \omega) \tau_\mathbf{y} u_k^*(\mathbf{0}, \omega)\, d\mu(\omega),\end{aligned}$$

where $\tau_\mathbf{y}$ is translation by $\mathbf{y}$ as defined in Section II.E. Thus,

$$\Gamma_{jk}(\mathbf{y} + \mathbf{h}, \mathbf{y}) = \int_\Lambda u_j(\mathbf{h}, \omega)\, u_k^*(\mathbf{0}, \omega)\, d\mu \circ \tau_\mathbf{y}^{-1}(\omega),$$

and, since μ is homogeneous, $\mu \circ \tau_\mathbf{y}^{-1} = \mu$. Hence,

$$\Gamma_{jk}(\mathbf{y} + \mathbf{h}, \mathbf{y}) = \int_\Lambda u_j(\mathbf{h}, \omega)\, u_k^*(\mathbf{0}, \omega)\, d\mu(\omega) = \Gamma_{jk}(\mathbf{h}, \mathbf{0}),$$

showing that $\Gamma(\mathbf{x}, \mathbf{y})$ is only a function of $\mathbf{x} - \mathbf{y}$.

Similarly, let $\mathbf{f}$ be the mean of μ and f_j its jth component. Then

$$\begin{aligned}f_j(\mathbf{x}) &= \int_\Lambda u_j(\mathbf{x}, \omega)\, d\mu(\omega) = \int_\Lambda \tau_\mathbf{x} u_j(\mathbf{0}, \omega)\, d\mu(\omega)\\ &= \int_\Lambda u_j(\mathbf{0}, \omega)\, d\mu \circ \tau_\mathbf{x}^{-1}(\omega) = \int_\Lambda u_j(\mathbf{0}, \omega)\, d\mu(\omega)\\ &= f_j(\mathbf{0}).\end{aligned}$$

Thus f_j is constant for all j.

Definition 5.1. If the RVF is admissible and homogeneous, the $(q \times q)$-matrix $R = \|R_{jk}\|$ of Lemma 5.2 is called the *correlation matrix*. Thus,

$$R_{jk}(\mathbf{h}) = \int_\Lambda u_j(\mathbf{x}+\mathbf{h}, \omega)\, u_k^*(\mathbf{x}, \omega)\, d\mu(\omega) \tag{5.9}$$

for any $\mathbf{x} \in X$.

In the *metrically transitive* case to be discussed in Section V.F, for $X = \mathbf{R}^n$, we have

$$R_{jk}(\mathbf{h}) = \lim_{A\to\infty} (2A)^{-n} \int_{-A}^{A} \cdots \int_{-A}^{A} u_j(\mathbf{x}+\mathbf{h}, \omega)\, u_k^*(\mathbf{x}, \omega)\, dx \tag{5.10}$$

and so, in this case, (5.9) is equivalent to Wiener's definition a.e.

The following result characterizes the correlation matrix of an admissible homogeneous RVF (see Birkhoff and Kampé de Fériet [8]).

Theorem 5.1. *A $(q \times q)$-matrix function $R(\mathbf{h}) = \|R_{jk}(\mathbf{h})\|$ is the covariance of an admissible homogeneous random vector field in Λ if and only if it is continuous and positive definite.*

In the homogeneous case, we can define the spectral matrix measure of the RVF. For it follows from Theorem 5.1 and Bochner's theorem that any continuous positive definite matrix function is the Fourier–Lebesgue transform of a matrix of measures $S = \|S_{jk}\|$ defined on the Borel sets in wave-vector space $X' = \mathbf{Z}^s\mathbf{R}^{n-s}$. That is,

$$R_{jk}(\mathbf{h}) = \int_{X'} e^{i\mathbf{l}\cdot\mathbf{h}}\, dS_{jk}(\mathbf{l}). \tag{5.11}$$

In general, the measures S_{jk} are complex valued. For a careful discussion of this result, we refer the reader to [8, Part C, §§1–3].

Definition 5.2. An admissible homogeneous RVF on Λ will be said to have the (complex) *spectral matrix measure* $S = \|S_{jk}\|$ if and only if its correlation matrix satisfies (5.11). [Note that, by Fourier–Stieltjes transform theory, there is at most one $S_{jk}(\mathbf{l})$ satisfying (5.11) for given $R_{jk}(\mathbf{h})$, hence for given μ.]

A *Hermitian matrix measure* is a matrix of σ-additive set functions on a Borel field $\mathscr{B}$ over a set M which is defined and nonnegative definite for all Borel sets B in $\mathscr{B}$. That is,

$$\sum_{jk} S_{jk}(B)\, \xi_j \xi_k^* \geq 0 \tag{5.12}$$

for all $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_q) \in \mathbf{C}^q$ and $B \in \mathscr{B}$.

Theorem 5.2. *The class of continuous positive definite matrix functions on $X = \mathbf{K}^s\mathbf{R}^{n-s}$ is identical with the class of Fourier–Lebesgue transforms* (5.11) *of bounded Hermitian matrix measures on the Borel sets of wave-vector space $X' = \mathbf{Z}^s\mathbf{R}^{n-s}$.*

Proof. This is [8, Part C, Theorem 2].

Definition 5.3. The *energy spectrum* of an admissible homogeneous RVF on Λ is the trace of its spectral matrix measure and so is a positive measure on the Borel sets of wave-vector space. The trace of the correlation matrix is called the *energy correlation*.

Compact case

The case of $X = \mathbf{K}^n$ of a (compact) torus was already considered briefly in Section IV.B. In this case, we can write

$$\mathbf{u}(\mathbf{x}, \omega) = \sum \mathbf{f}(\mathbf{k}, \omega) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{5.13}$$

where the summation is over all $\mathbf{k} \in X' = \mathbf{Z}^n$ (where $\mathbf{Z}$ is the integers). If we suppose that $\mathbf{f}(\mathbf{k}, \omega)$ and $\mathbf{f}(\mathbf{l}, \omega)$ are independently distributed, then $\mathbf{u}$ is homogeneous and the correlation matrix is given by, for $\mathbf{h} \in X$,

$$R_{jk}(\mathbf{h}) = \sum_{\mathbf{l}\in\mathbf{Z}^n} e^{i\mathbf{l}\cdot\mathbf{h}} \int_\Lambda f_j(\mathbf{l}, \omega) f_k^*(\mathbf{l}, \omega)\, d\mu(\omega). \tag{5.14}$$

Hence, we obtain a *discrete* spectrum for which

$$S_{jk}(B) = \sum_{\mathbf{l}\in B} c_{jk}(\mathbf{l}), \tag{5.15}$$

where

$$c_{jk}(\mathbf{l}) = \int_\Lambda f_j(\mathbf{l}, \omega) f_k^*(\mathbf{l}, \omega)\, d\mu(\omega) \tag{5.16}$$

and $B \subset X' = \mathbf{Z}^n$.

Notice that the energy spectrum in this case is

$$\begin{aligned} E(\mathbf{l}) &= \sum_{j=1}^{q} S_{jj}(\mathbf{l}) = \sum_{j=1}^{q} c_{jj}(\mathbf{l}) \\ &= \int_\Lambda \sum_j |f_j(\mathbf{l}, \omega)|^2\, d\mu(\omega) = \int_\Lambda |\mathbf{f}(\mathbf{l}, \omega)|^2\, d\mu(\omega), \end{aligned} \tag{5.17}$$

and this agrees with the earlier definition (4.6) for $t = 0$.

B. Extension to Tempered Distributions

In the next four sections, we shall establish analogues of the results of Section V.A in the space $S'(X)$ of vector-valued tempered distributions. Our results will play an essential role in Section VI, where the evolution of the spectrum of a random tempered distribution (RTD) under the action of a system of the form (3.1) will be determined. Our exposition will follow [10, Chapter IV]; some of the most technical arguments will be deferred to Appendix D.

Many of the underlying ideas and techniques are most easily grasped in the special case $X = \mathbf{R}$ of an ordinary *stochastic process*, and have been treated in this context by Gel'fand and Vilenkin [22, Chapter III], with special emphasis on Wiener processes. The extension of these ideas and techniques to (generalized) RVF ("random fields") is sketched in Sections 5.5 and 5.6 of Chapter III of this treatise where consequences of assuming homogeneity and isotropy are stated (cf. also Dudley [19]).

Because tempered distributions are not ordinary point functions in a locally Euclidean space, one cannot in general define the covariance matrix of a RTD by (5.5). Instead (cf. [22, pp. 247, 298]), in general one must define its entries as continuous bilinear functionals on the Cartesian product $\mathscr{S}(X) \times \mathscr{S}(X)$ of the space $\mathscr{S}(X)$ of *test functions* (infinitely differentiable and rapidly decaying at infinity) with itself. (See Section II.B.)

In the important special case that an RTD is in Λ_2 with probability one, the covariance as defined below (and in [22]) can be interpreted as an ordinary covariance as described by (5.5); see Section V.E.

Suppose, then, that μ is an admissible probability measure on $S'(X)$ as defined in Section II.E. We will say, in this case, that μ defines a *random tempered distribution* (RTD). As was noted in Section II.B, $S'(X) = \mathscr{S}'(X) \times \cdots \times \mathscr{S}'(X)$ (q copies), and so we can compose μ with the projection P_i on the ith coordinate to get a probability measure μ_i on $\mathscr{S}'(X)$. Now μ_i is an admissible measure on $\mathscr{S}'(X)$, for if $\phi \in \mathscr{S}(X)$, then certainly

$$\begin{aligned}\int_{\mathscr{S}'(X)} |U(\phi)|^2\, d\mu_i(U) &= \int_{S'(X)} |P_i T(\phi)|^2\, d\mu(T),\\ &= \int_{S'(X)} |T_i(\phi)|^2\, d\mu(T)\\ &\le \int_{S'(X)} |T(\phi)|^2\, d\mu(T),\end{aligned} \tag{5.18}$$

where T_i is the ith component of $T \in S'(X)$ and $U \in \mathscr{S}'(X)$. This inequality shows that $\phi \in L^2(\mu_i)$ and that the inclusion mapping $\mathscr{S}'(X) \subset L^2(\mu_i)$ is continuous.

Remark. The formidable-looking integrals in (5.18) are actually ordinary Lebesgue integrals with respect to the measure μ on the function "space" of all mappings $\phi: T \to T(\phi)$. This is because each such mapping takes $S'(X)$ into $\mathbf{C}^q$, whence the mapping $T \mapsto |T(\phi)|^2$ is real valued.

Definition 5.4. The *mean* of the RTD μ is the element $M \in S'(X)$ such that if $\phi \in \mathscr{S}(X)$, then

$$M(\phi) = \int_{S'(X)} T(\phi)\, d\mu(T), \tag{5.19}$$

where the integral is taken componentwise. That is,

$$M_i(\phi) = \int_{S'(X)} T_i(\phi)\, d\mu(T) = \int_{S'(X)} T_i(\phi)\, d\mu_i(T_i),$$

where M_i is the ith component of M. (For this definition, and the definition of the covariance matrix which follows, see [22, p. 298].)

Lemma 5.3. *The mean exists as a well-defined element of $S'(X)$.*

Proof. By the Schwarz inequality

$$|M(\phi)|^2 \le \int_{S'(X)} |T(\phi)|^2\, d\mu(T) \int_{S'(X)} d\mu(T) = \int_{S'(X)} |T(\phi)|^2\, d\mu(T) = \|\phi\|_2{}^2,$$

where $\| \ \|_2$ is the norm in $L^2(\mu)$. Since μ is admissible, we know that $\phi_n \to 0$ (weakly or strongly, it is the same in the Montel space $\mathscr{S}(X)$ by [52, p. 358, Corollary 2]) implies that $\|\phi_n\|_2 \to 0$. Hence M is sequentially continuous at zero for the weak or strong topology. Since M is clearly linear, it is thus continuous for sequential convergence in the Fréchet space $\mathscr{S}(X)$ and so must be continuous for the strong (metrizable) topology by [52, Proposition 8.5].

Definition 5.5. The *covariance matrix* of the RTD μ is the $(q \times q)$-matrix of continuous "sesquilinear" forms $\Gamma_{jk}(\phi, \psi)$ defined for $\phi, \psi \in \mathscr{S}(X)$ by

$$\Gamma_{jk}(\phi, \psi) = \int_{S'(X)} T_j(\phi)\, T_k(\psi)^*\, d\mu(T), \tag{5.20}$$

where T_k is the kth component of the vector-valued distribution T and z^* is the complex conjugate of z. (Since T_j and T_k are linear functionals on $\mathscr{S}(X)$, $\Gamma_{jk}(\phi, \psi)$ is linear in ϕ and conjugate linear in ψ, which is what "sesquilinear" means; see, for example, [52, p. 60].)

Lemma 5.4. *Γ_{jk} is a well-defined continuous sesquilinear form on $\mathscr{S}(X) \times \mathscr{S}(X)$ in the strong topology.*

Proof. The Γ_{jk} are clearly sesquilinear. We check that they are well defined and continuous:

$$\begin{aligned}|\Gamma_{jk}(\phi, \psi)|^2 &= \left|\int_{S'(X)} T_j(\phi)\, T_k(\psi)^*\, d\mu(T)\right|^2 \\ &\leq \int_{S'(X)} |T_j(\phi)|^2\, d\mu(T) \int_{S'(X)} |T_k(\psi)|^2\, d\mu(T) \\ &\leq \int_{S'(X)} |T(\phi)|^2\, d\mu(T) \int_{S'(X)} |T(\psi)|^2\, d\mu(T) \\ &\leq \|\phi\|_2{}^2 \|\psi\|_2{}^2.\end{aligned}$$

Thus Γ_{jk} is certainly well defined and sequentially continuous in each variable separately. But a form continuous in each variable separately on a product of Fréchet spaces is jointly continuous, i.e., continuous as a mapping on the product topology by [52, p. 354]. Thus Γ_{jk} is contnuous in the strong product topology on $\mathscr{S}(X) \times \mathscr{S}(X)$.

If we view $T(\phi)$ as a column vector, and let $T(\phi)^H$ denote the conjugate transpose of $T(\phi)$, then we have the following formula for the matrix Γ:

$$\Gamma(\phi, \psi) = \int_{S'(X)} T(\phi)\, T(\psi)^H\, d\mu(T),$$

where $T(\phi)\, T(\psi)^H$ is matrix multiplication and the integral is taken componentwise.

We compute some elementary properties of the covariance matrix. Clearly Γ is Hermitian symmetric:

$$\Gamma_{jk}(\phi, \psi) = \Gamma_{kj}(\psi, \phi)^*. \tag{5.21}$$

Furthermore, $\Gamma(\phi, \phi)$ is nonnegative definite for $\phi \neq 0$. For if $\mathbf{z}$ is a complex q-vector, and if $\mathbf{z}^H$ denotes the conjugate transpose of $\mathbf{z}$, and $\mathbf{z}T(\phi) = \sum_1^q z_j T_j(\phi)$, then

$$\begin{aligned}\mathbf{z}\Gamma(\phi, \phi)\mathbf{z}^H &= \int_{S'(X)} \mathbf{z}T(\phi)\, T(\phi)^H \mathbf{z}^H\, d\mu(T) = \int_{S'(X)} \mathbf{z}T(\phi)\, [\mathbf{z}T(\phi)]^*\, d\mu(T) \\ &= \int_{S'(X)} |\mathbf{z}T(\phi)|^2\, d\mu(T) \geq 0.\end{aligned} \tag{5.22}$$

More generally, Γ is of *positive type* in the sense that if $\phi_1, \ldots, \phi_r$ are elements of $\mathscr{S}(X)$, then the $(qr \times qr)$-matrix Q, whose $(q \times q)$-submatrices are given by $\Gamma(\phi_\alpha, \phi_\beta)$ $(1 \leq \alpha, \beta \leq r)$, is Hermitian definite. To see this, let

$$\mathbf{Z} = (\mathbf{Z}_1, \cdots, \mathbf{Z}_r) = (z_{11}, \ldots, z_{q1}; \ldots; z_{1r}, \ldots, z_{qr}) \in \mathbf{C}^{qr}$$

be the row vector obtained by concatenating the row vectors $\mathbf{Z}_\beta = (z_{1\beta}, \ldots, z_{q\beta})$, $1 \leq \beta \leq r$. Then $\mathbf{Z}Q\mathbf{Z}^H$ is a scalar, and we have the following inequality:

$$\begin{aligned}\mathbf{Z}Q\mathbf{Z}^H &= \sum_{\alpha,\beta} \mathbf{Z}_\alpha \Gamma(\phi_\alpha, \phi_\beta) \mathbf{Z}_\beta{}^H \\ &= \sum_{\alpha,\beta} \int_{S'(X)} \mathbf{Z}_\alpha T(\phi_\alpha)\, T(\phi_\beta)^H \mathbf{Z}_\beta{}^H\, d\mu(T) \\ &= \sum_{\alpha,\beta} \int_{S'(X)} \mathbf{Z}_\alpha T(\phi_\alpha)\, [\mathbf{Z}_\beta T(\phi_\beta)]^H\, d\mu(T) \\ &= \int_{S'(X)} \sum_\alpha \mathbf{Z}_\alpha T(\phi_\alpha) \left[\sum_\beta \mathbf{Z}_\beta T(\phi_\beta)\right]^H d\mu(T) \\ &= \int_{S'(X)} \left|\sum_\alpha \mathbf{Z}_\alpha T(\phi_\alpha)\right|^2 d\mu(T) \geq 0.\end{aligned} \tag{5.23}$$

C. Homogeneous Random Tempered Distributions

In this section we study the consequences of assuming that the RTD defined by μ is spatially homogeneous. We will see that this again allows us to define a correlation matrix, just as in A.

We recall from Section II.E that a measure on $S'(X)$ is homogeneous when it is invariant under translations of the underlying domain X.

Lemma 5.5. *Let μ be admissible and homogeneous. Then the mean of μ is a constant complex vector.*

Proof. Let $M \in S'(X)$ be the mean of μ. Then if $\mathbf{h} \in X$, $M(\tau_\mathbf{h} \phi) = M(\phi)$ for all $\phi \in \mathscr{S}(X)$, where $\tau_\mathbf{h}$ is translation by $\mathbf{h}$ as defined in Section II.E. For

$$\begin{aligned}M(\tau_\mathbf{h} \phi) &= \int_{S'(X)} T(\tau_\mathbf{h} \phi)\, d\mu(T) \\ &= \int_{S'(X)} \tau_{-\mathbf{h}} T(\phi)\, d\mu(T) \\ &= \int_{S'(X)} U(\phi)\, d(\mu \circ \tau_{-\mathbf{h}}^{-1})(U).\end{aligned}$$

But $\tau_\mathbf{x}^{-1} = \tau_{-\mathbf{x}}$, and so $\mu \circ \tau_{-\mathbf{h}}^{-1} = \mu \circ \tau_\mathbf{h} = \mu$, since μ is homogeneous. Thus

$$M(\tau_h \phi) = \int_{S'(X)} U(\phi)\, d\mu(U) = M(\phi),$$

whence the mean M is invariant under space translation: $\tau_\mathbf{h} M = M$ for all $\mathbf{h} \in X$. It follows from [30, p. 337] that $M = \mathbf{c} \in \mathbf{C}^q$.

One would expect that the covariance, as well as the mean, of a homogeneous RTD has a special form just as for the space Λ. We fix j and k and consider the sesquilinear form Γ_{jk} mapping $\mathscr{S}(X) \times \mathscr{S}(X)$ to $\mathbf{C}$. Since μ is homogeneous, moreover a repetition of the calculation used to prove Lemma 5.5 yields

$$\Gamma_{jk}(\tau_{\mathbf{h}}\phi, \tau_{\mathbf{h}}\psi) = \Gamma_{jk}(\phi, \psi). \tag{5.24}$$

The next theorem asserts that, because of (5.24), Γ_{jk} is defined by a single tempered distribution $R_{jk} \in \mathscr{S}'(X)$.

Theorem 5.3. *Let μ be an admissible homogeneous RTD and let Γ be its covariance matrix. Then*

$$\Gamma_{jk}(\phi, \psi) = R_{jk}(\phi * \tilde{\psi}) \tag{5.25}$$

for $\phi, \psi \in \mathscr{S}(X)$ and some $R_{jk} \in \mathscr{S}'(X)$. Here $*$ denotes convolution and $\tilde{\psi}$ is the conjugate reverse of ψ; i.e., $\tilde{\psi}(\mathbf{x}) = \psi(-\mathbf{x})^*$.

Proof. We adapt the arguments given for the one-dimensional case in [22, Chapter III §3.3]. The proof divides naturally into two parts that we separate as lemmas.

Lemma 5.6. *There exists a tempered distribution $G \in \mathscr{S}'(X \times X)$ such that $\Gamma_{jk}(\phi, \psi) = G(\phi(\mathbf{x})\,\psi(\mathbf{y})^*)$. G is invariant under simultaneous translation of its first and last n arguments by the same vector.*

Proof. The kernel theorem for tempered distributions, [52, p. 534], states that any continuous bilinear form $H\colon \mathscr{S}(X) \times \mathscr{S}(X) \to \mathbf{C}$ can be written in the form $H(\phi, \psi) = G(\phi(\mathbf{x})\,\psi(\mathbf{y}))$ where $G \in \mathscr{S}'(X \times X)$ is a tempered distribution on $X \times X$. Hence there is a tempered distribution G such that

$$\Gamma_{jk}(\phi, \psi) = G(\phi(\mathbf{x})\,\psi(\mathbf{y})^*). \tag{5.26}$$

Now finite linear combinations of the form $\sum_k \phi_k(\mathbf{x})\,\psi_k(\mathbf{y})^*$, where $\phi, \psi \in \mathscr{S}(X)$, constitute the tensor product $\mathscr{S}(X) \otimes \mathscr{S}(X)$; moreover this set is dense in $\mathscr{S}(X \times X)$ by [52, p. 530]. From (5.24) and (5.26), furthermore,

$$G(\phi(\mathbf{x}+\mathbf{h})\,\psi(\mathbf{y}+\mathbf{h})^*) = G(\phi(\mathbf{x})\,\psi(\mathbf{y})^*).$$

Since $\mathscr{S}(X) \otimes \mathscr{S}(X)$ is dense in $\mathscr{S}(X \times X)$, we conclude that G is invariant under simultaneous translation of its first and last n arguments by the same vector $\mathbf{h} \in \mathbf{R}^n$. That is,

$$G(f(\mathbf{x}, \mathbf{y})) = G(f(\mathbf{x}+\mathbf{h}, \mathbf{y}+\mathbf{h})) \tag{5.27}$$

for all $f \in \mathscr{S}(X \times X)$. This is the required result.

The next lemma is essentially a corollary of (5.27).

Lemma 5.7. *Let $G \in \mathscr{S}'(X \times X)$ be invariant under simultaneous translation of its first and last n arguments by the same vector. Then there is an $H \in \mathscr{S}'(X)$ such that if $f(\mathbf{x}, \mathbf{y}) \in \mathscr{S}(X \times X)$, then*

$$G(f) = H(\textstyle\int_X f(\mathbf{x}, \mathbf{x}-\mathbf{y})\,d\mathbf{x}), \tag{5.28}$$

where H is viewed as being applied to the variable $\mathbf{y}$.

Proof. We transform variables in (5.27) so as to get a distribution independent of $\mathbf{x}$. Accordingly, consider the mapping $A\colon X \times X \to X \times X$ given by $A(\mathbf{x}, \mathbf{y}) = (\mathbf{x}+\mathbf{y}, \mathbf{x}-\mathbf{y})$. A is an isomorphism of the topological linear structure on $X \times X$. Thus A induces an isomorphism $B\colon \mathscr{S}(X \times X) \to \mathscr{S}(X \times X)$ defined by $Bf(\mathbf{x}, \mathbf{y}) = f(A(\mathbf{x}, \mathbf{y}))$. Then $F = G \circ B$ is again a continuous linear functional on $\mathscr{S}(X \times X)$. Furthermore, F is independent of the first n variables. For if $\mathbf{h} \in \mathbf{R}^n$ and $\mathbf{k} = (\mathbf{h}, \mathbf{0})$ is the $2n$-vector with $(h_1, \ldots, h_n) = \mathbf{h}$ in the first n places and 0 in the last n places, then for all $f \in \mathscr{S}(X \times X)$,

$$\begin{aligned}\tau_{\mathbf{k}} F(f(\mathbf{x}, \mathbf{y})) &= F(\tau_{-\mathbf{k}} f(\mathbf{x}, \mathbf{y})) = F(f(\mathbf{x}-\mathbf{h}, \mathbf{y})) = G(Bf(\mathbf{x}-\mathbf{h}, \mathbf{y}))\\ &= G(f(A(\mathbf{x}-\mathbf{h}, \mathbf{y}))) = G(f(\mathbf{x}+\mathbf{y}-\mathbf{h}, \mathbf{x}-\mathbf{y}-\mathbf{h}))\\ &= G(f(\mathbf{x}+\mathbf{y}, \mathbf{x}-\mathbf{y})) = G(f(A(\mathbf{x}, \mathbf{y})))\\ &= G(Bf(\mathbf{x}, \mathbf{y})) = F(f(\mathbf{x}, \mathbf{y})).\end{aligned}$$

Since F is independent of its first n variables, it can be written in the form [30, p. 333]

$$F(f(\mathbf{x}, \mathbf{y})) = R\Big(\int_X f(\mathbf{x}, \mathbf{y})\,d\mathbf{x}\Big),$$

where $R \in \mathscr{S}'(X)$ is applied to the components $(y_1, \ldots, y_n)$ of $\mathbf{y}$. Hence

$$R\Big(\int_X f(\mathbf{x}, \mathbf{y})\,d\mathbf{x}\Big) = F(f(\mathbf{x}, \mathbf{y})) = G(Bf(\mathbf{x}, \mathbf{y})),$$

and so

$$G(f(\mathbf{x}, \mathbf{y})) = R\Big(\int_X B^{-1}f(\mathbf{x}, \mathbf{y})\,d\mathbf{x}\Big) = R\Big(\int_X fA^{-1}(\mathbf{x}, \mathbf{y})\,d\mathbf{x}\Big).$$

Moreover A^{-1} maps $(\mathbf{x}, \mathbf{y})$ onto $((\mathbf{x}+\mathbf{y})/2, (\mathbf{x}-\mathbf{y})/2)$; hence

$$G(f(\mathbf{x}, \mathbf{y})) = R\Big(\int_X f((\mathbf{x}+\mathbf{y})/2, (\mathbf{x}-\mathbf{y})/2)\,d\mathbf{x}\Big).$$

Letting $\mathbf{w} = (\mathbf{x}+\mathbf{y})/2$ in the preceding integral, this becomes (for $H = 2R$)

$$G(f(\mathbf{x}, \mathbf{y})) = R\Big(2\int_X f(\mathbf{w}, \mathbf{w}-\mathbf{y})d\mathbf{w}\Big) = H\Big(\int_X f(\mathbf{w}, \mathbf{w}-\mathbf{y})\,d\mathbf{w}\Big),$$

thus establishing Lemma 5.7.

Now we easily construct the correlation matrix. If $\phi, \psi \in \mathscr{S}(X)$, then (5.26), (5.27), and (5.28) imply

$$\Gamma_{jk}(\phi, \psi) = R_{jk}\left(\int_X \phi(\mathbf{x})\,\psi(\mathbf{w}-\mathbf{x})^*\,d\mathbf{x}\right).$$

This can be written compactly as,

$$\Gamma_{jk}(\phi, \psi) = R_{jk}(\phi * \tilde{\psi}), \tag{5.29}$$

which proves Theorem 5.4.

Definition 5.6. The matrix $R = \|R_{jk}\|$ is called the *correlation matrix* of the homogeneous RTD.

D. Distributional Spectral Matrix Measures

In this section we will prove that a spectral matrix measure exists for any homogeneous admissible RTD. This result is stated without proof in [22, Chapter III, §5.5]; we shall adapt the argument used in [8, Part C, §§3,4] to prove an analogous result.

Definition 5.7. A measure v defined on the Borel sets of $X = \mathbf{R}^n$ will be called *tempered* when

$$\int_X (1 + |\mathbf{x}|^2)^{-t}\,dv(\mathbf{x}) < \infty \tag{5.30}$$

for some $t > 0$.

One sees without difficulty [22, p. 145] that a measure v is tempered if and only if the mapping

$$\phi \mapsto \int_X \phi(\mathbf{x})\,dv(\mathbf{x}), \qquad \phi \in \mathscr{S}(X), \tag{5.31}$$

is a continuous linear functional on $\mathscr{S}(X)$; i.e., a tempered distribution.

By a *Hermitian tempered matrix measure* we shall mean a Hermitian matrix measure whose entries are tempered measures.

Theorem 5.4. *Let μ be an admissible homogeneous probability measure on $S'(X)$ and let $\Gamma = \|\Gamma_{jk}\|$ be its covariance matrix. Then there exists a Hermitian tempered matrix measure $\sigma = \|\sigma_{jk}\|$ such that if $\phi, \psi \in \mathscr{S}(X)$, then*

$$\Gamma_{jk}(\phi, \psi) = \mathscr{F}[\sigma_{jk}](\phi * \tilde{\psi}), \tag{5.32}$$

where $\mathscr{F}$ denotes the inverse Fourier transform, $*$ is convolution, and $\tilde{\psi}$ is the conjugate of ψ reversed as defined in Section V.C.

Definition 5.8. The matrix measure σ associated with a homogeneous admissible measure μ on $S'(X)$ will be called the *spectral matrix measure* of μ.

Proof. Since μ is homogeneous, we know from Theorem 5.3 that there is a correlation matrix $R = \|R_{jk}\|$ such that

$$\Gamma_{jk}(\phi, \psi) = R_{jk}(\phi * \tilde{\psi}). \tag{5.33}$$

Further, from (5.22), we know that if $\phi \in \mathscr{S}(X)$ and $\mathbf{z} = (z_1, \ldots, z_q) \in \mathbf{C}^q$, then

$$\sum_{j,k=1}^{q} \Gamma_{jk}(\phi, \phi)\, z_j z_k^* \geq 0. \tag{5.34}$$

Combining these relations,

$$\sum_{j,k} R_{jk}(\phi * \tilde{\phi})\, z_j z_k^* \geq 0 \tag{5.35}$$

for all $\phi \in \mathscr{S}(X)$ and $\mathbf{z} \in \mathbf{C}^q$.

Now if we fix $\mathbf{z} \in \mathbf{C}^q$, and let $R \in \mathscr{S}'(X)$ be defined by

$$R(\psi) = \sum_{j,k=1}^{q} z_j z_k^*\, R_{jk}(\psi), \tag{5.36}$$

then R is a positive definite tempered distribution in the sense that

$$R(\phi * \tilde{\phi}) \geq 0 \tag{5.37}$$

for all $\phi \in \mathscr{S}(X)$.

The Bochner–Schwartz theorem [22, p. 152] asserts that a positive definite tempered distribution is the inverse Fourier transform of a positive tempered measure. Thus there is a measure σ, defined on the Borel sets of X and depending on $\mathbf{z}$, such that $R = \mathscr{F}[\sigma]$ in the sense that

$$R(\psi) = \mathscr{F}[\sigma](\psi) \tag{5.38}$$

for all $\psi \in \mathscr{S}(X)$. This result has been observed, for example, by Dudley [19].

In particular, letting $\mathbf{z} = (0, \ldots, 1, \ldots, 0)$, where the 1 is in the jth place, we see that there is a positive tempered measure σ_{jj}, $1 \leq j \leq q$, so that $R_{jj} = \mathscr{F}[\sigma_{jj}]$.

Now letting $z_j = 1 = z_k$ and $z_r = 0$ for $r \notin \{j, k\}$, we get from (5.38) a positive tempered measure v_{jk}. Similarly, letting $z_j = 1$ and $z_k = i = \sqrt{-1}$, we get a positive tempered measure γ_{jk} from (5.38). Thus,

$$\begin{aligned} \mathscr{F}[v_{jk}] &= R_{jj} + R_{jk} + R_{kj} + R_{kk}, \\ \mathscr{F}[\gamma_{jk}] &= R_{jj} - iR_{jk} + iR_{kj} + R_{kk}. \end{aligned} \tag{5.39}$$

Solving (5.39) for R_{jk} and R_{kj} in terms of R_{jj}, R_{kk}, ν_{jk}, and γ_{jk}, we are led to make the following definition:

$$\sigma_{jk} = \tfrac{1}{2}[\nu_{jk} + i\gamma_{jk} - (1+i)(\sigma_{jj} + \sigma_{kk})]. \tag{5.40}$$

Notice that this formula is consistent with the case $j = k$ also. As defined, σ_{jk} is a complex tempered measure.

We claim that the tempered matrix measure $\sigma = \|\sigma_{jk}\|$ has the desired properties. A simple calculation shows that

$$\bar{\mathscr{F}}[\sigma_{jk}] = R_{jk}. \tag{5.41}$$

Hence, using (5.33), if $\phi, \psi \in \mathscr{S}(X)$,

$$\Gamma_{jk}(\phi, \psi) = \bar{\mathscr{F}}[\sigma_{jk}](\phi * \tilde{\psi}). \tag{5.42}$$

To finish the proof of the theorem, we must show that σ is Hermitian definite. That is, we need to prove that $\mathbf{z}\sigma(E)\mathbf{z}^H \geq 0$ whenever $\mathbf{z} \in \mathbf{C}^q$ and E is a Borel set in X. Notice that it is clear from (5.40) that

$$\sigma_{jk}(E) = \sigma_{kj}(E)^* \tag{5.43}$$

whenever E is Borel in X.

Let $\mathbf{z} = (z_1, \ldots, z_q)$ be fixed in $\mathbf{C}^q$. To show that $\mathbf{z}\sigma(E)\mathbf{z}^H \geq 0$ for all Borel sets E in X is equivalent to showing that the tempered measure $\nu = \sum z_i z_j^* \sigma_{ij} \geq 0$. A tempered measure ν is positive when

$$\int_X f(\mathbf{x})\, d\nu(\mathbf{x}) \geq 0 \tag{5.44}$$

for all $f \geq 0$ in $\mathscr{S}(X)$. As is shown in [22, p. 150], the collection $Q = \{|\phi|^2 \mid \phi \in \mathscr{S}(X)\}$ is dense in $\{f \geq 0 \mid f \in \mathscr{S}(X)\}$. Hence, to show that $\nu \geq 0$, it suffices to show that

$$\int_X |\phi(\mathbf{x})|^2\, d\nu(\mathbf{x}) \geq 0 \tag{5.45}$$

for all $\phi \in \mathscr{S}(X)$. Now we compute

$$\begin{aligned}\int_X |\phi(\mathbf{x})|^2\, d\nu(\mathbf{x}) &= \sum_{j,k} z_j z_k^* \int_X \phi(\mathbf{x})\,\phi(\mathbf{x})^*\, d\sigma_{jk}(\mathbf{x}) = \sum_{j,k} z_j z_k^* [\sigma_{jk}](\phi\phi^*) \\ &= \sum_{j,k} z_j z_k^* \bar{\mathscr{F}}[\sigma_{jk}](\mathscr{F}(\phi\phi^*)) = \sum_{j,k} z_j z_k^* \bar{\mathscr{F}}[\sigma_{jk}](\mathscr{F}\phi * \widetilde{\mathscr{F}\phi}) \\ &= \sum_{j,k} z_j z_k^* \Gamma_{jk}(\mathscr{F}\phi, \mathscr{F}\phi) = \sum_{j,k} z_j z_k^* \Gamma_{jk}(\psi, \psi),\end{aligned}$$

where $\mathscr{F}\phi = \psi$. The latter quantity is nonnegative by (5.34). This shows that $\nu \geq 0$ and, since z was arbitrary, establishes that σ is Hermitian definite. The proof of the theorem is now complete.

E. Normal Measures with a Given Spectrum

In this section, the problem of characterizing Gaussian probability measures on Λ and $S'(X)$ is considered. We will show that in the case of a normal admissible homogeneous measure μ with mean zero, the spectral matrix measure defined in Section V.A for Λ and Section V.D for $S'(X)$ uniquely determines μ. We shall also discuss the relation between the spectrum in Λ and $S'(X)$.

For the case of Λ, we merely state the major results and rely on [8] for proofs and technical details.

Theorem 5.5. *Let $S = \|S_{jk}\|$ be a Hermitian matrix measure on the Borel sets of X, where each S_{jk} is a finite measure. Then there is a unique normal admissible homogeneous RVF on Λ with mean zero, whose spectral matrix measure is S.*

Proof. This result is a combination of [8, Theorem 2, p. 696] and [8, Theorem 3, p. 686].

The analogous result is true for random tempered distributions; see Theorem 5.7 below. To establish Theorem 5.7, we need the following technical result, to be proved in Appendix D.

Theorem 5.6. *Let $\Gamma = \|\Gamma_{jk}\|$ be a $q \times q$ covariance matrix; that is, let each Γ_{jk} be a continuous sesquilinear form on $\mathscr{S}(X) \times \mathscr{S}(X)$ and Γ of positive type in the sense of (5.23). Then there is a unique normal admissible mean zero probability measure μ on the Borel sets of $S'(X)$ whose covariance is Γ.*

Assuming Theorem 5.6, then, we shall prove

Theorem 5.7. *Let $S = \|S_{jk}\|$ be a Hermitian matrix measure and suppose each S_{jk} is a tempered measure. Then there is a unique normal admissible homogeneous mean zero RTD whose spectral matrix measure is S.*

Proof. Let $\nu = \|\nu_{jk}\|$ be a $(q \times q)$-matrix of tempered measures which is Hermitian definite. Define $H: \mathscr{S}(X) \times \mathscr{S}(X) \to \mathbf{C}^q$ as follows:

$$H_{jk}(\phi, \psi) = \bar{\mathscr{F}}[\nu_{jk}](\phi * \tilde{\psi}) \tag{5.46}$$

for $1 \leq j, k \leq q$. Then certainly each H_{ij} is continuous and sesquilinear.

The plan of the proof is now as follows. First, we will show that $H = \|H_{jk}\|$ satisfies the hypothesis of Theorem 5.6 (Lemma 5.8). Theorem 5.6 will then imply the existence of a normal admissible measure on $S'(X)$ with covariance H. We will next show (Lemma 5.9) that this measure is homogeneous; again, applying Theorem 5.6, a spectral matrix measure can be associated with it. It will follow that this spectral matrix measure is identical with the given one. From this, uniqueness will follow directly by the uniqueness statement of Theorem 5.6.

Lemma 5.8. *The matrix $H = \|H_{ij}\|$ is of positive type in the sense discussed in* (5.23).

Proof. Let $\phi_1, \ldots, \phi_r$ lie in $\mathscr{S}(X)$ and let $\mathbf{Z} = (\mathbf{Z}_1, \cdots, \mathbf{Z}_r)$, where $\mathbf{Z}_\beta = (z_{1\beta}, \ldots, z_{q\beta}) \in \mathbf{C}^q$. Then if Q denotes the $(qr \times qr)$-matrix $\|H(\phi_\alpha, \phi_\beta)\|$, we have (summing for $\alpha, \beta = 1, \ldots, r$), and letting F_α denote $\mathscr{F}\phi_\alpha$,

$$\begin{aligned}\mathbf{Z}Q\mathbf{Z}^* &= \sum_{\alpha,\beta} \mathbf{Z}_\alpha H(\phi_\alpha, \phi_\beta)\mathbf{Z}_\beta^* = \sum_{\alpha,\beta} \mathbf{Z}_\alpha \mathscr{F}[v](\phi_\alpha * \phi_\beta)\mathbf{Z}_\beta^* \\ &= \sum_{\alpha,\beta} \mathbf{Z}_\alpha[v](\mathscr{F}(\phi_\alpha * \tilde{\phi}_\beta))\mathbf{Z}_\beta^* = \sum_{\alpha,\beta} \mathbf{Z}_\alpha \int_X F_\alpha(\mathbf{x})F_\beta^*(\mathbf{x})\, dv(\mathbf{x})\mathbf{Z}_\beta^* \\ &= \sum_{\alpha,\beta}\sum_{k,l} z_{\alpha k} \int_X F_\alpha(\mathbf{x})\, F_\beta^*(\mathbf{x})\, dv_{kl}(\mathbf{x}) z_{\beta l}^* \\ &= \sum_{k,l} \int_X \sum_{\alpha,\beta} z_{\alpha k} F_\alpha(\mathbf{x})\, F_\beta^*(\mathbf{x}) z_{\beta l}^*\, dv_{kl}(\mathbf{x}),\end{aligned} \tag{5.47}$$

and letting $\psi_p = \sum_i z_{ip} F_i$, this reduces to the sum for $k, l = 1, \ldots, q$:

$$\mathbf{Z}Q\mathbf{Z}^* = \sum_{k,l} \int_X \psi_k(\mathbf{x})\, \psi_l^*(\mathbf{x})\, dv_{kl}(\mathbf{x}). \tag{5.48}$$

So to show that H is of positive type it suffices to show that

$$\sum_{k,l} \int_X \psi_k(\mathbf{x})\, \psi_l(\mathbf{x})^*\, dv_{kl}(\mathbf{x}) \geq 0 \tag{5.49}$$

whenever $\psi_1, \ldots, \psi_q \in \mathscr{S}(X)$. Now we first notice that if B is any bounded Borel set and $\mathbf{z} = (z_1, \ldots, z_q) \in \mathbf{C}^q$, then

$$\sum_{k,l} \int_B z_k z_l^*\, dv_{kl}(x) \geq 0, \tag{5.50}$$

since the matrix v is nonnegative definite. We call a complex functional $f\colon X \to \mathbf{C}$ which is a finite linear combination of characteristic functions of bounded Borel sets a *simple* function. From (5.50) we will see that (5.49) holds for simple functions.

For suppose $\psi_1, \ldots, \psi_q$ are simple functions. Without loss of generality, we can suppose that

$$\psi_j(\mathbf{x}) = \sum_{k=1}^{p} z_{kj}\, \chi_k(\mathbf{x}) \qquad (1 \leq j \leq q), \tag{5.51}$$

where χ_k denotes the characteristic function of the set A_k, $A_1, \ldots, A_p$ being a family of disjoint bounded Borel sets. Then

$$\begin{aligned}\sum_{k,l} \int_X \psi_k(\mathbf{x})\, \psi_l(\mathbf{x})^*\, dv_{kl}(\mathbf{x}) &= \sum_{k,l} \int_X \sum_{i=1}^{p} z_{ik}\, \chi_i(\mathbf{x}) \sum_{j=1}^{p} z_{jl}^* \chi_j(\mathbf{x})\, dv_{kl}(\mathbf{x}) \\ &= \sum_{i,j=1}^{p} \int_X \chi_i(\mathbf{x})\, \chi_j(\mathbf{x}) \sum_{k,l} z_{ik} z_{jl}^*\, dv_{kl}(\mathbf{x}).\end{aligned} \tag{5.52}$$

But

$$\chi_i(\mathbf{x})\, \chi_j(\mathbf{x}) = \delta_{ij}\, \chi_i(\mathbf{x}), \tag{5.53}$$

since $A_i \cap A_j = \varnothing$ for $i \neq j$. Thus the last sum becomes

$$\sum_{i=1}^{p} \int_X \chi_i(\mathbf{x}) \sum_{k,l} z_{ik} z_{il}^*\, dv_{kl}(\mathbf{x}) = \sum_{i=1}^{p} \left[\sum_{k,l} \int_{A_i} z_{ik} z_{il}^*\, dv_{kl}(\mathbf{x})\right] \geq 0 \tag{5.54}$$

since each summand is nonnegative by (5.50).

The simple functions are dense in $L^2(v_{kl})$ for all k and l by [45, Theorem 3.13]. Hence (5.49) holds for all functions in $\bigcap_{k,l} L^2(v_{kl})$. In particular, then, since each v_{kl} is tempered, (5.49) holds for all elements of $\mathscr{S}(X)$. So H is of positive type. This completes the proof of Lemma 5.8.

Applying Theorem 5.6, there exists a (normal) admissible measure μ with mean zero on $S'(X)$ whose covariance is H. We will show that μ is necessarily homogeneous.

Lemma 5.9. *The admissible measure μ of Theorem 5.6 is homogeneous.*

Proof. Let $\mathbf{h} \in X$. We must check that $\mu = \mu \circ \tau_{\mathbf{h}}$. Let $\sigma = \mu \circ \tau_{\mathbf{h}}$. We will use Theorem 5.6. First, σ has mean zero. For if M' is the mean of σ and M the mean of μ, then $M = 0$ and

$$\begin{aligned}M'(\phi) &= \int_{S'(X)} T(\phi)\, d\sigma(T) = \int_{S'(X)} T(\phi)\, d(\mu \circ \tau_{\mathbf{h}})(T) \\ &= \int_{S'(X)} T(\tau_{\mathbf{h}}\phi)\, d\mu(T) = M(\tau_{\mathbf{h}}\phi) = 0.\end{aligned} \tag{5.55}$$

Further, σ is normal; for let $\phi \neq 0$ in $\mathscr{S}(X)$. Then if E is a Borel set in $\mathbf{C}$,

$$\begin{aligned}\sigma(\phi; E) &= \sigma(C(\phi; E)) = \mu(\tau_{\mathbf{h}} C(\phi; E)) \\ &= \mu(C(\tau_{\mathbf{h}}\phi; E)) = \mu(\tau_{\mathbf{h}}\phi; E).\end{aligned} \tag{5.56}$$

Thus $\sigma(\phi; *) = \mu(\tau_{\mathbf{h}}\phi; *)$ and, since the latter is Gaussian on $\mathbf{C}$, so is the former. Finally, we show that the covariance Γ of σ is H:

$$\begin{aligned}\Gamma(\phi, \psi) &= \int_{S'(X)} T(\phi)\, T(\psi)^H \, d\sigma(T) = \int_{S'(X)} T(\phi)\, T(\psi)^H \, d(\mu \circ \tau_{\mathbf{h}})(T) \\ &= \int_{S'(X)} T(\tau_{\mathbf{h}}\phi)\, T(\tau_{\mathbf{h}}\psi)^H \, d\mu(T) = H(\tau_{\mathbf{h}}\phi, \tau_{\mathbf{h}}\psi) \\ &= \mathscr{F}[v](\tau_{\mathbf{h}}\phi * \widetilde{\tau_{\mathbf{h}}\psi}),\end{aligned} \tag{5.57}$$

where we have used the definition (5.46) in the last step. On the other hand,

$$\begin{aligned}\tau_{\mathbf{h}}\phi * \widetilde{\tau_{\mathbf{h}}\psi}(\mathbf{x}) &= \int_X \tau_{\mathbf{h}}\phi(\mathbf{x}-\mathbf{y})\, \widetilde{\tau_{\mathbf{h}}\psi}(\mathbf{y})\, d\mathbf{y} \\ &= \int_X \tau_{\mathbf{h}}\phi(\mathbf{x}-\mathbf{y})\, \tau_{\mathbf{h}}\psi(-\mathbf{y})^*\, d\mathbf{y} \\ &= \int_X \phi(\mathbf{x}-\mathbf{y}-\mathbf{h})\, \psi(-\mathbf{y}-\mathbf{h})^*\, d\mathbf{y}\end{aligned} \tag{5.58}$$

and, letting $\mathbf{w} = \mathbf{y} + \mathbf{h}$, this becomes

$$\begin{aligned}\tau_{\mathbf{h}}\phi * \widetilde{\tau_{\mathbf{h}}\psi}(\mathbf{x}) &= \int_X \phi(\mathbf{x}-\mathbf{w})\, \psi(-\mathbf{w})^*\, d\mathbf{w} \\ &= \int_X \phi(\mathbf{x}-\mathbf{w})\, \tilde{\psi}(\mathbf{w})\, d\mathbf{w} = \phi * \tilde{\psi}(\mathbf{x}).\end{aligned} \tag{5.59}$$

Hence,

$$\Gamma(\phi, \psi) = \mathscr{F}[v](\tau_{\mathbf{h}}\phi * \widetilde{\tau_{\mathbf{h}}\psi}) = \mathscr{F}[v](\phi * \tilde{\psi}) = H(\phi, \psi) \tag{5.60}$$

as required.

So μ and σ are both normal admissible measures with mean zero and covariance H. By the uniqueness part of Theorem 5.6, then, we must have $\mu = \sigma$, and this shows that μ is homogeneous and finishes the proof of Lemma 5.9.

Now we can finish the proof of Theorem 5.7. It remains to show that the spectral matrix measure of μ is v. If $\tilde{v}$ is the spectral matrix measure of μ, then, from Theorem 5.4 and the fact that H is the covariance of μ,

$$H_{jk}(\phi, \psi) = \mathscr{F}[\tilde{v}_{jk}](\phi * \tilde{\psi}) \tag{5.61}$$

for all $\phi, \psi \in \mathscr{S}(X)$. Combining (5.61) with (5.46),

$$\mathscr{F}[\tilde{v}_{jk}](\phi * \tilde{\psi}) = \mathscr{F}[v_{jk}](\phi * \tilde{\psi}) \tag{5.62}$$

or, what is the same,

$$0 = \mathscr{F}[v_{jk} - \tilde{v}_{jk}](\phi * \tilde{\psi}) = [v_{jk} - \tilde{v}_{jk}](\mathscr{F}(\phi * \tilde{\psi})). \tag{5.63}$$

One calculates easily that $\mathscr{F}(\phi * \tilde{\psi}) = \mathscr{F}\phi\mathscr{F}\psi^*$, where, as before, $\mathscr{F}\phi$ is the inverse Fourier transform of ϕ. Such elements are dense in $\mathscr{S}(X)$. Hence the tempered distribution $[v_{jk} - \tilde{v}_{jk}]$ vanishes on a dense subset of $\mathscr{S}(X)$ and so must be identically zero. Thus $v_{jk} = \tilde{v}_{jk}$ and so the spectral matrix measure of μ is v as required.

As an application of Theorems 5.5 and 57, we now consider the relation between the correlation and the spectral matrix measures defined in Sections V.A and V.D. We assume that μ is a normal admissible homogeneous probability measure on Λ so that, by Theorem 4.7, $\mu(\Lambda \cap S'(X)) = 1$. Since μ is admissible and homogeneous, both the correlation and spectral matrix measure exist.

Then μ can be considered as a probability measure on $S'(X)$ also since $\Lambda \cap S'(X)$ is a Borel subset of $S'(X)$. Notice that μ is admissible when considered as a measure on $S'(X)$. For by [8, Theorem 4, p. 687], the correlation matrix $R = \|R_{jk}\|$ of the RVF on Λ is uniformly continuous on X. Let $\Gamma = \|\Gamma_{jk}\|$ be the covariance matrix of the RVF μ on Λ. Then

$$\Gamma_{jk}(\mathbf{x}, \mathbf{y}) = R_{jk}(\mathbf{x}-\mathbf{y}) \qquad \text{for all} \quad \mathbf{x}, \mathbf{y} \in X.$$

Since a uniformly continuous function grows at most linearly, the integral

$$\int_X \int_X \Gamma_{jk}(\mathbf{x}, \mathbf{y})\, \phi(\mathbf{x})\, \psi^*(\mathbf{y})\, d\mathbf{x}\, d\mathbf{y} \tag{5.64}$$

converges absolutely for all $\phi, \psi \in \mathscr{S}(X)$. We apply Fubini's theorem to (5.64) and deduce that

$$\int_\Lambda \int_X u_j(\mathbf{x}, \omega)\, \phi(\mathbf{x})\, d\mathbf{x} \int_X u_k(\mathbf{y}, \omega)^*\, \psi(\mathbf{y})^*\, d\mathbf{y}\, d\mu(\omega) \tag{5.65}$$

is finite, where u_j is the jth component of the homogeneous RVF associated with μ. But

$$\int_X f(\mathbf{x})\, \phi(\mathbf{x})\, d\mathbf{x} = [f](\phi) \tag{5.66}$$

is the definition of an element $f \in S'(X) \cap \Lambda$ operating on $\phi \in \mathscr{S}(X)$. It follows that

$$\begin{aligned}\int_{S'(X)} T_j(\phi)\, T_k(\psi)^*\, d\mu(T) &= \int_{S'(X) \cap \Lambda} T_j(\phi)\, T_k(\psi)^*\, d\mu(T) \\ &= \int_{S'(X) \cap \Lambda} [u_j](\phi) \cdot [u_k](\psi)^*\, d\mu\end{aligned} \tag{5.67}$$

is finite. Thus μ is an admissible measure on $S'(X)$. It is easy to check that μ is normal and homogeneous when considered as a probability measure on $S'(X)$ also.

Let $\nu = \|\nu_{jk}\|$ be the spectral matrix measure of μ (as a RVF on Λ). Then

$$R_{jk}(\mathbf{h}) = \int_X e^{i\mathbf{l}\cdot\mathbf{h}}\, d\nu_{jk}(\mathbf{l}). \tag{5.68}$$

Now ν_{jk} is a bounded measure by Theorem 5.2 and hence it is certainly tempered. Thus ν is a tempered Hermitian matrix measure and, by Theorem 5.7, there is a unique normal admissible homogeneous measure $\tilde{\mu}$ on $S'(X)$ with spectral matrix measure ν.

We claim that μ, considered as an admissible probability on $S'(X)$, also has spectral matrix measure $\nu = \|\nu_{jk}\|$. The Fourier–Stieltjes transform $R_{jk}(\mathbf{h})$ of the latter is thus a matrix of *point*-functions under our hypotheses. To establish our claim, we check that

$$\int_{S'(X)} T_j(\phi)\, T_k(\psi)^*\, d\mu(T) = \mathscr{F}[\nu_{jk}](\phi * \tilde{\psi}), \tag{5.69}$$

as in (5.32). Using the calculation (5.64)–(5.67), we obtain

$$I = \int_{S'(X)} T_j(\phi)\, T_k(\psi)^*\, d\mu(T) = \int_X \int_X \Gamma_{jk}(\mathbf{x}, \mathbf{y})\, \phi(\mathbf{x})\, \psi(\mathbf{y})^*\, d\mathbf{x}\, d\mathbf{y}. \tag{5.70}$$

Substituting from (5.68) in this we get

$$I = \int_X \int_X \int_X e^{i\mathbf{l}\cdot(\mathbf{x}-\mathbf{y})}\, d\nu_{jk}(\mathbf{l})\, \phi(\mathbf{x})\, \psi(\mathbf{y})^*\, d\mathbf{x}\, d\mathbf{y}. \tag{5.71}$$

Let $\mathbf{h} = \mathbf{x} - \mathbf{y}$, so that $\mathbf{y} = \mathbf{x} - \mathbf{h}$. Then

$$\begin{aligned}
I &= \int_X \int_X \int_X e^{i\mathbf{l}\cdot\mathbf{h}}\, \phi(\mathbf{x})\, \psi(\mathbf{x}-\mathbf{h})^*\, d\mathbf{x}\, d\mathbf{h}\, d\nu_{jk}(\mathbf{l}) \\
&= \int_X \int_X \int_X \phi(\mathbf{x})\, \tilde{\psi}(\mathbf{h}-\mathbf{x})\, d\mathbf{x}\, e^{i\mathbf{l}\cdot\mathbf{h}}\, d\mathbf{h}\, d\nu_{jk}(\mathbf{l}) \\
&= \int_X \int_X \phi * \tilde{\psi}(\mathbf{h})\, e^{i\mathbf{l}\cdot\mathbf{h}}\, d\mathbf{h}\, d\nu_{jk}(\mathbf{l}) \\
&= \int_X \mathscr{F}[\phi * \tilde{\psi}]\, d\nu_{jk}(\mathbf{l}) \\
&= [\nu_{jk}](\mathscr{F}[\phi * \tilde{\psi}]) \\
&= \mathscr{F}[\nu_{jk}](\phi * \tilde{\psi}),
\end{aligned} \tag{5.72}$$

which is the required result.

Corollary 5.1. *Let μ be an admissible homogeneous probability measure on Λ. Then $\mu(\Lambda \cap S'(X)) = 1$, and the spectral matrix measure of μ as a RVF on Λ is equal to its spectral matrix measure when considered as a RTD on $S'(X)$.*

Note that the preceding argument shows further that, if $\Gamma' = \|\Gamma'_{jk}\|$ is the covariance of the RTD μ, then also

$$\Gamma'_{jk}(\phi, \psi) = \int_X \int_X \Gamma_{jk}(\mathbf{x}, \mathbf{y})\, \phi(\mathbf{x})\, \psi(\mathbf{y})^*\, d\mathbf{x}\, d\mathbf{y}, \tag{5.73}$$

where $\Gamma = \|\Gamma_{jk}\|$ is the convariance of the RVF μ on Λ.

F. Metric Transitivity

A very important property of *homogeneous* random vector fields (HRVF) in *free space* $X = \mathbf{R}^n$ with *continuous* spectra is their metric transitivity. By this it is meant loosely that, for almost every sample function, "space averages are the same as ensemble averages." For completeness, we give now a brief discussion (mostly written before 1965) of this fact and some of its consequences; our proofs will be sketchy.

Indeed, they will be based on results of [8], whose discussion of metric transitivity was in turn based on old results of Maruyama.[20] Though it would be desirable to make a fresh exposition, especially since there has been no recent exposition of the ergodic theorem over n-parameter groups,[21] we have not seen fit to do this.

As in Section V.A, let μ be an admissible homogeneous probability measure μ on the Borel sets of the space $\Lambda = \Lambda_2(X, q)$. By Theorem 4.7, the resulting HRVF will be in $S'(X)$ with probability one, and so the results of Sections V.B–E will apply; however, we shall not use them. Let $\Gamma = \|\Gamma_{ij}\|$ be the covariance matrix of μ, and $S = \|S_{ij}\|$ the spectral matrix measure associated with Γ as in (5.5) and (5.11). Since μ is homogeneous, $\Gamma_{jl}(\mathbf{x}, \mathbf{y}) = R_{jl}(\mathbf{x} - \mathbf{y})$, where $R_{jl}(\mathbf{h})$ is the correlation matrix of (5.9).

If $X = \mathbf{R}^n$ and the spectral matrix measure $S_{jl}(K)$ is *absolutely continuous* so that, for any Borel set $K \subset X' = \mathbf{R}^n$,

$$S_{jl}(K) = \int_K \sigma_{jl}(\mathbf{k})\, dk_1 \cdots dk_n, \tag{5.74}$$

[20] *Mem. Fac. Sci. Kyusyu Univ.* **4** (1949), 49–106.

[21] In [8], reference is made to K. Ito, *J. Math. Soc. Japan* **3** (1951), 157–69, and G. Birkhoff and L. Alaoglu, *Ann. Math.* **41** (1940), 293–309. See also A. Calderón, *Ann. Math.* **58** (1953), 182–91; and Calvin C. Moore, *Amer. J. Math.* **88** (1966), 154–78.

then sample functions on $\mathbf{R}^n$ have the remarkable property of being *metrically transitive* in space with probability one [8, Part D, p. 330]. More precisely, let $\phi[\mathbf{u}_1, \ldots, \mathbf{u}_n]$ be any bounded Borel function. Then, for any nonzero vector $\mathbf{h}$,

$$\int_\Lambda \phi[\mathbf{u}(\mathbf{x}_1, \omega), \ldots, \mathbf{u}(\mathbf{x}_n, \omega)]\, d\mu(\omega)$$
$$= \lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} \phi[\mathbf{u}(\mathbf{x}_1 + s\mathbf{h}, \omega), \ldots, \mathbf{u}(\mathbf{x}_n + s\mathbf{h}, \omega)]\, ds, \tag{5.75}$$

with probability one in the sample space Λ_2. For *normal* measures, the hypothesis of boundedness can be replaced by

$$\int \exp[-\alpha(|\mathbf{u}_1|^2 + \cdots + |\mathbf{u}_n|^2)]\, \phi[\mathbf{u}_1, \ldots, \mathbf{u}_n]\, d\mathbf{u} < +\infty \qquad \text{for all} \quad \alpha > 0. \tag{5.76}$$

As a special case of (5.75), we obtain

Theorem 5.8 *Let μ be normal, admissible, and homogeneous. If the $S_{jl}(K)$ are absolutely continuous, then with μ-probability one the correlation matrix $R_{jl}(\mathbf{h})$ of (5.9) is the limit*

$$R_{jl}(\mathbf{h}) = \lim_{A\to\infty} (2A)^{-n} \int_{-A}^{A} \cdots \int_{-A}^{A} u_j(\mathbf{x} + \mathbf{h}, \omega)\, u_l^*(\mathbf{x}, \omega)\, dm(\mathbf{x}). \tag{5.77}$$

As a special case of (5.77), we have further with probability one

$$\lim_{A\to\infty} (2A)^{-n} \int_{-A}^{A} \cdots \int_{-A}^{A} |\mathbf{u}(\mathbf{x}, \omega)|^2\, dm = \sum_{j=1}^{q} R_{jj}(0). \tag{5.78}$$

(Here, as before, $dm = dx_1 \cdots dx_n$ is Lebesgue measure on X.) Hence almost all functions have the same Zaanen norm.

Moreover, we can now reconstruct $\sigma_{jl}(\mathbf{k})$ in (5.74) as the Fourier transform of $R_{jl}(\mathbf{h})$; this uniquely determines μ by [8, p. 686, Theorem 3], proving

Theorem 5.9. *Under the hypotheses of Theorem* 5.8, *μ can be reconstructed from almost every sample function.*

In other words, one can reconstruct the admissible probability measure μ of any HRVF with absolutely continuous spectral matrix from any sample function, almost surely.

For absolutely continuous $S_{jl}(K)$, assuming that $\mu(\Gamma) = 1$, we now *conjecture* an important extension of [8, Part D, Theorem 7]. [Incidentally, [8, Part D, Theorem 7] does not apply to the covariance $R_{jl}(\mathbf{h})$, since the product $u_j(\mathbf{x})\, u_l^*(\mathbf{x} + \mathbf{h})$ may be unbounded. However, we suspect that the conclusion is still valid.]

Theorem 5.10. *For a metrically transitive homogeneous random function, continuous with probability one, the regular probability measure can be reconstructed from any sample function by a countable sequence of periodically spaced measurements, with probability one.*

Remark. The preceding result, if true, establishes an almost surely valid process for determining μ as a limit of frequencies. It is similar to Borel's results about "normal" numbers, and implies that periodically spaced measurements almost surely form a "Kollektiv" in the sense of von Mises.

G. Spectrum and Smoothness: The Case $n = 1$

In Section III.C, we defined a regular Cauchy problem (3.1) with random initial values to be *statistically determinate* in a function space E when the orbits of an appropriate C_0-semigroup of Borel mappings of E represent (classical) solutions with probability one. We shall now consider sufficient conditions for this to be the case, i.e., for the orbits to represent *smooth* functions with probability one. Throughout, we shall impose arbitrarily (for mathematical convenience) the assumption of zero mean:

$$\int u_j(\mathbf{x}, t, \omega)\, d\mu(\omega) = 0 \qquad (t \geq 0,\ 1 \leq j \leq q), \tag{5.79}$$

which is implied for systems (3.1) if $\int u_j(\mathbf{x}, 0)\, d\mu = 0$.

We shall use in all cases the *separable version* of RVF in the sense of Doob [18, Chapter 2, §2]. Since "smooth" functions are at least continuous, we require that $\mu(\Gamma) = 1$ (Γ the set of essentially continuous functions defined in Section II.A), for which the separable version is trivially defined (see Section II.F). This is to avoid the ambiguity which is inherent otherwise if one considers solutions in the function spaces $\Lambda_p(X)$, $S'(X)$, etc., where they are determined only a.e. If $\mu(\Gamma) = 1$, we can select the continuous representative of each equivalence class of functions equal a.e.; this permits us to determine their smoothness properties.

For *normal* RVF, the probability distribution at any time t is determined [see [8] and, for a generalization to $S'(X)$, Theorem 5.7] by the covariance matrix

$$\Gamma_{jk}(\mathbf{x}, \mathbf{y}, t) = \int u_j(\mathbf{x}, t)\, u_k^*(\mathbf{y}, t)\, d\mu. \tag{5.80}$$

In the spatially *homogeneous case*, we saw in Lemma 5.2 of Section V.A that

$$\Gamma_{jk}(\mathbf{x}, \mathbf{y}, t) = R_{jk}(\mathbf{x} - \mathbf{y}, t); \tag{5.81}$$

hence, the differentiability (with probability one) of the HRVF of the separable version is determined by the spectral matrix measure $S_{jk}(K, t)$. More precisely, the smoothness of the samples of each jth component $u_j(\mathbf{x}, t, \omega)$ depends on the diagonal entries $R_{jj}(\mathbf{h}, t)$ and $S_{jj}(K, t)$. Hence the question of smoothness is reduced to the case $q = 1$ of random functions.

Fortunately, there are available numerous results concerning just this question. We treat the case $n = 1$ of homogeneous random function $u(x)$ of *one* space variable; the case of general n will be treated in Section V.H. We will also hold t fixed, and consider only differentiability in x.

For the case $n = 1$, Hunt [31] has shown that if the energy spectrum $S(K) = \mathscr{E}(K)$ of a normal homogeneous random function satisfies

$$\int_{-\infty}^{\infty} [\log(1 + |k|)]^{\beta}\, dS(k) < +\infty \qquad \text{for some} \quad \beta > 1, \tag{5.82}$$

then almost all sample functions are continuous, whence $\mu(\Gamma) = 1$ as desired. Belyaev [4] has shown that condition (5.82) on the spectrum can be replaced by the following condition on the correlation:

$$R(0) - R(h) = O([-\log|h|]^{-\alpha}) \qquad \text{for some} \quad \alpha > 1, \tag{5.83}$$

as $h \to 0$, which also implies almost sure continuity.

Belyaev has also proved a partial converse of Hunt's result. If the spectrum is absolutely continuous, and if the spectral density $\sigma(k)$ satisfies

$$\sigma(k) \geq C/[k(\log k)^2] \qquad (C > 0), \tag{5.84}$$

for all sufficiently large k, then a.e. sample function is *unbounded* in every finite interval. He has also proved a partial converse of (5.83): namely, if $R(h)$ is a concave function of h, and if

$$R(0) - R(h) \geq C/|\log|h|| \qquad \text{for some} \quad C > 0, \tag{5.85}$$

then again a.e. sample is unbounded on every finite interval. In this connection, we note also a remarkable dichotomy, established by Dobrushin [17]: for a stationary (i.e. homogeneous) normal random function, almost all samples of the separable version are *either continuous or unbounded*, on any finite interval. See further Belyaev [4].

From (5.82) one deduces that the condition

$$\int_{-\infty}^{+\infty} k^{2p}[\log(1 + |k|)]^{\beta}\, dS < +\infty \qquad \text{for some} \quad \beta > 1 \tag{5.86}$$

implies the existence of the derivative $R_p(t - s) = \partial^{2p}R(t - s)/\partial s^p\, \partial t^p$ which satisfies (5.83); thus almost all samples of the normal homogeneous random function $u(x, \omega)$ belong to $C^{(p)}$ in any finite interval and $d^p u(x, \omega)/dx^p$ admits $R_p(t - s)$ as correlation.[22]

Concerning the *analyticity* of samples of a homogeneous random function, Belyaev [3] has proved the following proposition. If $R(h)$ is analytic in the circle $|h| < r$, then almost all $u(x, \omega)$ are analytic in the strip $|\mathrm{Im}\{x\}| < r$—and hence analytic for real x. Moreover, this sufficient condition is also necessary in the case of a *normal* homogeneous random function. On the other hand [39, p. 137], it is known that $R(h)$ is analytic in $|h| < r$ if and only if, as $k \to +\infty$,

$$S([k, \infty]) + S([-\infty, -k]) = O(e^{-\alpha k}) \qquad \text{whenever} \quad 0 < \alpha < r. \tag{5.87}$$

It follows that, if (5.87) holds for *all* $\alpha > 0$, then almost every (a.e.) sample function $u(x, \omega)$ can be extended to an entire function in the complex z-plane.

It is also known [39, p. 141] that $R(h)$ is an entire function of order one and exponential type; i.e., $|R(h)| \leq Me^{a|h|}$ for some finite positive constants M and a, if and only if the energy spectrum is *bounded*. Analogous necessary and sufficient conditions on $S(K)$ for $R(h)$ to be of other orders and types have been given by Ramachandran [44]. [As mentioned earlier, the results stated in this section all refer to the *separable version* of the measure in question.]

H. Spectrum and Smoothness: The General Case

Recently, Delporte has extended the preceding results[23] to scalar functions of several variables [16].

We suppose that $u(\mathbf{x}, \omega)$ is a normal random function satisfying (5.79) on the n-cube $[0, 1] \times \cdots \times [0, 1]$, and define

$$\begin{aligned} \gamma(\mathbf{x}, \mathbf{y}) &= \int |u(\mathbf{x}, \omega) - u(\mathbf{y}, \omega)|^2\, d\mu \\ &= \Gamma(\mathbf{x}, \mathbf{x}) + \Gamma(\mathbf{y}, \mathbf{y}) - 2\Gamma(\mathbf{x}, \mathbf{y}). \end{aligned} \tag{5.88}$$

Then [16, p. 198] if, for all $\mathbf{x}$ in the n-cube and all sufficiently small $\sigma > 0$ we have

$$\gamma(\mathbf{x}, \mathbf{x} + \mathbf{h}) < C|\log \sigma|^{-1-\varepsilon} \qquad \text{for} \quad |\mathbf{h}| \leq \sigma, \tag{5.89}$$

then a.e. sample (of the separable version) of $u(\mathbf{x}, \omega)$ is continuous in the cube $[0, 1] \times [0, 1] \times \cdots \times [0, 1]$.

[22] This result has been generalized by Delporte [16] to nonhomogeneous normal random functions.

[23] He has also considered random functions which are neither stationary nor normal, but we shall restrict our attention to the normal case.

If the normal random function is stationary ("homogeneous"), (5.89) reduces to

$$R(\mathbf{0}) - R(\mathbf{h}) \leq D \cdot |\log \sigma|^{-1-\varepsilon} \qquad (D = C/2), \tag{5.90}$$

for all $|\mathbf{h}| \leq \sigma$ if $\sigma > 0$ is small enough. Since $u(\mathbf{x}, \omega)$ is stationary and normal on $X = \mathbf{K}^m\mathbf{R}^{n-m}$ and (5.90) holds near $\mathbf{0}$, $u(\mathbf{x}, \omega)$ is almost surely continuous near $\mathbf{0}$ and hence, by stationarity, almost surely continuous on X. The same is true if the energy spectrum satisfies the n conditions

$$\int [\log(1 + |k_i|)]^\beta \, dS(\mathbf{k}) < +\infty \qquad (i = 1, 2, \ldots, n) \tag{5.91}$$

as follows from [16, p. 199] and the inequality

$$[\log(1 + |\mathbf{k}|)]^\beta \leq C_\beta \sum_{i=1}^{n} [\log(1 + |k_i|)]^\beta,$$

where C_β is a constant depending only on β and n, We define

$$\Gamma^{(j)}(\mathbf{x}, \mathbf{y}) = \partial^2\Gamma(\mathbf{x}, \mathbf{y})/\partial x_j \, \partial y_j. \tag{5.92}$$

Delporte has proved further [16, p. 3556] that if, for all sufficiently small $\sigma > 0$ and all $\mathbf{x}$,

$$\Gamma^{(j)}(\mathbf{x}, \mathbf{x}) + \Gamma^{(j)}(\mathbf{x} + \mathbf{h}, \mathbf{x} + \mathbf{h}) - 2\Gamma^{(j)}(\mathbf{x}, \mathbf{x} + \mathbf{h}) \leq C|\log \sigma|^{-1-\varepsilon}, \tag{5.93}$$

then $u(\mathbf{x}, \omega)$ is continuous a.e. and has a continuous derivative $\partial u/\partial x_j$ in any finite interval.

In the case of stationary (i.e., homogeneous) normal random function, (5.92) and (5.93) reduce to

$$R^{(j)}(\mathbf{h}) = -\partial^2[R(\mathbf{h})]/\partial h_j^2 \tag{5.94}$$

and

$$R^{(j)}(\mathbf{0}) - R^{(j)}(\mathbf{h}) \leq C(\log|\mathbf{h}|)|^{-\alpha} \qquad (\alpha > 1), \tag{5.95}$$

respectively. Thus, by homogeneity, if (5.94) and (5.95) are satisfied for $j = 1, \ldots, n$, then almost all sample functions are continuously differentiable [in symbols, $u(\mathbf{x}, \omega) \in C^1(X)$ with probability one].

The extension of the preceding results to normal HRVF is easy. Due to (5.91), the qn conditions

$$\int_{\mathbf{R}^n} |\log(1 + |k_i|)|^\beta \, dS_{jj}(\mathbf{k}) \leq +\infty \qquad (\beta > 1)$$
$$(j = 1, 2, \ldots, q, \; i = 1, 2, \ldots, n) \tag{5.96}$$

imply that almost every sample of $u_j(\mathbf{x}, \omega)$ $(j = 1, 2, \ldots, q)$ is continuous; it is obvious that the single condition

$$\int_{\mathbf{R}^n} |\log(1 + k)|^\beta d\mathscr{E}(k) < +\infty \qquad (\beta > 1),$$
$$k = \left(\sum_{i=1}^{n} k_i^2\right)^{1/2}, \qquad d\mathscr{E}(k) = \sum_{j=1}^{q} dS_{jj}(k), \tag{5.97}$$

implies the continuity of almost every sample of the normal HRVF $\mathbf{u}(\mathbf{x}, \omega)$.

By the same reasoning, the condition

$$\int_{\mathbf{R}^r} k^2 |\log(1 + k)|^\beta \, d\mathscr{E}(\mathbf{k}) < +\infty \qquad (\beta = 1), \tag{5.98}$$

implies that almost every sample of the normal HRVF $\mathbf{u}(\mathbf{x}, \omega)$ belongs to $C^1(X)$.

Introducing the trace of the correlation tensor,

$$M = \sum_j R_{jj}(\mathbf{x} - \mathbf{y}), \tag{5.99}$$

and putting

$$M^{(i)} = \sum_j \frac{\partial^{2i} M}{\partial x_j{}^i \, \partial y_j{}^i}, \tag{5.100}$$

the condition

$$M^{(i)}(\mathbf{x} + \mathbf{h}, \mathbf{x} + \mathbf{h}) + M^{(i)}(\mathbf{x}, \mathbf{x}) - 2M^{(i)}(\mathbf{x} + \mathbf{h}, \mathbf{x}) = O[(-\ln|\mathbf{h}|)^\alpha] \qquad (\alpha > 1), \tag{5.101}$$

where $i = 0$ or 1, implies that almost every sample of $\mathbf{u}(\mathbf{x}, \omega)$ belongs to $C^{(i)}$ in any finite interval.[24]

VI. Evolution of the Spectral Matrix Measure

A. Evolution of the Initial Probability

In the analysis of the evolution in time of a system with random initial values, it is important to understand how its probability distribution evolves in time under the action of its governing DE's (3.1). In this part, we will relate this evolution to the evolution of the associated spectral matrix measure. The spaces Λ_p and $D'(X)$ are ill-suited to this problem because, for example, the Cauchy problem for the heat equation is not well set there (see Section III.A). The most appropriate general setting seems to be the space $S'(X)$. We

[24] For conditions (5.92), (5.98), and (5.101), see J. Delporte, *C. R. Acad. Sci. Paris* **268A** (1965), 3554–3557, or Delporte [16].

shall therefore assume that (3.1) is *regular* in the sense of Section III.A, and that μ is an *admissible* probability measure on $S'(X)$ in the sense of Section V.A. Under these assumptions, we shall deduce in this part the evolution in time of μ *and* of the associated spectral matrix measure. Our results and methods will be based on [10].

First, by Schultz's Theorem 3.1, we know that (3.1) determines a solution semigroup $\{T_t\}$; and that every mapping $T_t: S'(X) \to S'(X)$ is one–one. Further, T_t is Borel since it is continuous. Also the space $S'(X)$ is a standard Borel space as defined in Section II.C by Theorem 2.1. Thus T_t is a 1–1 measurable mapping of the standard Borel space $S'(X)$ into itself. Hence by [43, Theorem 2.4, p. 135], T_t is a Borel isomorphism of $S'(X)$ with the Borel subset $B = T_t(S'(X))$ of $S'(X)$.

Let μ be an admissible probability measure on $S'(X)$, and let

$$\mu_t(A) = \mu(T_t^{-1}(A)) \tag{6.1}$$

for all Borel sets A in $S'(X)$. Then μ_t is a probability measure on the Borel sets $\mathscr{B}$ of $S'(X)$ which gives measure one to B.

The measures μ_t give the statistical dynamics of the system. For, if $\mathbf{u}(\mathbf{x}, \omega)$ is the random tempered distribution having measure μ at $t = 0$, and we let $\mathbf{u}$ evolve in time under the action of Eq. (3.1), then we get a random tempered distribution $\mathbf{u}(\mathbf{x}, t, \omega)$ for each t which is distributed with probability μ_t on $S'(X)$.

To analyze the evolution of the measure μ in the light of the results of Section V, we need the following theorem.

Theorem 6.1. *Let μ be a normal admissible homogeneous probability measure on $S'(X)$ with mean zero, and let $\{T_t\}$ be the solution semigroup of a regular Cauchy problem. Then the associated Borel probability measures μ_t defined in* (6.1) *are also normal, admissible, homogeneous, and have mean zero. Their Lebesgue completions will thus be regular.*

Proof. Homogeneity of μ_t follows by Theorem 4.5. Normality follows from Theorem 4.8.

We prove that μ_t is admissible. Let $\phi \in \mathscr{S}(X)$. As before (e.g., in Section II.B), we let $\mathscr{F}$ denote the Fourier transform operator and $\bar{\mathscr{F}}$ its inverse. We evaluate

$$\int_{S'(X)} |V(\phi)|^2 \, d\mu_t(V) = \int_{S'(X)} |V(\phi)|^2 \, d(\mu \circ T_t^{-1}(V))$$
$$= \int_{S'(X)} |T_t V(\phi)|^2 \, d\mu(V),$$

where the preceding integrals are ordinary Lebesgue integrals by Section II.E and the remark after Eq. (5.18), and the last expression is finite by assumption.

We also know how T_t operates from Theorem 3.1:

$$T_t V(\phi) = (\bar{\mathscr{F}} M_t * V)(\phi) = (M_t \mathscr{F} V)(\bar{\mathscr{F}}\phi).$$

Thus the ith component of the vector $T_t V(\phi)$ is

$$T_t V(\phi)_i = \sum_{j=1}^{q} M_t^{ij} \mathscr{F} V_j(\bar{\mathscr{F}}\phi) = \sum_{j=1}^{q} \mathscr{F} V_j(M_t^{ij} \bar{\mathscr{F}}\phi) = \sum_{j=1}^{q} V_j(\mathscr{F} M_t^{ij} \bar{\mathscr{F}}\phi),$$

where V_j is the jth component of the vector-valued distribution V. We let $\theta_{ij} = \mathscr{F} M_t^{ij} \bar{\mathscr{F}}\phi$. Then $\theta_{ij} \in \mathscr{S}(X)$ depends continuously and linearly on ϕ for all i and j. Further, the ith component of $T_t(\phi)$ is $\sum_j V_j(\theta_{ij})$. Now we can estimate:

$$\int_{S'(X)} |T(\phi)|^2 \, d\mu_t(T) = \int_{S'(X)} |T_t V(\phi)|^2 \, d\mu(V) = \sum_{i=1}^{q} \int_{S'(X)} |T_t V(\phi)_i|^2 \, d\mu(V)$$
$$\leq q \sum_{i,j} \int_{S'(X)} |V_j(\theta_{ij})|^2 \, d\mu(V) \leq q \sum_{i,j} \int_{S'(X)} |V(\theta_{ij})|^2 \, d\mu(V).$$

The (i,j)th summand here is continuous in θ_{ij} since μ is admissible. Also, θ_{ij} depends continuously on ϕ. This shows that

$$\int_{S'(X)} |T(\phi)|^2 \, d\mu_t(T)$$

exists and is continuous at zero. From this it is easy to see that the above integral is continuous everywhere on $\mathscr{S}(X)$.

Last, we show μ_t has mean zero. This is an easy consequence of linearity. Let M denote the mean of μ_t. If $\phi \in \mathscr{S}(X)$, then

$$M(\phi) = \int_{S'(X)} T(\phi) \, d\mu_t(T) = \int_{S'(X)} T_t V(\phi) \, d\mu(V).$$

We saw above that the ith component of $T_t V(\phi)$ is given by $\sum_j V_j(\theta_{ij})$, where $\theta_{ij} = \mathscr{F} M_t^{ij} \bar{\mathscr{F}}\phi$ and V_j is the jth component of V. Therefore, if M_i is the ith component of M, then

$$M_i(\phi) = \sum_j \int_{S'(X)} V_j(\theta_{ij}) \, d\mu(V).$$

But since the mean of μ is zero, and $\theta_{ij} \in \mathscr{S}(X)$,

$$\int_{S'(X)} V(\theta_{ij}) \, d\mu(V) = 0$$

and hence each component of the last integral is zero. Thus each summand in $M_i(\phi)$ is zero and so $M_i(\phi) = 0$. Since ϕ and i were arbitrary, we see that $M = 0$. This completes the proof of Theorem 6.1.

B. Evolution of the Spectral Matrix Measure: I

We showed in Section VI.A that if an initial homogeneous admissible normal measure μ with mean zero evolves under the action of a regular system (3.1), then the measure μ_t is also homogeneous, admissible, and normal with mean zero at any later time $t > 0$.

We can associate a covariance matrix Γ^t with such a measure μ_t, and, by Theorem 5.6, μ_t is uniquely determined by Γ^t. Further, since the μ_t are homogeneous, we know from Theorem 5.4 that the action of Γ^t is given in terms of the Fourier transform of a tempered matrix measure ν_t, as in (5.32).

We have now developed enough machinery to establish an unpublished conjecture made by one of us several years ago. Namely, we will show that the evolution of a normal admissible homogeneous probability measure with mean zero under a regular equation is characterized by the motion of its spectral matrix under a certain differential equation.

Theorem 6.2. *Let μ be a normal homogeneous, admissible probability measure and let $P(D)$ be a regular differential operator. Let ν_0 be the spectral matrix measure associated with μ by Theorem 5.4, and consider the initial value problem defined by*

$$d\nu/dt = P(i\mathbf{k})\nu + \nu[P^*(i\mathbf{k})]^T \qquad [\nu(0) = \nu_0], \tag{6.2}$$

where, for each t, $\nu(t)$ is a $q \times q$ matrix of tempered measures on X. Then there is a unique solution $\nu(t)$ of (6.2); moreover, if μ_t is the measure derived from μ as in Theorem 6.1 then $\nu(t)$ is its spectral matrix measure.

The proof is fairly long and will be broken into two parts. In this section, we will establish that the spectral matrix measure ν_t of μ_t does satisfy (6.2). In the next section, the question of uniqueness will be considered.

Proof. Let $\{T_t\}$, $t \geq 0$, be the C_0-semigroup solving (3.1) and let μ_t be the measure induced by T_t as in (6.1). Let Γ^t be the covariance matrix of the admissible homogeneous zero-mean measure μ_t. Recall that the (i,j) entry of Γ^t is defined by

$$\Gamma^t_{ij}(\phi, \psi) = \int_{S'(X)} U_i(\phi)\, U_j(\psi)^*\, d\mu_t(U), \tag{6.3}$$

where $\phi, \psi \in \mathscr{S}(X)$, and U_k is the kth component of the vector-valued distribution U. Now we define the sesquilinear mapping $K_t : S(X) \times S(X) \to \mathbf{C}$ as follows; let f and g be elements of $S(X)$ and $t \geq 0$. Then, in the notation $\langle U \cdot f\rangle$ defined in (2.9),

$$K_t(f, g) = \int_{S'(X)} \langle U \cdot f\rangle \langle U \cdot g\rangle^*\, d\mu_t(U). \tag{6.4}$$

One checks easily as in Lemma 5.4 of Section V.B that K_t is well defined, sesquilinear, and continuous in the topology on $S(X)$. This follows from the admissibility of μ_t (Theorem 6.1); for, in particular, if f is the vector field whose kth component is $\delta_{ik}\phi$ and g is the vector field whose kth component is $\delta_{jk}\psi$, where $\phi, \psi \in \mathscr{S}(X)$, then

$$K_t(f, g) = \Gamma^t_{ij}(\phi, \psi). \tag{6.5}$$

Hence, by linearity, K_t can be expressed as a sum of entries from the matrix Γ^t and so is clearly continuous and sesquilinear.

Notice that K_t can be written in another way. Namely,

$$K_t(f, g) = \int_{S'(X)} \langle U \cdot f\rangle \langle U \cdot g\rangle^*\, d\mu_t(U)$$
$$= \int_{S'(X)} \langle T_t V \cdot f\rangle \langle T_t V \cdot g\rangle^*\, d\mu(V) \tag{6.6}$$

and, applying (3.8),

$$K_t(f, g) = \int_{S'(X)} \langle (\bar{\mathscr{F}} M_t * V) \cdot f\rangle \langle (\bar{\mathscr{F}} M_t * V) \cdot g\rangle^*\, d\mu(V)$$
$$= \int_{S'(X)} \langle M_t \mathscr{F} V \cdot \bar{\mathscr{F}} f\rangle \langle M_t \mathscr{F} V \cdot \bar{\mathscr{F}} g\rangle^*\, d\mu(V), \tag{6.7}$$

where M_t is multiplication by $\exp[tP(i\mathbf{k})]$ as discussed in Section III.B. Letting $L_t = M_t^T$ be the transpose of M_t, this can be rewritten as

$$\int_{S'(X)} \langle V \cdot \mathscr{F} L_t \bar{\mathscr{F}} f\rangle \langle V \cdot \mathscr{F} L_t \bar{\mathscr{F}} g\rangle^*\, d\mu(V) = K_0(\mathscr{F} L_t \bar{\mathscr{F}} f, \mathscr{F} L_t \bar{\mathscr{F}} g). \tag{6.8}$$

Comparing the above formulas, we see that

$$K_t(f, g) = K_0(\mathscr{F} L_t \bar{\mathscr{F}} f, \mathscr{F} L_t \bar{\mathscr{F}} g), \tag{6.9}$$

where $L_t = M_t^T$. Let $N_t = \mathscr{F} L_t \bar{\mathscr{F}}$ so that N_t maps $S(X)$ linearly and continuously into itself. We use (6.9), which expresses K_t in terms of the action of K_0 on the flows N_t in $S(X)$. This allows us to compute the evolution of K_t in time, and hence, using (6.5), the evolution of Γ^t. To show that the spectral matrix ν_t of μ_t evolves according to (6.2), we use Fourier transforms twice: first to define the semigroups $\{T_t\}$, and again to pass from covariance to spectral matrix.

In detail, fix $t > 0$ and $f, g \in S(X)$; then form the differential quotient:

$$\begin{aligned}(1/h)[K_{t+h}(f,g) - K_t(f,g)] &= (1/h)[K_0(N_{t+h}f, N_{t+h}g) - K_0(N_t f, N_t g)]\\ &= (1/h)[K_0(N_{t+h}f, N_{t+h}g) - K_0(N_t f, N_{t+h}g)\\ &\quad + K_0(N_t f, N_{t+h}g) - K_0(N_t f, N_t g)]\\ &= K_0((1/h)(N_{t+h}f - N_t f), N_{t+h}g) + K_0(N_t f, (1/h)(N_{t+h}g - N_t g)).\end{aligned} \tag{6.10}$$

But of course, if $f \in S(X)$, then

$$(1/h)(N_{t+h}f - N_t f) = \mathscr{F}[(1/h)(M_{t+h}^T - M_t^T)]\overline{\mathscr{F}}f \tag{6.11}$$

and, for $w \in S(X)$,

$$(1/h)(M_{t+h} - M_t)w(\mathbf{k}) \to P(i\mathbf{k})\, M_t w(\mathbf{k}) \quad \text{as} \quad h \to 0 \tag{6.12}$$

in $S(X)$, since the multipliers $\{M_t\}$ solve (3.7). Further, since M_t is a C_0-semigroup (see the proof of Theorem 3.1),

$$M_{t+h}w \to M_t w \quad \text{as} \quad h \to 0 \tag{6.13}$$

for $w \in S(X)$. So from (6.10)–(6.13), it follows that

$$\begin{aligned}&\lim_{h\to 0}(1/h)[K_{t+h}(f,g) - K_t(f,g)]\\ &\quad = K_0(\mathscr{F}P(i\mathbf{k})^T L_t\overline{\mathscr{F}}f, \mathscr{F}L_t\overline{\mathscr{F}}g) + K_0(\mathscr{F}L_t\overline{\mathscr{F}}f, \mathscr{F}P(i\mathbf{k})^T L_t\overline{\mathscr{F}}g).\end{aligned} \tag{6.14}$$

Also by the well-known properties of the Fourier transform in $S'(X)$ [20, p. 94],

$$\mathscr{F}P(i\mathbf{k})^T L_t\overline{\mathscr{F}}f = P(-D)^T\mathscr{F}L_t\overline{\mathscr{F}}f. \tag{6.15}$$

Thus

$$\begin{aligned}\partial K_t(f,g)/\partial t &= \lim_{h\to 0}(1/h)[K_{t+h}(f,g) - K_t(f,g)]\\ &= K_0(P(-D)^T\mathscr{F}L_t\overline{\mathscr{F}}f, \mathscr{F}L_t\overline{\mathscr{F}}g) + K_0(\mathscr{F}L_t\overline{\mathscr{F}}f, P(-D)^T\mathscr{F}L_t\overline{\mathscr{F}}g)\\ &= \int_{S'(X)}\langle V\cdot P(-D)^T\mathscr{F}L_t\overline{\mathscr{F}}f\rangle\langle V\cdot\mathscr{F}L_t\overline{\mathscr{F}}g\rangle^*\, d\mu(V)\\ &\quad + \int_{S'(X)}\langle V\cdot\mathscr{F}L_t\overline{\mathscr{F}}f\rangle\langle V\cdot P(-D)^T\mathscr{F}L_t\overline{\mathscr{F}}g\rangle^*\, d\mu(V).\end{aligned} \tag{6.16}$$

We calculate the first integral above[25]:

$$\begin{aligned}&\int_{S'(X)}\langle V\cdot P(-D)^T\mathscr{F}L_t\overline{\mathscr{F}}f\rangle\langle V\cdot\mathscr{F}L_t\overline{\mathscr{F}}g\rangle^*\, d\mu(V)\\ &\quad = \int_{S'(X)}\langle P(D)V\cdot\mathscr{F}L_t\overline{\mathscr{F}}f\rangle\langle V\cdot\mathscr{F}L_t\overline{\mathscr{F}}g\rangle^*\, d\mu(V)\\ &\quad = \int_{S'(X)}\langle(\overline{\mathscr{F}}M_t * P(D)V)\cdot f\rangle\langle(\overline{\mathscr{F}}M_t * V)\cdot g\rangle^*\, d\mu(V)\\ &\quad = \int_{S'(X)}\langle P(D)[\overline{\mathscr{F}}M_t * V]\cdot f\rangle\langle(\overline{\mathscr{F}}M_t * V)\cdot g\rangle^*\, d\mu(V)\\ &\quad = \int_{S'(X)}\langle P(D)T_t V\cdot f\rangle\langle T_t V\cdot g\rangle^*\, d\mu(V)\\ &\quad = \int_{S'(X)}\langle T_t V\cdot P(-D)^T f\rangle\langle T_t V\cdot g\rangle^*\, d\mu(V)\\ &\quad = K_t(P(-D)^T f, g).\end{aligned}$$

Similarly,

$$\int_{S'(X)}\langle V\cdot\mathscr{F}L_t\overline{\mathscr{F}}f\rangle\langle V\cdot P(-D)^T\mathscr{F}L_t\overline{\mathscr{F}}g\rangle^*\, d\mu(V) = K_t(f, P(-D)^T g).$$

Combining these two formulas with (6.16), we obtain

$$\partial K_t(f,g)/\partial t = K_t(P(-D)^T f, g) + K_t(f, P(-D)^T g). \tag{6.17}$$

We now specialize f and g, choosing *fixed* indices i, j in the range $1, \ldots, q$. For simplicity, we do not indicate notationally the dependence of f or g on this choice, but designate by f the element of $S(X)$ whose entries are all zero except in the ith row, where they are all $\phi \in \mathscr{S}(X)$. Likewise, $g \in S(X)$ has all zero rows except for the jth, which is ψ. Then the lth rows of $P(-D)^T f$ and $P(-D)^T g$ are $P_{il}(-D)\phi$ and $P_{jl}(-D)\psi$, respectively, for $l = 1, \ldots, q$. Finally, let $\phi_{kl} \in S(X)$ be the element which is $P_{kl}(-D)\phi$ in the kth row and zero in all other rows, and let $\psi_{kl} \in S(X)$ be the element equal to $P_{kl}(-D)\psi$ in the kth row and zero in other rows, where $1 \le k, l \le q$. Then

$$P(-D)^T f = \sum_{k=1}^{q}\phi_{ik}, \qquad P(-D)^T g = \sum_{k=1}^{q}\psi_{jk}. \tag{6.18}$$

We apply (6.5) to get

$$K_t(P(-D)^T f, g) = \sum_{k=1}^{q} K_t(\phi_{ik}, g) = \sum_{k=1}^{q}\Gamma_{kj}^t(P_{ik}(-D)\phi, \psi) \tag{6.19}$$

[25] Here and elsewhere, we shall often write $P(-D)^T$ for the matrix transpose $[P(-D)]^T = P^T(-D)$.

and similarly

$$K_t(f, P(-D)^T g) = \sum_{k=1}^{q} \Gamma_{ik}^t(\phi, P_{jk}(-D)\psi). \tag{6.20}$$

Since μ_t is homogeneous, we know by Theorem 5.4 that the action of Γ^t is given by a tempered spectral matrix measure $v^t = \|v_{ij}^t\|$ as follows:

$$\Gamma_{ij}^t(\phi, \psi) = \mathscr{F}[v_{ij}^t](\phi * \tilde{\psi}). \tag{6.21}$$

Hence

$$\begin{aligned} K_t(P(-D)f, g) &= \sum_{l=1}^{q} \mathscr{F}[v_{lj}^t](P_{il}(-D)\phi * \tilde{\psi}) = \sum_{l=1}^{q} \mathscr{F}[v_{lj}^t](P_{il}(-D)(\phi * \tilde{\psi})) \\ &= \sum_{l=1}^{q} [v_{lj}^t](\mathscr{F}(P_{il}(-D)(\phi * \tilde{\psi}))) = \sum_{l=1}^{q} [v_{lj}^t](P_{il}(\mathrm{i}k)\mathscr{F}(\phi * \tilde{\psi})) \\ &= \sum_{l=1}^{q} [P_{il}(\mathrm{i}k)v_{lj}^t](\mathscr{F}(\phi * \tilde{\psi})) = \mathscr{F}\left[\sum_{l=1}^{q} P_{il}(\mathrm{i}k)v_{lj}^t\right](\phi * \tilde{\psi}). \end{aligned} \tag{6.22}$$

A similar calculation yields

$$K_t(f, P(-D)^T g) = \mathscr{F}\left[\sum_{l=1}^{q} P_{jl}^*(\mathrm{i}k)v_{il}^t\right](\phi * \tilde{\psi}). \tag{6.23}$$

Combining (6.23), (6.22), and (6.17), we see that

$$\partial K_t(f, g)/\partial t = \mathscr{F}\left[\sum_{l=1}^{q} P_{il}(\mathrm{i}k)v_{lj}^t\right](\phi * \psi) + \mathscr{F}\left[\sum_{l=1}^{q} P_{jl}^*(\mathrm{i}k)v_{il}^t\right](\phi * \psi). \tag{6.24}$$

Also, using (6.5),

$$\begin{aligned} \partial K_t(f, g)/\partial t &= \lim_{h\to 0} (1/h)(K_{t+h}(f, g) - K_t(f, g)) \\ &= \lim_{h\to 0} (1/h)(\Gamma_{ij}^{t+h}(\phi, \psi) - \Gamma_{ij}^t(\phi, \psi)) \\ &= \lim_{h\to 0} (1/h)[\mathscr{F}[v_{ij}^{t+h}](\phi * \tilde{\psi}) - \mathscr{F}[v_{ij}^t](\phi * \tilde{\psi})] \\ &= \lim_{h\to 0} [(1/h)([v_{ij}^{t+h}] - [v_{ij}^t])](\mathscr{F}(\phi * \tilde{\psi})). \end{aligned} \tag{6.25}$$

Putting (6.24) and (6.25) together,

$$\lim_{h\to 0} [(1/h)([v_{ij}^{t+h}] - [v_{ij}^t])](\hat{\phi}\hat{\psi}^*) = \left[\sum_{l=1}^{q} P_{il}(\mathrm{i}k)v_{lj}^t\right](\hat{\phi}\hat{\psi}^*) + \left[\sum_{l=1}^{q} P_{jl}^*(\mathrm{i}k)v_{il}^t\right](\hat{\phi}\hat{\psi}^*), \tag{6.26}$$

where $\hat{\phi} = \mathscr{F}\phi$ denotes the inverse Fourier transform of ϕ. In (6.26), we have proved just what we want to prove when the limit exists, as it must for the subset M of elements of the form $\hat{\phi}\hat{\psi}^*$, which is dense since $M \supset D(X)$. To complete the proof, we show that the differential quotients constitute an *equicontinuous* subset of $\mathscr{S}'(X)$ for $|h| \leq 1$, say; it will follow that the limit (6.26) exists on all of $\mathscr{S}(X)$.

Lemma 6.1. *The collection $\{U_h\}_{-1\leq h\leq 1}$, defined by*

$$U_h(\phi) = \begin{cases} (1/h)([v_{ij}^{t+h}] - [v_{ij}^t])(\phi), & h \neq 0, \\ \left(\sum_l P_{il}(\mathrm{i}k)[v_{lj}^t] + \sum_l P_{jl}^*(\mathrm{i}k)[v_{il}^t]\right)(\phi), & h = 0, \end{cases} \tag{6.27}$$

for $\phi \in \mathscr{S}(X)$, is equicontinuous in $\mathscr{S}'(X)$.

Proof. First we show that the sesquilinear forms $\{B_h\}_{-1\leq h\leq 1}$ given by

$$B_h(\phi, \psi) = \begin{cases} (1/h)(\Gamma_{ij}^{t+h}(\phi, \psi) - \Gamma_{ij}^t(\phi, \psi)), & h \neq 0, \\ \sum_l \Gamma_{lj}^t(P_{il}(-D)\phi, \psi) + \sum_l \Gamma_{il}^t(\phi, P_{jl}(-D)\psi), & h = 0, \end{cases}$$

are equicontinuous in the space of sesquilinear forms on $\mathscr{S}(X)$. To see this, let $I = [-1, 1]$ and consider the following statements:

(a) For fixed $\psi \in \mathscr{S}(X)$, the linear mappings $L_h(\phi) = B_h(\phi, \psi)$ are equicontinuous in $\mathscr{S}'(X)$ for $h \in I$.

(b) For fixed $\phi \in \mathscr{S}(X)$, the linear functionals $L_h'(\psi) = B_h(\phi, \psi^*)$ are equicontinuous in $\mathscr{S}'(X)$ for $h \in I$.

Since the collections $\{L_h\}$ and $\{L_h'\}$ are weakly bounded for $h \in I$, and $\mathscr{S}(X)$ is barreled, the Banach–Steinhaus theorem [52, Theorem 33.2] implies that (a) and (b) are valid statements. Thus we have a family of forms on the metrizable linear space $\mathscr{S}(X)$ satisfying (a) and (b) and so, by [52, Theorem 34.1], the family is an equicontinuous subset of the space of all forms on $\mathscr{S}(X)$.

Finally, the family of tempered distributions $\{U_h\}_{-1\leq h\leq 1}$ in (6.27) is the continuous image of the family $\{B_h\}_{-1\leq h\leq 1}$ via the correspondence in Theorem 5.4. Hence $\{U_h\}_{-1\leq h\leq 1}$ is an equicontinuous subset of $\mathscr{S}'(X)$ and the proof of Lemma 6.1 is complete.

Elements of the form $\mathscr{F}(\phi * \tilde{\psi})$, where ϕ and ψ are in $\mathscr{S}(X)$, are dense in $\mathscr{S}(X)$. For elements ρ of this form, we saw above that, in the notation of (6.27),

$$\lim_{h\to 0} U_h(\rho) = U_0(\rho). \tag{6.28}$$

Now, as in [52, Proposition 32.5], an equicontinuous family that converges on a dense subset converges weakly everywhere. Applying Lemma 6.1, we get that $U_h \to U_0$ weakly in $\mathscr{S}'(X)$. But weak and strong convergence are the same in $\mathscr{S}'(X)$, so we may conclude that

$$\lim_{h\to 0} (1/h)([v_{ij}^{t+h}] - [v_{ij}^t]) = \left[\sum_{l=1}^{q} P_{il}(i\mathbf{k})v_{lj}^t\right] + \left[\sum_{l=1}^{q} P_{jl}(i\mathbf{k})^* v_{il}^t\right] \tag{6.29}$$

in $\mathscr{S}'(X)$ for each i and j. If we write (6.29) in matrix form, we get

$$\partial v^t/\partial t = P(i\mathbf{k})v^t + v^t\overline{P(i\mathbf{k})}^T. \tag{6.30}$$

So the spectral matrix measure does satisfy Eq. (6.2). Note that $v^0 = v$ is the correct initial value as well.

C. Evolution of the Spectral Matrix Measure: II

The uniqueness portion of Theorem 6.2 follows from abstract consideration of Eq. (6.2) itself. That is, consider the initial value problem

$$\partial u/\partial t = P(i\mathbf{k})u + uP(i\mathbf{k})^H \qquad [u(0) = v] \tag{6.31}$$

in the space $M_q(\mathscr{S}'(X))$ of $(q \times q)$-matrices whose entries are tempered distributions.

According to [20, Theorem 6] [with $\Phi = \Phi_1 = E = M_q(\mathscr{S}(X))$ the space of $(q \times q)$-matrices whose elements lie in $\mathscr{S}(X)$], (6.31) has a unique solution if there exists a solution of the modified adjoint problem on each finite interval $[0, T]$. But because the duality between $M_q(\mathscr{S}(X))$ and $M_q(\mathscr{S}'(X))$ is given by

$$(u, \eta) = \sum_{i,j=1}^{q} u_{ij}(\eta_{ij}), \tag{6.32}$$

the modified adjoint problem is easy to solve. For the adjoint of the operator $P(i\mathbf{k})$: $u \mapsto P(i\mathbf{k})u$ from $M_q(\mathscr{S}'(X))$ to $M_q(\mathscr{S}'(X))$ is the operator $P(i\mathbf{k})^T$: $M_q(\mathscr{S}(X)) \to M_q(\mathscr{S}(X))$ which maps η to $P(i\mathbf{k})^T\eta$. Similarly, the adjoint of the operator that maps u to $uP^*(i\mathbf{k})^T$ is the operator that maps η to $\eta P(i\mathbf{k})^*$. Hence, to demonstrate uniqueness, we need only prove the following lemma.

Lemma 6.2. *Given $T > 0$ and $\gamma \in M_q(\mathscr{S}(X))$, there is a solution to*

$$\begin{aligned} \partial\eta/\partial t &= -P(i\mathbf{k})^T\eta - \eta P^*(i\mathbf{k}) \qquad (0 \le t \le T) \\ \eta(T) &= \gamma. \end{aligned} \tag{6.33}$$

Proof. This is quite easy. For, since P is regular, both $\exp[-(t-T)P^T(i\mathbf{k})]$ and $\exp[-(t-T)P^*(i\mathbf{k})]$ have entries that are multipliers in $\mathscr{S}(X)$ for $t \le T$ (see Section III.B). We let

$$\eta(t) = \exp[-(t-T)P(i\mathbf{k})^T]\gamma \exp[-(t-T)P^*(i\mathbf{k})] \tag{6.34}$$

for $0 \le t \le T$. One checks without difficulty that this does indeed solve (6.33). The proof of the lemma and of Theorem 6.2 is now complete.

We point out that the unique solution of (6.2) guaranteed to exist by Theorem 6.2 is given by the formula

$$v(t) = \exp[tP(i\mathbf{k})]v_0 \exp[tP^*(i\mathbf{k})]^T. \tag{6.35}$$

The proof of this is a simple calculation. Formula (6.35) is well defined in $M_q(\mathscr{S}'(X))$ because P is a regular linear differential operator. The following corollary is also interesting.

Corollary 6.1. *Let $P(D)$ be a regular linear differential operator. If the initial value v of the initial value problem (6.2) is a Hermitian definite tempered matrix measure, then the unique solution $v(t)$ is, for each $t > 0$, a Hermitian definite tempered matrix measure.*

Proof. Let Γ be the covariance associated with v via Section V.E and let μ be the normal homogeneous admissible measure on $S'(X)$ associated with Γ. Let μ_t be as defined in (6.1). Then v is the spectral matrix measure of μ and, by the last lemma, Eq. (6.2) has a unique solution $v(t) = v^t$, where v^t is the spectral matrix measure of μ_t and so is a Hermitian definite tempered matrix measure.

To review, Theorem 3.1 implies that any regular Cauchy problem (3.1) with initial value in the space $S'(X)$ of vector-valued tempered distributions defines a unique C_0-semigroup on $S'(X)$. If the initial values are random and given by a normal homogeneous admissible probability measure μ on the Borel sets of $S'(X)$, then the solutions at time $t > 0$ are again randomly distributed according to a normal homogeneous admissible probability measure μ_t on the Borel sets of $S'(X)$. Furthermore, this seemingly complicated situation in which μ is evolving under Eq. (3.1) is reduced first to the evolution of the covariance of μ and then to the evolution of the spectral matrix. Finally, the precise equation that governs the evolution of the spectral matrix and an integral of this equation is provided.

Thus for homogeneous linear evolution equations with constant coefficients and normally distributed random initial data, the development in time of the autonomous system considered is entirely determined by the ordinary DE (6.2), which governs the evolution of the spectrum. This result may have applications to linear models of the statistical mechanics of homogeneous media (e.g., of water waves). It would be interesting to extend it to *inhomogeneous* linear DE's with random initial data and inputs ("forcing functions") —e.g., to wind-generated water waves.

VII. Parabolic Problems

A. The Parabolic Case

The solutions of Cauchy problems with random initial values, whose existence for hyperbolic DE's was established by general methods in Section IV.C, are typically *generalized functions* (not even "weak" solutions) unless special assumptions are made on the initial smoothness (spectrum).

In the case of regular *parabolic* DE's, however, the Green's function is typically smooth for $t > 0$ (the high-frequency components of the spectrum die out rapidly), and so by the results of Sections V.F–G, solutions of (3.1) with random initial values are almost sure to be strong *classical* solutions, except possibly at $t = 0$. Partly for the sake of contrast, and partly to bring out what is *not* easily proved by the general methods described earlier, we will now apply *classical* methods to parabolic DE's with random initial values, restricting attention to the case $n = q = 1$ in order to be able to use the results of Section V.G.

When $n = q = 1$, (3.1) simplifies to

$$\partial u/\partial t = P(\partial/\partial x)u, \qquad \text{where} \quad P(\xi) = \sum_{m=0}^{p} (\alpha_m + i\beta_m)\xi^m. \tag{7.1}$$

We define, for k real,

$$\Lambda(k) = \text{Re } P(ik) = \alpha_0 - \beta_1 k - \alpha_2 k^2 + \beta_3 k^3 + \cdots, \tag{7.2}$$

$$\Theta(k) = \text{Im } P(ik) = \beta_0 + \alpha_1 k - \beta_2 k^2 + \cdots. \tag{7.3}$$

One says that the DE (7.1) is of *parabolic type* when

$$\lim_{|k| \to +\infty} \Lambda(k) = -\infty. \tag{7.4}$$

(See [20, p. 191], where Shilov's and Petrowski's definitions, which agree when $n = q = 1$, are recalled.)

Thus the DE (7.1) is of parabolic type if and only if $\Lambda(k)$ is of even degree, the coefficient of the terms of higher order being negative[26]:

$$\Lambda(k) = A_0 + A_1 k + \cdots + A_{2s} k^{2s}, \qquad A_j \text{ real}, \quad A_{2s} < 0, \quad 2 \le 2s \le p. \tag{7.5}$$

This implies that there exists two real constant A and $B > 0$ such that

$$\Lambda(k) \le A - Bk^{2s} \qquad (\text{all real } k). \tag{7.6}$$

[26] The s used here and through Section VII.C is of course not the s of the definition $X = \mathbf{K}^s\mathbf{R}^{n-s}$; here and below, we have $X = \mathbf{R}$, which frees s and n for other uses.

One has also

$$\sup_{-\infty < k < +\infty} \Lambda(k) = \sigma < +\infty. \tag{7.7}$$

The Fourier techniques used below are based on

Lemma 7.1 *If the DE (7.1) is parabolic, then, for any polynomial $S(k)$ and $t > 0$,*

$$S(k)e^{tP(ik)} \in L^r(X) \tag{7.8}$$

for all $r \ge 1$.

Proof.

$$|S(k)e^{tP(ik)}| = |S(k)|e^{t\Lambda(k)} \le |S(k)| \exp(At - Btk^{2s}).$$

B. Green's Function

Obviously, $e^{ikx + tP(ik)}$ is a solution of the DE (7.1) for each k; we will define, for $t > 0$,

$$G(x, t) = \int_{-\infty}^{+\infty} e^{ikx + tP(ik)}\, dk \tag{7.9}$$

and show that this defines in the half-plane $t > 0$ the fundamental solution of (7.1) [20, p. 285].

Observe first that, for $t > 0$, $G(x, t)$ is, as a function of x, the Fourier transform of $e^{tP(ik)}$, which by Lemma 7.1 belongs to L^r for all $r \ge 1$. The continuity of $G(x, t)$ in x follows immediately from the known properties of Fourier transforms, and also (from the Riemann–Lebesgue theorem)

$$\lim_{|x| \to +\infty} G(x, t) = 0 \qquad (t > 0). \tag{7.10}$$

There is one case when $G(x, t)$ reduces (as a function of x) to a well-known class of functions:

$$\Theta(k) = 0, \tag{7.11}$$

since $e^{tP(ik)} = e^{t\Lambda(k)} > 0$, and $G(x, t)$ is, for each $t > 0$, a (in general complex) *correlation function* or, what is the same, $G(x, t)/G(0, t)$ is a characteristic function [39]. Then all the known properties [39] of these functions apply immediately; we note only the most important:

(a) $|G(x, t)| \le G(0, t)$;
(b) $G(-x, t) = G^*(x, t)$;

(c) $G(x, t)$ is continuous and positive definite in x [i.e., for any finite set $\{x_1, \ldots, x_n\}$, the quadratic Hermitian form $\sum_{j,k} y_j y_k^* G(x_j - x_k, t)$ is non-negative definite];

(d) $G(x, t)$ is an entire function of x which is [44, p. 1243] of order $\delta = 2s/(2s-1)$.

We will establish the fundamental properties of $G(x, t)$ in the general case

$$\Theta(k) \neq 0. \tag{7.12}$$

Theorem 7.1. (a) *For $t > 0$, $G(x, t)$ has continuous derivatives of all orders given by*

$$\frac{\partial^{m+n}}{\partial x^m \partial t^n} G(x, t) = \int_{-\infty}^{+\infty} e^{ikx + tP(ik)} (ik)^m [P(ik)]^n \, dk. \tag{7.13}$$

(b) *In $t > 0$, $G(x, t)$ is an analytic function of (x, t) and for each $t > 0$ an entire function of x.*

(c) *$G(x, t)$ is, in $t > 0$, a classical solution of* (7.1).

Proof. For $0 < \theta \leq t$ one deduces from (7.6)

$$|e^{ikx + tP(ik) - tA}(ik)^m [P(ik)]^n| \leq \exp(-B\theta k^{2s}) |k|^m |P(ik)|^n;$$

the right side being independent of (x, t) and belonging to L, one can apply the rule of derivation in a Lebesgue integral, and thus (a) is proved.

Computing $\partial G/\partial t$ and $P(D)G$ from (7.1) one immediately proves (c); in symbols, $\partial G/\partial t = P(D)G$ for $t > 0$.

Next, we deduce from (7.13), for any $t_0 > 0$,

$$\sum_{m,n} \frac{x^m}{m!} \frac{(t - t_0)^n}{n!} \frac{\partial^{m+n}}{\partial x^m \partial t^n} G(0, t_0) = \sum_{m,n} \int_{-\infty}^{+\infty} e^{t_0 P(ik)} \frac{x^m (ik)^m}{m!} \frac{(t - t_0)^n [P(ik)]^n}{n!} dk.$$

But, the series

$$\sum_{m,n} \frac{x^m (ik)^m}{m!} \frac{(t - t_0)^n [P(ik)]^n}{n!},$$

being equal to $e^{ikx + (t - t_0)P(ik)}$, is uniformly convergent in $|x| < \rho_1$, $|t - t_0| < \rho_2$, $|k| < N$. Thus, for $|x| < +\infty$, $|t - t_0| < t_0$, one has

$$\sum_{m,n} \frac{x^m}{m!} \frac{(t - t_0)^n}{n!} \frac{\partial^{m+n}}{\partial x^m \partial t^n} G(0, t_0) = \int_{-\infty}^{+\infty} e^{ikx + tP(ik)} \, dk = G(x, t),$$

which proves (b).

Theorem 7.2. *For any fixed $t > 0$, and $p, m, n = 0, 1, 2, \ldots$,*

$$x^p \frac{\partial^{m+n}}{\partial x^m \partial t^n} G(x, t) \in L \cap L^2$$

and one has the inversion formula

$$\frac{d^p}{dk^p} [e^{tP(ik)} (ik)^m [P(ik)]^n] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ikx} (-ix)^p \frac{\partial^{m+n}}{\partial x^m \partial t^n} G(x, t) \, dx. \tag{7.14}$$

Proof. By Lemma 7.1, we have

$$d^p \{e^{tP(ik)} (ik)^m [P(ik)]^n\} / dk^p \in L \cap L^2. \tag{7.15}$$

Now fix m and n, and consider the sequence of pth derivatives in (7.15), $p = 0, 1, 2, \ldots$. We prove

Lemma 7.2. *If every $f^{(p)}(k)$ is in $L \cap L^2$, then the inverse Fourier transform*

$$F(x) = \int_{-\infty}^{+\infty} e^{ikx} f(k) \, dk \tag{7.16}$$

of $f(k)$ satisfies, for all p,

(a) *$x^p F(x) \in L \cap L^2$, and*

(b) *$f^{(p)}(k) = (1/2\pi) \int_{-\infty}^{+\infty} e^{-ikx} (-ix)^p F(x) \, dx$.*

Proof. Since every $f^{(p)}(k) \in L$, one deduces first

$$(-ix) F(x) = \int_{-\infty}^{+\infty} e^{ikx} f'(k) \, dk \tag{7.17}$$

from (7.16) by integrating by parts and noting that $f(k)$ and $f'(k) \in L$ imply $\lim_{|k| \to \infty} f(k) = 0$. By induction, applying the same reasoning to $f'(k)$ and $f''(k)$, one has

$$(-ix)^p F(x) = \int_{-\infty}^{+\infty} e^{ikx} f^{(p)}(k) \, dk. \tag{7.18}$$

Now, because $f^{(p)}(k) \in L^2$ by Plancherel's theorem, we have

$$(-ix)^p F(x) \in L^2 \qquad (p = 0, 1, 2, \ldots). \tag{7.19}$$

Further, applying to the pairs: $F(x), xF(x)$; $xF(x), x[xF(x)]$; ... the classical result [51, p. 92] that

$$\phi(x) \quad \text{and} \quad x\phi(x) \in L^2 \quad \text{imply} \quad \phi(x) \in L, \tag{7.20}$$

we have $x^p F(x) \in L$, $p = 0, 1, 2, \ldots$ Thus (a) is proved from (7.19) and (7.20); next, due to (7.18), (b) is proved by applying the Fourier inversion formula.

Corollary 7.1. *One has*

$$\lim_{t\downarrow 0} \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikx}(-ix)^p\, G(x, t)\, dx = \delta_{0p} \qquad (p = 0, 1, 2, \ldots). \tag{7.21}$$

Proof. Apply (7.14) for $m = n = 0$ and make $t = 0$ in the left-hand member.

Note that (7.21) is typical for the Dirac delta-function, whose "Fourier transform" is precisely the constant 1.

Remark 1. For later use remark that, if we set

$$\gamma(k, x, t) = e^{ikx + tP(ik)}, \tag{7.22}$$

the pair of inverse Fourier transforms (7.13) and (7.14) with $p = 0$ can be written, for any $t > 0$ and $m, n = 0, 1, 2, \ldots,$

$$\frac{\partial^{m+n}}{\partial x^m\, \partial t^n} G(x, t) = \int_{-\infty}^{+\infty} \frac{\partial^{m+n}}{\partial x^m\, \partial t^n}\gamma(k, x, t)\, dk \tag{7.23}$$

and

$$e^{-ikx}\frac{\partial^{m+n}}{\partial x^m\, \partial t^n}\gamma(k, x, t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iky}\frac{\partial^{m+n}}{\partial y^m\, \partial t^n} G(y, t)\, dy. \tag{7.24}$$

Theorem 7.3. *One has, uniformly in x,*

$$\lim_{t\uparrow+\infty} e^{-\sigma t}\frac{\partial^{m+n}}{\partial x^m\, \partial t^n} G(x, t) = 0 \qquad (m, n = 0, 1, 2, \ldots), \tag{7.25}$$

where σ is defined in (7.7).

Proof. From (7.13) we deduce

$$e^{-\sigma t}\frac{\partial^{m+n}}{\partial x^m\, \partial t^n} G(x, t) \le \int_{-\infty}^{+\infty} e^{t[\Lambda(k)-\sigma]}|k|^m |P(ik)|^n\, dk.$$

Now we have

(a) $\lim_{t\uparrow+\infty} e^{t[\Lambda(k)-\sigma]}|k|^m |P(ik)|^n = 0,$

expect possibly for a finite number of values of k for which $\Lambda(k) = \sigma$; and, by Lemma 7.1, for any $\theta > 0$ and $t \ge \theta$:

(b) for $t \ge \theta > 0$, $e^{t[\Lambda(k)-\sigma]} \le e^{\theta[\Lambda(k)-\sigma]}$, and $e^{\theta[\Lambda(k)-\sigma]}|k|^m|P(ik)|^n \in L$.

Thus the result follows by Lebesgue's dominated convergence theorem.

Remark 2. For $G(x, t)$, one can easily establish the bound

$$|G(x, t)| \le e^{At}K/t^{1/2s}, \qquad K = 2\int_0^{\infty} \exp(-Bk^{2s})\, dk, \tag{7.26}$$

simply by applying inequalities (7.6–7.9),

$$|G(x, t)| \le \int_{-\infty}^{\infty} \exp(At - Btk^{2s})\, dk,$$

and replacing k by $kt^{1/2s}$ in the integral. [Incidentally, (7.26) implies

$$\lim_{t\uparrow+\infty} e^{-At} G(x, t) = 0,$$

but this result is weaker than (7.25) because $A \ge \sigma$.]

Theorem 7.4. *The function $G(x, t)$ has the semigroup property: for any $t_1 > 0$, $t_2 > 0$,*

$$G(x, t_1 + t_2) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} G(y, t_1)G(x - y, t_2)\, dy. \tag{7.27}$$

Proof. Apply the convolution theorem to the functions $G(x, t_1)$, $G(x, t_2)$, and $G(x, t_1 + t_2)$, whose Fourier transforms are $e^{t_1P(ik)}$, $e^{t_2P(ik)}$, and $e^{(t_1+t_2)P(ik)}$; the computation is legitimate since all the functions involved belong to $L \cap L^2$.

C. Construction of Solutions

Let μ be a homogeneous normal probability measure on $\Lambda_2 = \Lambda_2(\mathbf{R})$. From the corollary to Theorem 4.7 in Section IV.D, we know that, if

$$v(x, \omega) \in \Lambda_2(\mathbf{R}) \tag{7.28}$$

denotes the sample function associated with the measure μ, then, with probability one,

$$v(x, \omega)/(1 + x^2)^r \in L_2(\mathbf{R}) \qquad (\text{all } r > \tfrac{1}{2}), \tag{7.29}$$

and so

$$v(x, \omega)/(1 + x^2)^s \in L = L_1(\mathbf{R}) \qquad (\text{all } s > 1). \tag{7.30}$$

Our first main result will be

Theorem 7.5. *If $v(x, \omega)$ is a homogeneous normal random function in $\Lambda_2 = \Lambda_2(\mathbf{R})$, then with probability one,*

$$u(x, t, \omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} v(y, \omega)\, G(x - y, t)\, dy \tag{7.31}$$

represents a classical solution of (7.1) *for $t > 0$.*

Proof. First we will prove that (7.31) exists with probability one for $t > 0$, or, more generally,

Lemma 7.3. *For any $m, n = 0, 1, 2, \ldots$, all the integrals*

$$u_{m,n}(x, t, \omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} v(y, \omega)\frac{\partial^{m+n}}{\partial x^m\, \partial t^n} G(x - y, t)\, dy \tag{7.32}$$

exist with probability one for $t > 0$.

Proof. Let us write

$$u_{m,n}(x, t, \omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{v(y, \omega)}{1 + y^2}(1 + y^2)\frac{\partial^{m+n}}{\partial x^m\, \partial t^n} G(x - y, t)\, dy.$$

Now from (7.29) (taking $r = 1$), we have $v(y, \omega)/(1 + y^2) \in L^2$ with probability one; and, by Theorem 7.2,

$$(1 + y^2)\frac{\partial^{m+n}}{\partial x^m\, \partial t^n} G(x - y, t) \in L^2. \tag{7.33}$$

Thus the conclusion follows.

Now let us introduce the Fourier transform:

$$V(k, \omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikx}\frac{v(x, \omega)}{1 + x^2}\, dx. \tag{7.34}$$

Due to (7.29) with $r = 1$, this integral exists as a Plancherel transform and

$$V(k, \omega) \in L^2. \tag{7.35}$$

From (7.34) we deduce by a change of variables that

$$e^{-ikx}\, V(-k, \omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iky}\frac{v(x - y, \omega)}{1 + (x - y)^2}\, dy. \tag{7.36}$$

But from Theorem 7.2 we obtain

$$e^{-ikx}\, \Gamma(k, x, t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iky}[1 + (x - y)^2]\, G(y, t)\, dy \tag{7.37}$$

when [according to the definition (7.22) of $\gamma(k, x, t)$]

$$\Gamma(k, x, t) = \gamma(k, x, t) - \frac{d^2}{dk^2}\gamma(k, x, t). \tag{7.38}$$

Now let us apply to the two pairs of Fourier transforms (all in L^2)

$$F(k) = e^{-ikx}\, \Gamma(k, x, t), \qquad G(k) = e^{-ikx}\, V(-k, \omega)$$

$$f(y) = [1 + (x - y)^2]\, G(y, t), \qquad g(y) = \frac{v(x - y, \omega)}{1 + (x - y)^2}$$

the Parseval formula [51, p. 50, (2.1.1)]

$$\int_{-\infty}^{+\infty} F(k)\, G(-k)\, dk = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(y)\, g(y)\, dy.$$

We obtain, using the definition (7.33), the new expression

$$u(x, t, \omega) = \int_{-\infty}^{+\infty} \Gamma(k, x, t)\, V(k, \omega)\, dk, \tag{7.39}$$

where the integral exists with probability one for $t > 0$.

A glance at formula (7.38) shows that, for each k, $\Gamma(k, x, t)$ is a *classical solution* of the DE (7.1) because

$$\gamma(k, x, t) = e^{ikx + tP(ik)}$$

is, for each k, a classical solution of (7.1). Thus, in order to prove that $u(x, t, \omega)$ is a solution of (7.1), we have only to prove that

$$\frac{\partial^{m+n}}{\partial x^m\, \partial t^n} u(x, t, \omega) = \int_{-\infty}^{+\infty} \frac{\partial^{m+n}}{\partial x^m\, \partial t^n}\Gamma(k, x, t)\, V(k, \omega)\, dk. \tag{7.40}$$

If one observes that

$$\frac{\partial^{m+n}}{\partial x^m\, \partial t^n}\Gamma(k, x, t) = \frac{\partial^{m+n}}{\partial x^m\, \partial t^n}\gamma - \frac{d^2}{dk^2}\frac{\partial^{m+n}}{\partial k^m\, \partial t^n}\gamma$$

one sees that this derivative is a linear combination of terms of the form $S(k)e^{ikx + tP(ik)}$, where $S(k)$ is a polynomial. But by (7.6) the modulus of this term is dominated by $|S(k)|\exp(At - Btk^{2s})$; thus, for $t \geq \theta > 0$,

$$e^{-At}\, \partial^{m+n}\Gamma(k, x, t)/\partial x^m\, \partial t^n$$

is dominated by the function $|S(k)|\exp(-B\theta k^{2s}) \in L^2$ independently of (x, t). Moreover

$$e^{-At}\frac{\partial^{m+n}}{\partial x^m\, \partial t^n}\Gamma(k, x, t)\, V(k, \omega)$$

is dominated by $|S(k)|\exp(-B\theta k^{2s})\, V(k, \omega)$, which belongs to L (because $V \in L^2$) and is independent of (x, t). In conclusion, the rule of differentiation in a Lebesgue integral can be applied and (7.40) is proved.

D. Behavior as $t \downarrow 0$

Finally let us consider convergence as $t \downarrow 0$ to the initial value of the solution $u(x, t, \omega)$.

From (7.38) one deduces easily

$$\lim_{t \downarrow 0} \Gamma(k, x, t) = (1 + x^2)e^{ikx}. \tag{7.41}$$

This suggests proving that

$$\begin{aligned} \lim_{t \downarrow 0} u(x, t, \omega) &= \int_{-\infty}^{+\infty} \lim_{t \downarrow 0} \Gamma(k, x, t)\, V(k, \omega)\, dk \\ &= (1 + x^2) \int_{-\infty}^{+\infty} e^{ikx}\, V(k, \omega)\, dk. \end{aligned} \tag{7.42}$$

Theorem 7.6. *If the Fourier transform $V(k, \omega)$ is Lebesgue integrable, then*

$$\lim_{t \downarrow 0} u(x, t, \omega) = v(x, \omega) \qquad \textit{for all} \quad x. \tag{7.43}$$

Proof. If $V(k, \omega) \in L$, then the integral on the right side of (7.42) has the value

$$\int_{-\infty}^{+\infty} e^{ikx}\, V(k, \omega)\, dk = v(x, \omega)/(1 + x^2), \tag{7.44}$$

by Lebesgue's dominated convergence theorem. This is simply the inversion formula of the Fourier integral (7.34), whence the conclusion follows by (7.42), with probability one.

But if $v(x, \omega)$ satisfies (7.44), its Fourier transform $V(k, \omega)$ belongs to L^2 but not, in general, to L; a necessary (but not sufficient) condition for (7.44) is that $v(x, \omega)$ be continuous [which explains how (7.44) can hold for *all* x] and that $v(x, \omega)/(1 + x^2) \to 0$ if $|x| \to +\infty$. In the general case, when $V(k, \omega) \notin L$, we can prove the weaker result:

$$\lim_{t \downarrow 0} \int_a^b \frac{u(x, t, \omega)}{1 + x^2}\, dx = \int_a^b \frac{v(x, \omega)}{1 + x^2}\, dx \tag{7.45}$$

for any interval $[a, b]$.

First, due to (7.35), by a known result of Fourier–Plancherel's transform, we have [51, p. 74]

$$\int_0^\xi \frac{v(x, \omega)}{1 + x^2}\, dx = \int_{-\infty}^{+\infty} \frac{e^{ik\xi} - 1}{ik}\, V(k, \omega)\, dk. \tag{7.46}$$

Next from (7.39) we deduce

$$\begin{aligned} \int_0^\xi \frac{u(x, t, \omega)}{1 + x^2}\, dx &= \int_0^\xi \left[\int_{-\infty}^{+\infty} \Gamma(k, x, t)\, V(k, \omega)\, dk \right] \frac{dx}{1 + x^2} \\ &= \int_{-\infty}^{+\infty} \left[\int_0^\xi \frac{\Gamma(k, x, t)}{1 + x^2}\, dx\, V(k, \omega) \right] dk, \end{aligned} \tag{7.47}$$

the inversion of the two integrals being permissible through Fubini's theorem because $\Gamma(k, x, t)\, V(k, \omega)/(1 + x^2)$ is integrable. By direct computation from (7.38) we obtain

$$\Gamma(k, x, t) = (1 + x^2)\gamma(k, x, t) + t\eta(k, x, t).$$

Therefore, $\lim_{t \downarrow 0} \Gamma(k, x, t)/(1 + x^2) = e^{ikx}$, whence we deduce

$$\lim_{t \downarrow 0} \int_0^\xi \frac{\Gamma(k, x, t)}{1 + x^2}\, dx = \frac{e^{ik\xi} - 1}{ik}.$$

By application of dominated convergence in the integral, due to the fact that $\Gamma(k, x, t)/(1 + x^2)$ is dominated by a linear combination of terms of the form $S(k)\exp(At - Btk^{2s})$, we have

$$\lim_{t \downarrow 0} \int_0^\xi \frac{u(x, t, \omega)}{1 + x^2}\, dx = \int_{-\infty}^{+\infty} \frac{e^{ik\xi} - 1}{ik}\, V(k, \omega)\, dk. \tag{7.48}$$

From (7.46) and (7.48) we immediately deduce (7.43).

Appendix A: Borel Sets in the Relevant Function Spaces

The function spaces Γ, Λ_p, S', and D' play a fundamental role in our paper. Here, we prove that these spaces have standard "Borel structures," and that their Borel sets are generated in all cases by the Khinchine "window sets." To prove our first result, it would probably be simplest to first show that our function spaces are all standard T_1-spaces; as in Section II.C, this implies function spaces are standard Borel structures. Instead, we shall discuss their that they have standard Borel structures directly, partly because this exhibits more clearly the purely probabilistic (i.e., combinatorial) nature of the final conclusions, and partly because the concept of "Borel structures" is better established than our new notion of a "standard T_1-space," introduced in Section II.C. To avoid confusion, we begin by defining our terms precisely.

A *Borel space* is a set M together with a σ-field $\mathscr{B}$ of subsets of M. A Borel space $(M, \mathscr{B})$ is *countably generated* when there is a countable subclass of $\mathscr{B}$ which generates the σ-field $\mathscr{B}$, and *separated* when each singleton $\{x\}$, where $x \in M$, is an element of $\mathscr{B}$. (Thus, in the language of lattice theory, a "separated Borel space" is an atomic Borel algebra.) A Borel space $(M, \mathscr{B})$ is called *standard* when: (i) it is separated, and (ii) there is a Polish space Y such that $\mathscr{B}$ and the Borel sets of Y, $\mathscr{B}_Y$, are isomorphic as Borel algebras.

A topological linear space (TLS) that has a base for neighborhoods of 0 consisting of convex sets is called *locally convex*. A locally convex TLS is a *Fréchet space* when it is metrizable and complete. In a locally convex TLS, an absorbing, balanced convex, closed subset is called a barrel. Such a space is called barreled when every barrel is a neighborhood of zero. A locally convex Hausdorff TLS is called a *Montel space* when it is barreled and every closed bounded subset is compact. The *dual* of any TLS F is the collection of all continuous linear functionals defined on F, and is denoted F'. The *weak topology* on a TLS F is the topology having the intervals $|f_i(x)| < \varepsilon$ ($i \in I$, some finite set, $f_i \in F'$ fixed, and $x \in F$) as a neighborhood subbasis of 0. It is the weakest topology on F for which all the elements of F' are continuous. The *weak-star topology* on F' is, dually, the weakest topology on F' that makes all the elements of F (considered as elements of F'') continuous. The strong topology on F' is the topology of uniform convergence on bounded subsets of F. A TLS F is called *reflexive* when it coincides with its strong second dual F''. The *polar* of a subset A of F is denoted A^0 and is defined as

$$A^0 = \{f \in F' \mid |f(x)| \le 1 \quad \text{for all} \quad x \in A\}.$$

Dually, the polar of a subset B of F' is

$$B^0 = \{x \in F \mid |f(x)| \le 1 \quad \text{for all} \quad f \in B\}.$$

Remark. If F is reflexive, then the weak-star and the weak topology on F' are identical. Since all the function spaces considered in this paper are reflexive, their weak and weak-star topologies are the same.

We will need several well-known properties of some of the systems defined above. We state these now. These results will be referred to in the form $K(n)$, where K is a capital letter and n a positive integer. For example, the second result of Part B below is identified by the reference B(2).

A. Results on Polish spaces [13, pp. 195–196] and Section II.C.
   (1) A closed subset of a Polish space is Polish.
   (2) An open subset of a Polish space is Polish.

   (3) A countable product of Polish spaces is Polish.
   (4) The topological direct sum of a countable number of Polish spaces is Polish.

B. Results on standard Borel spaces [40, §3]; [43, Chapter 1, §§2,3 and Chapter 5, §2].
   (1) A subset B of a standard Borel space $(A, \mathscr{B})$ is standard in the relative Borel structure if and only if it is a Borel set. In this case, the Borel subsets of B are precisely the intersections of Borel subsets of A with B.
   (2) An injective Borel measurable function f from a standard Borel space X to a countably generated separated Borel space Y is a Borel isomorphism onto the Borel subset $f(X)$ of Y.
   (3) Countable products and countable sums of standard Borel spaces are standard.

C. Results on a continuous image of a Polish space in a Hausdorff space [13, pp. 197–206].
   (1) If X is a Polish space and f a continuous injection of X into the Hausdorff space Y, then $f(X)$ is a Borel subset of Y.
   (2) A 1–1 continuous image of a Polish space in a Hausdorff space is standard.

Comment. Concerning C(1), [13, p. 206] states a less general result: merely that any 1–1 continuous image of a Polish space in a metrizable space is standard. This is because a *Souslin space* is defined in a restrictive way [13, p. 197] as a continuous metrizable image of a Polish space. We shall use here the more general definition of [52], that a Souslin space is a continuous image of a Polish space in a Hausdorff space. It is tedious but straightforward to verify that what is proved in [13, pp. 197–205] can also be proved in the more general context of a Souslin space. Probably the most difficult step is to construct a "strict sifting" of a Souslin space. This is carried out in detail, for example, in [52, Appendix, p. 552]. The statement of C(2) is easily deduced from C(1).

D. General results on topological linear spaces.
   (1) Banach–Alaoglu theorem: If U is a neighborhood of 0 in a TLS F, then the polar of U is weak-star compact in F', (and hence an equicontinuous subset of F') [37, p. 155].
   (2) An equicontinuous subset of the dual of a separable locally convex TLS is weak-star metrizable [37, p. 164, J(a)].
   (3) A convex subset of a locally convex TLS is closed if and only if it is weakly closed [37, p. 154].

(4) A subset of a locally convex TLS is bounded if and only if it is weakly bounded [37, p. 155].

E. Results on Montel spaces.

(1) The dual of a Montel space, equipped with the strong topology, is a Montel space [52, p. 376].
(2) A Montel space is reflexive [52, p. 376].
(3) On bounded subsets of a Montel space, the initial topology and the weak topology coincide [52, p.376].
(4) Closed bounded subsets of a Montel space are compact [30, p. 231].
(5) Products and strict inductive limits of Montel spaces are Montel [30, p. 240].

F. Results on spaces of distributions.

(1) The space S of all C^∞ vector fields rapidly decreasing at infinity is a separable Fréchet–Montel space [30, pp. 116, 137, 240].
(2) The space D of all C^∞ vector fields with compact support is a strict inductive limit of separable Fréchet–Montel spaces [30, pp. 16, 154, 240, 241].

An important property for the theory of regular probability measures on such function spaces as Γ, Λ_p, S', and D' is that the Borel structure be standard (see Section I.C). We shall now show that all the preceding spaces have this property.

Theorem A1. *The Borel structures of Λ_p, Γ, and S' are standard. They are all Borel subsets of the standard Borel space D' of Schwartz distributions.*

The proof will proceed via a series of lemmas. First note that, since Γ and Λ_p are complete separable metrizable spaces, they are both Polish and so automatically standard Borel spaces [cf. C(2) above].

Lemma A1. *Let F be a separable Fréchet space. If V is a neighborhood of 0, in F, then the polar of V, $B = V^0$, is a compact metric space when considered as a subspace of the dual F' endowed with the weak-star topology.*

Proof. By the Banach–Alaoglu theorem [D(1)], $B = V^0$ is compact for the weak-star topology. In particular, B is equicontinuous and, by $D(2)$, since F is separable, B is metrizable in the topology induced on it as a subset of the weak dual. Thus B is a compact metric space as required.

Corollary A1. *In a separable Fréchet space F, the polar in F' of any neighborhood of 0 is a Polish space under the weak-star topology of F'.*

Proof. A compact metric space is complete and separable.

Lemma A2. *The weak or strong dual of a separable Fréchet–Montel space is a standard Borel space.*

Proof. Let $\{V_n\}$ be a countable basis for neighborhoods of 0 in the separable Fréchet–Montel space F. Let $B_n = V_n{}^0$. By the corollary to Lemma A1, the B_n are Polish spaces in the weak-star topology. The strong dual of a Montel space is reflexive and a Montel space by E(1) and E(2). Thus F' is a Montel space in its strong topology and so the weak-star and strong topologies agree on bounded subsets of F' by E(3). Hence the B_n, being bounded, are Polish spaces in either the strong or the weak-star topology.

Now $F' = \bigcup_{n=1}^\infty B_n$, for if $f \in F'$, then $\{f\}^0$ is a neighborhood of 0 in F, making $\{f\}^0 \supset V_k$ for some k, and hence $f \in \{f\}^{00} \subset B_k$. Set $D_n = B_n \backslash \bigcup_{k<n} B_k$. Then each D_n is an open subset of the Polish space B_n and so is Polish itself A(2). Further, the D_n are disjoint and their union is F'.

Let $G = \oplus_{n=1}^\infty D_n$ be the topological direct sum of the D_n. Then G is a Polish space by A(4), and the map $h: G \to F'$ obtained by injecting each D_n as a subset of F' is bijective and continuous for the strong and, hence, for the weak-star topology on F'. Thus F', in the strong or the weak-star topology, can be realized as a 1–1 continuous image of a Polish space; hence, F' is standard by C(1).

Lemma A2 implies that S' is a standard Borel space, since S is a separable Fréchet–Montel space by F(1). Note also that its proof implies more generally that the weak-star dual of any separable Fréchet space is standard.

Lemma A3. *Let F be a strict inductive limit of separable Fréchet–Montel spaces $\{F_n\}_{n=1}^\infty$. Then its weak dual and its strong dual are both standard Borel spaces.*

Proof. For each n, let F_n' be the weak (respectively, strong) dual of F_n. By Lemma A2, F_n' is a standard Borel space. Moreover F' is realized as the inverse limit of the system $\{F_n', g_n\}$, where $g_n : F_n' \to F_{n-1}'$ is the dual of the inclusion mapping $F_{n-1} \to F_n$. Hence F' is isomorphic, as a topological linear space, to the closed subspace K of the product $\prod_n F_n'$ of elements (f_k) for which $g_n(f_n) = f_{n-1}$. In particular, F' is homeomorphic to a closed subset of the standard Borel space $\prod_n F_n'$ by B(3), and, since a Borel subset of a standard Borel space is standard by B(1), F' is a standard Borel space.

Lemma A3 shows that D' is a standard Borel space, since D is the strict inductive limit of the separable Fréchet-Montel spaces $D(K_N) = C^\infty$ of vector fields with support in the compact "cylinders" K_N of $\mathbf{x} \in X = \mathbf{K}^s\mathbf{R}^{n-s}$ with $|x_j| < N$, $j = s+1, \ldots, n$.

Now an injective Borel measurable mapping of a standard Borel–Hausdorff space X into a standard Borel–Hausdorff space Y is a Borel isomorphism onto a Borel subset of Y, by B(2). In particular, the canonical insertion of S' into D' is continuous, and hence a Borel isomorphism of S' with a Borel subset of D' which we shall also denote by S'.

The spaces Λ_p and Γ can also be embedded in D'. For if $\mathbf{f}(\mathbf{x}) \in \Lambda_p$, say, we let $[\mathbf{f}]$ be defined on $\boldsymbol{\phi} \in D$ by

$$[\mathbf{f}](\boldsymbol{\phi}) = \int_X [\mathbf{f}(\mathbf{x}) \cdot \boldsymbol{\phi}(\mathbf{x})]\, dm(\mathbf{x}),$$

where $[\mathbf{f}(\mathbf{x}) \cdot \boldsymbol{\phi}(\mathbf{x})]$ is the standard inner product of the vector fields $\mathbf{f}$ and $\boldsymbol{\phi}$ and $dm(\mathbf{x})$ is Lebesgue measure on our underlying (connected, locally Euclidean, Abelian group) manifold X. This is well defined since $\boldsymbol{\phi}$ is continuous with compact support and $\mathbf{f} \in \Lambda_p \subseteq \Lambda_1$. One easily checks that $[\mathbf{f}] \in D'$ and that the mapping $\mathbf{f} \to [\mathbf{f}]$ is a continuous linear injection. Thus Λ_p can also be realized as a Borel subset of D'. Similar remarks apply to Γ. The remarks of the last paragraph now apply and the proof of Theorem A1 is complete.

In a standard Borel space B, $C \subseteq B$ is a Borel subset of B if and only if C is standard. In this case, the Borel subsets of B are precisely the intersections with C of Borel subsets of B by B(1). Consequently, a set in $\Lambda_p \cap S'$ (both considered as subsets of D') is Borel in S' if and only if it is a Borel subset of Λ_p. The Borel subsets of $\Lambda_p \cap S'$ generated by the relative topology induced by Λ_p are the same as the Borel subsets of $\Lambda_p \cap S'$ generated by the relative topology inherited from S'.

Remark. In Lemmas A2 and A3, we proved more than we stated. We actually showed that both S' and D' are not only standard but actually one–one continuous images of Polish spaces. This stronger fact will be used in Appendix B.

We now come to the second major result of this Appendix.

Theorem A2. *On the spaces Γ, Λ_p, S', and D', the weak and strong Borel structures are identical.*

Proof. The proof depends on two facts: the strong Borel structures on these spaces are standard (Theorem A1) and their dual spaces are weak-star separable. For Γ and Λ_p, weak-star separability follows from the separability of these spaces. For S', weak-star separability follows from [37, p. 164, J(b)] and the fact [F(1)] that S is a separable Fréchet space. For D', the result is obtained by applying [37, p. 164, J(b)] to the separable Fréchet spaces $D(K_n)$ defined after the proof of Lemma A3. Therefore D' with the weak-star topology is the inverse limit of a countable set of separable spaces and hence is itself separable. To complete the proof of Theorem A2, we next establish the following result.

Lemma A4. *Let E be a locally convex Hausdorff TLS whose dual is weak-star separable and whose strong Borel structure is standard. Then the Borel sets of E are the same in the weak as in the strong topology of E.*

Proof. Let $\{f_j\}_{j=1}^{\infty}$ be a countable dense subset of E'. Let $\mathscr{W}$ be the denumerable collection of sets of the form

$$W = \{x \in E \,|\, \alpha < f_j(x) < \beta\},$$

where α and β are rational numbers. (If E is a complex TLS, we must take real and imaginary parts; otherwise the proof is the same.)

We first show that $\mathscr{W}$ is a separating family in E in the sense that, given $x \neq y$ in E, there is an element $W \in \mathscr{W}$ such that $x \in W$ and $y \notin W$. By linearity, it is enough to show that, for any $z \neq 0$ in E, there exists a set $W \in \mathscr{W}$ with $0 \in W$ and $z \notin W$. Suppose on the contrary that $z \in W$ for all $W \in \mathscr{W}$ such that $0 \in W$. Then in particular the sets

$$W_N{}^j = \{x \in E \,|\, -1/N < f_j(x) < 1/N\}$$

all contain z. Hence, $f_j(z) = 0$ for all j. Since the f_j are weak-star dense in E', it follows that $f(z) = 0$ for all $f \in E'$. Thus $z = 0$, a contradiction.

Let $\mathscr{C}$ be the σ-field generated by $\mathscr{W}$. Then $(E, \mathscr{C})$ is separated (since the family $\mathscr{W}$ is a separating family) as a Borel space. Let $\mathscr{B}$ be the σ-field of strong Borel sets in E. Obviously $\mathscr{B} \supset \mathscr{C}$. We will show equality holds and so finish proving the lemma, and hence the theorem.

Let i be the identity map from E with the Borel structure $\mathscr{B}$ to E with the Borel structure $\mathscr{C}$. Then i is an injective Borel mapping from a standard Borel space to a countably generated and separated Borel space. By B(2), i is a Borel isomorphism and so $\mathscr{B} = \mathscr{C}$ as required.

The collection $\mathscr{W}$ in the last proof is a subcollection of the cylinder sets in E. Hence, it is a corollary of the proof of Lemma A4 that if a locally convex TLS is standard and has a separable dual, then the window sets generate the Borel sets. Since the spaces Γ, Λ_p, S', and D' of primary interest for our paper are standard and have separable duals, we can extract from the above the following final result of this Appendix.

Theorem A3. *In the spaces Γ, Λ_p, S', and D', the (weak or strong) Borel sets are generated by a countable subcollection of the Khinchine window sets.*

Remark. Since this was written, we have learned that Fernique [11] has obtained results implying our Theorems A1 and A2, using slightly different definitions.

Appendix B: Regular Probability Measures

B1. Basic Definitions

The purpose of this Appendix is to correlate our definition of a "regular" probability measure in Section II.D with various related but more general definitions, due to Carathéodory [14], Mourier [41], Halmos [25], Berberian ([5, 6]), Gnedenko and Kolmogoroff [23], and Gel'fand and Vilenkin [22]. The following terminology is basic.

Definition B1. A *measure* on a set E is a σ-additive, nonnegative real function[27] on a σ-field $\mathscr{B}$ of sets $B \subset E$. Such a measure is called *complete* (or "closed") when $\mu A = 0$ and $B \subset A$ imply $B \in \mathscr{B}$, and a *probability* measure when $\mu E = 1$.

A probability measure μ on a space E thus has two important properties not possessed by general measures: μ is finite and E is measurable.

These definitions follow Halmos [25, §7]; what we here call a "complete" measure, Gnedenko and Kolmogoroff include in their definition of a "measure" [23, p. 17]. It is known ([25, p. 55, Theorem B], [5, Chapter 1]) that every measure μ has a smallest complete extension γ, which we call the *Lebesgue completion* of μ.

Recall that the collection of Borel sets of a topological space X is the σ-field (σ-algebra) $\mathscr{B}$ generated by the closed subsets of X. In [8] (and here), we have defined a *regular* measure to be one which is the Lebesgue completion of a measure defined on $\mathscr{B}$. We shall now compare our definition with those given by other authors.

B2. Carathéodory Regularity

Carathéodory, to whom the term "regular" measure is due, was concerned primarily with locally compact separable metric spaces. In such spaces, every open set is a countable union of compact sets. As his basic concept was that of an *outer measure*, some explanation is needed to correlate his definition with ours.

[27] In the text, we have occasionally used more general measures, e.g., signed and complex measures in Section V. For these, see Rudin [45, Chapter 6]; many of the remarks below generalize easily to signed and complex measures.

Carathéodory [14, p. 238, I–III] defined an *outer measure* on a set E as an isotone, σ-subadditive function μ^* defined for *all* $S \subset E$, such that $\mu^*\varnothing = 0$. For any such outer measure μ^*, define $\mathscr{M} = \mathscr{M}(\mu^*)$ by the condition that $M \subset E$ belongs to $\mathscr{M}$ if and only if

$$\mu^*(A) = \mu^*(A \cap M) + \mu^*(A \cap M') \qquad \text{for all} \quad A \subset E. \tag{B1}$$

Then $\mathscr{M}$ is a σ-algebra of subsets of E, and the restriction μ of μ^* to $\mathscr{M}$ is a complete measure. Carathéodory [14, p. 258, V] postulated the condition

$$\mu^*(A) = \inf_{C \supset A,\, C \in \mathscr{M}} \mu^*(C) \qquad \text{for all} \quad A \subset E, \tag{B2}$$

which is purely measure-theoretic. Halmos [25, p. 52] calls such an outer measure a *regular outer measure*. Such outer measures are precisely those which are induced by a measure on some σ-ring of subsets of the underlying set E.

In addition, Carathéodory [14, p. 239, IV] postulated the following condition relating measurability to the topology of E:

$$\text{If} \quad \inf_{x \in A,\, y \in B} d(x, y) > 0, \qquad \text{then} \quad \mu^*(A \cup B) = \mu^*(A) + \mu^*(B). \tag{B3}$$

He called an outer measure satisfying (B2) *and* (B3) "regular"; we shall call the associated measure *Carathéodory regular*; it is always complete. We next prove[28] that, in the locally compact spaces of interest to Carathéodory, his notion of regularity and ours are closely related.

Theorem B1. *Let E be a separable locally compact metric space. A σ-finite measure on E is Carathéodory regular if and only if it is an extension of a regular measure on E.*

Proof. First suppose μ is an extension of a regular measure on E. Let $\mathscr{B}$ be the Borel algebra of all μ-measurable sets; then $\mathscr{B}$ contains all Lebesgue measurable subsets of E; i.e., all subsets which differ from a Borel set by a set of μ-measure zero. If we define the associated outer measure μ^* by

$$\mu^*(A) = \inf_{B \supset A,\, B \in \mathscr{B}} \mu(B), \qquad \text{where} \quad A \subset E,$$

then, since μ is complete (and σ-finite), the class $\mathscr{M}$ of μ^*-measurable sets coincides with $\mathscr{B}$ [25, p. 56]. Further, μ^* satisfies (B2) [25, p. 50]. Thus to

[28] This corrects the result of [8, p. 666, Lemma 1].

show that μ is Carathéodory regular, we need only verify that (B3) holds for μ^*. Let A and B be subsets of E satisfying the conditions of (B3). Then their closures $\bar{A}$ and $\bar{B}$ are disjoint. Let A_1, B_1, and C_1 be elements of $\mathscr{B}$ such that $A_1 \supset A$, $B_1 \supset B$, $C_1 \supset A \cup B$, and

$$\mu(A_1) = \mu^*(A), \qquad \mu(B_1) = \mu^*(B), \qquad \mu(C_1) = \mu^*(A \cup B).$$

Such sets exist since (B2) holds and $\mu = \mu^*$ on $\mathscr{B}$. Let $A' = A_1 \cap \bar{A} \cap C_1$ and $B' = B_1 \cap \bar{B} \cap C_1$; A' and B' lie in $\mathscr{B}$ and they satisfy

$$\begin{aligned} A \subset A' \subset A_1 &\Rightarrow \mu^*(A) = \mu(A') \\ B \subset B' \subset B_1 &\Rightarrow \mu^*(B) = \mu(B') \\ A \cup B \subset A' \cup B' \subset C_1 &\Rightarrow \mu^*(A \cup B) \\ &= \mu(A' \cup B'). \end{aligned}$$

Further, since $\bar{A} \cap \bar{B} = \varnothing$, $A' \cap B' = \varnothing$ also, so $\mu(A' \cup B') = \mu(A') + \mu(B')$. Putting all these facts together, we obtain

$$\mu^*(A \cup B) = \mu(A' \cup B') = \mu(A') + \mu(B') = \mu^*(A) + \mu^*(B),$$

as required.

Conversely, let μ^* be an outer measure on E defining a Carathéodory-regular measure μ. Let $\mathscr{M}$ be the collection of μ^*-measurable sets. Since μ is known to be complete, it suffices to show that any closed subset of E lies in $\mathscr{M}$. For then $\mathscr{M}$ contains all Borel sets, and so, by completeness, all sets which differ from Borel sets by sets of μ^* outer measure zero. Hence μ will be an extension of a measure regular in our sense.

We first establish the following fact for μ^*:

$$A_n \uparrow A \subset E \qquad \text{implies} \quad \mu^*(A) = \sup \mu^*(A_n). \tag{B4}$$

Certainly $\mu^*(A) \geq \sup \mu^*(A_n)$. By (B2), given $\varepsilon > 0$, there is an element $B_n^\varepsilon \in \mathscr{M}$ such that $B_n^\varepsilon \supset A_n$ and $\mu(B_n^\varepsilon) \leq \mu^*(A_n) + \varepsilon$ for each n. Let $C_n^\varepsilon = \bigcap_{k \geq n} B_k^\varepsilon$. Since $A_k \supset A_n$ for $k \geq n$, $C_n^\varepsilon \supset A_n$ and $C_n^\varepsilon \in \mathscr{M}$ with $\mu(C_n^\varepsilon) \leq \mu^*(A_n) + \varepsilon$. Further, $C_1^\varepsilon \subset C_2^\varepsilon \subset \cdots$. We let $C(\varepsilon) = \bigcup_{n=1}^\infty C_n^\varepsilon \supset A$. By the countable additivity of μ, $\mu(C(\varepsilon)) = \sup \mu(C_n^\varepsilon) \leq \sup \mu^*(A_n) + \varepsilon$. We let $C = \bigcap_{n=1}^\infty C(1/n)$. Then $C \in \mathscr{M}$, $C \supset A$, and $\mu(C) \leq \sup \mu^*(A_n) + 1/k$ for all k. This shows that $\mu^*(A) \leq \mu(C) \leq \sup \mu^*(A_n)$, and we may conclude that $\mu^*(A) = \sup \mu^*(A_n)$.

Now let C be a closed subset of E. Certainly $\mu^*(A) \leq \mu^*(A \cap C) + \mu^*(A \cap C')$ for any set A. To show that $C \in \mathscr{M}$, we must prove the reverse inequality. Since E is a separable locally compact metric space, the open set $U = C' = E \backslash C$ can be written as a countable union of compact sets which can be taken to be increasing; $U = \bigcup_{n=1}^\infty D_n$. Since D_n is compact and C closed, and $D_n \cap C = \varnothing$, they lie at positive distances from one another. Thus

$$\begin{aligned} \mu^*(A) &\geq \mu^*(A \cap (C \cup D_n)) = \mu^*((A \cap C) \cup (A \cap D_n)) \\ &= \mu^*(A \cap C) + \mu^*(A \cap D_n), \end{aligned}$$

since $A \cap C$ and $A \cap D_n$ satisfy the hypothesis of (B3). Since $D_n \uparrow C'$, $A \cap D_n \uparrow A \cap C'$ and hence $\mu^*(A \cap C') = \sup \mu^*(A \cap D_n)$ by (B4). This shows that

$$\mu^*(A) \geq \mu^*(A \cap C) + \mu^*(A \cap C')$$

as required. The proof is complete.

B3. L-MEASURES

The following definition reduces to that of Mourier [41] when E is a Banach space. Like Carathéodory's definition and unlike our definition of a regular measure, it fails to associate a unique probability measure to a stationary Gaussian process with given continuous spectral matrix measure ("power spectrum").

Definition B2. Let E be a locally convex topological linear space and let E' be its dual and μ a measure on E. Then μ is an *L-measure* when each element of E' is μ-measurable.

In the following result, E stands for any of the spaces Γ, Λ_p, S', or D'.

Theorem B2. *Let μ be an L-measure on E. Then the completion $\tilde{\mu}$ of μ is an extension of a regular measure on E.*

Proof. Since $\tilde{\mu}$ extends μ and μ is an L-measure, so is $\tilde{\mu}$. On the other hand, by Theorem A2, the weak and strong Borel sets of E are the same, and, by Theorem A3, they are generated by a countable collection $\mathscr{W}$ of window sets. Moreover, since $\tilde{\mu}$ is an L-measure, the Borel algebra (σ-field) $\tilde{\mathscr{B}}$ of all $\tilde{\mu}$-measurable sets of E must contain $\mathscr{W}$; hence, it contains the σ-field (Borel subalgebra) of sets generated by $\mathscr{W}$. Hence $\tilde{\mathscr{B}}$ must contain *all* (weakly) Borel sets of E.

We take the opportunity now to emphasize an essential difference between measures "regular" in our very restrictive sense, and most other notions of regularity, including those discussed in this and later sections: Theorems B1 and B2 are typical. They assert that, for some theoretical purposes, a measure is satisfactory if and only if it is an extension of a measure which is "regular" in our sense. (The relevant "regular" measure is of course unique.)

As an important example, as was proved in [8, Part B], a stationary separable Gaussian random function with given autocorrelation function determines a *unique* probability measure regular in our sense. For the more general notions of regularity due to Carathéodory and Mourier, this result does *not* hold. Similar remarks apply to perfect measures and to (probability) measures regular in the sense of Halmos and Berberian, as we shall show in the next two sections.

B4. Halmos and Berberian

Where Carathéodory regularity refers to *extensions* of "regular" (probability) measures as defined by us, the emphasis of Halmos [25] and Berberian ([5, 6]) in their discussions of "regularity" is on *restrictions* of such measures. Perhaps because their deepest concern is with sets of finite Haar measure in a locally compact topological group E, they do not even require E itself to be measurable, a requirement which is certainly appropriate for probability measures.

For their purposes, it seems most satisfactory to deal with completions of *Baire measures* defined on the σ-ring $\mathscr{B}_2$ of *Baire* sets generated by the compact G_δ sets.[29] They show [25, p. 228] that any such measure is automatically regular in that it is both "inner" and "outer" regular in the following sense.

Definition B3. Let E be a topological space, and $\mathscr{B}$ a σ-ring of subsets of E. A measure μ on $\mathscr{B}$ is *inner regular* when, for any $B \in \mathscr{B}$,

$$\mu B = \sup\{\mu C \mid C \subset B,\ C \in \mathscr{B},\ C \text{ compact}\}. \tag{B5}$$

It is *outer regular* when, for any $B \in \mathscr{B}$,

$$\mu B = \inf\{\mu U \mid B \subset U,\ U \in \mathscr{B},\ U \text{ open}\}. \tag{B6}$$

Completions of measures on the σ-ring $\mathscr{B}_1$ of strongly Borel sets generated by the compact sets are inner regular if and only if they are outer regular [6, p. 137]; whereas the Lebesgue completion of a measure on the set $\mathscr{B}(Y)$ of all the Borel sets B of a topological space Y, though "regular" in our sense, may be neither inner nor outer regular.

However, there are important cases when measures regular in our sense are also regular in the above sense. For example, a probability measure μ regular in our sense on any Polish space P is both inner and outer regular. For if A is a μ-measurable set, then $B_1 \subset A \subset B_2$, where B_1, B_2 are Borel sets and $\mu(B_2 \backslash B_1) = 0$. By the results of Neveu [42, p. 64], then

$$\begin{aligned}\mu(A) = \mu(B_2) &= \inf\{\mu(U) \mid U \text{ open},\ U \supset B_2\}\\ &\geq \inf\{\mu(U) \mid U \text{ open},\ U \supset A\} \geq \mu(A)\end{aligned}$$

so that equality must hold everywhere; that is,

$$\mu(A) = \inf\{\mu(U) \mid U \text{ open},\ U \supset A\}.$$

Similarly, we see that inner regularity holds,

$$\begin{aligned}\mu(A) = \mu(B_1) &= \sup\{\mu(K) \mid K \text{ compact},\ K \subset B_1\}\\ &\leq \sup\{\mu(K) \mid K \text{ compact},\ K \subset A\} \leq \mu(A),\end{aligned}$$

and so we are able to conclude that

$$\mu(A) = \sup\{\mu(K) \mid K \text{ compact},\ K \subset A\}, \tag{B7}$$

as desired.

Thus, since Γ and Λ_p are Polish spaces, a probability measure on one of these spaces which is regular in our sense will be regular also in the sense of Halmos and Berberian.

In Appendix A, we showed that S' and D' are standard Borel spaces. As we remarked there, we actually showed more: we demonstrated that S' and D' were 1–1 continuous images of Polish spaces. We next establish a general lemma.

Lemma B1. *Let P be any Polish space, and let X be any Hausdorff space which is a continuous image of P. Then any probability measure on X which is "regular" (in our sense) is also both inner and outer regular, hence regular in the sense of Halmos and Berberian.*

Proof. Suppose P is a Polish space and f a continuous bijection of P with the Hausdorff space X. Then f is a Borel isomorphism. For the continuity of f implies that the inverse image of any element of $\mathscr{B}_X$, the σ-field of Borel sets of X, is an element of $\mathscr{B}_P$, the σ-field of Borel sets of P. And by result C(1) of Appendix A and the fact that any closed subset of a Polish space is Polish, the image under f of any closed subset of P is a Borel set in X. It follows at once that the image under f of any Borel set in P is Borel in X. Finally, since f is a bijection, f preserves all the Boolean operations on sets.

Now let μ be a regular probability measure on X and let $\mathscr{B}$ be the σ-algebra of Lebesgue measurable sets for μ. The function $f^{-1}: X \to P$ is Borel measurable, so we can define the probability measure μ_f on P by

$$\mu_f(A) = \mu(f(A)) \qquad \text{for} \quad A \in \mathscr{B}_P = f^{-1}(\mathscr{B}_X). \tag{B8}$$

[29] The assumption of [6, p. 137], that $\mu(S) = \mu S$ is finite for compact G_δ, is trivially satisfied by probability measures. What we here call a "strongly Borel" set, Halmos and Berberian call a "Borel" set.

Letting $\mathscr{B}' = f^{-1}(\mathscr{B})$ and $\mathscr{M}$ be the σ-field of sets M which are Lebesgue measurable with respect to μ_f, we claim that $\mathscr{B}' = \mathscr{M}$.

If $B \subset A$, where $A \in \mathscr{B}_P$ and $\mu_f(A) = 0$, then $f(A) \in \mathscr{B}_X$ and $\mu(f(A)) = \mu_f(A) = 0$. Since $f(B) \subset f(A)$, $f(B)$ is a subset of a set of μ-measure zero and so lies in $\mathscr{B}$. Now f is a bijection, so $B = f^{-1}(f(B)) \in f^{-1}(\mathscr{B}) = \mathscr{B}'$. We may conclude that $\mathscr{M} \subset \mathscr{B}'$. Conversely, if $B' \in \mathscr{B}'$, then $B' = f^{-1}(B)$, where $B \in \mathscr{B}$. Since $B \in \mathscr{B}$, we have $B_1 \subset B \subset B_2$, where B_1, B_2 are in $\mathscr{B}_X$ and $\mu(B_2 \backslash B_1) = 0$. Let $B_i' = f^{-1}(B_i)$ for $i = 1, 2$. Then B_1', B_2' lie in B_P, $B_1' \subset B' \subset B_2'$, and $\mu_f(B_2' \backslash B_1') = \mu(B_2 \backslash B_1) = 0$. Thus $B' \in \mathscr{M}$, and so we have the reverse inclusion $\mathscr{B}' \subset \mathscr{M}$.

Thus the Lebesgue completion $\tilde{\mu}_f$ of μ_f is simply the extension of μ_f obtained by using (B8) for all A in the Borel algebra $\mathscr{A} = f^{-1}(\mathscr{B})$ instead of just for A in $\mathscr{B}_P = f^{-1}(\mathscr{B}_X)$. The measure $\tilde{\mu}_f$ so defined is, hence, a regular probability measure on the Polish space P. By our earlier calculations, $\tilde{\mu}_f$ is both inner and outer regular. From this fact we now deduce that μ is both inner and outer regular. For $B \in \mathscr{B}$, certainly

$$\sup\{\mu(K) \mid K \subset B, K \text{ compact}\} \le \mu(B). \tag{B9}$$

But $\tilde{\mu}_f$ is inner regular, so, if $A = f^{-1}(B)$, then

$$\begin{aligned}\mu(B) = \tilde{\mu}_f(A) &= \sup\{\mu_f(K) \mid K \text{ compact}, K \subset A\} \\ &= \sup\{\mu(f(K)) \mid K \text{ compact}, K \subset A = f^{-1}(B)\}.\end{aligned}$$

Now since f is continuous, the collection $\{f(K) \mid K \text{ compact}, K \subset A\}$ is contained in the collection $\{K \mid K \text{ compact}, K \subset B\}$. Thus

$$\sup\{\mu(f(K)) \mid K \subset A \text{ compact}\} \le \sup\{\mu(K) \mid K \text{ compact}, K \subset B\}.$$

The last two relations give us the reverse to inequality (B9), whence we conclude that

$$\mu(B) = \sup\{\mu(K) \mid K \text{ compact}, K \subset B\}; \tag{B10}$$

i.e., that μ is inner regular.

It now follows at once that μ is outer regular. For let $\varepsilon > 0$ and $A \in \mathscr{B}$ be given. Let $B = X \backslash A$. Then $B \in \mathscr{B}$ and so, by the inner regularity of μ, there is a compact set $K \subset B$ with $\mu(B) - \mu(K) < \varepsilon$. Now K is closed since X is Hausdorff. Let $U = X \backslash K$. Then U is open and, since $K \subset B = X \backslash A$, $U \supset A$. Furthermore, $\mu(U) - \mu(A) = \mu(U \backslash A) = \mu(B \backslash K) = \mu(B) - \mu(K) < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude

$$\inf\{\mu(U) \mid U \text{ open}, U \supset A\} \le \mu(A).$$

The reverse inequality is trivial and outer regularity follows. This finishes the proof of the lemma; since S' and D' are indeed 1–1 continuous images of Polish spaces, we can conclude.

Theorem B3. *Any probability measure on Γ, Λ_p, S', or D' which is "regular" in our sense is both inner and outer regular.*

Conversely, the (Hausdorff) spaces Γ, Λ_p, S', and D' are all *σ-compact*, i.e., unions of countably many compact sets. This is evident from Lemma A1 and the proof of Lemma A2 in Appendix A. But now, if μ is any *inner regular* measure on any σ-compact Hausdorff space E, then μ is defined on all compact and hence (E being σ-compact) on all closed sets of E. Hence μ is defined on all Borel sets of E and, if (Lebesgue) complete, μ is an extension of a measure on E which is regular in our sense. This proves

Theorem B4. *Any complete measure on Γ, Λ_p, S', or D' which is inner regular is an extension of a measure which is "regular" in our sense.*

B5. Perfect Measures

Around 1948, several disturbing measure-theoretic examples came to light. One of these concerned a probability measure μ on a Borel field $\mathscr{B}$ of subsets of a space X, and two real-valued μ-measurable functions f and g such that

$$\mu(f^{-1}(A) \cap g^{-1}(B)) = \mu(f^{-1}(A))\mu(g^{-1}(B))$$

for every pair of *Borel sets* A and B in $\mathbf{R}$, but not for any two subsets of $\mathbf{R}$ for which the three probabilities are defined. For an account of these and other pathological possibilities, see Blackwell [9] and the references given there.

Carathéodory's notion of regularity, which was later refined by Halmos and others into the concepts of inner and outer regularity, is sufficient to avoid the pathologies alluded to above. However, these notions have the defect of not being purely set-theoretic: they are relative to an assumed *topology* on the underlying space E.

To avoid pathological possibilities in a purely set-theoretic context, Gnedenko and Kolmogoroff [23] introduced the following notion of a *perfect* measure. Let $(X, \mathscr{B}, \mu)$ be a measure space, and denote the extended real line by $\overline{\mathbf{R}}$.

Definition B4. The complete measure μ is called *perfect* if, whenever $f: X \to \overline{\mathbf{R}}$ is a Borel function, the measure $f[\mu] = \mu \circ f^{-1}$ defined on the Borel sets of $\overline{\mathbf{R}}$ is an outer regular measure in the sense of (B6) above.

Notice that this definition is independent of any topology X may possess. According to Gnedenko and Kolmogoroff [23, p. 18], if X is a Polish space, if $\mathscr{B}$ is a σ-field that contains all Borel sets of X, and if μ is a probability measure on $\mathscr{B}$ which is complete in the sense of Lebesgue, then μ is "perfect" if and only if it is outer regular on X in the sense of (B6). It follows from the results of Section B4 that, if μ is a probability measure on a Polish space that is regular in our sense, then μ is perfect. For as we showed, such a measure is outer regular, and hence perfect from Gnedenko and Kolmogoroff's result stated above. Thus regular measures on Γ and Λ_p are perfect. We now prove that the same is true on S' and D', making use of the fact, proved in Appendix A, that S' and D' are 1–1 continuous images of Polish spaces.

If μ is a measure regular on a Hausdorff space X which is a 1–1 image of a Polish space P under a continuous function f, then the measure $\mu_f = \mu \circ f^{-1}$ is also regular in our sense as we remarked in Section B4. But μ_f is defined on the Polish space P and therefore is perfect since it is regular. Now μ is the measure induced on X by the function f. Since f is measurable, and μ_f is perfect, it follows trivially from the definition of perfect measure that μ is perfect. Thus we have proved the following result.

Theorem B5. *If a probability measure on any of the spaces Γ, Λ_p, S', or D' is regular in our sense, then it is a perfect measure.*

Thus, due to the results of Gnedenko and Kolmogoroff about perfect measures, we know that our regular probability measures on the function spaces Γ, Λ_p, S', and D' avoid the pathological possibilities described by Blackwell.

Appendix C: Cauchy Problems Which Are Statistically but Not Deterministically Well Set

We describe here some Cauchy problems which are statistically well set without being deterministically well set. Sharper and much more general results can be proved; our aim here is simplicity.

Consider the vibrating string equation in the form

$$u_t = v, \qquad v_t = u_{xx}, \tag{C1}$$

on the unit circle **K**. The system (C1) defines an abstract Cauchy problem [26, p. 387] on the Hilbert space $\mathbf{H} = L^2(\mathbf{K}) \times L^2(\mathbf{K})$ of all square-integrable function-pairs $(u(x), v(x))$ under the norm

$$\|(u, v)\| = \left\{\int_{\mathbf{K}} [|u(x)|^2 + |v(x)|^2]\, dx\right\}^{1/2}. \tag{C2}$$

The infinitesimal generator for this Cauchy problem is the unbounded linear operator

$$L = \begin{bmatrix} 0 & I \\ D^2 & 0 \end{bmatrix}, \qquad \text{where } D = \frac{\partial}{\partial x}. \tag{C3}$$

A (strong) *solution* for given initial $\mathbf{w}^0 = (u^0, v^0) \in \mathbf{H}$ is a continuous curve in $\mathbf{H}$,

$$\mathbf{w}(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}, \tag{C4}$$

whose derivative exists and satisfies

$$\mathbf{w}'(t) = \lim_{\Delta t \to 0} [\mathbf{w}(t + \Delta t) - \mathbf{w}(t)]/\Delta t = B\, \mathbf{w}(t)$$

for all $t \geq 0$, and for which

$$\lim_{t \downarrow 0} \mathbf{w}(t) = \begin{bmatrix} u(0) \\ v(0) \end{bmatrix}. \tag{C5}$$

By the Riesz–Fischer theorem, $\mathbf{H}$ can be identified with the space of all square-summable pairs of Fourier series

$$\mathbf{w}(x) = \begin{bmatrix} u(x) \\ v(x) \end{bmatrix} = \begin{bmatrix} a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \\ \alpha_0 + \sum_{k=1}^{\infty} (\alpha_k \cos kx + \beta_k \sin kx) \end{bmatrix}. \tag{C6}$$

Formally, the solutions of (C1) have the form

$$\mathbf{w}(t) = \mathbf{w}(x, t) = \begin{bmatrix} (a_0 + \alpha_0 t) + \sum_{k=1}^{\infty} \langle a_k(t) \cos kx + b_k(t) \sin kx \rangle \\ \alpha_0 \qquad + \sum_{k=1}^{\infty} \langle \alpha_k(t) \cos kx + \beta_k(t) \sin kx \rangle \end{bmatrix}, \tag{C7}$$

where, for $k > 0$,

$$\begin{aligned} a_k(t) &= a_k \cos kt + (\alpha_k/k) \sin kt \\ b_k(t) &= b_k \cos kt + (\beta_k/k) \sin kt \\ \alpha_k(t) &= -k a_k \sin kt + \alpha_k \cos kt \\ \beta_k(t) &= -k b_k \sin kt + \beta_k \cos kt. \end{aligned} \tag{C7$'$}$$

Evidently, the (pure) Cauchy problem defined by (C1) is *not deterministically well set* in **H** because, as was observed in [7, Part V, Example 4], the amplification factors of the Fourier components of solutions are unbounded; hence, the mappings $T_t = e^{tL}$ have unbounded norms and so are *not* continuous (do *not* define a C_0-semigroup).

We shall now define a class of regular measures μ on **H** for which the system

(C1) does, nevertheless, define a statistically well-set Cauchy problem. To this end, let M be the Borel subset of elements of the form (C6) for which

$$\sum k^4(a_k^2 + b_k^2) < \infty, \qquad \sum k^2(\alpha_k^2 + \beta_k^2) < \infty, \tag{C8}$$

where $\sum$ means $\sum_{k=1}^{\infty}$. Then M is a dense subset of **H** as defined above.

Theorem C1. *The Cauchy problem (C1) is statistically determinate in* **H** *for any initial regular probability measure* μ *with* $\mu(M) = 1$.

Proof. The Fourier series (C7)–(C7′), which satisfy (C8), yield strong, classical solutions of (C1); for the details, see, for example, A. N. Tychonoff and A. A. Samarskii, "Partial Differential Equations of Mathematical Physics," Chapter 2, Holden-Day, 1964. This proves existence; uniqueness is easily shown since the sin kx components of any *weak* solution of the Cauchy problem in **H** must satisfy, for the L of (C3),

$$b_k'(t) = \beta_k(t), \qquad \beta_k'(t) = -k^2 b_k(t) \qquad (k = 1, 2, 3, \ldots), \tag{C9}$$

and so $b_k(t)$ and $\beta_k(t)$ must be given by (C7′). Similar formulas hold for the cos kx components. (Actually, the sine components by themselves give the mixed Cauchy problem of the vibrating string.) This completes the proof of Theorem C1.

We now come to the main point: the construction of a *nonvoid subset* of the probability measures of Theorem C1 for which the initial value problem C1 is *statistically well set*. For simplicity, we shall do this only for the *mixed* Cauchy problem defined by the pure vibrating string, thus setting the coefficients of cos kx equal to zero.

Specifically, let $\{\sigma_k\}$ and $\{\tau_k\}$ be sequences of positive numbers such that

$$\sum k^4 \sigma_k^2 < \infty \qquad \text{and} \qquad \sum k^2 \tau_k^2 < \infty. \tag{C10}$$

For example, we might take $\sigma_k = \tau_k = k^{-3}$. We then consider the *sine series with random coefficients* [33] defined [cf. (C6)] by the initial conditions

$$\mathbf{w}(x) = \begin{bmatrix} u(x) \\ v(x) \end{bmatrix} = \begin{bmatrix} \sum b_k \sin kx \\ \sum \beta_k \sin kx \end{bmatrix}, \tag{C11}$$

where b_k and β_k are independent real Gaussian random variables with means $\overline{b_k} = \overline{\beta_k} = 0$ and variances $\overline{b_k^2} = \sigma_k^2$ and $\overline{\beta_k^2} = \tau_k^2$, respectively.

These conditions define a normal probability measure on **H**; for the details, see [43, Chapter 6]. Moreover, if $M \subset$ H is the set defined by (C7), then $\mu(M) = 1$ and the Cauchy problem is statistically determinate for the reasons given above. The measure μ is regular; moreover, since each σ_k and τ_k is positive, $\mu(U) > 0$ for any open subset $U \subset$ **H**.

Theorem C2. *The (mixed) initial value problem for the vibrating string is statistically well set in* **H** *for any normal probability measure* μ *which satisfies* (C10) *and is constructed as above, and* M *as defined by* (C8).

To complete the proof that the preceding (mixed) Cauchy problem is statistically well set, we need only show that (3.20) holds. Further, since $\mu(U) > 0$ for any open set $U \subset$ **H**, the set W of measure zero referred to in Section III.D is empty. We shall now prove (3.20), slightly changing the notation of (C11) (in which now $a_k = \alpha_k = 0$), by replacing the b_k of (C11) as α_k.

Let $t > 0$, $\varepsilon > 0$, and $\eta > 0$ be given, and let

$$\mathbf{w}_0(x) = \begin{bmatrix} \sum \alpha_k \sin kx \\ \sum \beta_k \sin kx \end{bmatrix}$$

be a fixed initial value in M. We shall show that, for $\delta > 0$ sufficiently small,

$$\mu(\|\mathbf{w}(x, t) - \mathbf{w}_0(x, t)\| \geq \eta \mid \|\mathbf{w}(x) - \mathbf{w}_0(x)\| < \delta) < \varepsilon, \tag{C12}$$

where $\mathbf{w}_0(x, t)$ is the unique solution of the Cauchy problem (C1) with initial value $\mathbf{w}_0(x)$ and, similarly for $\mathbf{w}(x, t)$, the solution associated with the *random* initial condition

$$\mathbf{w}(x) = \begin{bmatrix} \sum a_k \sin kx \\ \sum b_k \sin kx \end{bmatrix} = \begin{bmatrix} u(x) \\ v(x) \end{bmatrix},$$

under the probability measure μ. Here we can assume that $\mathbf{w} \in M$ since $\mu(M) = 1$, and hence $\mathbf{w}(x, t)$ is well defined by (C7). Also $\mu(A \mid B) = \mu(A \cap B)/\mu(B)$ is the conditional probability of the set A given the set B, which exists since any open set has positive probability in this case.

But now, in **H**, by definition,

$$\|\mathbf{w}(x) - \mathbf{w}_0(x)\|^2 = \sum (a_k - \alpha_k)^2 + \sum (b_k - \beta_k)^2$$

and

$$\|\mathbf{w}(x, t) - \mathbf{w}_0(x, t)\|^2 = \sum [(a_k - \alpha_k)\cos kt + k^{-1}(b_k - \beta_k)\sin kt]^2 + \sum [(b_k - \beta_k)\cos kt - k(a_k - \alpha_k)\sin kt]^2.$$

From these relations, one sees easily that it is sufficient in proving (C12) to show that, for $\delta > 0$ small enough,

$$\mu\left(\sum k^2(a_k - \alpha_k)^2 \geq \zeta \mid \sum [(a_k - \alpha_k)^2 + (b_k - \beta_k)^2] \leq \delta^2\right) < \varepsilon, \tag{C13}$$

where ζ can be taken to be $\eta^2/2$ for δ sufficiently small. In the following, which is the key to the examples constructed here, we show that this can be done, thus completing the proof of Theorem C2.

To simplify the formulas, let $Y = \sum k^2(a_k - \alpha_k)^2$ and $Z = \sum[(a_k - \alpha_k)^2 + (b_k - \beta_k)^2]$. Let μ_δ denote the probability μ conditioned on the set $B_\delta =$

$\{Z \le \delta^2\}$. In this notation, (C13) becomes: for $\delta > 0$ sufficiently small, prove that

$$\mu_\delta(Y \ge \zeta) < \varepsilon. \tag{C14}$$

To prove this, let $\mathbf{E}_\delta$ denote expectation with respect to μ_δ, i.e., the conditional expectation on B_δ. We will now show that $\mathbf{E}_\delta(Y) \to 0$ as $\delta \downarrow 0$. It will then be easy to deduce (C14) for δ sufficiently small, completing the proof of Theorem C2.

Lemma C1. $\mathbf{E}_\delta(Y)$ *converges to zero as* $\delta \downarrow 0$.

Proof. By the monotone convergence theorem, $\mathbf{E}_\delta(Y) = \sum k^2 \mathbf{E}_\delta([a_k - \alpha_k]^2)$. Now clearly,

$$\mathbf{E}_\delta([a_k - \alpha_k]^2) \le \delta^2$$

and

$$\mathbf{E}_\delta([a_k - \alpha_k]^2) \le \mathbf{E}([a_k - \alpha_k]^2) = \sigma_k^2 + \alpha_k^2.$$

The first inequality holds since $[a_k - \alpha_k]^2 \le \delta^2$ on B_δ and the second because the restriction to a sphere with center $\mathbf{w}_0$ can only decrease the expected value of $[a_k - \alpha_k]^2$. More precisely, the second inequality results from Fubini's theorem combined with the inequality

$$\int_{X \le a} X\,dv \le P(X \le a) \int_0^\infty X\,dv, \tag{C15}$$

for a nonnegative random variable X on the real line and regular probability measure on $[0, \infty]$. The inequality (C15) asserts the plausible result that the expectation of X, restricted to the set $\{X \le a\}$, is less than the expectation of X. This follows since

$$\int_0^\infty X\,dv = P(X \le a)\left[\frac{1}{P(X \le a)} \int_{X \le a} X\,dv\right] + P(X > a)\left[\frac{1}{P(X > a)} \int_{X > a} X\,dv\right]. \tag{C16}$$

Clearly,

$$\frac{1}{P(X \le a)} \int_{X \le a} X\,dv \le \frac{1}{P(X > a)} \int_{X > a} X\,dv,$$

so (C16) becomes

$$\int_0^\infty X\,dv \ge [P(X \le a) + P(X > a)] \frac{1}{P(X \le a)} \int_{X \le a} X\,dv,$$

from which (C15) follows since $P(x \le a) + P(x > a) = 1$.

Let $\lambda > 0$ be given. By (C8) and (C10), we can choose N so large that

$$\sum_{N+1}^\infty k^2 \sigma_k^2 < \lambda \quad \text{and} \quad \sum_{N+1}^\infty k^2 \alpha_k^2 < \lambda.$$

Then if $\delta^2 < \lambda/(\sum_1^N k^2)$, we have

$$\begin{aligned}\mathbf{E}_\delta(Y) &= \sum_1^N k^2 \mathbf{E}_\delta([a_k - \alpha_k]^2) + \sum_{N+1}^\infty k^2 \mathbf{E}_\delta([a_k - \alpha_k]^2)\\ &\le \sum_1^N k^2 \delta^2 + \sum_{N+1}^\infty k^2(\sigma_k^2 + \alpha_k^2) \le 3\lambda,\end{aligned}$$

and, since λ can be made arbitrarily small, Lemma C1 follows.

Now we can prove (C14) very easily. Apply Markov's inequality (Loève, "Probability Theory," 3rd ed., p. 158) to the random variable Y and the probability μ_δ, obtaining

$$\mu_\delta(Y \ge \zeta) \le (1/\zeta)\mathbf{E}_\delta(Y). \tag{C17}$$

The left side of (C17) is exactly the left side of (C14), while the right side of (C17) tends to zero with δ by Lemma C1; hence, it can be made less than ε for δ small enough. This finishes the proof of Theorem C2.

Actually, the assumption that each coordinate function b_k or β_k is nondegenerate Gaussian is not necessary. In fact, by using Theorem 3.3 in Section III.D and the regularity of μ, we need only assume (C10) and that the coordinate functions are independent random variables.

Finally, the proof of Theorem C2 has a straightforward extension to the pure Cauchy problem for (C1) on $L_2(\mathbf{K})$.

Appendix D: Existence of Normal Measures with Given Covariance

We now establish Theorem 5.6 from Section V.E, for tempered distributions over Euclidean n-space $\mathbf{R}^n = X$; for convenience we first restate this result.

Theorem 5.6. *Let* $\Gamma = \|\Gamma_{jk}\|$ *be a* $(q \times q)$*-matrix of continuous sesquilinear forms on* $\mathcal{S}(X) \times \mathcal{S}(X)$*, which is of positive type in the sense of* (5.23). *Then there exists a unique normal admissible probability measure* μ *on the Borel sets of* $S'(X)$*, whose mean is zero and whose covariance is* Γ.

Proof. For $q = 1$, and the "weak" Borel sets of $D'(X)$, uniqueness is proved by Gel'fand and Vilenkin in [22, pp. 248–50]; their method is to show that if μ and ν are two such probability measures, then $\mu(S) = \nu(S)$ on any cylinder set S and hence in the σ-field generated by cylinder sets—which is the σ-field of weak Borel sets.

The same argument can be used for $S'(X)$ when $q \geq 1$; it proves that $\mu(S) = \nu(S)$ for any cylinder set in $S'(X)$. Since the weak and strong Borel sets are the same in $S'(X)$ (see Appendix A), this proves the uniqueness statement of Theorem 5.6. (See also [18, p. 71] for the case $X = \mathbf{R}$.)

When $q = 1$, Gel'fand and Vilenkin also prove the existence of such a probability μ in $\mathscr{D}'(X)$; we sketch the (trivial) adaptation of their argument to $\mathscr{S}'(X)$. First, define μ on the cylinder sets of $\mathscr{S}'(X)$ in the only possible way [22, p. 337]. The remarks of [22, p. 339] explain how to extend this definition to a possibly degenerate covariance Γ, i.e., one for which equality is possible in (5.22) for some nonzero ϕ, $\mathbf{z}$. Then observe that, by the nuclearity of $\mathscr{S}(X)$ [52] and a deep result of Minlos [22, p. 315], μ can be consistently extended to a countably additive probability on the weak (=strong) Borel sets of $\mathscr{S}'(X)$.

Similar arguments can be applied when $q > 1$. Let $\Gamma = \|\Gamma_{jk}\|$ be a given $q \times q$ continuous, sesquilinear form on $\mathscr{S}(X) \times \mathscr{S}(X)$ in the strong topology. But, now $S'(X)$ is the dual space of (continuous linear functionals on) $S(X)$, and $S(X)$ is the space of column q-vectors $f, g, \ldots$ with components $f_j, g_k, \ldots$ in $\mathscr{S}(X)$. Hence the sum

$$K(f, g) = \sum_{j=1}^{q} \sum_{k=1}^{q} \Gamma_{jk}(f_j, g_k) \tag{D1}$$

is a continuous sesquilinear mapping $K: S(X) \times S(X) \to \mathbf{C}$ of nonnegative type, since Γ is continuous, sesquilinear, and of nonnegative (positive semidefinite) type. [Sesquilinearity and nonnegativity follow immediately on specializing $\mathbf{z}$ and $\mathbf{Z}$ in (5.22) and (5.23), respectively.]

These observations reduce the problem to the one-dimensional case. In this case, we again define a measure μ on the cylinder sets of $S'(X)$ as on [22, p. 337], with respect to the K of (D1)—or on the cylinder sets of a suitable closed subspace S_0 if K is degenerate; see again [22, p. 339]. Since $\mathscr{S}(X)$ is nuclear, so is $S(X)$. Therefore $S'(X)$ is the dual of a nuclear space and the criterion of Minlos [22, p. 310] applies once more. (In the degenerate case, we must use the fact that any quotient space of a nuclear space is nuclear and the remarks of [22, p. 339] to justify the application of Minlos's criterion.) Thus, to prove that μ extends to a countably additive probability on the weak (=strong) Borel sets of $S'(X)$ whose covariance is Γ, we need only show that, given $A > 0$, $\varepsilon > 0$, there is a neighborhood U of 0 in $S(X)$ such that, for all $f \in U$,

$$\mu(f; E) = \mu(\{T \in S'(X) \mid |\langle T \cdot f \rangle| \geq A\}) < \varepsilon, \tag{D2}$$

where $E = \{z \in \mathbf{C} \mid |z| \geq A\}$. This asserts the continuity of $\mu(f; E)$ at 0 for $f \in S(X)$. Note that (D2) is easily established in the present context since Γ, and therefore K, is assumed to be continuous for the topology of $S(X)$; the proof consists of writing down $\mu(f; E)$ in coordinate form.

This implies the existence of a unique probability measure μ defined on the Borel sets of $S'(X)$ such that

$$K(f, g) = \int_{S'(X)} \langle U \cdot f \rangle \langle U \cdot g \rangle^* \, d\mu(U). \tag{D3}$$

Here $\langle U \cdot f \rangle = \sum U_j(f_j)$ as in (2.9); i.e., $\langle\ \rangle$ expresses the duality between $S(X)$ and $S'(X)$. Further, μ is normal and has mean zero (and is regular and admissible), considered as a probability measure on the space of f, g [which is one copy of $S'(X)$].

It remains to transfer these properties from $K(f, g)$ to $\Gamma_{jk}(\phi, \psi)$, thus proving the following lemma.

Lemma D1. *The probability measure μ in (D3) is admissible, normal, has mean zero, and yields a continuous sesquilinear covariance matrix Γ.*

To achieve this transfer, we let $\phi, \psi \in \mathscr{S}(X)$ be given, together with j and k, and define $f, g \in S(X)$ by

$$f_j = \phi, \quad f_l = 0 \quad \text{if} \quad l \neq j, \qquad g_k = \psi, \quad g_l = 0 \quad \text{if} \quad l \neq k. \tag{D4}$$

Then, from (D1),

$$K(f, g) = \Gamma_{jk}(\phi, \psi); \tag{D5}$$

moreover, by (D3), we also have

$$K(f, g) = \int_{S'(X)} \langle U \cdot f \rangle \langle U \cdot g \rangle^* \, d\mu(U) = \int_{S'(X)} U_j(\phi)\, U_k(\psi)^* \, d\mu(U). \tag{D6}$$

Combining (D6) and (D5),

$$\Gamma_{jk}(\phi, \psi) = \int_{S'(X)} U_j(\phi)\, U_k(\psi)^* \, d\mu(U),$$

showing at once that μ is admissible and has covariance Γ.

Again letting $f \in S(X)$ be as in (D4), we see that, since μ has mean zero for the pairing $\langle\ \rangle$, then

$$0 = \int_{S'(X)} \langle U \cdot f \rangle \, d\mu(U) = \int_{S'(X)} U_j(\phi) \, d\mu(U), \tag{D7}$$

and, since $\phi \in \mathscr{S}(X)$ was arbitrary, we see that μ has mean zero.

Finally, we show that μ is normal. First μ is normal for the pairing $\langle\ \rangle$. Hence if $f^1, \ldots, f^r$ are elements of $S(X)$, then the measure $\mu(f^1, \ldots, f^r; *)$ defined on Borel sets E of $\mathbf{C}^r$ by

$$\mu(f^1, \ldots, f^r; E) = \mu(\{U \in S'(X) \mid (\langle U \cdot f^1 \rangle, \ldots, \langle U \cdot f^r \rangle) \in E\}) \qquad \text{(D8)}$$

is normally distributed. Now let $\phi \neq 0$ in $\mathcal{S}(X)$ and take $r = q$ and

$$f^j = \begin{bmatrix} 0 \\ \vdots \\ \phi \\ \vdots \\ 0 \end{bmatrix} \quad j\text{th row}, \quad i = 1, \ldots, q. \qquad \text{(D9)}$$

Then

$$\begin{aligned} &\{U \in S'(X) \mid (\langle U \cdot f^1 \rangle, \ldots, \langle U \cdot f^q \rangle) \in E\} \\ &\quad = \{U \in S'(X) \mid (U_1(\phi), \ldots, U_q(\phi)) \in E\}. \end{aligned} \qquad \text{(D10)}$$

Hence the following measure has a multivariate normal distribution in $\mathbf{C}^q$:

$$\begin{aligned} \mu(f^1, \ldots, f^q; E) &= \mu(\{U \in S'(X) \mid (U_1(\phi), \ldots, U_q(\phi)) \in E\}) \\ &= \mu(\phi; E), \end{aligned} \qquad \text{(D11)}$$

which shows that μ is normal. This finishes the proof of the lemma and the proof of Theorem 5.6.

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