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An Initial- and Boundary-Value Problem for a Model Equation for Propagation of Long Waves

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An initial- and boundary-value problem for a model equation for small-amplitude long waves is shown to be well-posed. The model has the form $u_t + u_x + uu_x - \nu u_{xx} - \alpha^2 u_{xxt} = 0$, where $x \in [0, 1]$ and $t > 0$. The solution $u = u(x, t)$ is specified at $t = 0$ and on the two boundaries $x = 0$ and $x = 1$. Unique classical solutions are shown to exist, which depend continuously on variations of the specified data within appropriate function classes.

1. INTRODUCTION

An initial- and boundary-value problem for a model equation for unidirectional propagation of waves is investigated here. The equation in question, which incorporates nonlinear, dispersive and dissipative effects, has been suggested as a model for surface water waves in a uniform channel (cf. [1, 2, 4, 5, 10]). It has the form

$$u_t + u_x + uu_x - \nu u_{xx} - \alpha^2 u_{xxt} = 0, \tag{1.1a}$$

for $x, t \in [0, 1] \times \mathbb{R}^+$. The equation is subjected to the auxiliary conditions

$$\begin{aligned} &u(0, t) = h(t), \qquad u(1, t) = g(t), \qquad && 0 \leqslant t, \\ &u(x, 0) = f(x), && 0 \leqslant x \leqslant 1. \end{aligned} \tag{1.1b}$$

Here $\alpha > 0$ and $\nu \geqslant 0$ and the consistency conditions

$$u(0, 0) = f(0) = h(0) \qquad \text{and} \qquad u(1, 0) = f(1) = g(0) \tag{1.2}$$

are also imposed.

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For the application to surface water waves, the dependent variable u represents the location of the free surface, relative to its undisturbed position, while the independent variables x and t are proportional to distance along the channel and to elapsed time, respectively. The constant ν expresses the relative importance of the nonlinear and dissipative effects and α plays a similar role regarding nonlinear and dispersive effects. Various hypotheses go into the derivation of (1.1a) as a model equation for water waves. These will play no role in the mathematical theory, except that a certain smoothness of the initial and boundary data will be assumed. This smoothness is entirely appropriate to the physical situation being modeled. Naturally, the specific hypotheses underlying the derivation of Eq. (1.1a) will be important when comparison with experimental observations is attempted. The theory developed here will cover the wave regime relevant to such observations. As regards comparison with experimental data, note that a simple change of variables allows the apparently more general equation

$$u_t + \beta u_x + \gamma u u_x - \mu u_{xx} - \delta^2 u_{xxt} = 0,$$

posed on $0 \leqslant x \leqslant l$, to be reduced to the form given in (1.1).

It will be shown that the problem (1.1) possesses a unique classical solution, which depends continuously on variations of the data f, g, and h within their respective function classes. Moreover, the solution depends continuously on $\nu \geqslant 0$. Thus a non-dissipative model is recovered in the limit as ν tends to zero.

A similar program has already been carried out for the pure initial-value problem for (1.1a), with $\nu = 0$, in [2], with $\nu \geqslant 0$, in [5], and with $\nu = 0$, for the initial- and boundary-value problem posed on the quarter-plane $x, t \geqslant 0$ in [3]. The periodic initial-value problem for this equation has also attracted attention (cf. Medeiros and Menzala [8], Showalter [12], and the Appendix of [3]). For the initial- and boundary-value problem in hand, existence and uniqueness of solutions has been demonstrated by Showalter [12] and by Medeiros and Miranda [9]. These works consider variously the cases $\nu = 0$ and $\nu \geqslant 0$, with more general nonlinear terms and with a forcing term added, but with homogeneous boundary conditions $h = g = 0$. The present analysis could incorporate broader classes of nonlinearities and a forcing term without essential difficulty, but the resulting complication appears not to be worth the gain in generality.

It should be pointed out that weaker forms of some of our results can be obtained via the following observation. Let $v(x, t) = xg(t) + (1 - x) h(t)$ and let $w = u - v$, where u is a solution of (1.1a)–(1.1b). Then w satisfies

$$w_t + w_x + w w_x + (vw)_x - \nu w_{xx} - \alpha^2 w_{xxt} = \xi, \tag{1.3}$$

where

$$\xi = \xi(x, t) = -[v_t + v_x + v v_x], \tag{1.4}$$

and $w(0, t) = w(1, t) = 0$. Conversely, a solution w of (1.3)–(1.4) determines a solution u of (1.1). The variable-coefficient problem (1.3)–(1.4) does not fall within the scope of [9], but Showalter's theory [12] very nearly covers this situation (the problem lying in the term vw_x, where v depends explicitly on x; cf. (3.11) in [12]). The arguments in [12], which are based on Showalter's general theory for Sobolev equations [11], can be extended to establish the existence of a weak solution of (1.3)–(1.4), and hence of a weak solution of (1.1). This solution may be further inferred to be such that each term in (1.1) lies in $L_2(0, 1)$, for almost every t, using Theorem 6 in [11]. The physical regime that (1.1) is intended to model does not encompass the formation of singularities such as shock formation or wave breaking. Hence classical solutions are to be expected, and are provided by the rather specific approach presented here. Moreover, the continuous dependence of the solution, within smooth function classes, on all the data, including ν and α, follows readily from the methods developed below, and consequently they have been favored over the general theory currently available.

The two-point boundary-value problem for (1.1a) suggested here is worth studying for several reasons. First, any numerical scheme for (1.1a) must inevitably be posed on a bounded domain, even though the pure initial-value problem or the quarter-plane problem may be in view. Hence the scheme will in fact be an approximation to a problem of the form (1.1) with some specified boundary conditions (the periodic initial-value problem is different in this facet). Second, as explained already in [3] and [4], the initial- and boundary-value problem is more amenable to direct comparison with experiments performed with water waves in a channel than either the pure initial-value problem or the periodic initial-value problem.A disturbance created at one end of a channel by a wavemaker may be measured at several stations down the channel. Two of these measurements may be used for g and h in (1.1) (with $f \equiv 0$, say, corresponding to the liquid being initially at rest). A numerical prediction may then be made using the model (1.1). This prediction may then be compared with measurements taken at a third station situated between the two stations used to determine g and h, and the outcome used to judge the accuracy of the model. The model written here is suitable for modeling the propagation of waves moving to the right. Consequently, it is appropriate to have $h(t)$ determined by a measurement taken near the wavemaker and to allow the experiment to run until the distrurbance has reached the right-hand end of the channel. Taking the right end of the channel at $x = 1$, in suitably scaled coordinates, it follows that $g(t) \equiv 0$ throughout the experiment. Note incidentally that the possibility of having nonhomogeneous boundary conditions is crucial for the application of (1.1a) to the experimental situation envisaged here.

It is noteworthy that the Korteweg–de Vries equation, with or without the dissipative term $-\nu u_{xx}$, is not well-posed with the conditions given in (1.1b). A third boundary condition must be specified (e.g., u_x could be specified at $x = 0$ or $x = 1$). If, as described above, a limited-time experiment is in question, in which the disturbance created by the wavemaker does not reach the station down the channel used to determine g, so that $g(t) = 0$ for all the relevant $t \geq 0$, then an obvious candidate for a third boundary condition would be $u_x(1, t) = 0$, for $t \geq 0$.

The dissipative term $-\nu u_{xx}$ incorporated in (1.1a) is only a model term. The proper dissipation for shallow water waves in a uniform channel, consistent with the level of approximation already inherent in the modeling of the nonlinear and dispersive effects, has been discussed by Kakutani and Matsuuchi [7]. Its nonlocal form requires a more refined analysis, particularly for the boundary-value problem studied here. If the disturbance has most of its energy concentrated in a single wavelength, as is the case in the periodic generation of small-amplitude long waves by a wavemaker located at one end of a channel, the damping given here can be argued to be quite a good approximation to the dissipation provided more exactly in [7], and consequently to the actual dissipation experienced by the waves. This point has been discussed in detail in [4], where comparisons of the model (1.1a) with experimental data are reported. Comparisons of solutions of (1.1a), in case $\nu = 0$, with experimental data have also been made by Hammack [6].

The plan of the paper is as follows. In Section 2, the mathematical notation to be used subsequently is introduced. Section 3 is devoted to a local existence and uniqueness theorem for (1.1), and to the regularity theory for solutions. In Section 4, the local solution established in Section 3 is extended to a global solution defined for all $t \geq 0$. The last section contains the results of continuous dependence.

2. Notation

Throughout the paper, all functions are assumed to be real-valued. We will denote by C^k, or by $C^k(a, b)$ the Banach space of k-times continuously differentiable functions defined on $[a, b]$, with the norm

$$\|f\|_{C^k} = \sup_{\substack{a \leq x \leq b,\\ 0 \leq j \leq k}} |f^{(j)}(x)| .$$

The L_2-norm of a function f which is square-integrable on $[0, 1]$ is denoted by $\|f\|$. For $m = 0, 1, 2,\ldots$, $H^m = H^m(0, 1)$ is the Sobolev space consisting of those L_2 functions whose first m generalized derivatives also lie in L_2, with the usual norm,

$$\|f\|_m = \left(\sum_{j=0}^{m} \|f^{(j)}\|^2\right)^{1/2}.$$

There will also be occasion to use the spaces $C(0, T; X)$, for $X = C^k$ or

$X = H^m$. In general if X is any Banach space then $C(0, T; X)$ is the Banach space of continuous functions u: $[0, T] \to X$ with the norm

$$\| u \|_{C(0,T;X)} = \sup_{0 \leq t \leq T} \| u(t) \|_X .$$

The abbreviation $\mathscr{C}_T^{k,l}$ will be employed for the functions u: $[0, 1] \times [0, T] \to \mathbb{R}$ such that $\partial_t{}^i \partial_x{}^j u \in C(0, T; C^0)$ for $0 \leq j \leq k$ and $0 \leq i \leq l$. This space of functions will carry the norm

$$\| u \|_{\mathscr{C}_T^{k,l}} = \sum_{\substack{0 \leq j \leq k \\ 0 \leq i \leq l}} \| \partial_t{}^i \partial_x{}^j u \|_{C(0,T;C^0)} .$$

The space $\mathscr{C}_T^{0,0}$ will be abbreviated simply $\mathscr{C}_T$.

An inequality that will be used several times below is contained in the following lemma (cf. Sobolev [13]).

LEMMA 1. *If $f \in H^1(0, 1)$ then f is equal almost everywhere to a bounded continuous function $\tilde{f}$ and there is a constant c_* such that*

$$\| \tilde{f} \|_{C^0} \leq c_* \| f \|_1 . \tag{2.1}$$

It follows that if $u \in C(0, T; H^1)$, then u is equal almost everywhere to a bounded continuous function $\tilde{u} \in C(0, T; C^0)$ and that

$$\sup_{\substack{0 \leq x \leq 1 \\ 0 \leq t \leq T}} | \tilde{u}(x, t) | \leq c_* \| u \|_{C(0,T;H^1)} . \tag{2.2}$$

Subsequently, whenever a function $f \in H^1(0, 1)$, or $u \in C(0, T; H^1)$, presents itself, it will be tacitly understood that the continuous representative has been selected from the equivalence class of functions equal almost everywhere to the given function.

3. LOCAL EXISTENCE THEORY

A solution of (1.1) is shown to exist over a positive time interval. The regularity and uniqueness of this solution are examined. The existence of a solution is established by converting the differential equation into an integral equation and applying the contraction mapping theorem to the integral equation. The regularity then follows from the fact that any solution of the integral equation is exactly as smooth as the data afford. The argument thus closely parallels that worked out in detail in [2] for the pure initial-value problem and we may therefore justifiably omit many of the detailed calculations.

Write (1.1a) as

$$(1 - \alpha^2\partial_x^2)\, u_t = -u_x - uu_x + \nu u_{xx}.$$

Invert the operator $1 - \alpha^2\partial_x^2$, subject to the boundary conditions $u_t(0, t) = h'(t)$ and $u_t(1, t) = g'(t)$ implied by (1.1b), to obtain formally

$$u_t(x, t) = h'(t)\phi_1(x) + g'(t)\phi_2(x) - \int_0^1 P(x, \xi)\,[u_\xi(\xi, t) + u(\xi, t)\, u_\xi(\xi, t)]\, d\xi + \nu \int_0^1 P(x, \xi)\, u_{\xi\xi}(\xi, t)\, d\xi, \tag{3.1}$$

where

$$\phi_1(x) = \frac{\sinh((1-x)/\alpha)}{\sinh(1/\alpha)}, \qquad \phi_2(x) = \frac{\sinh(x/\alpha)}{\sinh(1/\alpha)}, \tag{3.2}$$

and

$$P(x, \xi) = \frac{1}{2\alpha[\exp(2/\alpha) - 1]}\{-\exp((x + \xi)/\alpha) + \exp(|x - \xi|/\alpha) + \exp((2 - |x - \xi|)/\alpha) - \exp((2 - (x + \xi))/\alpha)\}. \tag{3.3}$$

Note, for later use, that $\phi_1, \phi_2 \geqslant 0$ and that

$$\phi_1(0) = 1, \quad \phi_1(1) = 0, \quad \sup_{0 \leqslant x \leqslant 1} \phi_1(x) = 1,$$
$$\phi_2(0) = 0, \quad \phi_2(1) = 1, \quad \sup_{0 \leqslant x \leqslant 1} \phi_2(x) = 1. \tag{3.4}$$

Since $P(x, 1) = P(x, 0) = 0$, for all $x \in [0, 1]$, the first integral on the right side of (3.1) may be integrated by parts to reach the integral

$$\int_0^1 K(x, \xi)\,[u(\xi, t) + \tfrac{1}{2}u^2(\xi, t)]\, d\xi, \tag{3.5}$$

where

$$K(x, \xi) = \frac{1}{2\alpha^2[\exp(2/\alpha) - 1]}\{-\exp((x + \xi)/\alpha) + \exp((2 - (x + \xi))/\alpha) - \operatorname{sgn}(x - \xi)\exp(|x - \xi|/\alpha) + \operatorname{sgn}(x - \xi)\exp((2 - |x - \xi|)/\alpha)\}. \tag{3.6}$$

Similarly the second integral on the right side of (3.1) may be written

$$-\frac{4\exp(1/\alpha)\,\nu}{2\alpha^2[\exp(2/\alpha) - 1]}\{\sinh(1/\alpha)\, u(x, t) - \sinh((1 - x)/\alpha)\, h(t) - \sinh(x/\alpha)\, g(t)\} + \frac{\nu}{\alpha^2}\int_0^1 P(x, \xi)\, u(\xi, t)\, d\xi, \tag{3.7}$$

by two integrations by parts. Putting this together, and with further simplifications, yields

$$\begin{aligned}
u_t(x,t) &+ \frac{\nu}{\alpha^2}u(x,t)\\
&= \Big(h'(t) + \frac{\nu}{\alpha^2}h(t)\Big)\phi_1(x) + \Big(g'(t) + \frac{\nu}{\alpha^2}g(t)\Big)\phi_2(x) \\
&\quad + \int_0^1 K(x,\xi)\Big(u(\xi,t) + \tfrac{1}{2}u^2(\xi,t)\Big)\,d\xi + \frac{\nu}{\alpha^2}\int_0^1 P(x,\xi)\,u(\xi,t)\,d\xi.
\end{aligned} \tag{3.8}$$

Now view (3.8) as an ordinary differential equation, in the temporal variable, of the form

$$u_t + \frac{\nu}{\alpha^2}u = \phi(x,t).$$

This may be solved explicitly for u, and upon substituting the expression for ϕ implied by (3.8), one obtains the integral equation

$$u(x,t) = \exp(-\nu t/\alpha^2)f(x) + \phi_1(x)\tilde{h}(t) + \phi_2(x)\tilde{g}(t) + B(u)(x,t), \tag{3.9}$$

where

$$\begin{aligned}
B(u)(x,t) &= \int_0^t\int_0^1 \exp(-\nu(t-\tau)/\alpha^2)K(x,\xi)\,[u(\xi,\tau) + \tfrac{1}{2}u^2(\xi,\tau)]\,d\xi\,d\tau \\
&\quad + \frac{\nu}{\alpha^2}\int_0^t\int_0^1 \exp(-\nu(t-\tau)/\alpha^2)P(x,\xi)\,u(\xi,\tau)\,d\xi\,d\tau
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
\tilde{h}(t) &= h(t) - \exp(-\nu t/\alpha^2)h(0),\\
\tilde{g}(t) &= g(t) - \exp(-\nu t/\alpha^2)g(0).
\end{aligned} \tag{3.11}$$

This is the integral equation promised earlier. Note that any classical solution of (1.1) does indeed satisfy this integral equation, since all the steps leading to the derivation of (3.9) may then be justified.

PROPOSITION 1. *Let* $f \in C^0(0,1)$ *and* $g, h \in C^0(0,T)$ *for some* $T > 0$. *Then there is an* $S = S(\|f\|_{C^0}, \|g\|_{C^0}, \|h\|_{C^0}, \alpha, \nu, T)$ *in* $(0, T]$ *and a unique function* u *in* $C(0,S; C^0(0,1))$ *that satisfies the integral equation* (3.9). *Moreover, for any* $T_1 \leqslant T$ *there is at most one solution of* (3.9) *in* $C(0,T_1; C^0(0,1))$.

Proof. For v in $C(0,S;C^0)$ define

$$Av(x,t) = \exp(-\nu t/\alpha^2)f(x) + \phi_1(x)\tilde{h}(t) + \phi_2(x)\tilde{g}(t) + Bv(x,t), \tag{3.12}$$

where B is as in (3.10). Let $\mathscr{C} = C(0, S; C^0)$, where S remains to be chosen. Note that, if $v \in \mathscr{C}$, then $Av \in \mathscr{C}$ also. Moreover, if $v, w \in \mathscr{C}$, a straightforward estimate shows that

$$\begin{aligned} &| Bw(x, t) - Bv(x, t)| \\ &\quad \leqslant c_1 S \| w - v \|_{\mathscr{C}} (1 + \tfrac{1}{2}(\| w \|_{\mathscr{C}} + \| v \|_{\mathscr{C}})) + c_2 S \| w - v \|_{\mathscr{C}}, \end{aligned} \tag{3.13}$$

where

$$c_1 = c_1(\alpha) = \sup_{0 \leqslant x \leqslant 1} \int_0^1 | K(x, \xi)| \, d\xi = \frac{\coth(1/\alpha) - \operatorname{csch}(1/\alpha)}{\alpha},$$

and

$$c_2 = c_2(\alpha, \nu) = \sup_{0 \leqslant x \leqslant 1} \frac{\nu}{\alpha^2} \int_0^1 | P(x, \xi)| \, d\xi = \frac{\nu}{\alpha^2} (1 - \operatorname{sech}(1/2\alpha)).$$

It follows, by taking the supremum in (3.13) for x, t in $[0, 1] \times [0, S]$, that

$$\| Aw - Av \|_{\mathscr{C}} \leqslant \| w - v \|_{\mathscr{C}} \{[c_1(1 + \tfrac{1}{2}(\| w \|_{\mathscr{C}} + \| v \|_{\mathscr{C}})) + c_2] \, S\}. \tag{3.14}$$

In particular, for $w \in \mathscr{C}$,

$$\| Aw \|_{\mathscr{C}} \leqslant \| f \|_{C^0} + 2 \| h \|_{C^0(0,S)} + 2 \| g \|_{C^0(0,S)} + \| Aw - A\theta \|_{\mathscr{C}}, \tag{3.15}$$

where $\theta(x, t) \equiv 0$. The inequalities (3.14) and (3.15) are enough to justify an application of the contraction mapping theorem. Consider A as a mapping of the ball B_R of radius R about zero in $\mathscr{C}$. Define

$$r(S) = \| f \|_{C^0} + 2(\| h \|_{C^0(0,S)} + \| g \|_{C^0(0,S)}).$$

Then if $v, w \in B_R$, (3.14) implies that

$$\| Aw - Av \|_{\mathscr{C}} \leqslant S(c_1(1 + R) + c_2) \| w - v \|_{\mathscr{C}} \equiv \eta(R, S) \| w - v \|_{\mathscr{C}}. \tag{3.16}$$

Hence (3.15) implies

$$\| Aw \|_{\mathscr{C}} \leqslant r(S) + R\eta(R, S). \tag{3.17}$$

The contraction mapping theorem applies to A, considered as a mapping of B_R to itself provided R and S can be chosen so that $\eta(R, S) < 1$ and $r(S) + R\eta(R, S) \leqslant R$. These may be simultaneously satisfied by choosing $R = 2r(S)$ and then choosing S small enough that $\eta \leqslant \frac{1}{2}$. That is, choose S in $(0, T]$ such that

$$S \leqslant \frac{1}{2(c_1(1 + 2r(S)) + c_2)}. \tag{3.18}$$

The question of uniqueness is easily settled. Suppose there were two distinct solutions u and v of (3.9) in $C(0, T_1; C^0)$. Since u and v are continuous there is a point t_0 in $[0, T_1)$ such that $u \equiv v$ for $0 \leqslant t \leqslant t_0$ and on no interval $[t_0, t_0 + \epsilon]$ is this still true if $\epsilon > 0$. Hence the integral equation

$$w(x, t) = w_0(x, t) + [\exp(-\nu t/\alpha^2) - \exp(-\nu t_0/\alpha^2)]\, f(x) + \phi_1(x)\, [\tilde{h}(t) - \tilde{h}(t_0)] + \phi_2(x)\, [\tilde{g}(t) - \tilde{g}(t_0)] + \tilde{B}w(x, t) = \tilde{A}w(x, t),$$

where

$$w_0(x, t) = u(x, t_0) + \int_0^{t_0} \gamma(t, \tau) \int_0^1 K(x, \xi)\, [u(\xi, \tau) + \tfrac{1}{2}u^2(\xi, \tau)]\, d\xi\, d\tau + \frac{\nu}{\alpha^2} \int_0^{t_0} \gamma(t, \tau) \int_0^1 P(x, \xi)\, u(\xi, \tau)\, d\xi\, d\tau,$$

$$\gamma(t, \tau) = \exp(-\nu(t - \tau)/\alpha^2) - \exp(-\nu(t_0 - \tau)/\alpha^2),$$

and

$$\tilde{B}w(x, t) = \int_{t_0}^{t} \exp(-\nu(t - \tau)/\alpha^2) \int_0^1 K(x, \xi)\, [w(\xi, \tau) + \tfrac{1}{2}w^2(\xi, \tau)]\, d\xi\, d\tau + \frac{\nu}{\alpha^2} \int_{t_0}^{t} \exp(-\nu(t - \tau)/\alpha^2) \int_0^1 P(x, \xi)\, w(\xi, \tau)\, d\xi\, d\tau,$$

has two distinct solutions, which we denote by u and v again, though they are in fact u and v restricted to $[t_0, T_1]$. Moreover, while these solutions agree at t_0, they do not agree identically in any neighborhood of t_0.

The existence argument presented above is easily adapted to show that, for R large enough and for $t_1 = t_1(R)$ close enough to t_0, $\tilde{A}$ is a contraction mapping of the ball $\tilde{B}_R$ of radius R centered at the zero function in $C(t_0, t_1; C^0)$. But if

$$R \geqslant \max\{\| u \|_{C(0,T_1;C^0)}, \| v \|_{C(0,T_1;C^0)}\},$$

then $\tilde{A}$ has the two distinct fixed points u and v in $\tilde{B}_R$. This contradiction forces the conclusion $u \equiv v$ on $[0, T_1]$, and the proposition is established.

Let $u \in C(0, T; C^0)$ be a solution of the integral equation (3.9). Suppose f, g, and h are only continuous, but not differentiable. Is it possible to conclude that u is smoother for $t > 0$?

Suppose indeed that u is continuously differentiable with respect to x and t. In any case, if $u \in C(0, T; C^0)$, it is easy to see that $B(u)$ is continuously differentiable with respect to both x and t. Hence $u - Bu$ is differentiable with respect to x and t. But according to (3.9)

$$u - Bu = \exp(-\nu t/\alpha^2)\, f(x) + \phi_1(x)\, \tilde{h}(t) + \phi_2(x)\, \tilde{g}(t),$$

where ϕ_1 and ϕ_2 are linearly independent C^∞ functions which are non-zero on (0, 1). It follows that f, g and h are C^1 functions, a contradiction. This argument may be used to show that generally the solutions of (3.9) can be no more regular in x than f, and no more regular in t than g and h (see, however, the remark at the end of this section).

So the only hope for a smoother solution is to assume smoother data. Here is a result of further regularity for solutions of (3.9).

PROPOSITION 2. *Let $f \in C^2(0, 1)$ and g, $h \in C^1(0, T)$ satisfy the compatibility conditions* (1.2). *Then any solution u in $C(0, T; C^0)$ of the integral equation* (3.9) *lies in $\mathscr{C}_T^{2,1}$ and is the unique classical solution of the initial- and boundary-value problem* (1.1) *on the interval* $[0, T]$.

Proof. Since u is a continuous function, Au is differentiable with respect to t. Since $u = Au$, it follows that u_t exists and is given by (3.8). Hence $u_t \in \mathscr{C}_T$ since g' and h' lie in $C^0(0, T)$ and u is in $\mathscr{C}_T$.

By dividing the interval of spatial integration at $\xi = x$ and considering each of the resulting pieces separately, it is verified that u_x exists and is given by

$$\begin{aligned} u_x(x, t) &= \exp(-\nu t/\alpha^2) f'(x) + \phi_1'(x)\,\tilde{h}(t) + \phi_2'(x)\,\tilde{g}(t) \\ &\quad + \frac{1}{\alpha^2}\int_0^t \exp(-\nu(t-\tau)/\alpha^2)\left[u(x,\tau) + \tfrac{1}{2}u^2(x,\tau)\right] d\tau \\ &\quad - \frac{1}{\alpha^2}\int_0^t\int_0^1 \exp(-\nu(t-\tau)/\alpha^2)\, M(x,\xi)\left[u(\xi,\tau) + \tfrac{1}{2}u^2(\xi,\tau)\right] d\xi\, d\tau \\ &\quad - \frac{\nu}{\alpha^3}\int_0^t\int_0^1 \exp(-\nu(t-\tau)/\alpha^2)\, Q(x,\xi)\, u(\xi,\tau)\, d\xi\, d\tau, \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} M(x,\xi) = \frac{1}{2\alpha(\exp(2/\alpha)-1)} &\{\exp((x+\xi)/\alpha) + \exp((2-(x+\xi))/\alpha) \\ &+ \exp(|x-\xi|/\alpha) + \exp((2-|x-\xi|)/\alpha)\} \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} Q(x,\xi) = \frac{1}{2\alpha(\exp(2/\alpha)-1)} &\{\exp((x+\xi)/\alpha) - \exp((2-(x+\xi))/\alpha) \\ &- \operatorname{sgn}(x-\xi)\exp(|x-\xi|/\alpha) + \operatorname{sgn}(x-\xi)\exp((2-|x-\xi|)/\alpha)\}. \end{aligned} \tag{3.21}$$

From (3.19) and the assumptions on f, g, and h, it is apparent that $u_x \in \mathscr{C}_T$. Once this is appreciated, then the right-hand side of (3.19) can be seen to be differentiable with respect to both x and t. Upon differentiating the right side of

(3.19) with respect to t, it appears that $u_{xt} \in \mathscr{C}_T$. Again dividing the spatial integrals at $\xi = x$, it is determined that

$$\begin{aligned} u_{xx}(x, t) = {} & \exp(-\nu t/\alpha^2) f''(x) + \phi_1'' \tilde{h} + \phi_2'' \tilde{g} \\ & + \frac{1}{\alpha^2} \int_0^t \exp(-\nu(t-\tau)/\alpha^2)\, \partial_x \left(u + \tfrac{1}{2}u^2\right) d\tau \\ & + \frac{1}{\alpha^2} \int_0^t \int_0^1 \exp(-\nu(t-\tau)/\alpha^2)\, K(x, \xi) \left(u + \tfrac{1}{2}u^2\right) d\xi\, d\tau \\ & + \frac{\nu}{\alpha^4} \int_0^t \int_0^1 \exp(-\nu(t-\tau)/\alpha^2)\, P(x, \xi)\, u(\xi, \tau)\, d\xi\, d\tau \\ & - \frac{\nu}{\alpha^4} \int_0^t \exp(-\nu(t-\tau)/\alpha^2)\, u(x, \tau)\, d\tau. \end{aligned}$$

This may be simplified, using the original integral equation for u and the relations $\alpha^2\phi_j'' = \phi_j$, $j = 1, 2$. With these observations u_{xx} may be expressed as

$$\begin{aligned} u_{xx}(x, t) = {} & \exp(-\nu t/\alpha^2) f''(x) + \frac{1}{\alpha^2} \int_0^t \exp(-\nu(t-\tau)/\alpha^2)\, [u_x + uu_x]\, d\tau \\ & + \frac{1}{\alpha^2} u(x, t) - \frac{1}{\alpha^2} \exp(-\nu t/\alpha^2) f(x) \\ & - \frac{\nu}{\alpha^4} \int_0^t \exp(-\nu(t-\tau)/\alpha^2)\, u(x, \tau)\, d\tau. \end{aligned} \tag{3.22}$$

Since u_x is in $\mathscr{C}_T$, so is the right side of (3.22). Hence u_{xx} is in $\mathscr{C}_T$. Additionally, u_{xx} is differentiable with respect to t, and upon performing this differentiation, simplifying the resulting expression, and multiplying by α^2, it is verified that u is a classical solution of the differential equation (1.1a).

From (3.9) it is clear that $u(x, 0) = f(x)$ since $\tilde{h}(0) = \tilde{g}(0) = 0$. Also, since $K(0, \xi) = K(1, \xi) = 0$ and $P(0, \xi) = P(1, \xi) = 0$, for $0 \leqslant \xi \leqslant 1$, and because of (3.4),

$$u(0, t) = \exp(-\nu t/\alpha^2)\,(f(0) - h(0)) + h(t),$$

and

$$u(1, t) = \exp(-\nu t/\alpha^2)\,(f(1) - g(0)) + g(t).$$

Thus the boundary conditions in (1.1b) are satisfied by virtue of the compatibility conditions (1.2).

The uniqueness assertion follows from the uniqueness of the solutions of the integral equation, and the fact that any classical solution of (1.1) satisfies the associated integral equation in the form (3.9).

COROLLARY. *Let $f \in C^l(0, 1)$ and $g, h \in C^k(0, T)$ where $l \geqslant 2$ and $k \geqslant 1$.*

Suppose f, g, and h satisfy the compatibility conditions (1.2). *Then any u in* $\mathscr{C}_T$ *that satisfies the integral equation* (3.9) *corresponding to f, g, and h lies in* $\mathscr{C}_T^{l,k}$ *and is the unique classical solution of* (1.1) *corresponding to f, g, and h.*

This is a simple extension of the methods employed in Proposition 2. Note that further compatibility conditions on f, g, and h are not needed for the further regularity of the solution. This is due to the presence of the mixed spatial–temporal derivative in the model equation. In the special case when $\nu = 0$, the solution of (1.1), corresponding to data f, g and h as in the last corollary, has the additional property that

$$\partial_t^j u \in C(0, T; C^{l+1})$$

for $1 \leqslant j \leqslant k$. This follows from examination of (3.8), with ν set to zero.

4. Global Existence Theory

Here the result of local existence, established in the last section, is extended. The method is to iterate the local existence result. This is effective because of the following *a priori* bound on the growth of the solution.

Proposition 3. *Let* $f \in C^2(0, 1)$ *and* $g, h \in C^1(0, T)$ *and suppose the compatibility conditions* (1.2) *are valid for f, g and h. Let u be the classical solution of* (1.1) *corresponding to the data f, g and h. Then*

$$\| u \|_1 \leqslant C \| f \|_1 + D \tag{4.1}$$

for $0 \leqslant t \leqslant T$, *where C and D are positive constants depending on* α, *T and on the* $C^1(0, T)$*-norms of g and h.*

Remark. In the special case where $g \equiv 0$, bounds that grow in time only as rapidly as the energy supplied from the left-hand boundary may be obtained. This is a point which may be of some interest in the application of the model equation (1.1a), but we pass over this aspect here.

Proof. Let $v(x, t) = xg(t) + (1 - x)\, h(t)$ and let $w = u - v$. Then w satisfies (1.3)–(1.4) with

$$w(x, 0) = f(x) - [xf(1) + (1 - x) f(0)] \tag{4.2}$$

and

$$w(0, t) = w(1, t) \equiv 0. \tag{4.3}$$

Multiply Eq. (1.3) by w and integrate the resulting identity over [0, 1]. Integration by parts, and using (4.3), yields

$$\frac{1}{2}\frac{d}{dt}\int_0^1 [w^2 + \alpha^2 w_x^2]\,dx + \nu\int_0^1 w_x^2\,dx$$
$$= \int_0^1 \xi w\,dx - \frac{1}{2}\int_0^1 v_x w^2\,dx \leqslant \|\xi\|\,\|w\| + \tfrac{1}{2}(|g(t)| + |h(t)|)\,\|w\|^2. \tag{4.4}$$

For $\eta \in H^1(0, 1)$, define the auxiliary norm

$$|||\eta|||_1 = \left\{\int_0^1 [\eta^2 + \alpha^2\eta_x^2]\,dx\right\}^{1/2}.$$

Then for $\eta \in H^1(0, 1)$,

$$\frac{1}{c(\alpha)}\|\eta\|_1 \leqslant |||\eta|||_1 \leqslant c''(\alpha)\,\|\eta\|_1, \tag{4.5}$$

where $c(\alpha) = \max\{1, \alpha^{-1}\}$ and $c''(\alpha) = \max\{1, \alpha\}$. The inequality (4.4) yields

$$\frac{1}{2}\frac{d}{dt}|||w(\cdot, t)|||_1^2 \leqslant A\,|||w(\cdot, t)|||_1 + B\,|||w(\cdot, t)|||_1^2, \tag{4.6}$$

where, from (1.4),

$$A = A(h, g, T)$$
$$= \sup_{0 \leqslant t \leqslant T} \|\xi\| \leqslant \|g\|_{C^1(0,T)} + \|h\|_{C^1(0,T)} + (\|g\|_{C^0(0,T)} + \|h\|_{C^0(0,T)})^2$$

and

$$B = B(h, g, T) = \tfrac{1}{2}(\|g\|_{C^0(0,T)} + \|h\|_{C^0(0,T)}).$$

From (4.6) it follows that

$$|||w(\cdot, t)|||_1 \leqslant \tfrac{1}{2}\,|||w(\cdot, 0)|||_T\, e^{Bt} + AB^{-1}(e^{Bt} - 1),$$

for $0 \leqslant t \leqslant T$. An application of (4.5) gives

$$\|w(\cdot, t)\|_1 \leqslant c'(\alpha)\,e^{Bt}\,\|w(\cdot, 0)\|_1 + c(\alpha)\,AB^{-1}(e^{Bt} - 1),$$

for $0 \leqslant t \leqslant T$, where $c'(\alpha) = \max\{\alpha, \alpha^{-1}\}$ and $c(\alpha)$ is defined below (4.5). Since $w = u - v$, the finishing touch is supplied by the triangle inequality;

$$\|u(\cdot, t)\|_1 \leqslant C\,\|f\|_1 + D,$$

for $0 \leqslant t \leqslant T$, where

$$C = C(\alpha, g, h, T) = c'(\alpha)\, e^{BT}$$

and

$$D = D(\alpha, g, h, T)$$
$$= c'(\alpha)\, e^{BT} \| v(\cdot, 0)\|_1 + c(\alpha) AB^{-1}(e^{BT} - 1) + \sup_{0 \leqslant t \leqslant T} \| v(\cdot, t)\|_1 .$$

This establishes the proposition.

The main result of existence may now be proved. The result is stated for f in $C^2(0, 1)$ and g, h in $C^1(0, T)$, but the corollary at the end of Section 3 shows that if f, g, and h are smoother, then so is the solution obtained herein.

THEOREM 1. *Let $f \in C^2(0, 1)$ and $g, h \in C^1(0, T)$ where $T > 0$. Suppose that f, g, and h satisfy the compatibility conditions* (1.2). *Then there exists a unique classical solution in $\mathscr{C}_T^{2,1}$ of the initial- and boundary-value problem* (1.1).

Proof. Uniqueness has already been dealt with in Proposition 2. Existence may be established by iteration of the local existence theorem proved in Proposition 1. Such a method is effective because of the *a priori* bound (4.1). This bound shows that on any finite time interval $[0, T]$, $\| u(\cdot, t)\|_1$, and hence $\| u(\cdot, t)\|_{C^0}$, is uniformly bounded. But g and h are uniformly bounded, on $[0, T]$, by assumption. Therefore the function

$$r_t(S) = \| u(\cdot, t)\|_{C^0} + 2(\| h \|_{C^0(t, t+S)} + \| g \|_{C^0(t, t+S)}),$$

analogous to the function $r(S)$ defined above (3.17), is uniformly bounded for $0 \leqslant t \leqslant T$ and $t \leqslant t + S \leqslant T$. But $r_t(S)$ determines, via (3.18), a lower bound on how far a solution, defined already on $[0, t]$, can be extended by an application of the local existence result in Proposition 1. As $r_t(S)$ is bounded above, so from (3.18) this extension length is bounded below by a positive constant. It follows that the solution may be extended to $[0, T]$ by a finite number of applications of Proposition 1.

5. CONTINUOUS DEPENDENCE RESULTS

An important aspect of a model equation for waves where singularities in the flow are not expected to develop is the solution's continuous dependence on the prescribed data. Such a property is crucual if laboratory measurements are to be compared to numerical approximations of solutions of the model. The following theorem shows that the present model has a satisfactory property of continuous dependence of its solutions on the data. When combined with Theorem 1, this continuous dependence result confirms that the model is well-posed in the classical sense.

THEOREM 2. *The mapping* $U: (\nu, f, g, h) \mapsto u$ *that maps* $\nu \geqslant 0$ *and the data for* (1.1b) *into the corresponding solution of* (1.1a) *is continuous from* $\mathbb{R}^+ \times C^2(0, 1) \times C^1(0, T) \times C^1(0, T)$ *into the solution Banach space* $\mathscr{C}_T^{2,1}$.

Remarks. Let $\mathscr{X}$ denote the product $C^2(0, 1) \times C^1(0, T) \times C^1(0, T)$. Then $\mathbb{R} \times \mathscr{X}$ is given the product Banach-space structure with

$$\|(\nu, f, g, h)\|_{\mathbb{R}\times\mathscr{X}} = |\nu| + \|f\|_{C^2(0,1)} + \|g\|_{C^1(0,T)} + \|h\|_{C^1(0,T)}.$$

The space $\mathscr{C}_T^{2,1}$ is defined in Section 2. The proof given below actually shows that U is a local Lipschitz mapping between $\mathbb{R} \times \mathscr{X}$ and $\mathscr{C}_T^{2,1}$. Note that continuous dependence of the solutions of (1.1) on the parameter α may be established by the methods appearing in this section, provided α varies in the range $\{\alpha > 0\}$. The nondispersive limit $\alpha \to 0$ is more complicated.

Proof. Let $(\nu_i, f_i, g_i, h_i) \in \mathbb{R}^+ \times \mathscr{X}$, for $i = 1, 2$. It is to be shown that $U(\nu_1, f_1, g_1, h_1) - U(\nu_2, f_2, g_2, h_2)$ is small in $\mathscr{C}_T^{2,1}$ provided that (ν_1, f_1, g_1, h_1) is close to (ν_2, f_2, g_2, h_2) in $\mathbb{R} \times \mathscr{X}$. By the triangle inequality, it is enough to show that $U(\nu_1, f_1, g_1, h_1) - U(\nu_1, f_2, g_2, h_2)$ and $U(\nu_1, f_2, g_2, h_2) - U(\nu_2, f_2, g_2, h_2)$ are both small in $\mathscr{C}_T^{2,1}$. That is, variations of ν in $\mathbb{R}^+$ and of (f, g, h) in $\mathscr{X}$ may be considered separately. (Indeed, variations in each piece of data could be considered separatelly, but it does not simplify matters to break up variations in f, g, and h.)

First, variations of f, g, h, in $\mathscr{X}$ are considered. Let $\nu = \nu_1$ and let $u_i = U(\nu, f_i, g_i, h_i)$ for $i = 1, 2$. Let $w = u_1 - u_2$. Then w satisfies the variable-coefficient nonhomogeneous initial- and boundary-value problem

$$w_t + w_x + ww_x + (u_2 w)_x - \nu w_{xx} - \alpha^2 w_{xxt} = 0 \qquad \text{for } 0 \leqslant x \leqslant 1,\ 0 \leqslant t \leqslant T \tag{5.1a}$$

with

$$\begin{aligned} &w(x, 0) = f(x) && \text{for } 0 \leqslant x \leqslant 1, \\ &w(0, t) = h(t) \quad \text{and} \quad w(1, t) = g(t) && \text{for } 0 \leqslant t \leqslant T. \end{aligned} \tag{5.1b}$$

Here $f = f_1 - f_2$, $g = g_1 - g_2$, and $h = h_1 - h_2$. It follows that w satisfies an integral equation, analogous to (3.9),

$$\begin{aligned} w(x, t) &= \exp(-\nu t/\alpha^2) f(x) + \phi_1(x)\tilde{h}(t) + \phi_2(x)\tilde{g}(t) \\ &+ \int_0^t \int_0^1 \exp(-\nu(t-\tau)/\alpha^2) K(x, \xi) [w(\xi, \tau) + \tfrac{1}{2}w^2(\xi, \tau) + u_2(\xi, \tau) w(\xi, \tau)] \, d\xi \, d\tau \\ &+ \frac{\nu}{\alpha^2} \int_0^t \int_0^1 \exp(-\nu(t-\tau)/\alpha^2) P(x, \xi) w(\xi, \tau) \, d\xi \, d\tau, \end{aligned} \tag{5.2}$$

where $\tilde{h}(t) = h(t) - \exp(-\nu t/\alpha^2)\, h(0)$ and $\tilde{g}(t) = g(t) - \exp(-\nu t/\alpha^2)\, g(0)$. Suppose now that

$$\|(f, g, h)\|_{\mathscr{X}} = \|f\|_{C^2(0,1)} + \|g\|_{C^1(0,T)} + \|h\|_{C^1(0,T)} \leqslant \epsilon, \tag{5.3}$$

where $\epsilon > 0$ and, without loss of generality, $\epsilon \leqslant 1$.

It will first be shown that if ϵ is small, then w is small in $C(0, T; H^1(0, 1))$. This preliminary fact leads quickly to the desired result. Toward this end, again let $v(x, t) = xg(t) + (1 - x)\, h(t)$ and let $\eta = w - v$. Then η satisfies

$$\eta_t + \eta_x + (u_2\eta)_x + \eta\eta_x - \nu\eta_{xx} - \alpha^2\eta_{xxt} + (v\eta)_x = \xi, \tag{5.4}$$

and

$$\begin{aligned} &\eta(x, 0) = f(x) - v(x, 0) \qquad &&\text{for } 0 \leqslant x \leqslant 1, \\ &\eta(0, t) = \eta(1, t) = 0, &&\text{for } 0 \leqslant t \leqslant T. \end{aligned} \tag{5.5}$$

Here

$$\xi(x, t) = -(v_t + v_x + (u_2 v)_x + v v_x). \tag{5.6}$$

If (5.4) is multiplied by η and the resulting identity integrated over $[0, 1]$, then, after integrations by parts, the following relation emerges.

$$\frac{1}{2}\frac{d}{dt}\int_0^1 [\eta^2 + \alpha^2\eta_x^2]\, dx + \nu\int_0^1 \eta_x^2\, dx = -\tfrac{1}{2}\int (v + u_2)_x\, \eta^2\, dx + \int_0^1 \eta\xi\, dx. \tag{5.7}$$

Now $v_x = g - h$, and, from Theorem 1, it is known that u_2 lies in $\mathscr{C}_T^{2,1}$. Because $\epsilon \leqslant 1$,

$$\sup_{\substack{0 \leqslant x \leqslant 1 \\ 0 \leqslant t \leqslant T}} \tfrac{1}{2}\, |(v + u_2)_x| \leqslant 1 + \tfrac{1}{2}b = M,$$

where the notation

$$b = \|u_2\|_{\mathscr{C}_T^{2,1}}$$

has been introduced. An explicit calculation of ξ, in terms of g and h, and a straightforward use of (5.3) leads to the inequality

$$\|\xi\| \leqslant \epsilon[5 + 4b] = \epsilon N.$$

Making use of the last two inequalities, and of the auxiliary norm introduced above (4.5), in (5.7) yields

$$\frac{1}{2}\frac{d}{dt}|||\eta(\cdot, t)|||_1^2 \leqslant M\, |||\eta(\cdot, t)|||_1^2 + \epsilon N\, |||\eta(\cdot, t)|||_1 .$$

It follows that

$$|||\,\eta(\cdot, t)|||_1 \leqslant |||\,\eta(\cdot, 0)|||_1\, e^{Mt} + \epsilon N M^{-1}(e^{Mt} - 1),$$

for $0 \leqslant t \leqslant T$. Using (5.3) again, and the definition of v, one checks that, for any t in $[0, T]$,

$$|||\, v(\cdot, t)|||_1 \leqslant \epsilon[2 + 2\alpha] = \epsilon L.$$

Thus, for $0 \leqslant t \leqslant T$,

$$\|\, w(\cdot, t)\|_1 \leqslant c'(\alpha)\, e^{MT}\, \| f \|_1 + c(\alpha)\, \epsilon[L + NM^{-1}(e^{MT} - 1) + Le^{MT}],$$

where the equivalence of norms expressed in (4.5) has been used once again and where $c'(\alpha) = \max\{\alpha, \alpha^{-1}\}$. Since $\| f \|_{C^2} \leqslant \epsilon$, $\| f \|_1 \leqslant 2^{1/2}\epsilon$, and so for $0 \leqslant t \leqslant T$,

$$\|\, w(\cdot, t)\|_1 \leqslant \epsilon Q,$$

where

$$Q = c'(\alpha)\, 2^{1/2} \exp(MT) + c(\alpha)\,[L(1 + \exp(MT)) + NM^{-1}(\exp(MT) - 1)].$$

Note that Q is not dependent on ϵ in $(0, 1]$. This shows that w can be made small in $C(0, T; H^1)$ by making ϵ small, that is, by keeping the two sets of data close together in $\mathscr{X}$. Hence from (2.2), w can be made small in $\mathscr{C}_T = C(0, T; C^0)$ by making ϵ small, and in fact

$$\|\, w \|_{\mathscr{C}_T} \leqslant \epsilon Q_*\,, \tag{5.8}$$

where $Q_* = c_* Q$.

Once this latter fact is appreciated, the integral equation (5.2) may be used to show that w may be made small in $\mathscr{C}_T^{2,1}$ by making ϵ small. For example, from (5.2),

$$w_t = \phi_1 \left(h' + \frac{\nu}{\alpha^2} h\right) + \phi_2 \left(g' + \frac{\nu}{\alpha^2} g\right) + \int_0^1 K(x, \xi) \left(w + \tfrac{1}{2} w^2 + u_2 w\right) d\xi$$
$$+ \frac{\nu}{\alpha^2} \int_0^1 P(x, \xi)\, w\, d\xi - \frac{\nu}{\alpha^2}\, w.$$

It follows that

$$\|\, w_t \|_{\mathscr{C}_T} \leqslant \left(1 + \frac{\nu}{\alpha^2}\right) (\| g \|_{C^1(0,T)} + \| h \|_{C^1(0,T)})$$
$$+ c_1(\alpha) \|\, w \|_{\mathscr{C}_T} (1 + \tfrac{1}{2} \|\, w \|_{\mathscr{C}_T} + \|\, u_2 \|_{\mathscr{C}_T})$$
$$+ \frac{\nu}{\alpha^2} \|\, w \|_{\mathscr{C}_T} + c\,(\alpha, \nu) \|\, w \|_{\mathscr{C}_T}\,,$$

where c_1 and c_2 are defined below (3.13). Hence using (5.8),

$$\| w_t \|_{\mathscr{C}_T} \leqslant \epsilon Q_1 , \tag{5.9}$$

where

$$Q_1 = 2\left(1 + \frac{\nu}{\alpha^2}\right) + c_1(\alpha) Q_* \left(1 + \tfrac{1}{2} Q_* + b\right) + Q_* \left(\frac{\nu}{\alpha^2} + c(\alpha, \nu)\right),$$

is a constant depending on α, T, b and ν.

Using the integral equation again, an expression for w_x may be derived, analogous to (3.19), and a bound on $\| u_x \|_{\mathscr{C}_T}$ of the form ϵQ_2 obtained, using (5.8). Differentiating the expression for w_x with respect to t, and using (5.8) and (5.9), it is seen that $\| w_{xt} \|_{\mathscr{C}_T} \leqslant \epsilon Q_3$. Using the integral equation once more to derive an expression for w_{xx}, and taking account of the information already in hand, it follows that $\| w_{xx} \|_{\mathscr{C}_T} \leqslant \epsilon Q_4$. Here, and below, the constants Q_i depend on α, T, b and ν. Finally, using the differential equation, a similar bound is found for $\| w_{xxt} \|_{\mathscr{C}_T}$. It thus follows that

$$\| u_1 - u_2 \|_{\mathscr{C}_T^{2,1}} \leqslant \epsilon Q . \tag{5.10}$$

A similar, but simpler, argument shows that $U(\nu_1, f_2, g_2, h_2) - U(\nu_2, f_2, g_2, h_2)$ is small in $\mathscr{C}_T^{2,1}$ provided $| \nu_1 - \nu_2 |$ is small. Letting $u = U(\nu_1, f_2, g_2, h_2)$ and $v = U(\nu_2, f_2, g_2, h_2)$, and $y = u - v$, then

$$\begin{aligned} y_t + y_x + y y_x + (vy)_x - \nu_1 y_{xx} - \alpha^2 y_{xxt} & \\ = (\nu_1 - \nu_2)\, v_{xx} \qquad \text{for } 0 \leqslant x \leqslant 1, \quad 0 \leqslant t \leqslant T, & \end{aligned} \tag{5.11}$$

and

$$\begin{aligned} y(x, 0) &= 0, \qquad 0 \leqslant x \leqslant 1, \\ y(0, t) = y(1, t) &= 0, \qquad 0 \leqslant t \leqslant T. \end{aligned} \tag{5.12}$$

Upon multiplying (5.11) by y and integrating over [0, 1], there appears, after several integrations by parts, and using (5.12) to evaluate the boundary terms so obtained,

$$\begin{aligned} \tfrac{1}{2} \frac{d}{dt} \int_0^1 (y^2 + \alpha^2 y_x^2)\, dx + \nu_1 \int_0^1 y_x^2\, dx & \\ = (\nu_1 - \nu_2) \int_0^1 y v_{xx}\, dx + \int_0^1 v y y_x\, dx & \\ \leqslant | \nu_1 - \nu_2 | \, \| v \|_{\mathscr{C}_T^{2,1}} \| y \| + \| v \|_{\mathscr{C}_T^{2,1}} \| y \|_1^2 & \\ \leqslant b_1 (| \nu_1 - \nu_2 | \, \| y \|_1 + \| y \|_1^2), & \end{aligned}$$

where $b_1 = \| v \|_{\mathscr{C}_T^{2,1}}$. Since $y(x, 0) \equiv 0$, it follows that

$$||| y(\cdot, t) |||_1 \leqslant | \nu_1 - \nu_2 | (\exp(b_1 c^2(\alpha) t) - 1)/c(\alpha)$$

for $0 \leqslant t \leqslant T$. This shows that

$$\| y \|_{\mathscr{C}_T} \leqslant | \nu_1 - \nu_2 | Q_6 , \tag{5.13}$$

where Q_6 depends on α, ν_i, T and on b_1. Now use of an integral equation derived for (5.11), in analogy with (3.9) and (5.2), allows (5.13) to be extended to a bound of the form

$$\| y \|_{\mathscr{C}_T^{2,1}} \leqslant | \nu_1 - \nu_2 | Q_7 . \tag{5.14}$$

Combining (5.10) and (5.14), it follows that

$$\| U(\nu_1, f_1, g_1, h_1) - U(\nu_2, f_2, g_2, h_2) \|_{\mathscr{C}_T^{2,1}} \leqslant \epsilon Q_5 + | \nu_1 - \nu_2 | Q_7 \tag{5.15}$$

and the desired continuous dependence result is established.

As a corollary of the method outlined above in the proof of Theorem 2, a more general continuous dependence result emerges.

COROLLARY. *The mapping* $U: (\nu, f, g, h) \mapsto u$ *that maps* $\nu \geqslant 0$ *and the data for* (1.1b) *into the corresponding solution of* (1.1a) *is continuous from* $\mathbf{R}^+ \times C^l(0, 1) \times C^k(0, T) \times C^k(0, T)$ *into* $\mathscr{C}_T^{l,k}$, *where* $l \geqslant 2$ *and* $k \geqslant 1$.

6. Conclusion

A model for the propagation of unidirectional small-amplitude long waves, which accounts for the small effects of nonlinearity, dispersion and dissipation, has been confirmed to have a satisfactory mathematical theory of existence and uniqueness of solutions. Moreover, the model was shown to be robust in that small changes in the presented data, or in the dissipative or dispersive parameters, ν and α, lead only to small changes in the corresponding solutions of the model. The continuous dependence of solutions on ν is especially pleasing, since this parameter may not be precisely known in an experimental situation.

It should be emphasized that solutions exist on unrestricted time intervals. For waves in the regime where it is hoped such a model could apply, it is not anticipated that singularities will develop in the flow. Consequently, a theorem of temporally local existence of solutions would have rather less interest. The global existence theorem proved in Section 4 is a consequence of the *a priori* bound derived in Proposition 3. If the data are restricted in size, corresponding to the ranges where the model may be expected to apply, and if the right-hand boundary condition is $g(t) \equiv 0$, better bounds on solutions may be obtained. These may be relevant when a practical numerical scheme for the model is proposed and analyzed. Such results are best worked out in the context of a particular application of the model, and will be exposed in [4], where the model will be tested against data collected in laboratory experiments on surface water waves.

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