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SIAM J. MATH. ANAL.
Vol. 14, No. 6, November 1983

00036-1410/83/1406-0003 $01.25/0

THE KORTEWEG–DE VRIES EQUATION, POSED IN A QUARTER-PLANE*

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Abstract. An initial- and boundary-value problem for the Korteweg–de Vries equation is shown to be well-posed. The considered problem may serve as a model for unidirectional propagation of plane waves generated by a wavemaker in a uniform medium. Such models apply in regimes in which nonlinear and dispersive effects are of comparable small order.

AMS-MOS subject classification (1980). Primary 35B45, 35B65, 35C15, 35Q20, 45G10, 76B15

Key words. Korteweg–de Vries equation, surface water waves, existence, uniqueness and regularity, initial- and boundary-value problems, waves generated by a wavemaker

1. Introduction. The Korteweg–de Vries equation, originally suggested in connection with a certain regime of surface water waves, has been derived as a model for unidirectional propagation of small-amplitude long waves in a number of physical systems. Because of the range of its potential application, and because of its very interesting mathematical properties, this equation has been the object of prolific study in the last few years. These studies have generally concentrated on aspects of the pure initial-value problem,

$$u_t+u_x+uu_x+u_{xxx}=0, \tag{1.1}$$

$$u(x,0)=f(x), \tag{1.2}$$

for $x\in\mathbf{R}$ and $t\geq 0$, say. Equation (1.1) is a version of the Korteweg–de Vries equation in which the dependent and independent variables are nondimensional, but unscaled. The initial data f in (1.2) typically decays to zero at infinity, or is taken to be a periodic function, though these do not exhaust the theory thus far existent (cf. Bona and Schonbek [7] and Menikoff [20]). For comprehensive descriptions of results pertaining to the KdV equation, as (1.1) will be named subsequently, the reader may consult the review articles of Benjamin [3], Jeffrey and Kakutani [14], Lax [17], Miura [21], [22] and Scott, Chu and McLaughlin [24].

The applicability of the KdV equation in a particular context depends on many factors. Among the more universal of these is that the waves be unidirectional and essentially one-dimensional in character. It must generally be the case that, at least locally, the nonlinear and dispersive terms, uu_x and u_{xxx}, respectively, represent small corrections to the basic one-way propagator $u_t+u_x=0$ (cf. [4, §2]). In attempting to assess the performance of the KdV equation as a model for waves in a particular system, the pure initial-value problem may not be particularly convenient. There might be difficulty associated with determining the entire wave profile accurately at a given instant of time. One method of obtaining unidirectional waves to test the appurtenance of KdV is to generate waves at one end of a homogeneous stretch of the medium in question and to allow them to propagate into the initially undisturbed medium beyond the wavemaker. (Figure 1 shows an instance of this situation in the case of surface waves in a channel. For this system x is proportional to distance along the channel, t is

*Received by the editors August 15, 1981.
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proportional to elapsed time and the dependent variable η is the deviation of the liquid's surface from its equilibrium position at the point x at time t. Here the dependent variable has been denoted η since u is usually reserved for a velocity in fluid flow contexts.) During the time when the waves propagate freely, it may be expected that KdV can apply. Of course any real medium will have finite extent, and once the waves have been influenced by another boundary the experiment should cease, as far as KdV is concerned. In such an experiment it may be comparatively easy to measure the passage of the generated waves at a fixed location at or away from the wavemaker. If this is the case, the generated waves can be determined, at or near the wavemaker, and at another station further away from the wavemaker. One could imagine using the measurement nearest the wavemaker as data for the KdV equation. It may then be possible to predict, perhaps numerically, the behavior of the waves further from the wavemaker on the basis of the KdV equation, and to compare the prediction with the measurements made well away from the wavemaker.

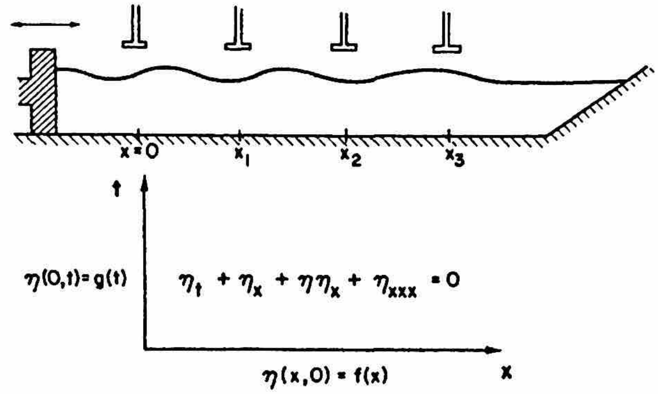

FIG. 1. *Sketch of the experimental configuration and the proposed mathematical model.*

The major accomplishment of the theory presented here is the demonstration that the program, just described, can, in principle, be carried out. Let us agree to fix the zero of the spatial coordinate x, which is along the direction of propagation, at the station nearest the wavemaker where a measurement is to be taken. Then the mathematical problem that accompanies the above discussion is expressed as the following initial- and boundary-value problem (cf. again Fig. 1).

$$\begin{aligned} &u_t+u_x+uu_x+u_{xxx}=0 && \text{for } x,t\geq 0,\\ &u(x,0)=f(x) && \text{for } x\geq 0,\\ &u(0,t)=g(t) && \text{for } t\geq 0. \end{aligned} \tag{1.3}$$

According to the above general discussion, it could be warranted to take $f\equiv 0$ and to assume that g, which is determined experimentally, is consistent with small-amplitude long-wavelength waves. These assumptions will play no role in the theory developed here.

All that will be required is that f and g exhibit smoothness, which is entirely appropriate to the use of KdV as a model equation, and that f decay to zero at infinity appropriately. The smoothness requirement extends to the origin, and results in a certain compatibility that must be satisfied between f and g. These conditions will be spelled out presently.

The same initial- and boundary-value problem has been analyzed for the alternative equation, proposed by Peregrine [23] and Benjamin et al. [4],

$$u_t+u_x+uu_x-u_{xxt}=0, \tag{1.4}$$

in [5]. Results related to those established in the latter reference will be derived and used in the attack on (1.3). The connection between KdV and (1.4) is a regularized version of problem (1.3), namely,

$$\begin{aligned} &u_t+u_x+uu_x+u_{xxx}-\varepsilon u_{xxt}=0 && \text{for } x,t\geq 0,\\ &u(x,0)=f(x) && \text{for } x\geq 0,\\ &u(0,t)=g(t) && \text{for } t\geq 0, \end{aligned} \tag{1.5}$$

where $\varepsilon>0$. The regularized problem (1.5) intervenes in a substantial way in the existence theory for (1.3) developed here. The regularized differential equation appearing in (1.5) is the same tool used already in [7] and [8] in discussions of various pure initial-value problems for KdV. The general outline of the theory herein is patterned after that developed in [8]. The technical difficulties presented by the nonhomogeneous boundary condition $u(0,t)=g(t)$, for $t\geq 0$, require a more delicate analysis than that effected in the last-quoted reference.

The present theory may be considered an extension of the earlier work of Ton [27] and Bona and Heard [6]. Ton's paper undertook the study of the problem,

$$\begin{aligned} &u_t+uu_x\pm u_{xxx}=0, && x,t>0,\\ &u(x,0)=f(x), && x\geq 0,\\ &u(0,t)=0, && t\geq 0. \end{aligned} \tag{1.6}$$

If the minus sign appears in front of the dispersive term, then the extra boundary condition $u_x(0,t)=0$, for $t>0$, is appended. For problem (1.6), with the positive sign taken, the methods exemplified in Lions' text [18], combined with the regularization used by Temam [26] in an early paper on the periodic initial-value problem for KdV, are used to obtain global existence of weak solutions and local existence of classical solutions. (The interval of existence is proportional to the inverse of $\|f\|_6$, in the notation to be introduced in §2.)

Actually, problem (1.6) is not an appropriate model for water waves in a uniform channel, as is suggested in [27]. For the differential equation in (1.6) is written in travelling coordinates, and consequently the boundary condition, if it is to correspond to observations of the disturbance at a fixed position in the channel, should be applied, not at $(0,t)$, for $t\geq 0$, but rather at $(-t,t)$, for $t\geq 0$. This awkwardness is easily obfuscated by the inclusion of the extra linear term u_x in the differential equation, an addition without serious consequence as regards Ton's mathematical proofs. A more serious objection to the theory developed in [27] is that the homogeneous boundary condition $u(0,t)=0$, for $t\geq 0$, is not well-suited to model waves generated by a wavemaker at one end of a uniform stretch of medium, as already explained. Moreover, for problems of long-wave propagation, it is not anticipated that the flow will develop singularities, and consequently it is expected that the model equation should have a global theory of classical solutions, corresponding to suitably smooth data. These drawbacks in the earlier theory are here shown to be methodological, and not inherently a property of the model equation.

In [6], a local existence theory for (1.3) is developed, using the methods of Kato [16]. The boundary data is required to be mildly smooth, but otherwise arbitrary. For technical reasons, this theory has not, thus far, yielded solutions defined globally in time.

It is worth drawing attention to several comparisons which have been made with experimentally obtained data, pertaining to the originally conceived application of the KdV equation to small-amplitude surface water waves. We cite the studies of Zabusky and Galvin [31] and Hammack and Segur [13], and of Hammack [12] using (1.4). These studies all used pure initial-value problems for their theoretical predictions, even though the experimental configuration was exactly as described earlier in justifying the further study of the initial- and boundary-value problem considered here. That is, a uniform channel of water, initially at rest, had waves generated at one end by a wavemaker. The waves propagated down the channel and their passage was recorded at various stations along the channel. Entailed in each of these studies was a transformation of data measured over time, at a fixed location, to data measured spatially at a fixed instant of time. The approximate transformations used in the above-quoted studies introduce errors, which can be analyzed. In fact, the forthcoming work [10] addresses this issue in some detail, and consequently it is not taken up here, except to report that quite significant errors, particularly as regards the phase speed, can be expected when using the approach of converting the boundary-value problem to a pure initial-value problem.

It is also worth noting that, at least for surface water waves, damping effects need to be considered. Such effects were introduced, in an ad hoc way, in [12] and [13], and more systematically in [10]. An additional term that models the damping due to the boundary layers on the bottom and sides of a uniform channel of shallow water, at the level of approximation entailed in the KdV equation, has been derived carefully by Kakutani and Matsuuchi [15]. The incorporation of such dissipative terms in the initial- and boundary-value problem (1.3) is under study, but will not be addressed here.

The paper is organized as follows. Section two sets out the notation and terminology to be used subsequently and presents a sample of the main results in the paper. In §3 the regularized problem (1.5) is considered, and is shown to admit a satisfactory theory when ε is fixed and positive. A priori ε-independent bounds for solutions of the regularized problem are derived in §§4 and 5. Passage to the limit $\varepsilon \downarrow 0$ is effected in §6, where smooth solutions of the initial- and boundary-value problem (1.3) are shown to exist. The paper concludes with some commentary concerning aspects not covered in the present study.

2. Preliminaries and statement of the main result. For an arbitrary Banach space X, the associated norm will be denoted $\|\cdot\|_X$. The following spaces will intervene in the subsequent analysis.

If Ω is a bounded domain in $\mathbf{R}^n$, then $C^j(\bar{\Omega})$ denotes the space of real-valued functions which have classical derivatives up to order j in Ω, and whose derivatives, up to order j, extend to a continuous function on $\bar{\Omega}$. If $j=0$, $C^0(\bar{\Omega})$ will be denoted simply $C(\bar{\Omega})$. The norm on $C(\bar{\Omega})$ is

$$\|f\|_{C(\bar{\Omega})}=\sup_{x\in\Omega}|f(x)|,$$

and the norm on $C^j(\bar{\Omega})$ is

$$\|f\|_{C^j(\bar{\Omega})}=\sum_{|\alpha|\leq j}\|\partial^\alpha f\|_{C(\bar{\Omega})}, \tag{2.1}$$

where $\alpha=(\alpha_1,\cdots,\alpha_n)$ is a multi-index of nonnegative integers, $|\alpha|=\alpha_1+\cdots+\alpha_n$, and

$$\partial^\alpha f(x)=\frac{\partial^{|\alpha|}f(x)}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}.$$

The notation ∂_x^r for $\partial^r/\partial x^r$ and ∂_t^r for $\partial^r/\partial t^r$ will be employed throughout when it is convenient. If Ω is unbounded, $C_b^j(\overline{\Omega})$ is defined exactly as in the case that Ω is bounded except that the function and its derivatives are required to be bounded. The norm is again defined by (2.1).

The space $C^\infty(\overline{\Omega}) = \cap_j C^j(\overline{\Omega})$ will be used, but its usual Fréchet-space topology will not be needed. $\mathcal{D}(\Omega)$ is the subspace of $C^\infty(\overline{\Omega})$ of functions with compact support in Ω. Its dual space, $\mathcal{D}'(\Omega)$, is the space of Schwartz distributions on Ω.

If Ω is open in $\mathbf{R}^n$, then $C^j(\Omega)$ is the continuous real-valued functions defined on Ω and possessing classical derivatives up to order j which are continuous on Ω. No restrictions are placed on the behavior of the functions near the boundary of Ω. This class can also be given a natural Fréchet-space topology, but this topology will not figure in the developments here. Naturally, $C^\infty(\Omega) = \cap_j C^j(\Omega)$.

If $T > 0$, we will systematically use the abbreviation $C(0, T)$ for $C([0, T])$. Similarly, $C^m(0, T)$ will stand for $C^m([0, T])$.

For any real p in the range $[1, \infty)$, $L^p(\Omega)$ denotes the collection of real-valued Lebesgue measurable pth-power absolutely integrable functions defined on Ω. As usual, $L^\infty(\Omega)$ denotes the essentially bounded real-valued functions defined on Ω. These spaces get their usual norms,

$$\|f\|_{L^p(\Omega)} = \left\{ \int_\Omega |f(x)|^p dx \right\}^{1/p},$$

for $1 \leq p < \infty$, and

$$\|f\|_{L^\infty(\Omega)} = \underset{x \in \Omega}{\text{essential supremum}} |f(x)|.$$

If $1 \leq p \leq \infty$, and $m \geq 0$ is an integer, let $W^{m,p}(\Omega)$ be the Sobolev space of $L^p(\Omega)$-functions whose distributional derivatives up to order m also lie in $L^p(\Omega)$. The norm on $W^{m,p}(\Omega)$ is

$$\|f\|^p_{W^{m,p}(\Omega)} = \sum_{|\alpha| \leq m} \|\partial^\alpha f\|^p_{L^p(\Omega)}.$$

When $p = 2$, $W^{m,p}(\Omega)$ will be denoted $H^m(\Omega)$. This is a Hilbert space, and $H^0(\Omega) = L^2(\Omega)$. For $s > 0$, not necessarily an integer, $H^s(\Omega)$ is defined by interpolation. For $s > 0$, $H_0^s(\Omega)$ is the closure in $H^s(\Omega)$ of $\mathcal{D}(\Omega)$. For $s > 0$, $H^{-s}(\Omega)$ is the dual of $H_0^s(\Omega)$ with respect to the pairing which is the extension by continuity of the usual $L^2(\Omega)$-inner product. The noninteger-order Sobolev spaces only intrude at one point in our analysis, and then only in the interest of sharpness. Details concerning these spaces may be found in Lions and Magenes' work [19] or in Stein's text [25], for example. The notation $H^\infty(\Omega) = \cap_j H^j(\Omega)$ will be used for the C^∞-functions on Ω, all of whose derivatives lie in $L^2(\Omega)$.

Finally, $H^s_{\text{loc}}(\Omega)$ is the set of real-valued functions f defined on Ω such that, for each $\varphi \in \mathcal{D}(\Omega)$, $\varphi f \in H^s(\Omega)$. This space is equipped with the weakest topology such that all of the mappings $f \to \varphi f$, for $\varphi \in \mathcal{D}(\Omega)$, are continuous from $H^s_{\text{loc}}(\Omega)$ into $H^s(\Omega)$. With this topology, $H^s_{\text{loc}}(\Omega)$ is a Fréchet space (cf. Treves [28]). Let $\mathbf{R}^+$ denote the positive real numbers, $(0, \infty)$. A simple but pertinent example of the localized Sobolev spaces is $H^s_{\text{loc}}(\mathbf{R}^+)$. Interpreting the foregoing definitions in this special case, $g \in H^s_{\text{loc}}(\mathbf{R}^+)$ if and only if $g \in H^s(0, T)$, for all finite $T > 0$. Moreover, $g_n \to g$ in $H^s_{\text{loc}}(\mathbf{R}^+)$ if and only if $g_n \to g$ in $H^s(0, T)$, for each $T > 0$. Here and below, the abbreviation $H^s(0, T)$ has been used for $H^s((0, T))$.

In the analysis of the quarter-plane problem (1.3), the spaces $H^s(\Omega)$ will occur often, with s a positive integer and $\Omega=\mathbf{R}^+$ or $\Omega=(0,T)$. Because of their frequent occurrence, it is convenient to abbreviate their norms. Thus let

(2.2a) $$\|\cdot\|_s=\|\cdot\|_{H^s(\mathbf{R}^+)} \quad\text{and}\quad |\cdot|_{s,T}=\|\cdot\|_{H^s(0,T)}.$$

If $s=0$, the subscript s will be omitted altogether. So

(2.2b) $$\|\cdot\|=\|\cdot\|_{L^2(\mathbf{R}^+)} \quad\text{and}\quad |\cdot|_T=|\cdot|_{0,T}.$$

Some special cases of the Sobolev embedding theorems will be used occasionally and are worth recalling here. Let I be an open interval on the real line, not necessarily bounded. If $s>1/2+m$, where m is a nonnegative integer, then

(2.3) $$H^s(I)\subset C_b^m(\bar{I}),$$

algebraically, and continuously with respect to the norms on these two spaces. (More precisely, an element in $H^s(I)$ is, after possible modification on a set of Lebesgue measure zero, a C^m-function on I, all of whose derivatives up to order m are uniformly continuous on I, and so may be extended to $\bar{I}$.) In the special case where $I=\mathbf{R}^+$ and $s=k$, a positive integer, it is also useful to recall that if $f\in H^k(\mathbf{R}^+)$, then,

(2.4) $$f(x),f'(x),\cdots,f^{(k-1)}(x)\to 0 \quad \text{as } x\to+\infty.$$

An inequality that will find use is the following, valid for $f\in H^1(\mathbf{R}^+)$. According to (2.3), such a function is bounded and continuous on $\mathbf{R}^+$, and furthermore,

(2.5) $$\|f\|_{C_b(\mathbf{R}^+)}\le\sqrt{2}\,(\|f\|\|f'\|)^{1/2}.$$

This inequality, which is sharp in fact, follows from the observation that, for any $y\in\mathbf{R}^+$, and $f\in H^1(\mathbf{R}^+)$,

$$\begin{aligned} f^2(y)&=-2\int_y^\infty f(x)f'(x)\,dx\le 2\left\{\int_y^\infty f^2(x)\,dx\cdot\int_y^\infty [f'(x)]^2dx\right\}^{1/2} \\ &\le 2\|f\|\|f'\|. \end{aligned}$$

Spaces will be needed to describe the evolution in time of the spatial structure. If X is a Banach space, $1\le p\le\infty$, and $-\infty\le a<b\le\infty$, then $L^p(a,b;X)$ denotes the space of measurable functions $u:(a,b)\to X$ whose norms are pth-power integrable (essentially bounded, if $p=\infty$). These are Banach spaces in their own right, with the norms

$$\|u\|_{L^p(a,b;X)}=\left\{\int_a^b\|u(t)\|_X^p\,dt\right\}^{1/p} \quad \text{for } p<\infty,$$

and

$$\|u\|_{L^\infty(a,b;X)}=\underset{t\in(a,b)}{\text{essential supremum}}\,\{\|u(t)\|_X\}.$$

The subspace of $L^\infty(a,b;X)$ of continuous and bounded functions $u:[a,b]\to X$ is denoted $C_b(a,b;X)$. (In case a and b are both finite, the subscript b, for "bounded", is dropped.)

These spaces all possess localized versions. The only one appearing here is the space $L^\infty_{\text{loc}}(\bar{\mathbf{R}}^+;X)$ of measurable maps $u:\bar{\mathbf{R}}^+\to X$ which are essentially bounded on any compact subset of $\bar{\mathbf{R}}^+$.

Finally, if X is still an arbitrary Banach space, we may consider the X-valued distributions $\mathcal{D}'(a,b;X)$ on the interval (a,b). Formally, $\mathcal{D}'(a,b;X)$ is the set of linear and continuous maps of $\mathfrak{D}(a,b)$ into X. If $T\in\mathfrak{D}'(a,b;X)$, its distributional derivative is defined by

$$\frac{dT}{dt}(\varphi)=-T(\varphi'),$$

for $\varphi\in\mathfrak{D}(a,b)$. Thus, if $f\in L^p(a,b;X)$, then f may be viewed as an X-valued distribution via the definition

$$f(\varphi)=\int_a^b f(t)\varphi(t)\,dt,$$

for $\varphi\in\mathfrak{D}(a,b)$. The integral is, of course, X-valued, and converges since φ has compact support. Thus, "temporal" derivatives of $L^p(a,b;X)$-functions may always be defined, at least in the distributional sense. There is a considerable theory pertaining to when distributional derivatives are in fact classically defined. Some of these results will be called upon later. Specific uses of this theory will be referenced precisely, but the reader may consult [18], [19], [25] or [28] for general commentary concerning such issues.

The following is a special case of the main result of this paper. It serves simultaneously to give orientation and define the goals of the paper.

THEOREM. *Consider the initial- and boundary-value problem* (1.3) *and suppose that the data* f,g *has* $f\in H^4(\mathbf{R}^+)$ *and* $g\in H^2_{\mathrm{loc}}(\mathbf{R}^+)$. *Suppose that* f *and* g *satisfy the compatibility conditions,*

$$g(0)=f(0),$$
$$g_t(0)=-\left(f_{xxx}(0)+f(0)f_x(0)+f_x(0)\right).$$

Then there exists a unique solution u *in* $L^\infty_{\mathrm{loc}}(\mathbf{R}^+;H^4(\mathbf{R}^+))$ *of* (1.3) *corresponding to the data* f *and* g.

Remarks. By the term "solution", we will always mean, in the first instance, a solution in the sense of distributions on the quarter-plane. The term *classical solution* is reserved for a function which is continuous and continuously differentiable the requisite number of times, and which satisfies the differential equation pointwise everywhere, and the initial and the boundary conditions pointwise.

Note that since $g\in H^2_{\mathrm{loc}}(\mathbf{R}^+)$, $g\in C^1(0,T)$, for any $T>0$. Also, $f\in H^4(\mathbf{R}^+)$ implies $f\in C_b^3(\overline{\mathbf{R}}^+)$. In consequence, the compatibility conditions are both well-defined. The first compatibility condition simply expresses the continuity of the solution u at the origin. The second condition would necessarily hold for a classical solution.

The theorem above is a part of Theorem 6.2 below. There it will also be established that if $f\in H^{3k+1}(\mathbf{R}^+)$ and $g\in H^{k+1}_{\mathrm{loc}}(\mathbf{R}^+)$, where k is a positive integer, and if corresponding higher order compatibility conditions hold, then the solution u lies in the class $L^\infty_{\mathrm{loc}}(\mathbf{R}^+;H^{3k+1}(\mathbf{R}^+))$. In particular, if $k\geq 2$, it is easily inferred that u is a classical and global solution of the quarter-plane problem for the KdV equation.

3. Theory relating to the regularized problem. In this section, interest will be focused entirely on the regularized initial- and boundary-value problem (1.5), repeated here for convenience.

(3.1a) $$u_t+u_x+uu_x+u_{xxx}-\varepsilon u_{xxt}=0 \quad \text{for } x,t\geq 0,$$

with

(3.1b) $$u(x,0)=f(x) \quad \text{for } x\geq 0,$$
$$u(0,t)=g(t) \quad \text{for } t\geq 0.$$

For consistency, the restriction

(3.2) $$u(0,0)=f(0)=g(0)$$

will be imposed throughout the discussion. For the present, the positive parameter ε will be treated as a fixed constant, in the range $(0,1]$, say. Following the development in [8], let

(3.3) $$v(x,t)=\varepsilon u(\varepsilon^{1/2}(x-t),\varepsilon^{3/2}t).$$

It is immediately verified that u is a smooth solution of (3.1) if and only if v is a smooth solution of the problem

(3.4a) $$v_t+(1+\varepsilon)v_x+vv_x-v_{xxt}=0 \quad \text{in } \bar{\Omega},$$

and

(3.4b) $$\begin{aligned} v(x,0)&=F(x) \quad \text{for } x\geq 0,\\ v(t,t)&=G(t) \quad \text{for } t\geq 0. \end{aligned}$$

Here $\Omega=\{(x,t)\colon t>0 \text{ and } x>t\}$, $F(x)=\varepsilon f(\varepsilon^{1/2}x)$, and $G(t)=\varepsilon g(\varepsilon^{3/2}t)$. The dependence of F and G on ε is suppressed, since ε is viewed as fixed here. Of course (3.2) implies and is implied by

(3.5) $$F(0)=G(0).$$

The initial- and boundary-value problem (3.4) is somewhat peculiar, owing to the domain (a sector of angle $\pi/4$) in which it is posed (cf. Fig. 2).

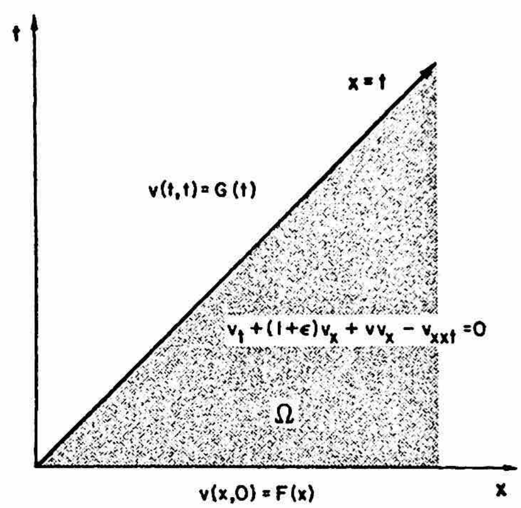

FIG. 2. *The regularized problem, after the change of variables.*

Related initial- and boundary-value problems have been analyzed by passing to an associated integral equation. This method proves to be effective in the present circumstances.

To convert (3.4) into an integral equation, proceed formally as follows. Write (3.4) as

$$v_t-v_{xxt}=-(1+\varepsilon)v_x-vv_x,$$

and, for fixed $x \geq t$, integrate this relation over the temporal interval $(0,t)$. There appears

$$w - w_{xx} = S \quad \text{for } x > t, \tag{3.6}$$

where

$$w(x,t) = v(x,t) - F(x),$$

and

$$S(x,t) = -\int_0^t [(1+\varepsilon)v_x(x,s) + v(x,s)v_x(x,s)]\,ds.$$

The solution of (3.6) may be expressed in the form

$$w(x,t) = \alpha e^{-x} + \frac{1}{2}\int_t^\infty e^{-|x-\xi|}S(\xi,t)\,d\xi, \tag{3.7}$$

by the variation of constants formula. Of course $\alpha = \alpha(t)$, and it has been assumed tacitly that S and w are bounded. If $t \geq 0$, then at $x = t$,

$$G(t) - F(t) = v(t,t) - F(t) = w(t,t) = \alpha(t)e^{-t} + \frac{1}{2}\int_t^\infty e^{-|t-\xi|}S(\xi,t)\,d\xi.$$

Hence,

$$\alpha(t) = e^t\left\{G(t) - F(t) - \frac{1}{2}\int_t^\infty e^{-|t-\xi|}S(\xi,t)\,d\xi\right\}. \tag{3.8}$$

The result of (3.7) and (3.8) is that

$$\begin{aligned} v(x,t) = F(x) + e^{-(x-t)}(G(t) - F(t)) - \frac{1}{2}e^{-(x-t)}\int_t^\infty e^{-|t-\xi|}S(\xi,t)\,d\xi \\ + \frac{1}{2}\int_t^\infty e^{-|x-\xi|}S(\xi,t)\,d\xi. \end{aligned}$$

Since $\xi \geq t$, this simplifies to

$$v(x,t) = F(x) + e^{-(x-t)}(G(t) - F(t)) + \int_t^\infty M(x-t,\xi-t)S(\xi,t)\,d\xi, \tag{3.9}$$

where

$$M(y,z) = \frac{1}{2}\left[\exp(-|y-z|) - \exp(-(y+z))\right]. \tag{3.10}$$

Replacing S by its definition in terms of v, and integrating once by parts, (3.9) may be expressed in the form

$$\begin{aligned} v(x,t) = F(x) + e^{-(x-t)}(G(t) - F(t)) \\ + \int_t^\infty K(x-t,\xi-t)\int_0^t\left[(1+\varepsilon)v(\xi,s) + \frac{1}{2}v^2(\xi,s)\right]ds\,d\xi, \end{aligned} \tag{3.11}$$

where

$$K(y,z) = \frac{1}{2}\left[\exp(-y-z) + \operatorname{sgn}(y-z)\exp(-|y-z|)\right]. \tag{3.12}$$

The boundary term that appears in the integration by parts vanishes because $e^{-|x-\xi|} = e^{-(x+\xi)+2t}$ when $\xi = t$ and $x \geq t$. Notice that $K(0,\xi - t) \equiv 0$, so that $v(t,t) = G(t)$, for all $t \geq 0$. Note also that $v(x,0) = F(x)$, provided the consistency condition (3.5) holds.

Equation (3.11) is the desired integral equation. It has been derived formally, and thus far its relation to solutions of (3.4) is not rigorously established. Our object now is to make a rigorous connection between solutions of the integral equation and solutions of (3.4), and to show that the integral equations possesses solutions, at least for small time intervals.

Turning to the second objective first, let $T>0$ and let $\mathcal{C}_T$ be the Banach space of bounded continuous functions defined on the closure of the set

$$\Omega_T=\{(x,t): t\in(0,T) \text{ and } x>t\}.$$

$\mathcal{C}_T$ is equipped with the supremum norm. Let A denote the operator that maps a function $w\in\mathcal{C}_T$ into the function

$$(3.13)\quad \begin{aligned}(Aw)(x,t)&=F(x)+e^{-(x-t)}(G(t)-F(t))\\ &\quad+\int_t^\infty K(x-t,\xi-t)\int_0^t\left[(1+\varepsilon)w(\xi,s)+\frac{1}{2}w^2(\xi,s)\right]ds\,d\xi,\end{aligned}$$

defined for $(x,t)\in\bar{\Omega}_T$. Because the kernel K is integrable, and assuming that F and G are bounded and continuous, it is plain that $Aw\in\mathcal{C}_T$ also. Existence of a solution of the integral equation (3.11) will be provided by showing that, for T small enough, A is a contraction mapping of a ball centered at the zero function in $\mathcal{C}_T$. The following estimate is the basis on which this assertion is established.

Let u and w be elements of $\mathcal{C}_T$. Consider the difference of their images under the operator A,

$$\begin{aligned}&Au(x,t)-Aw(x,t)\\ &\quad=\int_t^\infty K(x-t,\xi-t)\int_0^t\left[1+\varepsilon+\frac{1}{2}u(\xi,s)+\frac{1}{2}w(\xi,s)\right][u(\xi,s)-w(\xi,s)]\,ds\,d\xi.\end{aligned}$$

For t fixed in the interval $[0,T]$,

$$\begin{aligned}&\sup_{x\geq t}|Au(x,t)-Aw(x,t)|\\ &\quad\leq\sup_{x\geq t}\int_t^\infty|K(x-t,\xi-t)|\,d\xi\sup_{\xi\geq t}\int_0^t\left|1+\varepsilon+\frac{1}{2}u(\xi,s)+\frac{1}{2}w(\xi,s)\right||u(\xi,s)-w(\xi,s)|\,ds.\end{aligned}$$

But, for $x\geq t$,

$$\begin{aligned}\int_t^\infty|K(x-t,\xi-t)|\,d\xi&=\frac{1}{2}\int_t^\infty\left|e^{2t-(x+\xi)}+\operatorname{sgn}(x-\xi)e^{-|x-\xi|}\right|d\xi\\ &=\frac{1}{2}\int_x^\infty\left|e^{2t-(x+\xi)}-e^{x-\xi}\right|d\xi+\frac{1}{2}\int_t^x\left(e^{2t-(x+\xi)}+e^{\xi-x}\right)d\xi\\ &=1-e^{2(t-x)}\leq 1.\end{aligned}$$

Hence, as $0\leq t\leq T$,

$$\begin{aligned}\sup_{x\geq t}|Au(x,t)-Aw(x,t)|&\leq\sup_{\xi\geq t}\int_0^t\left|1+\varepsilon+\frac{1}{2}u(\xi,s)+\frac{1}{2}w(\xi,s)\right||u(\xi,s)-w(\xi,s)|\,ds\\ &\leq T\left[1+\varepsilon+\frac{1}{2}\left(\|u\|_{\mathcal{C}_T}+\|w\|_{\mathcal{C}_T}\right)\right]\|u-w\|_{\mathcal{C}_T}.\end{aligned}$$

It follows that

$$(3.14)\qquad \|Au-Aw\|_{\mathcal{C}_T}=\sup_{(x,t)\in\Omega_T}|Au(x,t)-Aw(x,t)|$$
$$\leq T\left[(1+\varepsilon)+\frac{1}{2}\left(\|u\|_{\mathcal{C}_T}+\|w\|_{\mathcal{C}_T}\right)\right]\|u-w\|_{\mathcal{C}_T}.$$

This inequality implies the desired result. Let $\theta(x,t)\equiv 0$ and set

$$(3.15)\qquad R(T)=2\|A\theta\|_{\mathcal{C}_T}\leq 4\|F\|_{C_b(\overline{\mathbf{R}}^+)}+2\|G\|_{C(0,T)}.$$

Let $B_T=\{w\in\mathcal{C}_T\colon \|w\|_{\mathcal{C}_T}\leq R(T)\}$ and let

$$(3.16)\qquad \Theta(T)=T[1+\varepsilon+R(T)].$$

Then it follows straightforwardly that, for u and w in B_T,

$$\|Au-Aw\|_{\mathcal{C}_T}\leq\Theta(T)\|u-w\|_{\mathcal{C}_T},$$

and

$$\|Au\|_{\mathcal{C}_T}\leq\|Au-A\theta\|_{\mathcal{C}_T}+\|A\theta\|_{\mathcal{C}_T}\leq\Theta(T)\|u\|_{\mathcal{C}_T}+\frac{1}{2}R(T)\leq\left[\Theta(T)+\frac{1}{2}\right]R(T).$$

Because of the last two inequalities, A will be a contraction mapping of B_T if $\Theta(T)\leq\frac{1}{2}$. Referring to (3.16), one appreciates immediately that, for fixed data F and G, this certainly holds for T sufficiently small. In fact, it is worth noting that, essentially because of the inequality in (3.15), for any $M>0$ we may take

$$(3.17)\qquad T=\min\left\{M,\frac{1}{2\left(1+\varepsilon+4\|F\|_{C_b(\overline{\mathbf{R}}^+)}+2\|G\|_{C(0,M)}\right)}\right\},$$

and have $\Theta(T)\leq\frac{1}{2}$. Thus (3.11) has a solution in $\mathcal{C}_T$, for T sufficiently small. This result is summarized formally in the following.

PROPOSITION 3.1. *Let $M>0$, $G\in C(0,M)$ and $F\in C_b(\overline{\mathbf{R}}^+)$ with $F(0)=G(0)$. Then there exists a positive constant*

$$T_0=T_0\left(\|F\|_{C_b(\overline{\mathbf{R}}^+)},\|G\|_{C(0,M)}\right)$$

such that for any T' with $0<T'\leq\min(T_0,M)$, there is a solution of (3.11) *in $\mathcal{C}_{T'}$. Moreover, for any $T\in(0,M]$, there is at most one solution of* (3.11) *in $\mathcal{C}_T$.*

Proof. The question of existence has already been settled. Suppose there are two distinct solutions v and w of (3.11) in $\mathcal{C}_T$. Since v and w are continuous, there is a $t_0\in[0,T)$ such that $v\equiv w$ on $\overline{\Omega}_{t_0}$, and on no domain Ω_t is this still true, if $t>t_0$. Let $U(x,t)=v(x,t)=w(x,t)$, in $\overline{\Omega}_{t_0}$. Define

$$U_0(x,t)=F(x)+e^{-(x-t)}(G(t)-F(t))$$
$$+\int_t^\infty K(x-t,\xi-t)\int_0^{t_0}\left[(1+\varepsilon)U(\xi,s)+\frac{1}{2}U^2(\xi,s)\right]ds\,d\xi,$$

for $(x,t)\in D=\{(x,t)\colon t_0\leq t\leq T \text{ and } x\geq t\}$. Plainly U_0 is bounded and continuous on D. Then the integral equation

$$u(x,t)=U_0(x,t)+\int_t^\infty K(x-t,\xi-t)\int_{t_0}^t\left[(1+\varepsilon)u(\xi,s)+\frac{1}{2}u^2(\xi,s)\right]ds\,d\xi$$
$$=\tilde{A}u(x,t),$$

defined on D, has two distinct solutions, which we denote by v and w again, though they are in fact v and w restricted to D. Moreover, while these two solutions agree at t_0, they do not agree identically in any neighborhood of t_0.

The existence argument presented above is easily adapted to show that, for R large enough and for $t_1 = t_1(R)$ near enough to t_0, $\tilde{A}$ is a contraction mapping of the ball $\tilde{B}_R$ of radius R centered at the zero function in $C_b(D_1)$, where

$$D_1 = \{(x,t): t_0 \le t \le t_1 \text{ and } x \ge t\}.$$

But if

$$R \ge \max\{\|v\|_{\mathcal{C}_T}, \|w\|_{\mathcal{C}_T}\},$$

then $\tilde{A}$ has two distinct fixed points v and w in $\tilde{B}_R$. This contradiction forces the conclusion $v \equiv w$ on Ω_T, and the proposition is established. □

It will be important in subsequent sections to have smooth solutions, up to the boundaries, of the regularized problem (3.1) at our disposal. This amounts to the program of relating solutions of the integral equation (3.11) to solutions of the transformed problem (3.4). The following result will be sufficient for our later needs.

PROPOSITION 3.2. *Suppose that* $F \in C_b^k(\overline{\mathbf{R}}^+)$ *and* $G \in C^m(0,T_0)$, *where* $k \ge 2$, $m \ge 1$, *and* $k \ge m$. *Suppose also* $F(0) = G(0)$. *Let* v *be a solution in* $\mathcal{C}_T$ *of the integral equation* (3.11), *where* $0 < T \le T_0$. *Then*

$$\partial_x^i \partial_t^j v \in \mathcal{C}_T, \quad \text{for } 0 \le j \le m \quad \text{and} \quad 0 \le i \le k+j. \tag{3.18}$$

Moreover, v is a classical solution of the transformed problem (3.4) *in* $\overline{\Omega}_T$. *Conversely, if v lies in $\mathcal{C}_T$ and is a classical solution of* (3.4) *on* $\overline{\Omega}_T$, *then v is a solution of the integral equation* (3.11) *over* $\overline{\Omega}_T$, *and so v satisfies* (3.18).

Remark. The partial derivatives in (3.18) may be defined at the boundary of Ω_T by the obvious one-sided differential quotients. The reader will appreciate that a function v defined on $\overline{\Omega}_T$ does not possess a classically defined partial derivative with respect to t at the point $(0,0)$. In case $j>0$ in (3.18), the condition $\partial_x^i \partial_t^j v \in \mathcal{C}_T$ connotes that this partial derivative exists classically in $\overline{\Omega}_T \setminus \{(0,0)\}$, is bounded and continuous there, and that it may be extended continuously to $\overline{\Omega}_T$.

Proof. First note that if $F \in C_b^k(\overline{\mathbf{R}}^+)$ and $G \in C^m(0,T)$, where $k \ge m$, then

$$v_0(x,t) = F(x) + e^{-x}e^t(G(t) - F(t)) \tag{3.19}$$

has $\partial_x^i \partial_t^j v_0 \in \mathcal{C}_T$, for $0 \le i \le k$ and $0 \le j \le m$. Also, since $v \in \mathcal{C}_T$, then

$$J(x,t) = \int_0^t \left[(1+\varepsilon)v(x,s) + \frac{1}{2}v^2(x,s)\right] ds \tag{3.20}$$

has $J_t \in \mathcal{C}_T$. A short calculation using Leibniz' rule confirms that

$$\begin{aligned} v_t(x,t) &= \partial_t v_0(x,t) - K(x-t,0)J(t,t) \\ &\quad + \int_t^\infty \partial_t[K(x-t,\xi-t)]\, J(\xi,t)\, d\xi + \int_t^\infty K(x-t,\xi-t) J_t(\xi,t)\, d\xi. \end{aligned}$$

Simplifying,

$$\begin{aligned} v_t(x,t) &= \partial_t v_0(x,t) - e^{-(x-t)}J(t,t) \\ &\quad + \int_t^\infty e^{2t-(x+\xi)} J(\xi,t)\, d\xi + \int_t^\infty K(x-t,\xi-t) J_t(\xi,t)\, d\xi. \end{aligned} \tag{3.21}$$

Thus $v_t \in \mathcal{C}_T$.

By dividing the range of spatial integration at $\xi=x$, it is readily seen that $v_x\in\mathcal{C}_T$, and that

$$(3.22)\quad v_x(x,t)=\partial_x v_0(x,t)+K_-(x-t,x-t)J(x,t)-K_+(x-t,x-t)J(x,t) + \int_t^\infty L(x-t,\xi-t)J(\xi,t)\,d\xi,$$

where

$$(3.23)\quad L(y,z)=-\frac{1}{2}\{\exp(-|y-z|)+\exp(-y-z)\},$$
$$K_\pm(x-t,x-t)=\lim_{\xi\to x\pm}K(x-t,\xi-t),$$

and $\xi\to x+$ means $\xi\downarrow x$ while $\xi\to x-$ means $\xi\uparrow x$. Thus it appears that

$$(3.24)\quad v_x(x,t)=\partial_x v_0(x,t)+J(x,t)+\int_t^\infty L(x-t,\xi-t)J(\xi,t)\,d\xi.$$

Since $k\geq 2$, $\partial_x v_0$ may be differentiated with respect to x. Moreover, since $v_x\in\mathcal{C}_T$, $J(x,t)$ may be differentiated with respect to x. And, the integral on the right side of (3.24) may be differentiated with respect to x. Performing the indicated differentiations, we see that

$$(3.25)\quad v_{xx}(x,t)=\partial_x^2 v_0(x,t)+J_x(x,t)+\int_t^\infty K(x-t,\xi-t)J(\xi,t)\,d\xi.$$

This representation shows plainly that $v_{xx}\in\mathcal{C}_T$. Formula (3.25) may be simplified by use of the original integral equation. Thus,

$$\begin{aligned}(3.26)\quad v_{xx}(x,t)&=\partial_x^2 v_0(x,t)+J_x(x,t)+(v(x,t)-v_0(x,t))\\ &=J_x(x,t)+v(x,t)+F_{xx}(x)-F(x)\\ &=\int_0^t[(1+\varepsilon)v_x(x,s)+v(x,s)v_x(x,s)]\,ds+v(x,t)+F_{xx}(x)-F(x).\end{aligned}$$

It is now clear that v_{xx} is differentiable with respect to t, and that

$$v_{xxt}(x,t)=(1+\varepsilon)v_x(x,t)+v(x,t)v_x(x,t)+v_t(x,t).$$

So, if $k\geq 2$ and $m\geq 1$, any solution v in $\mathcal{C}_T$ of the integral equation (3.11) is a classical solution, up to the boundary, of the transformed differential equation (3.4a). As already remarked, a continuous solution of (3.11) has $v(t,t)=G(t)$, for $0\leq t\leq T$, and has $v(x,0)=F(x)$, for $x\geq 0$, provided the consistency condition $F(0)=G(0)$ holds.

Further regularity of a $\mathcal{C}_T$-solution of the integral equation may be established by similar arguments. As this issue is important in our subsequent investigation, a little more detail is warranted.

First, if $m\geq 2$, then since $v_t\in\mathcal{C}_T$, it follows that every term on the right-hand side of (3.21) is differentiable with respect to t. Moreover, each of these derivatives lies in $\mathcal{C}_T$, as is easily verified. So $v_{tt}\in\mathcal{C}_T$. This argument may now be iterated, with the conclusion that $\partial_t^j v\in\mathcal{C}_T$, for $0\leq j\leq m$.

A similar argument, based on (3.26), may be used to show that $\partial_x^i v\in\mathcal{C}_T$, for $0\leq i\leq k$. Specifically,

$$(3.27)\quad \partial_x^{l+2}v(x,t)=\partial_x^l v(x,t)+\partial_x^{l+2}F(x)-\partial_x^l F(x) + \int_0^t\partial_x^{l+1}\left[(1+\varepsilon)v(x,s)+\frac{1}{2}v^2(x,s)\right]ds,$$

for $l=0,1,\cdots,k-2$.

Since $v_t \in \mathcal{C}_T$, it follows from (3.24) that $v_{xt} \in \mathcal{C}_T$ and that

$$v_{xt}(x,t)=\partial_t\partial_x v_0(x,t)+J_t(x,t)+e^{-(x-t)}J(t,t) + \int_t^\infty L(x-t,\xi-t)J_t(\xi,t)\,d\xi-\int_t^\infty e^{2t-(x+\xi)}J(\xi,t)\,d\xi. \tag{3.28}$$

Finally, by using the differential equation, the results already derived, and induction, mixed partial derivatives of the form $\partial_x^i \partial_t^j v$, where $j\geq 1$ and $i\geq 2$, are seen to lie in $\mathcal{C}_T$, provided that $j\leq m$ and $i\leq k+j$.

If, on the other hand, v is a bounded classical solution of the differential equation (3.4a) which satisfies the boundary conditions (3.4b), then necessarily $F(0)=G(0)$ because v is continuous at the origin. Moreover, in this case, each step of the formal construction leading from (3.1) to (3.4) is easily validated. In consequence, v is seen to satisfy (3.11). Hence by the argument just elucidated, pertaining to solutions of the integral equation (3.11), v satisfies the conditions of regularity in (3.18). This concludes the proof of the proposition. □

In our subsequent analysis, it will be convenient to have at our disposal smooth solutions of (3.1) which are not confined to $\overline{\mathbb{R}}^+\times[0,T]$ where T is small. This corresponds to providing smooth solutions of (3.4) on $\overline{\Omega}_{T'}$, where T' is given. It seems natural to iterate the local result propounded in Proposition 1. This will be effective as soon as an a priori bound on the L^∞-norm of a solution defined on $\overline{\Omega}_T$ is provided. More precisely, suppose a classical solution v of (3.4), defined on $\overline{\Omega}_T$ for some $T>0$, is in hand. And suppose the boundary data G is defined at least on $[0,T_0]$, where $T_0>T$. Consider a new initial- and boundary-value problem,

$$w_t+(1+\varepsilon)w_x+ww_x-w_{xxt}=0 \quad \text{for } (x,t) \text{ such that } t\geq T \text{ and } x\geq t, \tag{3.29}$$

with

$$w(x,T)=v(x,T) \quad \text{for } x\geq T,$$
$$w(t,t)=G(t) \quad \text{for } t\geq T.$$

The initial value of w is the terminal value of v. Just as for (3.4), (3.29) may be converted to an integral equation, which in all aspects is similar to (3.11). A solution to this integral equation may be inferred to exist on some domain of the form

$$\{(x,t)\colon T\leq t\leq T+\Delta T \text{ and } x\geq t\}.$$

Provided v and G are smooth enough, the solution w of the integral equation will provide a classical solution of (3.29). In this manner, v is extended to a solution of (3.4) on $\overline{\Omega}_{T+\Delta T}$. As in Proposition 1, a lower bound for the size of ΔT depends on the L^∞-norm of the data in (3.29). Specifically referring to (3.17),

$$\Delta T\geq \min\left\{T_0-T,\ \frac{1}{2\left[1+\varepsilon+4\|v(\cdot,T)\|_{C_b(T,\infty)}+2\|G\|_{C(T,T_0)}\right]}\right\}.$$

Suppose it is known that, for the given data F and G, any solution v of (3.4) defined on $\overline{\Omega}_T$, for some $T\leq T_0$, has the property that

$$\|v\|_{C_b(\overline{\Omega}_T)}\leq C=C(T_0,F,G).$$

Then a lower bound on ΔT can be imputed, and in consequence, after a finite number of steps, the solution may be extended to $\overline{\Omega}_{T_0}$. This conclusion is worth stating formally.

PROPOSITION 3.3. *Let $T_0>0$ be given, and $G\in C^m(0,T_0)$, $F\in C_b^k(\overline{\mathbf{R}}^+)$ with $F(0)=G(0)$, where $k\geq 2$, $m\geq 1$ and $k\geq m$. Suppose there is a constant C, dependent on T_0, F and G, such that for any solution w of* (3.4) *defined on $\overline{\Omega}_T$, where $T\leq T_0$,*

$$\|w\|_{C_b(\overline{\Omega}_T)}\leq C. \tag{3.30}$$

Then there exists a unique solution $v\in\mathcal{C}_{T_0}$ to (3.11), *which is also a classical solution of* (3.4) *and which satisfies the conditions of regularity expressed in* (3.18). *Moreover, v is defined locally as the fixed-point of a contraction mapping of the type in* (3.13), *by iterating the result of Proposition* 3.1 *a finite number of times.*

Provision of the relevant a priori bound is now considered. To this end, the following technical lemma is useful.

LEMMA 3.4. *Let $F\in C_b^k(\overline{\mathbf{R}}^+)$ and $G\in C^m(0,T_0)$ with $F(0)=G(0)$, where $k\geq 2$, $m\geq 1$ and $k\geq m$. Let v be a solution of* (3.4) *in $\mathcal{C}_{T_0}$. Let $0\leq p\leq k$ and suppose that*

$$\partial_x^j F(x)\to 0 \quad \text{as } x\to+\infty, \tag{3.31}$$

for $0\leq j\leq p$. Then

$$\partial_x^j\partial_t^i v(x,t)\to 0 \quad \text{as } x\to+\infty, \tag{3.32}$$

uniformly for $0\leq t\leq T_0$, for i,j such that $0\leq i\leq m$ and $0\leq j\leq p+i$.

Proof. Suppose it is determined that $v(x,t)\to 0$ as $x\to+\infty$, uniformly for $0\leq t\leq T_0$. Since v is a classical solution of (3.4) on $\overline{\Omega}_{T_0}$, it satisfies the integral equation (3.11) on $\overline{\Omega}_{T_0}$. Referring to formula (3.21) for v_t, it is clear that $v_t(x,t)\to 0$ as $x\to+\infty$, uniformly for $0\leq t\leq T_0$. If $m>1$, then upon differentiating (3.21) with respect to t and using the fact that v and v_t tend to 0 at $+\infty$, it is straightforwardly assured that $v_{tt}(x,t)\to 0$ as $x\to\infty$, uniformly for $0\leq t\leq T_0$. Continuing inductively, it follows that

$$\partial_t^i v(x,t)\to 0 \quad \text{as } x\to+\infty,$$

for $0\leq i\leq m$, uniformly for $0\leq t\leq T_0$.

Next, by considering formula (3.22), we see that if $p>0$, then $v_x(x,t)\to 0$ as $x\to+\infty$, uniformly for $0\leq t\leq T_0$. Then from (3.28), $v_{xt}(x,t)\to 0$ as $x\to+\infty$, uniformly for $0\leq t\leq T_0$. From the differential equation (3.4a), it is seen that $v_{xxt}(x,t)\to 0$, as $x\to+\infty$, uniformly for $0\leq t\leq T_0$. Continuing in the pattern of the proof of Proposition 3.2 leads to the conclusion that (3.32) holds.

The above analysis was all predicated on the desired result holding good for v itself. The lemma will therefore be established as soon as it is confirmed that (3.32) holds for $i=j=0$.

For $T>0$, let $\mathcal{C}_T^0$ be the closed subspace of $\mathcal{C}_T$ composed of those elements which converge to 0 at $+\infty$, uniformly for $0\leq t\leq T$. If $F(x)\to 0$, as $x\to+\infty$, then operators of the type exhibited in (3.13) map $\mathcal{C}_T^0$ into itself. Because a solution v of (3.4) is provided in $\mathcal{C}_{T_0}$, the uniqueness result of Proposition 3.1 implies that condition (3.30) holds. So v is obtained locally as a fixed-point of a contraction mapping of the form in (3.13). This fixed-point may be determined by iterating the operator on the zero function θ. The sequence $\{v_n\}_{n=1}^{\infty}$ thus generated ($v_1=A\theta$ and $v_{n+1}=Av_n$, for $n\geq 1$) lies in $\mathcal{C}_T^0$ and converges to v in $\mathcal{C}_T$. Therefore $v\in\mathcal{C}_T^0$. As a finite number of such steps are needed to recover v on $\overline{\Omega}_{T_0}$, it follows that $v\in\mathcal{C}_{T_0}^0$. This concludes the proof of the lemma. □

Attention is now turned fully toward derivation of a priori information concerning smooth solutions of (3.4) which imply (3.30). A bound that will suffice is the subject of the next proposition. The same bound will also be needed in §4. Because of this, it is especially convenient to derive the bound in the context of (3.1). Of course the reader will realize that the theory, thus far developed for (3.4), implies the existence of smooth solutions of the regularized problem (3.1), at least locally in time. This is simply a matter of tracing the inverse of the transformation (3.3) which led from (3.1) to (3.4). The precise result is spelled out in Theorem 3.8. For now, it is simply assumed that a classical solution of (3.1) is in hand.

PROPOSITION 3.5. *Let $f\in C_b^3(\overline{\mathbf{R}}^+)$, $g\in C^1(0,T)$, where $f(0)=g(0)$, and suppose $0<\varepsilon\le 1$. Let u be a classical solution of* (3.1), *up to the boundary, on $\overline{\mathbf{R}}^+\times[0,T]$. Suppose in addition that $f\in H^1(\mathbf{R}^+)$. Then for all $t\in[0,T]$, $u(\cdot,t)\in H^1(\mathbf{R}^+)$. Moreover, there are positive constants a_0 and a_1,*

$$a_0=a_0\left(\|f\|+\varepsilon^{1/2}\|f_x\|,|g|_{1,T}\right)$$

and

$$a_1=a_1\left(\|f\|_1,|g|_{1,T}\right),$$

depending continuously on their arguments, such that

$$\|u(\cdot,t)\|\le a_0 \tag{3.33}$$

and

$$\|u(\cdot,t)\|_1^2+\int_0^t\left[u_x^2(0,s)+\left(u_{xx}(0,s)-\varepsilon u_{xt}(0,s)\right)^2\right]ds\le a_1, \tag{3.34}$$

for $0\le t\le T$. These inequalities hold uniformly for ε in (0, 1].

Remark. While not stated explictly here or later, the various constants that appear in the development of our theory generally depend on T. Besides a direct dependence on T, a_0 and a_1 also depend indirectly on T via the $H^1(0,T)$-norm of g, $|g|_{1,T}$. The reader will quickly perceive that a_0 and a_1 may be presumed to depend monotonically on T, for given f and g. In fact, a_0 and a_1 may be assumed to depend monotonically on their arguments generally, but this will not be needed here.

Before proving the proposition, the following corollary result is stated. This is the result of central interest for the present section.

COROLLARY 3.6. *Let $F\in C_b^3(\overline{\mathbf{R}}^+)$ and $G\in C^1(0,T_0)$ with $F(0)=G(0)$. Suppose in addition that $F\in H^1(\mathbf{R}^+)$. Then there exists a constant C, dependent on $\|F\|_1$ and the $H^1(0,T_0)$-norm of G, such that any classical solution v of* (3.4) *defined on $\overline{\Omega}_T$, for $T\le T_0$, satisfies*

$$\|v\|_{C_b(\overline{\Omega}_T)}\le C.$$

Proof. Let v be a classical solution of (3.4) on $\overline{\Omega}_T$, for some $T\le T_0$. The inverse of the change of variables (3.3) is

$$u(x,t)=\varepsilon^{-1}v\left(\varepsilon^{-1/2}x+\varepsilon^{-3/2}t,\varepsilon^{-3/2}t\right). \tag{3.35}$$

Then u is a classical solution of (3.1a) on $\overline{\mathbf{R}}^+\times[0,T']$, where $T'=\varepsilon^{-3/2}T$, which satisfies the auxiliary conditions (3.16) where

$$f(x)=\varepsilon^{-1}F(\varepsilon^{-1/2}x)\quad\text{and}\quad g(t)=\varepsilon^{-1}G(\varepsilon^{-3/2}t). \tag{3.36}$$

Here $\varepsilon>0$ is fixed, and so f and g satisfy the hypotheses of Proposition 3.5. Hence the $H^1(\mathbf{R}^+)$-norm of u is bounded on $[0,T']$ by a constant that depends on $\|f\|_1$, and on the $H^1(0,T_0')$-norm $|g|_{1,T_0'}$ of g, say. Here, $T_0'=\varepsilon^{3/2}T_0$. Because of the basic inequality (2.1), it follows that u is bounded on $\overline{\mathbf{R}}^+\times[0,T']$ by a constant C dependent only on $\|f\|_1$ and $|g|_{1,T_0'}$. In particular, C does not depend on T' for T' in the range $[0,T_0']$.

But, v is defined from u via the transformation (3.3). Hence the desired result follows, and the corollary is established. □

Proof of Proposition 3.5. First note that since $f \in C_b^3(\overline{\mathbf{R}}^+) \cap H^1(\mathbf{R}^+)$, $f(x)$, $f'(x)$, $f''(x) \to 0$ as $x \to +\infty$ (cf. [9]). Let v be defined from u as in (3.3). Then by Lemma 3.4, $\partial_x^i \partial_t^j v(x,t) \to 0$, as $x \to +\infty$, uniformly for $0 \le t \le T$, for $0 \le j \le 1$ and $0 \le i \le 2+j$. Because u is recovered from v by (3.35), $\partial_x^\nu \partial_t^\mu u(x,t) \to 0$, as $x \to +\infty$, uniformly for $0 \le t \le T$, for μ and ν with $\mu + \nu \le 2$. Thus $u, u_x, u_t, u_{xx}, u_{xt}$ and u_{tt} tend to zero at $+\infty$, uniformly for $0 \le t \le T$.

Let $U(x,t) = g(t)e^{-x}$ and $w = u - U$. There is a constant c_* such that, for $0 \le t \le T$,

$$\|U(\cdot,t)\| \le |g(t)| \le c_* |g|_{1,T}.$$

So to prove (3.33), it is enough to establish a similar estimate for w. Now w satisfies the initial- and boundary-value problem

$$w_t + w_x + ww_x + w_{xxx} - \varepsilon w_{xxt} = \varphi - (wU)_x \quad \text{in } \overline{\mathbf{R}}^+ \times [0,T], \tag{3.37}$$

where $\varphi = -(U_t + U_x + UU_x + U_{xxx} - \varepsilon U_{xxt})$, and

$$\begin{aligned} w(x,0) &= f(x) - g(0)e^{-x} && \text{for } x \in \overline{\mathbf{R}}^+, \\ w(0,t) &= 0 && \text{for } t \in [0,T]. \end{aligned} \tag{3.38}$$

Multiply (3.37) by $2w$ and integrate the resulting expression over $(0,M) \times (0,t)$. There appears, after integrations by parts, and using the auxiliary conditions (3.38),

$$\begin{aligned} &\int_0^M [w^2(x,t) + \varepsilon w_x^2(x,t)]\,dx + \int_0^t w_x^2(0,s)\,ds \\ &= \int_0^M [w^2(x,0) + \varepsilon w_x^2(x,0)]\,dx \\ &\quad + \int_0^t \Big[-w^2(M,s) - \frac{2}{3} w^3(M,s) - 2w(M,s) w_{xx}(M,s) \\ &\qquad + w_x^2(M,s) + 2\varepsilon w(M,s) w_{xt}(M,s) - w^2(M,s) U(M,s) \Big]\,ds \\ &\quad + 2\int_0^t \int_0^M \varphi(x,s) w(x,s)\,dx\,ds - \int_0^t \int_0^M U_x(x,s) w^2(x,s)\,dx\,ds. \end{aligned} \tag{3.39}$$

Because $U(x,t) = g(t)e^{-x}$, it follows that

$$\|U\|_{C_b(\overline{\mathbf{R}}^+ \times [0,T])},\ \|U_x\|_{C_b(\overline{\mathbf{R}}^+ \times [0,T])} \le \|g\|_{C(0,T)} \le c_* |g|_{1,T}.$$

Similarly, since $\varepsilon \le 1$,

$$\|\varphi(\cdot,t)\| \le 2|g'(t)| + 2|g(t)| + g^2(t),$$

so that

$$\int_0^t \int_0^M \varphi^2(x,s)\,dx\,ds \le C_1(|g|_{1,T}),$$

for all $(M,t) \in \overline{\mathbf{R}}^+ \times [0,T]$. If

$$W_M(t) = \int_0^M [w^2(x,t) + \varepsilon w_x^2(x,t)]\,dx,$$

and if h_M denotes the supremum, over $[0,T]$, of the second integral on the right-hand side of (3.39), then the inequality

$$W_M(t) \le W_M(0) + h_M + C_1(|g|_{1,T}) + c_* |g|_{1,T} \int_0^t W_M(s)\,ds$$

emerges. Gronwall's lemma implies

$$W_M(t)\leq\left[W_M(0)+h_M+C_1(|g|_{1,T})\right]e^{c_* T|g|_{1,T}},$$

for $0\leq t\leq T$. Reference to (3.38) will convince the reader that $w(\cdot,0)\in H^1(\mathbf{R}^+)$. So $W_M(0)$ is bounded, as $M\to+\infty$. In fact,

$$W_M(0)\to\int_0^\infty\left[w^2(x,0)+\varepsilon w_x^2(x,0)\right]dx=W(0),$$

as $M\to+\infty$. Since u and u_x tend to zero as $x\to+\infty$, uniformly for $0\leq t\leq T$, so also do w and w_x. It follows that $h_M\to 0$ as $M\to+\infty$. Hence,

$$\overline{\lim_{M\to\infty}}\ W_M(t)\leq\left[W(0)+C_1(|g|_{1,T})\right]e^{c_* T|g|_{1,T}},$$

for all $t\in[0,T]$. Thus for each $t\in[0,T]$, $w(\cdot,t)\in H^1(\mathbf{R}^+)$, and

$$\|w\|\leq C_2\left(\|f\|+\varepsilon^{1/2}\|f_x\|,|g|_{1,T}\right),$$

for any ε in $(0,1]$. This is the desired bound on the $L^2(\mathbf{R}^+)$-norm of w, and so (3.33) is shown to be valid.

Now multiply the regularized equation (3.1a) by the combination $2\varepsilon u_{xt}-2u_{xx}-u^2$ and integrate the resulting relation over $\mathbf{R}^+\times(0,t)$. After integrations by parts, in which the fact that u and various of its derivatives vanish at $+\infty$ is used repeatedly, it is verified that

$$(3.40)\qquad \begin{aligned}(1+\varepsilon)&\int_0^\infty u_x^2(x,t)\,dx+\int_0^t\left[u_x^2(0,s)+H^2(s)\right]ds\\&=(1+\varepsilon)\int_0^\infty f_x^2(x)\,dx-\frac{1}{3}\int_0^\infty f^3(x)\,dx\\&\quad+\frac{1}{3}\int_0^\infty u^3(x,t)\,dx+\int_0^t\left[\frac{1}{3}g^3(s)+\varepsilon g_t^2(s)\right]ds-2\int_0^t g_t(s)u_x(0,s)\,ds,\end{aligned}$$

where

$$H(s)=u_{xx}(0,s)-\varepsilon u_{xt}(0,s)+\frac{1}{2}g^2(s).$$

Elementary inequalities, including (2.5), show that

$$\begin{aligned}\int_0^\infty u^3(x,t)\,dx&\leq\|u(\cdot,t)\|^2\|u(\cdot,t)\|_{C_b(\bar{\mathbf{R}}^+)}\leq\sqrt{2}\,\|u(\cdot,t)\|^{5/2}\|u_x(\cdot,t)\|^{1/2}\\&\leq\frac{1}{2}\|u_x(\cdot,t)\|^2+\|u(\cdot,t)\|^{10/3}.\end{aligned}$$

Putting together (3.40), the last observation, and the already established (3.33) yields,

$$\begin{aligned}&\|u_x(\cdot,t)\|^2+\int_0^t\left[u_x^2(0,s)+H^2(s)\right]ds\\&\qquad\leq 2a_0^{10/3}+2(1+\varepsilon)\|f_x\|^2-\frac{2}{3}\int_0^\infty f^3(x)\,dx+2\int_0^t\left[-\frac{1}{3}g^3(s)+(1+\varepsilon)g_t^2(s)\right]ds,\end{aligned}$$

where a_0 is the constant on the right of (3.33). Inequality (3.34) now follows, and the proposition is proved. □

A theorem of global existence of solutions of (3.1) and (3.4) is now in view. Its statement is postponed until after examination of one other aspect, of importance in the analysis in §§4 and 5. This aspect is embodied in the next proposition.

PROPOSITION 3.7. *Let* $F \in C_b^k(\overline{\mathbf{R}}^+) \cap H^k(\mathbf{R}^+)$ *and* $G \in C^m(0,T)$, *with* $F(0)=G(0)$, $k \geq 3$, $m \geq 1$ *and* $k \geq m$. *Let* v *be the solution of* (3.4) *defined in* $\mathcal{C}_T$. *Then there exists a constant C such that, for each* $t \in [0,T]$,

$$(*) \qquad \|\partial_x^i \partial_t^j v(\cdot,t)\|_{L^2((t,\infty))} \leq C,$$

provided that $0 \leq j \leq m$ *and* $0 \leq i \leq k+j$.

Proof. Throughout the demonstration, C will denote various constants which are independent of t in $[0,T]$. It will be convenient to introduce another condition, denoted $(*)_1$, which, for a function w defined on Ω_T, amounts to the requirement that $w(\cdot,t) \in H^1((t,\infty))$ for $t \in [0,T]$, and that

$$(*)_1 \qquad \|w(\cdot,t)\|_{H^1((t,\infty))} \leq C,$$

independently of t in $[0,T]$.

According to (3.33) and (3.34) in Lemma 3.5, $(*)_1$ holds for v itself. Thus v and v_x satisfy $(*)$. For one-dimensional domains, H^1 is an algebra, so that products of H^1 functions are again in H^1. Thus $(1+\varepsilon)v+\frac{1}{2}v^2$ satisfies $(*)_1$. Hence if, as before,

$$J(x,t)=\int_0^t\left[(1+\varepsilon)v(x,s)+\frac{1}{2}v^2(x,s)\right]ds,$$

then J satisfies $(*)_1$. So J and J_t satisfy $(*)_1$. It then follows from formula (3.21) that v_t satisfies $(*)_1$ as well. This observation may be used inductively to show that $\partial_t^i v$ satisfies $(*)_1$, for $0 \leq i \leq m$. Turning now to spatial derivatives, since $k>1$ formula (3.24) shows that v_x satisfies $(*)_1$. This means in particular that J_x satisfies $(*)_1$. Since $k>2$, then $F_{xx} \in H^1(\mathbf{R}^+)$, so, by reference to (3.26), one sees that v_{xx} satisfies $(*)_1$. Proceeding inductively, and using (3.27), it follows that $\partial_x^j v$ satisfies $(*)_1$ if $j \leq k-1$, and so $\partial_x^k v$ satisfies $(*)$.

From (3.28), v_{xt} is observed to satisfy $(*)_1$. The differential equation (3.4a) shows that v_{xxt} satisfies $(*)_1$. Using the differential equation, the results already in hand, and induction, mixed partial derivatives of the form $\partial_x^i \partial_t^j v$, where $j \geq 1$ and $i \geq 2$, are seen to satisfy $(*)_1$ when $j \leq m$ and $i \leq k+j-1$. Hence $\partial_x^i \partial_t^j v$ satisfies $(*)$ provided that $0 \leq j \leq m$ and $0 \leq i \leq k+j$. The desired results are now all established. □

It is worth summarizing the accomplishments of the present section. As the transformed problem (3.4) is only of transient interest, the theory is recapitulated in terms of the regularized problem (3.1). Thus the results stated now are consequences of the established propositions and the transformation (3.35) taking (3.4) to (3.1).

THEOREM 3.8. *Let* $\varepsilon>0$ *and* $T>0$ *be given. Suppose* $f \in C_b^k(\overline{\mathbf{R}}^+)$ *and* $g \in C^m(0,T)$ *with* $f(0)=g(0)$, $k \geq 3$, $m \geq 1$, *and* $k \geq m$. *Then there exists* $T_0>0$ *and a unique function u in* $C_b(\overline{\mathbf{R}}^+ \times [0,T_0])$ *which is a classical solution of the initial- and boundary-value problem* (3.1) *corresponding to the given f and g. Additionally,*

$$\partial_x^i \partial_t^j u \in C_b(\overline{\mathbf{R}}^+ \times [0,T]), \tag{3.41}$$

for i and j such that $0 \leq j \leq m$, $0 \leq i \leq k$, *and* $i+j \leq k$. *Moreover, if* $f \in H^r(\mathbf{R}^+)$, *where* $r \geq 1$, *then u may be extended to a solution of* (3.1) *on* $\overline{\mathbf{R}}^+ \times [0,T]$. *In that case, there is a constant C such that, for* $0 \leq t \leq T$,

$$\|\partial_x^i \partial_x^j u(\cdot,t)\| \leq C,$$

for i and j such that $0 \leq j \leq \min\{r,m\}$, $0 \leq i \leq r$, *and* $i+j \leq r$.

As a corollary to this theorem, the following result emerges. It is this corollary which will find explicit use in the upcoming sections.

COROLLARY 3.9. *Let $\varepsilon>0$ be given. Let $f\in H^{\infty}(\mathbf{R}^+)$ and $g\in C^{\infty}(\mathbf{R}^+)$, with $f(0)=g(0)$. Then there exists a unique solution u of* (3.1) *defined on the quarter-plane $\overline{\mathbf{R}}^+\times\overline{\mathbf{R}}^+$ which is bounded on finite time intervals and which corresponds to the data f and g. Moreover, $u\in C^{\infty}(\overline{\mathbf{R}}^+\times\overline{\mathbf{R}}^+)$ and, for each $k\geq 0$,*

$$\partial_x^i\partial_t^j u\in C(\overline{\mathbf{R}}^+;H^k(\mathbf{R}^+)), \tag{3.42}$$

for all $i,j\geq 0$.

Proof. The existence of global solutions follows immediately from the theorem and the uniqueness result. Also, for any $i,j\geq 0$, $k>0$, and $T>0$, $w=\partial_x^i\partial_t^j u$ is uniformly bounded in $H^k(\mathbf{R}^+)$, for $0\leq t\leq T$.

It remains only to check that the mapping $t\to w(\cdot,t)$ is continuous, from $[0,T]$ to $H^k(\mathbf{R}^+)$. But, in fact, $u\in L^{\infty}(0,T;H^k(\mathbf{R}^+))$ and $u_t\in L^{\infty}(0,T;H^k(\mathbf{R}^+))$. It follows immediately (cf. [19]) that $u\in C(0,T;H^k(\mathbf{R}^+))$. The corollary is now verified. □

4. Estimates in $H^3(\mathbf{R}^+)$ for the regularized problem. The purpose of this and the next section is to derive a priori bounds, which do not depend on ε, for solutions of the regularized initial- and boundary-value problem,

$$u_t+u_x+uu_x+u_{xxx}-\varepsilon u_{xxt}=0 \quad \text{in } \overline{\mathbf{R}}^+\times[0,T], \tag{4.1a}$$

and

$$\begin{aligned} u(x,0)&=f(x) \quad &&\text{for } x\in\overline{\mathbf{R}}^+,\\ u(0,t)&=g(t) \quad &&\text{for } t\in[0,T]. \end{aligned} \tag{4.1b}$$

Here T is a fixed positive real number, and the aspired-for bounds will hold independently of t in $[0,T]$.

Throughout this section it will be assumed that $f\in H^{\infty}(\mathbf{R}^+)$, $g\in C^{\infty}(0,T)$, and $f(0)=g(0)$. In consequence of Corollary 3.9, for any ε in $(0,1]$, there is a classical solution $u=u_\varepsilon$ of (4.1) which is such that

$$u\in C^{\infty}(\overline{\mathbf{R}}^+\times[0,T]),$$

and, for integers $j,k\geq 0$,

$$\partial_t^j u\in C(0,T;H^k(\mathbf{R}^+)).$$

Some preliminary relations, established via energy arguments, will be derived in a sequence of technical lemmas. These prefatory results will be combined to obtain ε-independent bounds for u within the function class $C(0,T;H^3(\mathbf{R}^+))$ and for u_t within the function class $C(0,T;H^1(\mathbf{R}^+))$.

As a start on this program, recall that from Proposition 3.5, there is a constant a_1, depending only on $\|f\|_1$ and $|g|_{1,T}$, such that, independently of ε in $(0,1]$,

$$\|u(\cdot,t)\|_1^2+\int_0^t\left[u_x^2(0,s)+(u_{xx}(0,s)-\varepsilon u_{xt}(0,s))^2\right]ds\leq a_1, \tag{4.2}$$

for all t in $[0,T]$. So, from (2.5) it follows that

$$\|u\|^2_{C_b(\overline{\mathbf{R}}^+\times[0,T])}\leq 2\sup_{0\leq t\leq T}\left\{\|u_x(\cdot,t)\|\|u(\cdot,t)\|\right\}\leq a_1, \tag{4.3}$$

and, because of the differential equation (4.1a),

$$\begin{aligned}\int_0^t(u_{xxx}(0,s)-\varepsilon u_{xxt}(0,s))^2ds&=\int_0^t(g_t(s)+u_x(0,s)+g(s)u_x(0,s))^2ds\\ &\leq c=c(\|f\|_1,|g|_{1,T}),\end{aligned} \tag{4.4}$$

for all t in $[0,T]$.

If u is the solution of (4.1) and $t\in[0,T]$, define

$$A^2(t)=\sup_{0\le s\le t}\left\{\|u(\cdot,s)\|_3^2+\varepsilon\|u_{xxxx}(\cdot,s)\|^2\right\}$$
$$+\int_0^t\left[u_{xxxx}^2(0,s)+u_{xxx}^2(0,s)+u_{xx}^2(0,s)+\varepsilon^2u_{xt}^2(0,s)+\varepsilon u_{xxt}^2(0,s)\right]ds,$$

and

$$B^2(t)=\sup_{0\le s\le t}\|u_t(\cdot,s)\|_1^2+\int_0^t u_{xt}^2(0,s)\,ds.$$

It will be shown that $A(t)$ and $B(t)$ are bounded on $[0,T]$, independently of ε small enough. The first step in obtaining this result is the following $H^2(\mathbf{R}^+)$-estimate.

LEMMA 4.1. *Let $T>0$, $f\in H^\infty(\mathbf{R}^+)$, $g\in H^\infty(0,T)$, with $f(0)=g(0)$. There exist positive constants ε_1, a_2 and c_1, where*

$$\varepsilon_1=\varepsilon_1(\|f\|_1,|g|_{1,T}),\qquad a_2=a_2(\|f\|_2+\varepsilon_1^{1/2}\|f_{xxx}\|,|g|_{1,T}),$$
$$c_1=c_1(\|f\|_1,|g|_{1,T}),$$

such that the solution u of (4.1) *corresponding to the data f and g satisfies*

$$\|u(\cdot,t)\|_2^2+\int_0^t\left[u_{xxx}^2(0,s)+u_{xx}^2(0,s)+\varepsilon^2u_{xt}^2(0,s)\right]ds$$
$$\le a_2+c_1\varepsilon\int_0^t A^2(s)B(s)\,ds-\frac{18}{5}\int_0^t u_{xx}(0,s)u_{xt}(0,s)\,ds,$$

provided that $t\in[0,T]$ and $\varepsilon\in(0,\varepsilon_1]$.

Remark. The presence of the last term on the right-hand side of the above inequality means that this estimate is not directly effective in bounding $\|u(\cdot,t)\|_2$, independently of ε.

Proof. For each t in $[0,T]$, define $V(t)$ as

$$V(t)=\int_0^\infty\left[\left(\frac{9}{5}-3\varepsilon u\right)u_{xx}^2-3uu_x^2+\frac{1}{4}u^4+\frac{9}{5}\varepsilon u_{xxx}^2\right]dx.$$

Multiply (4.1a) by $u^3-3u_x^2$, differentiate (4.1a) once with respect to x and multiply the result by $-6uu_x-\frac{18}{5}u_{xxx}$, add the two equations thus obtained, and integrate their sum over $\mathbf{R}^+\times(0,t)$. After several integrations by parts, there appears,

(4.5)

$$V(t)-V(0)+\frac{9}{5}\int_0^t\left[u_{xxx}^2(0,s)+u_{xx}^2(0,s)\right]ds$$

$$=\int_0^t\Big[\frac{1}{4}g^4(s)+\frac{1}{5}g^5(s)-3g(s)u_x^2(0,s)+g^3(s)u_{xx}(0,s)-\frac{9}{2}g^2(s)u_x^2(0,s)$$
$$-6g(s)u_x(0,s)u_{xxx}(0,s)+\frac{6}{5}g(s)u_{xx}^2(0,s)$$
$$-\frac{3}{5}u_x^2(0,s)u_{xx}(0,s)-\frac{18}{5}u_{xx}(0,s)u_{xt}(0,s)\Big]ds$$
$$+\varepsilon\int_0^t\big[6g_t(s)u_x(0,s)u_{xx}(0,s)+6g(s)u_x(0,s)u_{xxt}(0,s)$$
$$-3u_x^2(0,s)u_{xt}(0,s)-g^3(s)u_{xt}(0,s)\big]ds$$
$$+\varepsilon\int_0^t\int_0^\infty\left[3u_{xx}^2u_t+6u_tu_xu_{xxx}-3u^2u_xu_{xt}\right]dx\,ds.$$

Because of the relation (4.2), the first seven boundary terms on the right-hand side of (4.5) can be bounded in terms of the data f and g and a suitable small multiple of the two boundary integrals on the left-hand side of (4.5). Using (2.5) and (4.2), it follows that for any $\delta>0$,

$$(4.6)\quad \int_0^t u_x^2(0,s)u_{xx}(0,s)\,ds$$

$$\le \|u_x\|_{C_b(\bar{\mathbf{R}}^+\times[0,t])}\left(\int_0^t u_x^2(0,s)\,ds\int_0^t u_{xx}^2(0,s)\,ds\right)^{1/2}$$

$$\le \sqrt{2}\left\{\sup_{0\le s\le t}\left(\|u_x(\cdot,s)\|^{1/2}\|u_{xx}(\cdot,s)\|^{1/2}\right)\cdot\left(\int_0^t u_x^2(0,s)\,ds\int_0^t u_{xx}^2(0,s)\,ds\right)^{1/2}\right\}$$

$$\le a_1^3\delta^{-3}+\delta\left\{\sup_{0\le s\le t}\|u_{xx}(\cdot,s)\|^2+\int_0^t u_{xx}^2(0,s)\,ds\right\}.$$

Since

$$\int_0^t g_t(s)u_x(0,s)u_{xx}(0,s)\,ds\le \|u_x\|_{C_b(\bar{\mathbf{R}}^+\times[0,t])}|g_t|_T\left(\int_0^t u_{xx}^2(0,s)\,ds\right)^{1/2},$$

a similar bound holds for the term

$$\varepsilon\int_0^t g_t(s)u_x(0,s)u_{xx}(0,s)\,ds.$$

The estimate (4.2) also implies that

$$\varepsilon^2\int_0^t u_{xt}^2(0,s)\,ds\le 2\left\{\int_0^t (u_{xx}(0,s)-\varepsilon u_{xt}(0,s))^2ds+\int_0^t u_{xx}^2(0,s)\,ds\right\}$$
$$\le 2a_1+2\int_0^t u_{xx}^2(0,s)\,ds.$$

As a consequence, bounds similar to that in (4.6) obtain for the terms

$$\varepsilon\int_0^t u_x^2(0,s)u_{xt}(0,s)\,ds \quad\text{and}\quad \varepsilon\int_0^t g^3(s)u_{xt}(0,s)\,ds.$$

Making use of (4.4), the term,

$$\varepsilon\int_0^t g(s)u_x(0,s)u_{xxt}(0,s)\,ds,$$

may be bounded in the same way.

Still relying on (4.2) and (4.3), the term

$$3\int_0^\infty uu_x^2\,dx\le 3\|u\|_{C_b(\bar{\mathbf{R}}^+\times[0,t])}\int_0^\infty u_x^2\,dx\le 3a_1^{3/2}.$$

Hence,

$$\int_0^\infty\left(\frac{9}{5}-3\varepsilon u\right)u_{xx}^2\,dx\le V(t)+3a_1^{3/2}.$$

But, by (4.3), $|u|$ does not exceed the value $a_1^{1/2}$ on $\overline{\mathbf{R}}^+\times[0,T]$. Consequently, if $\varepsilon_1=(25a_1)^{-1/2}$, then for $0<\varepsilon\leq\varepsilon_1$,

$$\frac{6}{5}\int_0^\infty u_{xx}^2\,dx\leq V(t)+3a_1^{3/2},$$

for all t in $[0,T]$.

Therefore, if (4.5) and a suitable multiple of (4.2) are summed, and use is made of the above estimates, then for t in $[0,T]$ and ε in $(0,\varepsilon_1]$,

$$\|u(\cdot,t)\|_2^2+\int_0^t\left[u_{xxx}^2(0,s)+u_{xx}^2(0,s)+\varepsilon^2u_{xt}^2(0,s)\right]ds$$

$$\leq a_2-\frac{18}{5}\int_0^t u_{xx}(0,s)u_{xt}(0,s)\,ds$$

$$+\varepsilon\int_0^t\int_0^\infty\left[3u_{xx}^2u_t+6u_xu_{xxx}u_t-3u^2u_xu_{xt}\right]dx\,ds.$$

Here, the constant a_2 stems from $V(0)$ and from the various combinations of a_1 that appear in the foregoing estimates. The desired result now follows from the last relation, (4.2), and the definitions of $A(t)$ and $B(t)$. □

The estimate of the $H^2(\mathbf{R}^+)$-norm of the solution u of (4.1) given in Lemma 4.1 will be used in determining the following bound for $A(t)$.

LEMMA 4.2. *Let $T>0$, $f\in H^\infty(\mathbf{R}^+)$, $g\in C^\infty(0,T)$, with $f(0)=g(0)$. There exist positive constants a_3 and c_2, where*

$$a_3=a_3\left(\|f\|_3+\varepsilon_1^{1/2}\|f_{xxxx}\|,|g|_{2,T}\right)\quad\text{and}\quad c_2=c_2\left(\|f\|_1,|g|_{1,T}\right),$$

such that the solution of (4.1) *corresponding to f and g satisfies*

$$A^2(t)-\varepsilon c_2\left[A^3(t)+\varepsilon(1+A^2(t))B^2(t)\right]\leq a_3+\varepsilon^{1/2}c_2\int_0^tA^2(s)B(s)\,ds,$$

for all t in $[0,T]$ *and ε in* $(0,\varepsilon_1]$.

Remark. The ε_1 appearing in the above statement is that derived already in Lemma 4.1.

Proof. As in the proof of the last lemma, the desired result will be obtained from a technical "energy" argument. In the proof, various constants dependent on aspects of the data f and g will appear. These will generally be denoted simply by c, and this symbol's occurrence in different formulae is not taken to connote the same constant. Define, for each t in $[0,T]$,

$$W(t)=\int_0^\infty\left[\frac{108}{35}\left(\varepsilon u_{xxxx}^2+u_{xxx}^2\right)-\frac{36}{5}(u-\varepsilon u_{xx})u_{xx}^2\right.$$

$$\left.+6\left(u_x^2+\varepsilon u_{xx}^2\right)-\frac{1}{5}u^5-3\varepsilon u_x^4-\frac{36}{5}\varepsilon uu_{xxx}^2\right]dx.$$

Multiply (4.1a) by $12uu_x^2-\frac{36}{5}u_{xx}^2-u^4$, differentiate (4.1a) once with respect to x and multiply this by $12u^2u_x$, differentiate (4.1a) twice with respect to x and multiply this by $-\frac{216}{35}u_{xxxx}-\frac{72}{5}uu_{xx}$, add the three resulting equations and integrate their sum over

$\mathbf{R}^+\times(0,t)$. After many integrations by parts with respect to the spatial variable x, there appears,

(4.7)

$$W(t)-W(0)+\frac{108}{35}\int_0^t\left[u_{xxxx}^2(0,s)+u_{xxx}^2(0,s)\right]ds$$
$$=\int_0^t\Big[-\frac{36}{5}g(s)u_{xx}^2(0,s)+6g^2(s)u_x^2(0,s)-\frac{1}{5}g^5(s)-\frac{1}{6}g^6(s)$$
$$-g^4(s)(u_{xx}(0,s)-\varepsilon u_{xt}(0,s))+8g^3(s)u_x^2(0,s)$$
$$+12g^2(s)u_x(0,s)(u_{xxx}(0,s)-\varepsilon u_{xxt}(0,s))$$
$$-12g(s)u_x^2(0,s)(u_{xx}(0,s)-\varepsilon u_{xt}(0,s))+3u_x^4(0,s)-\frac{66}{5}g^2(s)u_{xx}^2(0,s)$$
$$+\frac{72}{5}u_x(0,s)u_{xx}(0,s)(u_{xxx}(0,s)-\varepsilon u_{xxt}(0,s))$$
$$-\frac{72}{5}u_x(0,s)u_{xxx}(0,s)(u_{xx}(0,s)-\varepsilon u_{xt}(0,s))$$
$$-\frac{144}{35}u_x(0,s)u_{xx}(0,s)u_{xxx}(0,s)$$
$$+\frac{144}{35}g(s)u_{xxx}^2(0,s)-\frac{72}{5}g(s)u_{xx}(0,s)u_{xxxx}(0,s)-\frac{36}{35}u_{xx}^3(0,s)$$
$$-\frac{216}{35}u_{xxx}(0,s)u_{xxt}(0,s)+\frac{72}{5}\varepsilon g(s)u_{xx}(0,s)u_{xxxt}(0,s)\Big]ds$$
$$+\varepsilon\int_0^t\int_0^\infty\Big[4u^3u_xu_{xt}+24uu_xu_{xx}u_{xt}+\frac{72}{5}u_{xx}u_{xxx}u_{xt}$$
$$+\frac{72}{5}u_xu_{xxxx}u_{xt}+12uu_tu_{xx}^2-\frac{36}{5}u_tu_{xxx}^2\Big]\,dx\,ds.$$

First note that, because of (4.2), there is a positive constant c, depending on $\|f\|_1$ and $|g|_{1,T}$, so that

$$\frac{108}{35}\left(\|u_{xxx}(\cdot,t)\|^2+\varepsilon\|u_{xxxx}(\cdot,t)\|^2\right)-\frac{72}{5}\varepsilon A^3(t)\le W(t)+c,$$

for all t in $[0,T]$. Also, in consequence of (2.1) and (4.2), there is another constant c, depending again on $\|f\|_1$ and $|g|_{1,T}$, such that, for any $\delta>0$,

$$\|u_x\|^2_{C_b(\bar{\mathbf{R}}^+\times[0,t])}\le 2\Big\{\sup_{0\le s\le t}\left(\|u_x(\cdot,s)\|\|u_{xx}(\cdot,s)\|\right)\Big\} \tag{4.8}$$
$$\le c\delta^{-1}+\delta\Big\{\sup_{0\le s\le t}\|u_{xx}(\cdot,s)\|^2\Big\}.$$

By an analogous argument,

$$\|u_{xx}\|^2_{C_b(\bar{\mathbf{R}}^+\times[0,t])}\le c\delta^{-3}+\delta\Big\{\sup_{0\le s\le t}\|u_{xxx}(\cdot,s)\|^2\Big\}. \tag{4.9}$$

Taken together with (4.2), these estimates imply that there is a constant c, depending on $\|f\|_1$ and $|g|_{1,T}$, such that for all $\delta>0$,

$$(4.10)\qquad \int_0^t u_x(0,s)u_{xx}(0,s)u_{xxx}(0,s)\,ds$$
$$\le \|u_{xx}\|_{C_b(\bar{\mathbf{R}}^+\times[0,t])}\left[\int_0^t u_x^2(0,s)\,ds\int_0^t u_{xxx}^2(0,s)\,ds\right]^{1/2}$$
$$\le c\left[\delta^{-3}+\int_0^t u_{xxx}^2(0,s)\,ds\right]+\delta\left\{\sup_{0\le s\le t}\|u_{xxx}(\cdot,s)\|^2\right\}.$$

By adding (4.7) and a suitable positive multiple $\alpha=\alpha(\sup_{0\le s\le t}\|u(\cdot,s)\|_1)$ of the inequality stated in Lemma 4.1, and using (4.2) and (4.3), bounds similar to those exhibited in (4.10) may be shown to hold for all the boundary terms on the right-hand side of (4.7) except for the last three. Choosing δ appropriately, it may thus be inferred that, for all ε in $(0,\varepsilon_1]$, and for all t in $[0,T]$,

$$(4.11)\qquad 3\left[A^2(t)-\varepsilon\int_0^t u_{xxt}^2(0,s)\,ds\right]-\frac{72}{5}\varepsilon A^3(t)$$
$$\le \tilde{a}_3+\tilde{c}_2\varepsilon^{1/2}\int_0^t A^2(s)B(s)\,ds$$
$$-\int_0^t\left[\frac{36}{35}u_{xx}^3(0,s)+\frac{216}{35}u_{xxx}(0,s)u_{xxt}(0,s)\right.$$
$$\left.-\frac{72}{5}\varepsilon g(s)u_{xx}(0,s)u_{xxxt}(0,s)+\frac{18}{5}\alpha u_{xx}(0,s)u_{xt}(0,s)\right]ds.$$

Here $\tilde{a}_3=\tilde{a}_3(\|f\|_3+\varepsilon_1^{1/2}\|f_{xxxx}\|,|g|_{2,T})$ and $\tilde{c}_2=\tilde{c}_2(\|f\|_1,|g|_{1,T})$.

To complete the proof of the lemma, it suffices to control suitably the boundary terms appearing on the right side of inequality (4.11). To this end, observe first that (4.2) and (4.9) imply

$$(4.12)\qquad \int_0^t u_{xx}^3(0,s)\,ds\le\|u_{xx}\|^2_{C_b(\bar{\mathbf{R}}^+\times[0,t])}\int_0^t|u_{xx}(0,s)|\,ds$$
$$\le\|u_{xx}\|^2_{C_b(\bar{\mathbf{R}}^+\times[0,t])}\int_0^t\left(|u_{xx}(0,s)-\varepsilon u_{xt}(0,s)|+\varepsilon|u_{xt}(0,s)|\right)ds$$
$$\le c(\delta^{-3}+\delta A^2(t))(1+\varepsilon^2B^2(t)),$$

for any $\delta>0$, where the constant c depends on $\|f\|_1$, $|g|_{1,T}$ and T. Next note that equation (4.1a) implies

$$-\int_0^t u_{xxx}(0,s)u_{xxt}(0,s)\,ds$$
$$=\int_0^t[g_t(s)+u_x(0,s)+g(s)u_x(0,s)-\varepsilon u_{xxt}(0,s)]u_{xxt}(0,s)\,ds.$$

Integration by parts in the temporal variable yields

$$\int_0^t[g_t(s)+u_x(0,s)+g(s)u_x(0,s)]u_{xxt}(0,s)\,ds$$
$$=[g_t(s)+u_x(0,s)+g(s)u_x(0,s)]u_{xx}(0,s)\Big|_{s=0}^{s=t}$$
$$-\int_0^t[g_{tt}(s)+u_{xt}(0,s)+g_t(s)u_x(0,s)+g(s)u_{xt}(0,s)]u_{xx}(0,s)\,ds.$$

From (4.1a) it also follows that

$$(4.13)\quad u_{xt}(0,s)=\varepsilon u_{xxxt}(0,s)-\left[u_{xx}(0,s)+u_x^2(0,s)+g(s)u_{xx}(0,s)+u_{xxxx}(0,s)\right].$$

Hence, due to (4.8) and (4.9), for any $\delta>0$ there is a constant c_δ, depending on δ, $\|f\|_1$ and $|g|_{2,T}$, such that

$$(4.14)\quad -\int_0^t u_{xxx}(0,s)u_{xxt}(0,s)\,ds$$
$$\le c_\delta-\varepsilon\int_0^t u_{xxt}^2(0,s)\,ds$$
$$+\delta\left[A^2(t)+\int_0^t u_{xxxx}^2(0,s)\,ds\right]-\varepsilon\int_0^t(1+g(s))u_{xx}(0,s)u_{xxxt}(0,s)\,ds.$$

Similarly, it follows from (4.8), (4.9) and (4.13) that, for any $\delta>0$,

$$(4.15)\quad -\int_0^t u_{xx}(0,s)u_{xt}(0,s)\,ds$$
$$\le c_\delta+\delta\left[A^2(t)+\int_0^t u_{xxxx}^2(0,s)\,ds\right]-\varepsilon\int_0^t u_{xx}(0,s)u_{xxxt}(0,s)\,ds,$$

where the constant c_δ depends on δ, $\|f\|_1$ and $|g|_{1,T}$. Combining (4.15) with (4.11), (4.12) and (4.14), and choosing δ in a perspicuous way, there appears,

$$(4.16)\quad 2A^2(t)-\varepsilon\hat{c}_2\left[A^3(t)+\varepsilon(1+A^2(t))B^2(t)\right]$$
$$\le\hat{a}_3+\varepsilon^{1/2}\hat{c}_2\int_0^t A^2(s)B(s)\,ds$$
$$-\varepsilon\int_0^t\left[\left(\frac{18}{5}\alpha+\frac{216}{35}\right)-\frac{288}{35}g(s)\right]u_{xx}(0,s)u_{xxt}(0,s)\,ds,$$

holding for all ε in $(0,\varepsilon_1]$ and t in $[0,T]$. Here,

$$\hat{a}_3=\hat{a}_3\left(\|f\|_3+\varepsilon_1^{1/2}\|f_{xxxx}\|,|g|_{2,T}\right)\quad\text{and}\quad\hat{c}_2=\hat{c}_2\left(\|f\|_1,|g|_{1,T}\right).$$

To estimate the boundary terms on the right-hand side of (4.16), use (4.9) again to deduce that, corresponding to any $\delta>0$ there is another constant c_δ, dependent on δ, $\|f\|_1$ and $|g|_{1,T}$, such that

$$(4.17)\quad -\varepsilon\int_0^t\left[\left(\frac{18}{5}\alpha+\frac{216}{35}\right)-\frac{288}{35}g(s)\right]u_{xx}(0,s)u_{xxxt}(0,s)\,ds$$
$$\le\delta^{-1}\left[\left(\frac{18}{5}\alpha+\frac{216}{35}\right)+\frac{288}{35}\|g\|_{C(0,T)}\right]^2\int_0^t u_{xx}^2(0,s)\,ds+\delta\varepsilon^2\int_0^t u_{xxxt}^2(0,s)\,ds$$
$$\le c_\delta+\delta A^2(t)+\delta\varepsilon^2\int_0^t u_{xxxt}^2(0,s)\,ds.$$

So, the only term still presenting difficulty is the final one in (4.17). To estimate this quantity, differentiate the regularized equation (4.1a) twice with respect to x, multiply the result by $2\varepsilon u_{xxxt}$ and integrate over $\mathbf{R}^+\times(0,t)$. The effect of these operations is to produce the relation

$$(4.18)\quad \varepsilon\left(\|u_{xxx}(\cdot,t)\|^2-\|u_{xxxx}(\cdot,t)\|^2\right)+\varepsilon^2\int_0^t u_{xxxt}^2(0,s)\,ds$$
$$=\varepsilon\left(\|f_{xxx}\|^2-\|f_{xxxx}\|^2\right)+\varepsilon\int_0^t u_{xxt}^2(0,s)\,ds$$
$$+2\varepsilon\int_0^t u_{xxxx}(0,s)u_{xxxt}(0,s)\,ds-2\varepsilon\int_0^t\int_0^\infty(uu_x)_{xx}u_{xxxt}\,dx\,ds.$$

The last integral on the right-hand side of (4.18) seems somewhat awkward. However, after integration by parts,

$$\begin{aligned}\int_0^t\int_0^\infty (uu_x)_{xx}u_{xxxt}\,dx\,ds &= \int_0^t\int_0^\infty (3u_xu_{xx}+uu_{xxx})u_{xxxt}\,dx\,ds\\ &= \int_0^\infty\left[\frac{1}{2}u(x,s)u_{xxx}^2(x,s)-u_{xx}^3(x,s)\right]dx\Big|_{s=0}^{s=t}\\ &\quad +3\int_0^t u_x(0,s)\left[u_{xxx}(0,s)u_{xt}(0,s)-u_{xx}(0,s)u_{xxt}(0,s)\right]ds\\ &\quad +\int_0^t\int_0^\infty\left(3u_{xx}u_{xxx}u_{xt}+3u_xu_{xxxx}u_{xt}-\frac{1}{2}u_{xxx}^2u_t\right)dx\,ds.\end{aligned}$$

Also, by (4.8) there is a constant c, dependent on $\|f\|_1$ and $|g|_{1,T}$, such that

$$\begin{aligned}\varepsilon\int_0^t u_x(0,s)u_{xxx}(0,s)u_{xt}(0,s)\,ds &\le \varepsilon^2B^2(t)\|u_x\|^2_{C_b(\bar{\mathbf{R}}^+\times[0,t])}+\int_0^t u_{xxx}^2(0,s)\,ds\\ &\le c\varepsilon^2(1+A^2(t))B^2(t)+A^2(t).\end{aligned}$$

And,

$$\begin{aligned}\varepsilon\int_0^t u_x(0,s)u_{xx}(0,s)u_{xxt}(0,s)\,ds &\le \varepsilon\|u_x\|^2_{C_b(\bar{\mathbf{R}}^+\times[0,t])}\int_0^t u_{xx}^2(0,s)\,ds+\varepsilon\int_0^t u_{xxt}^2(0,s)\,ds\\ &\le c\varepsilon A^3(t)+A^2(t).\end{aligned}$$

Referring to the definition of A and B below (4.4), and applying elementary estimates, it follows at once that

$$\begin{aligned}&2\varepsilon\int_0^t\int_0^\infty\left(3u_{xx}u_{xxx}u_{xt}+3u_xu_{xxxx}u_{xt}-\frac{1}{2}u_{xxx}^2u_t\right)dx\,ds\\ &\quad\le\int_0^t\left[6\varepsilon A^2(s)B(s)+6\varepsilon^{1/2}A^2(s)B(s)+\varepsilon A^2(s)B(s)\right]ds\\ &\quad\le 13\varepsilon^{1/2}\int_0^t A^2(s)B(s)\,ds.\end{aligned}$$

Here, and above, the restriction $\varepsilon\le 1$ is used. The last few relations combine with (4.18) to produce the inequality

$$\begin{aligned}\varepsilon^2\int_0^t u_{xxxt}^2(0,s)\,ds &\le c+c'\varepsilon^{1/2}\int_0^t A^2(s)B(s)\,ds\\ &\quad +c''\left\{A^2(t)+\varepsilon\left[A^3(t)+\varepsilon(1+A^2(t))B^2(t)\right]\right\}.\end{aligned}\tag{4.19}$$

If, in (4.17), δ is now chosen small enough, the desired inequality follows from (4.16), (4.17) and (4.19). This completes the proof of Lemma 4.2. □

To make effective use of Lemma 4.2, an estimate for $B(t)$ is needed. The following result will be sufficient.

LEMMA 4.3. *Let $T>0$, $f\in H^\infty(\mathbf{R}^+)$, $g\in H^\infty(0,T)$, with $f(0)=g(0)$. There are positive constants a_4 and c_3, with*

$$a_4=a_4(\|u_t(\cdot,0)\|_1,|g|_{2,T})\quad\text{and}\quad c_3=c_3(\|f\|_3,|g|_{1,T}),$$

such that the solution of (4.1) *corresponding to the data f and g satisfies the inequality*

$$B^2(t)\le a_4+c_3\int_0^t\left[(1+A(s))B^2(s)+\varepsilon B^3(s)\right]ds,$$

for all t in $[0,T]$ and ε in $(0,1]$.

Proof. Let $v(x,t)=u_t(x,t)$. Then v satisfies the variable-coefficient partial differential equation

(4.20)
$$v_t+v_x+(uv)_x+v_{xxx}-\varepsilon v_{xxt}=0,$$

holding for (x,t) in $\overline{\mathbf{R}}^+\times[0,T]$. Multiply (4.20) by $2v$ and integrate over $\mathbf{R}^+\times(0,t)$, where $t\in[0,T]$. Then, it follows that

(4.21)
$$\begin{aligned}\|v(\cdot,t)\|^2+\varepsilon\|v_x(\cdot,t)\|^2+\int_0^t v_x^2(0,s)\,ds\\ =\|v(\cdot,0)\|^2+\varepsilon\|v_x(\cdot,0)\|^2+\int_0^t(1+g(s))g_t^2(s)\,ds\\ +2\int_0^t g_t(s)[v_{xx}(0,s)-\varepsilon v_{xt}(0,s)]\,ds-\int_0^t\int_0^\infty u_x v^2\,dx\,ds.\end{aligned}$$

Next, multiply (4.20) by $2(\varepsilon v_{xt}-uv-v_{xx})$ and integrate again over $\mathbf{R}^+\times(0,t)$. This leads to

(4.22)
$$\begin{aligned}(1+\varepsilon)\|v_x(\cdot,t)\|^2-\int_0^\infty u(x,t)v^2(x,t)\,dx+\int_0^t\{v_x^2(0,s)+[v_{xx}(0,s)-\varepsilon v_{xt}(0,s)]^2\}\,ds\\ =(1+\varepsilon)\|v_x(\cdot,0)\|^2-\int_0^\infty f(x)v^2(x,0)\,dx+\int_0^t g_{tt}^2(s)[\varepsilon-g^2(s)]\,ds\\ -2\int_0^t g_{tt}(s)v_x(0,s)\,ds-2\int_0^t g(s)g_t(s)[v_{xx}(0,s)-\varepsilon v_{xt}(0,s)]\,ds\\ +\int_0^t\int_0^\infty(2uvv_x-v^3)\,dx\,ds.\end{aligned}$$

The underlying equation (4.1a) implies that

$$-\int_0^t\int_0^\infty v^3\,dx\,dx=\int_0^t\int_0^\infty v^2(u_x+uu_x+u_{xxx}-\varepsilon u_{xxt})\,dx\,ds.$$

The last term on the right side of this relation is potentially troublesome, but after integration by parts,

$$\varepsilon\int_0^t\int_0^\infty v^2u_{xxt}\,dx\,dx=-\varepsilon\int_0^t g_t^2(s)v_x(0,s)\,ds-2\varepsilon\int_0^t\int_0^\infty vv_x^2\,dx\,ds.$$

Also,

$$\int_0^\infty u(x,t)v^2(x,t)\,dx\le\|u\|_{C_b(\overline{\mathbf{R}}^+\times[0,t])}\|v(\cdot,t)\|^2\le c\|v(\cdot,t)\|^2,$$

where c depends on $\|f\|_1$ and $|g|_{1,T}$, as in (4.3). The desired result thus follows by adding an appropriate multiple of (4.21) to (4.22) and making the kind of estimates based on (4.2) that are, by now, familiar. □

Recapitulating the outcome of Lemmas 4.2 and 4.3, if u is the solution of (4.1) corresponding to initial data f and boundary data g, and A and B are the associated functionals defined below (4.4), then A and B are restricted by the system of inequalities

(4.23)
$$\begin{aligned}A^2(t)-\varepsilon c_2[A^3(t)+\varepsilon(1+A^2(t))B^2(t)]\le a_3+\varepsilon^{1/2}c_2\int_0^t A^2(s)B(s)\,ds,\\ B^2(t)\le a_4+c_3\int_0^t[(1+A(s))B^2(s)+\varepsilon B^3(s)]\,ds,\end{aligned}$$

holding for all t in $[0,T]$ and ε in $(0,\varepsilon_1]$. The constants ε_1, a_3, a_4, c_2 and c_3 have all been previously determined to depend simply on T, on various norms of f and g and on $\|u_t(\cdot,0)\|_1$. The system (4.23) will be exploited to obtain the following bound on u, which holds uniformly for ε sufficiently small.

LEMMA 4.4. *Let $T>0$, $f\in H^\infty(\mathbf{R}^+)$, $g\in H^\infty(0,T)$ be given with $f(0)=g(0)$. Let u be the solution of* (4.1) *corresponding to the data f and g. There are positive constants ε_2 and c_4, both depending on $\|f\|_4$, $|g|_{2,T}$ and $\|u_t(\cdot,0)\|_1$, such that for ε in $(0,\varepsilon_2]$ and t in $[0,T]$, both $A(t)$ and $B(t)$ are no larger than c_4.*

Proof. For each $M\in\mathbf{R}$ such that

$$M>\max(A(0),B(0)), \tag{4.24}$$

let

$$t_M=\inf\{t\in[0,T]: A(t)\geq M \text{ or } B(t)\geq M\},$$

with the understanding that if the set over which the infimum is taken is empty, then $t_M=T$. To establish the lemma, it suffices to show that $t_M=T$ for some M and all sufficiently small ε.

Observe that on the interval $[0,t_M)$, where M is supposed chosen as above, (4.23) implies that

$$\begin{aligned}
&[1-\varepsilon c_2M(1+\varepsilon M)]A^2(t)\leq a_3+\varepsilon^{1/2}c_2\int_0^t A^2(s)B(s)\,ds+c_2(\varepsilon M)^2, \\
&B^2(t)\leq a_4+c_3\int_0^t(1+A(s))B^2(s)\,ds+\varepsilon c_3TM^3.
\end{aligned} \tag{4.25}$$

For each M satisfying (4.24), choose $\varepsilon_2=\varepsilon_2(M)\in(0,\min(\frac{1}{2},\varepsilon_1))$ such that for all ε in $(0,\varepsilon_2)$,

$$1-c_2\varepsilon M(1+\varepsilon M)\geq\frac{1}{2},\quad c_2(\varepsilon M)^2\leq 1,\quad c_3\varepsilon TM^3\leq 1. \tag{4.26}$$

Further, let $A_1(t)=1+A(t)$. Then from (4.25), it follows that for all t in $[0,t_M)$ and for all ε in $(0,\varepsilon_2)$,

$$\begin{aligned}
&A_1^2(t)\leq 6+4a_3+4c_2\varepsilon^{1/2}\int_0^t A_1^2(s)B(s)\,ds, \\
&B^2(t)\leq 1+a_4+c_3\int_0^t A_1(s)B^2(s)\,ds.
\end{aligned}$$

Hence, in this range of t and ε, there are positive constants α,β and γ, independent of M, such that

$$\begin{aligned}
&A_1^2(t)\leq\frac{\alpha}{1-\varepsilon^{1/2}}+2\frac{\gamma}{\beta}\varepsilon^{1/2}\int_0^t A_1^2(s)B(s)\,ds, \\
&B^2(t)\leq\frac{\beta}{1-\varepsilon^{1/2}}+2\frac{\gamma}{\alpha}\int_0^t A_1(s)B^2(s)\,ds.
\end{aligned} \tag{4.27}$$

(First choose α and β large enough, and then choose γ large enough. Note then that α,β and γ only depend on the constants a_3, a_4, c_2 and c_3.) Define $\bar{A}_1$ and $\bar{B}$ to be the maximal solution of the system

$$\begin{aligned}
&\bar{A}_1^2(t)=\frac{\alpha}{1-\varepsilon^{1/2}}+2\frac{\gamma}{\beta}\varepsilon^{1/2}\int_0^t\bar{A}_1^2(s)\bar{B}(s)\,ds, \\
&\bar{B}^2(t)=\frac{\beta}{1-\varepsilon^{1/2}}+2\frac{\gamma}{\alpha}\int_0^t\bar{A}_1(s)\bar{B}^2(s)\,ds.
\end{aligned}$$

Then, $\bar{A}_1(t)\geq A_1(t)$ and $\bar{B}(t)\geq B(t)$, for all t for which $\bar{A}_1(t)$ and $\bar{B}(t)$ are finite. Moreover, $\bar{A}_1$ and $\bar{B}$ may be determined explicitly as,

$$\bar{A}_1(t)=\frac{\alpha}{1-\varepsilon^{1/2}e^{\gamma t}} \quad \text{and} \quad \bar{B}(t)=\frac{\beta e^{\gamma t}}{1-\varepsilon^{1/2}e^{\gamma t}},$$

whenever $\exp(\gamma t)<\varepsilon^{-1/2}$. Therefore, if M is chosen so that

$$M>2\max\{\alpha,\beta e^{\gamma T}\},$$

and then ε_2 is chosen so that, as well as satisfying (4.26),

$$1-\varepsilon_2^{1/2}e^{\gamma T}\geq\frac{1}{2},$$

then $t_M=T$ for all ε in $(0,\varepsilon_2]$. Taking $c_4=M$, the lemma is now established. □

The constants ε_2 and c_4 in Lemma 4.4 depend on $\|u_t(\cdot,0)\|_1$, since the constant a_4 in Lemma 4.3 had such a dependence. In order to control the size of $A(t)$ and $B(t)$, uniformly for small ε, some estimate of $\|u_t(\cdot,0)\|_1$ must be obtained in terms of the data f and g. An appropriate bound is forthcoming if the data satisfies the additional compatibility condition,

$$g_t(0)=-\left[f_x(0)+f(0)f_x(0)+f_{xxx}(0)\right]. \tag{4.28}$$

LEMMA 4.5. *Let $T>0$, $f\in H^\infty(\mathbf{R}^+)$, $g\in H^\infty(0,T)$ with $f(0)=g(0)$. Suppose the data f and g also satisfy* (4.28). *Then there is a constant a_5 depending on $\|f\|_4$ such that*

$$\|u_t(\cdot,0)\|_1\leq a_5,$$

for all ε in $(0,1]$, *where u is the solution of* (4.1) *corresponding to f and g.*

Proof. Let $\varphi(x)=-[f_x(x)+f(x)f_x(x)+f_{xxx}(x)]$. Then $u_t(\cdot,0)$ is a solution of the boundary-value problem

$$u_t(\cdot,0)-\varepsilon u_{xxt}(\cdot,0)=\varphi,$$
$$u_t(0,0)=g_t(0), \qquad \lim_{x\to\infty}u_t(x,0)=0.$$

Hence, $u_t(\cdot,0)$ is given by

$$u_t(x,0)=e^{-x/\varepsilon^{1/2}}g_t(0)+\int_0^\infty M_\varepsilon(x,\xi)\varphi(\xi)\,d\xi, \tag{4.29}$$

where, as in (3.10),

$$M_\varepsilon(x,\xi)=\frac{1}{2\varepsilon^{1/2}}\left[\exp(-|x-\xi|/\varepsilon^{1/2})-\exp(-(x+\xi)/\varepsilon^{1/2})\right].$$

It follows immediately from this representation that

$$\|u_t(\cdot,0)\|\leq\frac{\varepsilon^{1/4}}{2^{1/2}}|g_t(0)|+c\|\varphi\|,$$

where c is a constant which is independent of ε. Since $g_t(0)=\varphi(0)$, and because of the definition of φ, it is concluded there is a constant a depending on $\|f\|_4$ such that

$$\|u_t(\cdot,0)\|\leq a, \tag{4.30}$$

and this relation holds uniformly for ε in $(0,1]$. Differentiation of (4.29) with respect to x leads to the relation

$$u_{xt}(x,0) = -\frac{1}{\varepsilon^{1/2}}e^{-x/\varepsilon^{1/2}}g_t(0) + \frac{1}{2\varepsilon}\int_0^\infty e^{-(x+\xi)/\varepsilon^{1/2}}\varphi(\xi)\,d\xi$$
$$-\frac{1}{2\varepsilon}\int_0^x e^{(-x+\xi)/\varepsilon^{1/2}}\varphi(\xi)\,d\xi + \frac{1}{2\varepsilon}\int_x^\infty e^{(x-\xi)/\varepsilon^{1/2}}\varphi(\xi)\,d\xi.$$

Integrating the right-hand side by parts, there appears the formula

$$u_{xt}(x,0) = \frac{1}{\varepsilon^{1/2}}e^{-x/\varepsilon^{1/2}}[\varphi(0) - g_t(0)] + \int_0^\infty \tilde{M}_\varepsilon(x,\xi)\varphi_x(\xi)\,d\xi, \tag{4.31}$$

where

$$\tilde{M}_\varepsilon(x,\xi) = \frac{1}{2\varepsilon^{1/2}}\left[\exp(-|x-\xi|/\varepsilon^{1/2}) + \exp(-(x+\xi)/\varepsilon^{1/2})\right].$$

The integral on the right-hand side of (4.31) presents no difficulty. For it is readily verified that

$$\left\|\int_0^\infty \tilde{M}_\varepsilon(\cdot,\xi)\varphi_x(\xi)\,d\xi\right\| \le c\|\varphi_x\|,$$

where again c denotes a constant independent of ε and φ. The presumption (4.28) has the effect of eliminating the other, potentially troublesome term from the right-hand side of (4.31). Again taking account of the definition of φ, it follows that there is a constant $\tilde{a}$, depending on $\|f\|_4$, such that

$$\|u_{xt}(\cdot,0)\| \le \tilde{a}, \tag{4.32}$$

holding uniformly for ε in $(0,1]$. Taken together, (4.30) and (4.32) imply the desired result. □

Combining the imports of Lemmas 4.4 and 4.5 leads directly to the principal result of this section.

THEOREM 4.6. *Let $T>0$ be given, and let $f \in H^\infty(\mathbb{R}^+)$ and $g \in H^\infty(0,T)$ and suppose the compatibility conditions*

$$f(0) = g(0), \qquad g_t(0) + f_x(0) + f(0)f_x(0) + f_{xxx}(0) = 0$$

hold. Let u be the solution of the regularized initial- and boundary-value problem (4.1) *corresponding to the given data f and g. Then there is a constant a_6, depending on $\|f\|_4$ and $|g|_{2,T}$, such that*

$$\|u(\cdot,t)\|_3 + \|u_t(\cdot,t)\|_1 \le a_6,$$

for all t in $[0,T]$ and ε in $(0,\varepsilon_2]$. Here ε_2 is the positive constant arising in Lemma 4.4, *and so depends on $\|f\|_4$ and $|g|_{2,T}$ as well.*

Remarks. A somewhat stronger result than is stated in Theorem 4.6 is available from the foregoing analysis. This strengthened result has been eschewed, for simplicity and because it is not needed in what follows. Nevertheless, it is worth recording that

$$\varepsilon\|u_{xxxx}(\cdot,t)\|^2 + \int_0^T \left[u_{xxx}^2(0,s) + u_{xxxx}^2(0,s) + \varepsilon u_{xxt}^2(0,s) + u_{xt}^2(0,s)\right] ds \le (a_6)^2$$

as well, provided that ε lies in $(0,\varepsilon_2]$ and t lies in $[0,T]$. The constants ε_2 and a_6 are those specified in the statement of the last theorem.

The various constants appearing in the statements of results in this section may all be taken to depend continuously and monotonically on both T and the norms of the data that occur. This follows immediately upon examination of the presented proofs. Such an aspect is without crucial significance in what follows, and so will be passed over.

5. Higher-order estimates for the regularized problem. The derivation of ε-independent bounds for solutions of the regularized initial- and boundary-value problem (4.1) is continued in this section. The bounds established in §4 would be sufficient to establish an existence theory set in the space $L^\infty(0,T;H^4(\mathbb{R}^+))$ for the quarter-plane problem (1.3). Smoother solutions would be expected to obtain provided the initial and boundary data is appropriately restricted. A proof of such further regularity, presented in §6, is based on the additional estimates to be obtained in the present section.

The assumption that $f\in H^\infty(\mathbb{R}^+)$, $g\in H^\infty(0,T)$, and $f(0)=g(0)$ will continue to be enforced throughout this section. This hypothesis will be recalled informally by the stipulation that the data f and g is smooth and compatible. If j is a nonnegative integer, the notation

$$u^{(j)}=\partial_t^j u$$

will be convenient, and employed henceforth. This section consists of two technical lemmas, which lead directly to the principal goal, Theorem 5.3. The first technical result generalizes Lemma 4.4.

LEMMA 5.1. *Let $f\in H^\infty(\mathbb{R}^+)$ and $g\in H^\infty(0,T)$ be given, with $f(0)=g(0)$. Let u be the solution of* (4.1) *corresponding to the data f and g, and let k be a nonnegative integer. There is a constant*

$$b_1=b_1\Big(|g|_{k+2,T},\ \max_{0\le j\le k}\big\{\|u^{(j)}(\cdot,0)\|_4,\|u^{(j+1)}(\cdot,0)\|_1\big\}\Big),$$

depending continuously on its arguments, such that

$$\|u^{(k)}(\cdot,t)\|_3^2+\varepsilon\|u^{(k)}_{xxxx}(\cdot,t)\|^2 + \int_0^t\big\{[u^{(k)}_{xxx}(0,s)]^2+[u^{(k)}_{xxxx}(0,s)]^2+\varepsilon[u^{(k+1)}_{xx}(0,s)]^2\big\}\,ds\le b_1,$$

$$\|u^{(k+1)}(\cdot,t)\|_1^2+\int_0^t[u^{(k+1)}_x(0,s)]^2ds\le b_1,$$

for all t in $[0,T]$ and ε in $(0,\varepsilon_2]$. Here, ε_2 is specified in Lemma 4.4.

Proof. First note that for $k=0$, the desired result is implied by Lemma 4.4. The proof proceeds by induction on k. Let $k\ge 1$ be given, and suppose that the stated estimates hold for all nonnegative integers less than or equal to $k-1$. Let $v=u^{(k)}$, where u is the solution of the regularized initial- and boundary-value problem (4.1) corresponding to the given smooth and compatible data f and g. For t in $[0,T]$, define

$$A^2(t)=\sup_{0\le s\le t}\big\{\|v(\cdot,s)\|_3^2+\varepsilon\|v_{xxxx}(\cdot,s)\|^2\big\} + \int_0^t[v^2_{xxx}(0,s)+v^2_{xxxx}(0,s)+\varepsilon v^2_{xxt}(0,s)]\,ds$$

and

$$B^2(t)=\sup_{0\le s\le t}\big\{\|v_t(\cdot,s)\|_1^2\big\}+\int_0^t v^2_{xt}(0,s)\,ds.$$

The induction hypothesis implies that

$$\|u\|_{L^\infty(0,T;H^3(\mathbf{R}^+))},\ \|v\|_{L^\infty(0,T;H^1(\mathbf{R}^+))}\le c, \tag{5.1}$$
$$\|u\|_{L^\infty(0,T;W^{2,\infty}(\mathbf{R}^+))},\ \|v\|_{L^\infty(\mathbf{R}^+\times[0,T])}\le c,$$

where here, and in the remainder of this proof, c will denote various constants which all depend on the same variables as the constant b_1 given in the statement of the lemma, but which will always be independent of ε.

For any integer $j\ge 1$ the function $u^{(j)}$ satisfies the equation

$$u_t^{(j)}+u_x^{(j)}+\left(uu^{(j)}+h_j(u)\right)_x+u_{xxx}^{(j)}-\varepsilon u_{xxt}^{(j)}=0, \tag{5.2}$$

where

$$h_j(u)=\frac{1}{2}\sum_{i=1}^{j-1}\binom{j}{i}u^{(i)}u^{(j-i)}.$$

The induction hypothesis also implies that

$$\|h_k(u)\|_{L^\infty(0,T;W^{2,\infty}(\mathbf{R}^+))}\le c\|h_k(u)\|_{L^\infty(0,T;H^3(\mathbf{R}^+))}\le c. \tag{5.3}$$

The functions $A(t)$ and $B(t)$ will be estimated via an energy inequality derived from equation (5.2). Taking $j=k$, differentiate (5.2) once with respect to x, multiply by $-2v_{xxx}$ and integrate the resulting expression over $\mathbf{R}^+\times(0,t)$. The outcome of this process may be written

$$V_2(t)+\int_0^t\left[v_{xx}^2(0,s)+v_{xxx}^2(0,s)\right]ds \tag{5.4}$$
$$=V_2(0)-2\int_0^t v_{xt}(0,s)v_{xx}(0,s)\,ds+2\int_0^t\int_0^\infty\left[uv+h_k(u)\right]_{xx}v_{xxx}\,dx\,ds,$$

where $V_2(t)=\|v_{xx}(\cdot,t)\|^2+\varepsilon\|v_{xxx}(\cdot,t)\|^2$.

Inequalities (5.1) and (5.3) imply that

$$\int_0^t\int_0^\infty\left[uv+h_k(u)\right]_{xx}v_{xxx}\,dx\,ds\le c\left(1+\int_0^t\|v(\cdot,s)\|_3^2ds\right). \tag{5.5}$$

Because of (2.1) and (5.1), for any $\delta>0$, there is a constant c_δ such that for all t in $[0,T]$,

$$\|v\|_{L^\infty(0,t;W^{2,\infty}(\mathbf{R}^+))}\le c_\delta+\delta\left\{\sup_{0\le s\le t}\|v(\cdot,s)\|_3^2\right\}. \tag{5.6}$$

Combining (5.1), (5.2), (5.3), and (5.6), it follows that, for all $\delta>0$ and $t\in[0,T]$,

$$-\int_0^t v_{xt}(0,s)v_{xx}(0,s)\,ds$$
$$=\int_0^t\left\{v_{xx}(0,s)+\left[uv+h_k(u)\right]_{xx}(0,s)+v_{xxxx}(0,s)-\varepsilon v_{xxxt}(0,s)\right\}v_{xx}(0,s)\,ds$$
$$\le c_\delta+\delta\left\{\sup_{0\le s\le t}\|v(\cdot,s)\|_3^2+\int_0^t v_{xxxx}^2(0,s)\,ds\right\}-\varepsilon\int_0^t v_{xxxt}(0,s)v_{xx}(0,s)\,ds.$$

Together with (5.4) and (5.5) this implies that for all $\delta>0$ there is a constant c_δ such that

(5.7) $$V_2(t)+\int_0^t[v_{xx}^2(0,s)+v_{xxx}^2(0,s)]\,ds$$
$$\le c_\delta\left[1+\int_0^t A^2(s)\,ds\right]+\delta A^2(t)-2\varepsilon\int_0^t v_{xxxt}(0,s)v_{xx}(0,s)\,ds,$$

where A is defined above (5.1).

Next, differentiate (5.2), again with $j=k$, twice with respect to x, multiply by $-2v_{xxxx}$ and integrate over $\mathbf{R}^+\times(0,t)$. After suitable integrations by parts, there appears

(5.8) $$V_3(t)+\int_0^t[v_{xxx}^2(0,s)+v_{xxxx}^2(0,s)]\,ds$$
$$=V_3(0)-2\int_0^t v_{xxt}(0,s)v_{xxx}(0,s)\,ds+2\int_0^t\int_0^\infty[uv+h_k(u)]_{xxx}v_{xxxx}\,dx\,ds,$$

holding for all $t\in[0,T]$, and where

$$V_3(t)=\|v_{xxx}(\cdot,t)\|^2+\varepsilon\|v_{xxxx}(\cdot,t)\|^2.$$

Observe that

$$\int_0^t\int_0^\infty(uv)_{xxx}v_{xxxx}\,dx\,ds$$
$$=\int_0^t\int_0^\infty(uv_{xxx}+3u_xv_{xx}+3u_{xx}v_x+u_{xxx}v)v_{xxxx}\,dx\,ds$$
$$=-\int_0^t\left[\frac{1}{2}g(s)v_{xxx}^2(0,s)+3u_x(0,s)v_{xx}(0,s)v_{xxx}(0,s)\right.$$
$$\left.+3u_{xx}(0,s)v_x(0,s)v_{xxx}(0,s)+u_{xxx}(0,s)v(0,s)v_{xxx}(0,s)\right]ds$$
$$-\int_0^t\int_0^\infty\left[\frac{7}{2}u_xv_{xxx}^2+6u_{xx}v_{xx}v_{xxx}+4u_{xxx}v_xv_{xxx}+u_{xxxx}vv_{xxx}\right]dx\,ds.$$

The induction hypothesis and the fact that

$$\int_0^t\|v_x(\cdot,s)\|^2_{L^\infty(\mathbf{R}^+)}ds\le c\int_0^t A^2(s)\,ds$$

implies that there is a constant c such that

$$\int_0^t\int_0^\infty u_{xxx}v_xv_{xxx}\,dx\,ds\le\int_0^t\|v_x(\cdot,s)\|_{L^\infty(\mathbf{R}^+)}\|u_{xxx}(\cdot,s)\|\|v_{xxx}(\cdot,s)\|\,ds$$
$$\le c\int_0^t A^2(s)\,ds.$$

Also, it follows directly from the regularized equation (4.1a) that

$$u_{xxxx}=\varepsilon u_{xxx}^{(1)}-\left(uu_{xx}+u_x^2+u_{xx}+u_x^{(1)}\right).$$

Hence, from (5.1) and the induction hypothesis,

$$\int_0^t\int_0^\infty u_{xxxx}vv_{xxx}\,dx\,ds\le\int_0^t\|v(\cdot,s)\|_{L^\infty(\mathbf{R}^+)}\|u_{xxxx}(\cdot,s)\|\|v_{xxx}(\cdot,s)\|\,ds$$
$$\le c\int_0^t A^2(s)\,ds,$$

for all t in $[0,T]$. By (5.1) and the above estimates, it may now be concluded that

$$(5.9)\qquad \int_0^t\int_0^\infty (uv)_{xxx}v_{xxxx}\,dx\,ds\le c\left[1+\int_0^t A^2(s)\,ds+\int_0^t v_{xxx}^2(0,s)\,ds\right].$$

To estimate the rest of the third term on the right-hand side of (5.8), note that

$$\int_0^t\int_0^\infty (h_k(u))_{xxx}v_{xxxx}\,dx\,ds$$
$$=-\int_0^t (h_k(u))_{xxx}(0,s)v_{xxx}(0,s)\,ds-\int_0^t\int_0^\infty (h_k(u))_{xxxx}v_{xxx}\,dx\,ds.$$

Equation (5.2), once-differentiated with respect to x, is

$$u_{xxxx}^{(j)}=\varepsilon u_{xxx}^{(j+1)}-\left\{\left[uu^{(j)}+h_j(u)\right]_{xx}+u_{xx}^{(j)}+u_x^{(j+1)}\right\}.$$

Together with the induction hypothesis this relation implies that

$$\int_0^t \|(h_k(u))_{xxxx}(\cdot,s)\|^2 ds\le c\left[1+\varepsilon^2\int_0^t A^2(s)\,ds\right].$$

Therefore, using again the induction hypothesis and the estimate above, we may conclude that

$$(5.10)\qquad \int_0^t\int_0^\infty (h_k(u))_{xxx}v_{xxxx}\,dx\,ds\le c\left[1+\int_0^t A^2(s)\,ds+\int_0^t v_{xxx}^2(0,s)\,ds\right].$$

It remains to estimate the boundary term on the right-hand side of (5.8). The equation (5.2), with $j=k$ again, implies

$$-\int_0^t v_{xxt}(0,s)v_{xxx}(0,s)\,ds$$
$$=\int_0^t v_{xxt}(0,s)\{v_t(0,s)+v_x(0,s)+\left[uv+h_k(u)\right]_x(0,s)-\varepsilon v_{xxt}(0,s)\}\,ds.$$

Integrating by parts with respect to s yields the relation

$$\int_0^t v_{xxt}(0,s)\{v_t(0,s)+v_x(0,s)+\left[uv+h_k(u)\right]_x(0,s)\}\,ds$$
$$=v_{xx}(0,s)\{v_t(0,s)+v_x(0,s)+\left[uv+h_k(u)\right]_x(0,s)\}\Big|_{s=0}^{s=t}$$
$$-\int_0^t v_{xx}(0,s)\{v_{tt}(0,s)+v_{xt}(0,s)+\left[uv+h_k(u)\right]_{xt}(0,s)\}\,ds.$$

From (5.1), (5.3) and (5.6), and the fact that $v_{tt}(0,s)=g^{(k+2)}(s)$ and $v_t(0,s)=g^{(k+1)}(s)$, it thus appears that for any $\delta>0$ there is a constant c_δ such that

$$(5.11)\qquad -\int_0^t v_{xxt}(0,s)v_{xxx}(0,s)\,ds$$
$$\le c_\delta-\varepsilon\int_0^t v_{xxt}^2(0,s)\,ds+\delta A^2(t)-\int_0^t[1+g(s)]\,v_{xx}(0,s)v_{xt}(0,s)\,ds.$$

The estimates (5.8), (5.9), (5.10) and (5.11) and the identity

$$-v_{xt}=v_{xx}+\left[uv+h_k(u)\right]_{xx}+v_{xxxx}-\varepsilon v_{xxxt},$$

obtained from (5.2), now imply that, for all $\delta>0$, there is a constant c_δ such that for all $t\in[0,T]$,

$$V_3(t)+\int_0^t\left[v_{xxx}^2(0,s)+v_{xxxx}^2(0,s)+\varepsilon v_{xxt}^2(0,s)\right]ds$$
$$\leq c_\delta\left[1+\int_0^t A^2(s)\,ds+\int_0^t v_{xxx}^2(0,s)\,ds\right]+\delta A^2(t)$$
$$-2\varepsilon\int_0^t[1+g(s)]\,v_{xx}(0,s)v_{xxxt}(0,s)\,ds.$$

By adding this estimate and a suitable multiple of (5.7), and using the induction hypothesis again, it appears that for each $\delta>0$ there is a constant c_δ so that, for all t in $[0,T]$,

$$A^2(t)\leq c_\delta\left[1+\int_0^t A^2(s)\,ds\right]+\delta\varepsilon^2\int_0^t v_{xxxt}^2(0,s)\,ds. \tag{5.12}$$

Inequality (5.12) is not useful until the second integral is bounded. This may be accomplished by virtually the same argument as was used to bound the corresponding term appearing in the proof of Lemma 4.2. Differentiate (5.2), with $j=k$, twice with respect to x, multiply the result by $2\varepsilon v_{xxxt}$, and then integrate over $\mathbb{R}^+\times(0,t)$. This leads to the identity

$$\varepsilon\left\{\|v_{xxx}(\cdot,t)\|^2-\|v_{xxxx}(\cdot,t)\|^2\right\}+\varepsilon^2\int_0^t v_{xxxt}^2(0,s)\,ds \tag{5.13}$$
$$=\varepsilon\left\{\|v_{xxx}(\cdot,0)\|^2-\|v_{xxxx}(\cdot,0)\|^2\right\}+\varepsilon\int_0^t v_{xxt}^2(0,s)\,ds$$
$$+2\varepsilon\int_0^t v_{xxxx}(0,s)v_{xxxt}(0,s)\,ds-2\varepsilon\int_0^t\int_0^\infty[uv+h_k(u)]_{xxx}v_{xxxt}\,dx\,ds.$$

Since (5.2) implies that

$$\varepsilon\int_0^t\int_0^\infty[uv+h_k(u)]_{xxx}v_{xxxt}\,dx\,ds$$
$$=\int_0^t\int_0^\infty[uv+h_k(u)]_{xxx}\left\{v_{xxxx}+[uv+h_k(u)]_{xx}+v_{xx}+v_{xt}\right\}dx\,ds,$$

it follows from (5.9), (5.10) and the induction hypothesis that for all $t\in[0,T]$,

$$\varepsilon\int_0^t\int_0^\infty[uv+h_k(u)]_{xxx}v_{xxxt}\,dx\,ds$$
$$\leq c\left\{1+\int_0^t[A^2(s)+A(s)B(s)]\,ds+\int_0^t v_{xxx}^2(0,s)\,ds\right\}.$$

In consequence of (5.12) and (5.13) we therefore infer the existence of a constant c such that

$$A^2(t)\leq c\left\{1+\int_0^t[A^2(s)+A(s)B(s)]\,ds\right\}, \tag{5.14}$$

for all $t\in[0,T]$.

Next $B(t)$ will be estimated. Let $w=u^{(k+1)}$. By (5.2) w satisfies the equation

$$w_t+w_x+[uw+h_{k+1}(u)]_x+w_{xxx}-\varepsilon w_{xxt}=0. \tag{5.15}$$

Multiply this equation by $2w$ and integrate over $\mathbf{R}^+\times(0,t)$ to obtain

$$\begin{aligned}\|w(\cdot,t)\|^2+\varepsilon\|w_x(\cdot,t)\|^2+\int_0^t w_x^2(0,s)\,ds\\ =\|w(\cdot,0)\|^2+\varepsilon\|w_x(\cdot,0)\|^2+\int_0^t[1+g(s)]\,w^2(0,s)\,ds\\ +2\int_0^t w(0,s)[w_{xx}(0,s)-\varepsilon w_{xt}(0,s)]\,ds\\ -\int_0^t\int_0^\infty\{u_xw^2+2w[h_{k+1}(u)]_x\}\,dx\,ds.\end{aligned}$$

The induction hypothesis therefore implies that, for all $\delta>0$, there is a constant c_8 such that

$$\text{(5.16)}\quad \|w(\cdot,t)\|^2+\varepsilon\|w_x(\cdot,t)\|^2+\int_0^t w_x^2(0,s)\,ds \\ \leq c_8\left[1+\int_0^t B^2(s)\,ds\right]+\delta\int_0^t[w_{xx}(0,s)-\varepsilon w_{xt}(0,s)]^2ds,$$

for all $t\in[0,T]$. To complete the satisfactory estimation of $B(t)$, multiply (5.15) by $2(\varepsilon w_{xt}-uw-w_{xx})$ and integrate over $\mathbf{R}^+\times(0,t)$. This yields

$$\begin{aligned}\text{(5.17)}\quad (1+\varepsilon)\|w_x(\cdot,t)\|^2-\int_0^\infty w^2(x,t)u(x,t)\,dx\\ +\int_0^t\{w_x^2(0,s)+[w_{xx}(0,s)-\varepsilon w_{xt}(0,s)]^2\}\,ds\\ =(1+\varepsilon)\|w_x(\cdot,0)\|^2-\int_0^\infty w^2(x,0)f(x)\,dx\\ +\int_0^t[\varepsilon w_t^2(0,s)-g^2(s)w^2(0,s)]\,ds\\ -2\int_0^t w_t(0,s)w_x(0,s)\,ds-2\int_0^t g(s)w(0,s)[w_{xx}(0,s)-\varepsilon w_{xt}(0,s)]\,ds\\ +\int_0^t\int_0^\infty\{2uww_x-u_tw^2+2[h_{k+1}(u)]_x(w_{xx}+uw-\varepsilon w_{xt})\}\,dx\,ds.\end{aligned}$$

Integration by parts implies that

$$\begin{aligned}\int_0^t\int_0^\infty[h_{k+1}(u)]_xw_{xx}\,dx\\ =-\int_0^t[h_{k+1}(u)]_x(0,s)w_x(0,s)\,ds-\int_0^t\int_0^\infty[h_{k+1}(u)]_{xx}w_x\,dx\,ds,\end{aligned}$$

and that

$$\begin{aligned}\varepsilon\int_0^t\int_0^\infty[h_{k+1}(u)]_xw_{xt}\,dx\,ds=\varepsilon\int_0^\infty[h_{k+1}(u)]_x(x,s)w_x(x,s)\,dx\Big|_{s=0}^{s=t}\\ -\varepsilon\int_0^t\int_0^\infty[h_{k+1}(u)]_{xt}w_x\,dx\,ds.\end{aligned}$$

Hence, it follows from the induction hypothesis that for all $t\in[0,T]$,

$$\int_0^t\int_0^\infty [h_{k+1}(u)]_x(w_{xx}+uw-\varepsilon w_{xt})\,dx\,ds$$

$$\leq \varepsilon^2\|w_x(\cdot,t)\|^2+c\left\{1+\int_0^t[B^2(s)+A(s)B(s)]\,ds+\int_0^t w_x^2(0,s)\,ds\right\}.$$

Therefore, if (5.17) is added to a suitable multiple of (5.16), it follows that

$$B^2(t)\leq c\left\{1+\int_0^t[B^2(s)+A(s)B(s)]\,ds\right\} \tag{5.18}$$

for all $t\in[0,T]$ and all ε in $(0,\varepsilon_2]$. Here, without loss of generality, ε_2 has been presumed to be strictly less than 1.

From (5.14), (5.18) and Gronwall's lemma it now follows that there is a constant c such that

$$A(t),\,B(t)\leq c$$

for all $t\in[0,T]$. This completes the induction argument and hence the proof of Lemma 5.1. □

The bounds established in Lemma 5.1 are just what will be needed in §6, except that, so far as is known now, not all the arguments of the constant b_1 are independent of ε. To attain the goal for this section, it will suffice to give conditions on the data f and g which imply that $\|u^{(j)}(\cdot,0)\|_4$ and $\|u^{(j+1)}(\cdot,0)\|_1$, $0\leq j\leq k$, are bounded, independently of ε sufficiently small. This amounts to extending Lemma 4.5.

We have not succeeded in giving an absolutely straightforward generalization of Lemma 4.5 to the case $j>0$. However, by modifying the data, in an ε-dependent way, a result is obtained which is sufficient for our purposes in the next section. Before stating this lemma, some convenient notation is introduced.

Let $\varphi^{(0)}(x)=f(x)$, and for each integer $j\geq 1$ define functions $\varphi^{(j)}$ inductively by the recurrence

$$\varphi^{(j+1)}=-\left[\varphi_x^{(j)}+\varphi_{xxx}^{(j)}+\frac{1}{2}\left(\sum_{i=0}^{j}\binom{j}{i}\varphi^{(i)}\varphi^{(j-i)}\right)_x\right]. \tag{5.19}$$

Also, for nonnegative integers j, let

$$g^{(j)}(t)=\partial_t^j g(t).$$

Here is the result alluded to above.

LEMMA 5.2. *Let $f\in H^\infty(\mathbf{R}^+)$ and $g\in H^\infty(0,T)$ be given, with $f(0)=g(0)$. Let $k\geq 1$ be a given integer and suppose additionally that*

$$g^{(j)}(0)=\varphi^{(j)}(0)\quad \textit{for } j=1,2,\cdots,k.$$

Then there exists a family $\{g_\varepsilon\}_{0<\varepsilon\leq 1}$ in $H^\infty(0,T)$ such that

(i) $g_\varepsilon(0)=g(0)$ *and* $\lim_{\varepsilon\to 0}|g_\varepsilon-g|_{k+1,T}=0$;

(ii) *there exists a constant b_2, depending continuously on $\|f\|_{3k+1}$, such that*

$$\|u_\varepsilon^{(j)}(\cdot,0)\|_{3(k-j)+1}\leq b_2$$

for $0\leq j\leq k$ and all $\varepsilon\in(0,1]$, where u_ε denotes the solution of (4.1) *with initial data f and boundary data g_ε.*

Proof. First, two sequences of functions $\{\varphi_\varepsilon^{(j)}\}_{1\le j\le k}$ and $\{w_\varepsilon^{(j)}\}_{1\le j\le k}$ are introduced. These will be used momentarily to define the modified boundary data $g_\varepsilon(t)$. If j is an integer in the range $[0,k]$, let $\nu(j)=[3(k-j)/2]$ and define $w_\varepsilon^{(j)}$ and $\varphi_\varepsilon^{(j)}$ on $\mathbf{R}^+$ by $w_\varepsilon^{(0)}=\varphi_\varepsilon^{(0)}=f$ and, recursively for $j>0$,

$$\varphi_\varepsilon^{(j)}=-\left[\left(w_\varepsilon^{(j-1)}\right)_x+\left(w_\varepsilon^{(j-1)}\right)_{xxx}+\frac{1}{2}\sum_{i=0}^{j-1}\binom{j-1}{i}\left(w_\varepsilon^{(i)}w_\varepsilon^{(j-i-1)}\right)_x\right] \tag{5.20}$$

and

$$w_\varepsilon^{(j)}=\exp(-x/\varepsilon^{1/2})\sum_{i=0}^{\nu(j)}\varepsilon^i\left(\partial_x^{2i}\varphi_\varepsilon^{(j)}\right)(0)+\int_0^\infty M_\varepsilon(x,\xi)\varphi_\varepsilon^{(j)}(\xi)\,d\xi. \tag{5.21}$$

Here, as in the proof of Lemma 4.5,

$$M_\varepsilon(x,\xi)=\frac{1}{2\varepsilon^{1/2}}\left[\exp(-|x-\xi|/\varepsilon^{1/2})-\exp(-(x+\xi)/\varepsilon^{1/2})\right]$$

and

$$\tilde{M}_\varepsilon(x,\xi)=\frac{1}{2\varepsilon^{1/2}}\left[\exp(-|x-\xi|/\varepsilon^{1/2})+\exp(-(x+\xi)/\varepsilon^{1/2})\right].$$

Note that $w_\varepsilon^{(j)}$ has been determined as the solution of the boundary-value problem

$$v-\varepsilon v_{xx}=\varphi_\varepsilon^{(j)}, \tag{5.22}$$

with

$$v(0)=\lambda_\varepsilon^{(j)}\quad\text{and}\quad\lim_{x\to+\infty}v(x)=0,$$

where

$$\lambda_\varepsilon^{(j)}=\sum_{i=0}^{\nu(j)}\varepsilon^i\left(\partial_x^{2i}\varphi_\varepsilon^{(j)}\right)(0),$$

for $j=1,2,\cdots,k$.

By differentiating (5.21) the following identities are obtained, for all integers $r\ge 1$,

$$\left(\partial_x^{2r+1}w_\varepsilon^{(j)}\right)(x)=\exp(-x/\varepsilon^{1/2})\varepsilon^{-(r+1/2)}\left[\sum_{i=0}^{r}\varepsilon^i\left(\partial_x^{2i}\varphi_\varepsilon^{(j)}\right)(0)-\lambda_\varepsilon^{(j)}\right] \tag{5.23a}$$
$$+\int_0^\infty\tilde{M}_\varepsilon(x,\xi)\left(\partial_x^{2r+1}\varphi_\varepsilon^{(j)}\right)(\xi)\,d\xi$$

and

$$\left(\partial_x^{2r}w_\varepsilon^{(j)}\right)(x)=\exp(-x/\varepsilon^{1/2})\varepsilon^{-r}\left[\lambda_\varepsilon^{(j)}-\sum_{i=0}^{r-1}\varepsilon^i\left(\partial_x^{2i}\varphi_\varepsilon^{(j)}\right)(0)\right] \tag{5.23b}$$
$$+\int_0^\infty M_\varepsilon(x,\xi)\left(\partial_x^{2r}\varphi_\varepsilon^{(j)}\right)(\xi)\,d\xi.$$

Hence, there is a constant c, independent of $w_\varepsilon^{(j)}$, $\varphi_\varepsilon^{(j)}$ and ε, such that

$$\|w_\varepsilon^{(j)}\|_{3(k-j)+1}\le c\|\varphi_\varepsilon^{(j)}\|_{3(k-j)+1}, \tag{5.24}$$

for $0 \le j \le k$. Using (5.20), (5.24) and a simple inductive argument, it follows that there is a constant $b_2 = b_2(\|f\|_{3k+1})$ such that

$$\|w_\varepsilon^{(j)}\|_{3(k-j)+1}, \ \|\varphi_\varepsilon^{(j)}\|_{3(k-j)+1} \le b_2, \tag{5.25}$$

independently of ε in $(0,1]$ and j in $[0,k]$.

For each $\varepsilon \in (0,1]$ define modified boundary data $g_\varepsilon(t)$ by

$$g_\varepsilon(t) = g(t) + \sum_{j=1}^{k} \frac{t^j}{j!}\left[\lambda_\varepsilon^{(j)} - \varphi^{(j)}(0)\right].$$

Observe that $g_\varepsilon(0) = g(0)$. Also, since $g^{(j)}(0) = \varphi^{(j)}(0)$ by assumption,

$$g_\varepsilon^{(j)}(0) = \lambda_\varepsilon^{(j)}, \tag{5.26}$$

for $1 \le j \le k$.

Now let u_ε denote the solution of (4.1) with initial data f and boundary data g_ε. It follows inductively from (5.20), (5.22) and (5.26) that $u_\varepsilon^{(j)}(\cdot, 0) = w_\varepsilon^{(j)}$ for $0 \le j \le k$, and hence the desired bounds on $u_\varepsilon^{(j)}(\cdot, 0)$ follow from (5.25).

To complete the proof it is only required to check that

$$\lim_{\varepsilon \downarrow 0} |g_\varepsilon - g|_{k+1,T} = 0.$$

Because of the definition of g_ε, this is equivalent to showing that

$$\lim_{\varepsilon \downarrow 0} |\lambda_\varepsilon^{(j)} - \varphi^{(j)}(0)| = 0,$$

for $0 \le j \le k$. Referring to the definition of $\lambda_\varepsilon^{(j)}$ below (5.22), and keeping in mind the bounds in (5.25) and the simple inequality (2.5), we see that

$$\lambda_\varepsilon^{(j)} = \varphi_\varepsilon^{(j)}(0) + O(\varepsilon),$$

as $\varepsilon \downarrow 0$, for $0 \le j \le k$. More precisely,

$$|\lambda_\varepsilon^{(j)} - \varphi_\varepsilon^{(j)}(0)| \le c\varepsilon \|\varphi_\varepsilon^{(j)}\|_{3(k-j)+1} \le cb_2\varepsilon. \tag{5.27}$$

Hence it is enough to show that

$$\lim_{\varepsilon \downarrow 0} |\varphi_\varepsilon^{(j)}(0) - \varphi^{(j)}(0)| = 0,$$

for $0 \le j \le k$. This latter relation will be proved by establishing that the estimate

$$\|\varphi_\varepsilon^{(i)} - \varphi^{(i)}\|_{W^{3(k-i),\infty}(\mathbf{R}^+)} \le c\varepsilon^{1/4}, \tag{5.28}$$

holds for $0 \le i \le k$, where the constant $c = c(\|f\|_{3k+1})$.

The inequality (5.28) is proved by induction on i. For $i = 0$ and $i = 1$, (5.28) follows since $\varphi_\varepsilon^{(0)} = \varphi^{(0)} = f$ and $\varphi_\varepsilon^{(1)} = \varphi^{(1)}$. Assume (5.28) holds for $i \le j$, where $1 \le j < k$. In order to establish the result for $i = j+1$, note first that the definitions (5.19) and (5.20) imply that

$$\|\varphi_\varepsilon^{(j+1)} - \varphi^{(j+1)}\|_{W^{3(k-j-1),\infty}(\mathbf{R}^+)} \le c\left\{ \sup_{0 \le i \le j} \|w_\varepsilon^{(i)} - \varphi^{(i)}\|_{W^{3(k-i),\infty}(\mathbf{R}^+)} \right\},$$

where $c = c(\|f\|_{3k+1})$. Since

$$\|\varphi_\varepsilon^{(i)} - \varphi^{(i)}\|_{W^{3(k-i),\infty}(\mathbf{R}^+)} \le c\varepsilon^{1/4},$$

for $0 \le i \le j$, by the induction hypothesis, (5.28) will follow if it can be demonstrated that, for $0 \le i \le j$,

$$\|\varphi_\varepsilon^{(i)} - w_\varepsilon^{(i)}\|_{W^{3(k-i),\infty}(\mathbf{R}^+)} \le c\varepsilon^{1/4}, \tag{5.29}$$

where again $c = c(\|f\|_{k+1})$. The fact that $w_\varepsilon^{(i)}$ solves (5.22) means that

$$w_\varepsilon^{(i)}(x) - \varphi_\varepsilon^{(i)}(x) = \exp(-x/\varepsilon^{1/2})\left[\lambda_\varepsilon^{(i)} - \varphi_\varepsilon^{(i)}(0)\right] + \varepsilon\int_0^\infty M_\varepsilon(x,\xi)\partial_x^2\varphi_\varepsilon^{(i)}(\xi)\,d\xi.$$

Differentiating this relation with respect to x, in the same way that (5.21) was differentiated to yield (5.23a, b), and using (5.27), we readily obtain the estimate,

$$\|w_\varepsilon^{(i)} - \varphi_\varepsilon^{(i)}\|_{3(k-i)-1} \le c\varepsilon\|\varphi_\varepsilon^{(i)}\|_{3(k-i)+1},$$

where the constant c is independent of $w_\varepsilon^{(i)}$, $\varphi_\varepsilon^{(i)}$ and ε. The bounds expressed in (5.25) thus imply that

$$\|w_\varepsilon^{(i)} - \varphi_\varepsilon^{(i)}\|_{3(k-i)-1} \le c\varepsilon, \tag{5.30}$$

where $c = c(\|f\|_{3k+1})$. Also implied by (5.25), and the triangle inequality, is the estimate

$$\|w_\varepsilon^{(i)} - \varphi_\varepsilon^{(i)}\|_{3(k-i)+1} \le c, \tag{5.31}$$

where $c = c(\|f\|_{3k+1})$. Standard results in the interpolation-theory of Banach spaces now come to our rescue (cf. (2.5) and [19, Chap. 1]). Thus, if h denotes $\varphi_\varepsilon^{(i)} - w_\varepsilon^{(i)}$, then

$$\begin{aligned}\|h\|_{W^{3(k-i),\infty}(\mathbf{R}^+)} &\le \|h\|_{3(k-i)}^{1/2}\|h\|_{3(k-i)+1}^{1/2}\\ &\le c\|h\|_{3(k-i)-1}^{1/4}\|h\|_{3(k-i)+1}^{3/4} \le c\varepsilon^{1/4},\end{aligned}$$

where $c = c(\|f\|_{3k+1})$. This completes the induction argument in favor of (5.28), and thus finishes the proof of the lemma. □

The outcome of Lemmas 5.1 and 5.2 is conveniently collected in the following theorem. This is, in effect, a higher-order analogue of Theorem 4.6. In the statement of the theorem, ε_2 is the same positive constant that already appeared in Theorem 4.6.

THEOREM 5.3. *Let $T > 0$ and a positive integer k be given. Let $f \in H^\infty(\mathbf{R}^+)$ and $g \in H^\infty(0,T)$ and suppose that $g^{(j)}(0) = \varphi^{(j)}(0)$, for $0 \le j \le k$, where the functions $\varphi^{(j)}$ are related to f as in* (5.19). *Then there exists a family $\{g_\varepsilon\}_{0<\varepsilon\le\varepsilon_2}$ in $H^\infty(0,T)$ such that*

(i) $g_\varepsilon(0) = g(0)$, $\lim_{\varepsilon\downarrow 0}|g_\varepsilon - g|_{k+1,T} = 0$;

(ii) *there exists a constant $b_3 = b_3(\|f\|_{3k+1}, |g|_{k+1,T})$, depending continuously on its arguments, such that*

$$\begin{aligned}&\|u^{(j-1)}(\cdot,t)\|_3^2 + \varepsilon\|\partial_x^4 u^{(j-1)}(\cdot,t)\|^2 + \|u^{(j)}(\cdot,t)\|_1^2\\ &\quad + \int_0^t\left\{\left[\partial_x^4 u^{(j-1)}(0,s)\right]^2 + \left[\partial_x^3 u^{(j-1)}(0,s)\right]^2 + \left[\partial_x u^{(j)}(0,s)\right]^2\right.\\ &\qquad\qquad \left. + \varepsilon\left[\partial_x^2 u^{(j)}(0,s)\right]^2\right\} ds \le b_3\end{aligned}$$

holds for $1 \le j \le k$ and all ε in $(0, \varepsilon_2]$. Here, $u^{(j-1)}(x,t) = \partial_t^{j-1}u_\varepsilon(x,t)$ and u_ε denotes the solution of (4.1) *with initial data f and boundary data g_ε.*

6. Existence and uniqueness of solution. The major undertaking of this paper is to prove existence of smooth solutions of the quarter-plane problem for the KdV equation. Using the theory developed in §§3, 4 and 5, this task becomes comparatively simple. Recall that a function $u=u(x,t)$ is sought such that

$$u_t+u_x+uu_x+u_{xxx}=0 \quad \text{for } x,t>0, \tag{6.1a}$$

subject to the auxiliary conditions,

$$\begin{aligned} u(x,0)&=f(x) \quad \text{for } x\geq 0,\\ u(0,t)&=g(t) \quad \text{for } t\geq 0, \end{aligned} \tag{6.1b}$$

where f and g are given functions.

The issue of uniqueness of solutions of this initial- and boundary-value problem is especially straightforward to settle. As the uniqueness of solutions of (6.1) is useful later, it is established first.

THEOREM 6.1. *Let $T>0$ and $s>\frac{3}{2}$. Then, corresponding to given auxiliary data f and g, there is at most one solution of* (6.1) *in the function class $L^\infty(0,T;H^s(\mathbf{R}^+))$.*

Remarks. As usual in this paper, we mean, at the outset, by the word *solution* a distributional solution of (6.1a) for which the auxiliary conditions (6.1b) can be given a well-defined sense. Of course if u is a distributional solution of (6.1a) which is additionally known to lie in a class of smooth functions, it will follow that u is a classical solution of the differential equation. This point will be amplified later in this section.

Proof. Suppose that $u,v\in L^\infty(0,T;H^s(\mathbf{R}^+))$ are both solutions of (6.1) corresponding to the same data f and g. The $H^s(\mathbf{R}^+)$-norm of u and v is thus essentially bounded on $[0,T]$. In particular, for almost every t in $[0,T]$, $u(\cdot,t)$, $v(\cdot,t)\in H^s(\mathbf{R}^+)$. Invoking the Sobolev embedding results (cf. [19, Chap. 1]), it may therefore be supposed that, for almost every t in $[0,T]$, $u(\cdot,t)$, $u_x(\cdot,t)$, $v(\cdot,t)$ and $v_x(\cdot,t)$ are bounded and uniformly continuous functions on $\bar{\mathbf{R}}^+$. Moreover, u,u_x, v and v_x are essentially bounded on $\bar{\mathbf{R}}^+\times[0,T]$. From this it follows straightforwardly that both u and v converge, in $L^\infty(0,T)$, in the limit as $x\downarrow 0$. Thus the boundary value in (6.1b) is taken on meaningfully.

Let $w=u-v$ and $\chi=\frac{1}{2}(u+v)$. Then w is a distributional solution of the linear variable-coefficient differential equation

$$w_t+w_x+(\chi w)_x+w_{xxx}=0 \quad \text{in } \mathbf{R}^+\times(0,T), \tag{6.2a}$$

which satisfies the auxiliary conditions

$$w(x,0)=0 \quad \text{for } x\in\mathbf{R}^+, \qquad w(0,t)=0 \quad \text{for } t \text{ in } [0,T]. \tag{6.2b}$$

The boundary condition in (6.2b) holds at least in $L^\infty(0,T)$, whereas it will appear presently that the initial condition is valid at least in the sense that $\|w(\cdot,t)\|\to 0$, as $t\downarrow 0$.

Since $H^q(\mathbf{R}^+)$ is linearly and continuously embedded in $H^s(\mathbf{R}^+)$, for $q>s$, we may, without loss of generality, suppose that $s<3$ and let $r=3-s$. Note that $0<r<3/2$. Note also that w_x and $(\chi w)_x$ lie in $L^\infty(0,T;H^{s-1}(\mathbf{R}^+))$ and that w_{xxx} lies in $L^\infty(0,T;H^{-r}(\mathbf{R}^+))$. From (6.2a) it is thus apparent that w_t lies in $L^\infty(0,T;H^{-r}(\mathbf{R}^+))$.

The spaces $H_0^r(\mathbf{R}^+)$ and $H^{-r}(\mathbf{R}^+)$ are viewed as being in duality in the usual manner. The pairing between them is denoted by sharp brackets $\langle\cdot,\cdot\rangle$. (For a detailed exposition of these spaces, and the duality between them, the reader is urged to consult the first two chapters of Lions and Magenes [19].) Note especially that since, for almost every t in $[0,T]$, $w\in H^s(\mathbf{R}^+)$ and $w(0,t)=0$, it follows that $w\in H_0^r(\mathbf{R}^+)$, for almost every t in $[0,T]$. Thus $w\in L^\infty(0,T;H^s(\mathbf{R}^+)\cap H_0^r(\mathbf{R}^+))$. For this, it is crucial that $r<3/2$ of course. Otherwise a second boundary condition $w_x(0,t)=0$ would be implied by membership in $H_0^r(\mathbf{R}^+)$.

In this situation, it is a standard result (cf. [18, p. 71]) that $w\in C(0,T;L^2(\mathbf{R}^+))$, and that

$$\frac{1}{2}\frac{d}{dt}\|w(\cdot,t)\|^2=\langle w,w_t\rangle. \tag{6.3}$$

Thus, in particular, the initial value in (6.1b) or (6.2b) is taken on meaningfully. The right-hand side of (6.3) lies in $L^1(0,T)$. Hence $\|w(\cdot,t)\|^2$ is absolutely continuous, and upon integrating (6.3) over $[0,t]$, using the equation (6.2a) and the zero initial condition in (6.2b), there appears

$$\frac{1}{2}\|w(\cdot,t)\|^2=-\int_0^t\langle w,w_x+(\chi w)_x+w_{xxx}\rangle\,d\tau. \tag{6.4}$$

Since w_x and $(\chi w)_x$ are continuous square-integrable functions, for almost every t, and $w(0,t)=0$, it is straightforward that

$$\langle w,w_x\rangle=\int_0^\infty w(x,t)w_x(x,t)\,dx=0,$$

and that

$$\begin{aligned}\langle w,(\chi w)_x\rangle&=\int_0^\infty w(x,t)[\chi(x,t)w(x,t)]_x\,dx\\&=\frac{1}{2}\int_0^\infty w^2(x,t)\chi_x(x,t)\,dx\\&\le\|\chi_x\|_{L^\infty(\mathbf{R}^+\times(0,T))}\|w(\cdot,t)\|^2\le M\|w(\cdot,t)\|^2,\end{aligned}$$

where

$$M=\frac{1}{2}\|u+v\|_{L^\infty(0,T;H^s(\mathbf{R}^+))}.$$

In the last step, the fact that $s>\frac{3}{2}$ was vital. Finally, we claim that $\langle w,w_{xxx}\rangle\ge 0$, for almost every t in $[0,T]$. Fix t and let $h(\cdot)=w(\cdot,t)$. Then $h\in H^s(\mathbf{R}^+)\cap H_0^r(\mathbf{R}^+)$. Let $\tilde{h}$ be a function in $H^\infty(\mathbf{R}^+)$, say, such that

$$\partial_x^j\tilde{h}(0)=\partial_x^j h(0)\quad\text{for } 0\le j<s-\frac{1}{2}.$$

Then $h-\tilde{h}\in H_0^s(\mathbf{R}^+)$. Hence there is a sequence $\{\psi_n\}_1^\infty$ in $\mathfrak{D}(\mathbf{R}^+)$ such that $\psi_n\to h-\tilde{h}$ in the $H^s(\mathbf{R}^+)$-norm, as $n\to\infty$. Let $h_n=\psi_n+\tilde{h}$. The sequence $\{h_n\}_1^\infty$ has the following properties:

(i) $h_n\in H^\infty(\mathbf{R}^+)$ and $h_n(0)=0$, for all n;

(ii) $h_n\to h$ in $H^s(\mathbf{R}^+)$, as $n\to\infty$.

Then $\partial_x^3h_n\to\partial_x^3h$ in $H^{-r}(\mathbf{R}^+)$ and $h_n\to h$ in $H_0^r(\mathbf{R}^+)$, as $n\to\infty$. Hence,

$$\begin{aligned}\langle h,h_{xxx}\rangle=\lim_{n\to\infty}\langle h_n,\partial_x^3h_n\rangle&=\lim_{n\to\infty}\int_0^\infty h_n(x)\partial_x^3h_n(x)\,dx\\&=\lim_{n\to\infty}\left\{-\int_0^\infty\partial_xh_n(x)\partial_x^2h_n(x)\,dx\right\}\\&=\lim_{n\to\infty}\frac{1}{2}[\partial_xh_n(0)]^2\ge 0.\end{aligned}$$

Putting together the pieces, there appears

$$\|w(\cdot,t)\|^2 \le M\int_0^t \|w(\cdot,\tau)\|^2 d\tau,$$

for t in $[0,T]$. Gronwall's lemma thus implies that $\|w(\cdot,t)\| \equiv 0$ on $[0,T]$, whence $w=0$ and so $u=v$, as required. □

Attention is now turned to the existence theory. It is convenient to recall here the notation introduced in §5. Namely, if f is a given sufficiently smooth function defined on $\bar{\mathbf{R}}^+$, then set $\varphi^{(0)}=f$,

$$\varphi^{(1)}(x) = -\left(f(x)+\frac{1}{2}f^2(x)+f_{xx}(x)\right)_x, \tag{6.5a}$$

and inductively,

$$\varphi^{(j+1)}(x) = -\left(\varphi_x^{(j)}+\varphi_{xxx}^{(j)}+\frac{1}{2}\left(\sum_{i=0}^{j}\varphi^{(i)}\varphi^{(j-i)}\right)_x\right). \tag{6.5b}$$

Remember that $\varphi^{(j)}(0)=g^{(j)}(0)$, where $g^{(j)}(t)=\partial_t^j g(t)$ as before, is just the jth-order compatibility condition implied by the KdV equation (6.1a) for solutions that are sufficiently smooth at the origin (0,0). Here is the main result.

THEOREM 6.2. *Let k be a positive integer, $f \in H^{3k+1}(\mathbf{R}^+)$ and $g \in H_{\text{loc}}^{k+1}(\mathbf{R}^+)$. Suppose the $k+1$ compatibility conditions*

$$g^{(j)}(0)=\varphi^{(j)}(0) \quad \textit{for } 0\le j\le k,$$

hold, where $\varphi^{(j)}$ is defined above. Then there exists a unique solution u in $L_{\text{loc}}^{\infty}(\mathbf{R}^+; H^{3k+1}(\mathbf{R}^+))$ of (6.1) corresponding to the data f and g. In case $k>1$, u defines a classical solution, up to the boundary, of (6.1) in the quarter-plane $\mathbf{R}^+ \times \mathbf{R}^+$.

The proof of this result relies on the theory for the regularized problem developed in §§3, 4, and culminating in Theorem 5.3. To make use of the last-quoted result, the following technical lemma seems essential.

LEMMA 6.3. *Let f and g be as in Theorem 6.2. Then there exist sequences $\{f_N\}_1^\infty \subseteq H^\infty(\mathbf{R}^+)$ and $\{g_N\}_1^\infty \subseteq C^\infty(\mathbf{R}^+)$ such that*

(i) *$g_N^{(j)}(0)=\varphi_N^{(j)}(0)$ for $0\le j\le k$;*

(ii) *$f_N \to f$ in $H^{3k+1}(\mathbf{R}^+)$, $g_N \to g$ in $H_{\text{loc}}^{k+1}(\mathbf{R}^+)$.*

Here $\varphi_N^{(j)}$ is as defined in (6.5) with f_N replacing f and $g_N^{(j)}=\partial_t^j g_N$.

Proof. Let $\{f_N\}_1^\infty \subseteq H^\infty(\mathbf{R}^+)$ and $\{h_N\}_1^\infty \subseteq C^\infty(\mathbf{R}^+)$ satisfy condition (ii) in the statement of the lemma, relative to f and g, respectively. Define

$$a_j^N = h_N^{(j)}(0)-\varphi_N^{(j)}(0) \quad \text{for } 0\le j\le k,$$

where $h_N^{(j)}=\partial_t^j h_N$ and $\varphi_N^{(j)}$ is given as in (6.5). Then set

$$g_N(t)=h_N(t)-P_N(t),$$

where

$$P_N(t)=\sum_{j=0}^{k} a_j^N \frac{t^j}{j!}.$$

By construction, for $0\le j\le k$,

$$g_N^{(j)}(0)=h_N^{(j)}(0)-a_j^N=\varphi_N^{(j)}(0).$$

Moreover, $g_N \in C^\infty(\mathbf{R}^+)$, for each N. It remains to verify that $g_N \to g$ in $H^{k+1}_{\mathrm{loc}}(\mathbf{R}^+)$. This will be true if and only if $P_N \to 0$ in $H^{k+1}_{\mathrm{loc}}(\mathbf{R}^+)$. But, for $0 \le j \le k$,

$$\lim_{N\to\infty} a_j^N = \lim_{N\to\infty} \left[h_N^{(j)}(0) - \varphi_N^{(j)}(0)\right] = 0,$$

since f and g satisfy $k+1$ compatibility conditions. Let $T>0$ be given. Then

$$\|P_N\|_{H^{k+1}(0,T)} \le \sum_{j=0}^{k} \left|a_j^N\right| \frac{1}{j!} \|t^j\|_{H^{k+1}(0,T)} \le \sum_{j=0}^{k} M_j \left|a_j^N\right|,$$

where the constants M_j depend only on j and T. Since $a_j^N \to 0$, as $N \to +\infty$, for each j, it follows that

$$\|P_N\|_{H^{k+1}(0,T)} \to 0,$$

as $N \to +\infty$. Since $T>0$ was arbitrary, the lemma is established. □

The next step in the proof of Theorem 6.2 is to establish that solutions of (6.1) exist in case f and g happen to be infinitely smooth.

PROPOSITION 6.4. *Let there be given a positive number T and a positive integer k. Let $f \in H^\infty(\mathbf{R}^+)$ and $g \in H^\infty(0,T)$ satisfy $k+1$ compatibility conditions,*

$$g^{(j)}(0) = \varphi^{(j)}(0) \quad \text{for } 0 \le j \le k.$$

Then there exists a solution u of (6.1) in $L^\infty(0,T; H^{3k+1}(\mathbf{R}^+))$ corresponding to the data f and g. Moreover, there exists a constant

$$b = b\left(\|f\|_{3k+1}, |g|_{k+1,T}\right),$$

such that

$$\|u^{(j-1)}(\cdot,t)\|_3 + \|u^{(j)}(\cdot,t)\|_1 \le b, \tag{6.6}$$

for $1 \le j \le k$, where $u^{(j)} = \partial_t^j u$. The constant b depends continuously on its arguments.

Proof. The proposition follows from Theorem 5.3. More precisely, Theorem 5.3 provides the following. There is a $\delta>0$ and a family $\{g_\varepsilon\}_{0<\varepsilon\le\delta} \subseteq H^\infty(0,T)$ such that $g_\varepsilon(0) = f(0)$, and

$$|g_\varepsilon - g|_{k+1,T} \to 0 \quad \text{as } \varepsilon \downarrow 0.$$

Let u_ε be the solution of the regularized initial- and boundary-value problem (4.1), corresponding to the data f and g_ε. Then there is a constant $b = b(\|f\|_{3k+1}, |g|_{k+1,T})$ depending continuously on its arguments, but independent of ε in $(0,\delta]$, such that

$$\begin{aligned} &\|u^{(j-1)}(\cdot,t)\|_3^2 + \varepsilon\|u^{(j-1)}_{xxxx}(\cdot,t)\|^2 + \|u^{(j)}(\cdot,t)\|_1^2 \\ &\quad + \int_0^t \left\{\left[u^{(j-1)}_{xxxx}(0,s)\right]^2 + \left[u^{(j-1)}_{xxx}(0,s)\right]^2 + \left[u^{(j)}_x(0,s)\right]^2 + \varepsilon\left[u^{(j)}_{xx}(0,s)\right]^2\right\} ds \le b, \end{aligned} \tag{6.7}$$

for $0 \le j \le k$. (In (6.7), the subscript ε has been suppressed when writing u_ε.) And, from Corollary 3.9,

$$\partial_t^i u_\varepsilon \in C(0,T; H^m(\mathbf{R}^+)),$$

for all nonnegative integers i and m. Thus

$$\left\{\partial_t^j u_\varepsilon\right\}_{0<\varepsilon\le\delta} \text{ is bounded in } L^\infty(0,T; H^3(\mathbf{R}^+)),$$

for $0 \le j < k$, and

$$\left\{\partial_t^k u_\varepsilon\right\}_{0<\varepsilon\le\delta} \text{ is bounded in } L^\infty(0,T; H^1(\mathbf{R}^+)).$$

If H is any Hilbert space, then $L^\infty(0,T;H)$ is the dual of $L^1(0,T;H)$. (Here, H is identified with its dual space.) In consequence of this fact, the unit ball in $L^\infty(0,T;H)$ is compact, for the weak-star topology induced by $L^1(0,T;H)$. Hence, by taking a sequence from $(0,\delta]$ converging to 0, and passing progressively to further subsequences, we deduce the existence of a sequence $\{\varepsilon_n\}_1^\infty$, with $\varepsilon_n \downarrow 0$ such that if

$$u_n(x,t)=u_{\varepsilon_n}(x,t), \qquad n=1,2,3,\cdots,$$

then there are functions u and U_j in $L^\infty(0,T;H^3(\mathbf{R}^+))$, $0<j<k$, and a function U_k in $L^\infty(0,T;H^1(\mathbf{R}^+))$, such that

$$\begin{aligned}
(6.8)\qquad & u_n\to u && \text{weak-star in } L^\infty\big(0,T;H^3(\mathbf{R}^+)\big),\\
& \partial_t^j u_n\to U_j && \text{weak-star in } L^\infty\big(0,T;H^3(\mathbf{R}^+)\big) \quad \text{for } 0<j<k, \text{ and}\\
& \partial_t^k u_n\to U_k && \text{weak-star in } L^\infty\big(0,T;H^1(\mathbf{R}^+)\big),
\end{aligned}$$

as $n\to+\infty$. Since $u_n\to u$ weak-star in $L^\infty(0,T;H^3(\mathbf{R}^+))$, certainly $u_n\to u$ in $\mathfrak{D}'(0,T;H^3(\mathbf{R}^+))$. Hence $\partial_t^j u_n\to\partial_t^j u$, for all j, at least in the distributional sense. Because of (6.8), we may therefore identify U_j with $\partial_t^j u$, for $0<j\le k$.

Note also that if $\nabla u_n=(\partial_x u_n,\partial_t u_n)$, then $\{\nabla u_n\}_1^\infty$ comprises a bounded sequence in $L^\infty(0,T;H^1(\mathbf{R}^+))\times L^\infty(0,T;H^1(\mathbf{R}^+))$. Since $H^1(\mathbf{R}^+)\subset C_b(\mathbf{R}^+)$, this means that each component of $\{\nabla u_n\}_1^\infty$ is a sequence uniformly bounded in $L^\infty(\mathbf{R}^+\times(0,T))$. In consequence, $\{u_n\}_1^\infty$ forms an equicontinuous sequence, when restricted to any compact subset of $\bar{\mathbf{R}}^+\times[0,T]$. Hence for any $M>0$, $\{u_n\}_1^\infty$ is precompact in $C([0,M]\times[0,T])$, by the Ascoli–Arzela lemma. So by passing to still further subsequences, and finishing off with a Cantor diagonalization, it may be presumed that

$$u_n\to u \quad \text{as } n\to+\infty, \text{ uniformly on compact subsets of } \bar{\mathbf{R}}^+\times[0,T].$$

(More precisely, this argument leads to the conclusion that $u_n\to v$, uniformly on compact subsets of $\bar{\mathbf{R}}^+\times[0,T]$, as $n\to+\infty$. This in turn implies that $u_n\to v$ in $\mathfrak{D}'(\mathbf{R}^+\times(0,T))$ and thus leads to the identification $v=u$.) Exactly the same argument holds good for $\partial_t^j u_n$, provided $j<k$. Thus, for $0\le j<k$,

$$(6.9)\qquad \partial_t^j u_n\to\partial_t^j u \quad \text{as } n\to+\infty, \text{ uniformly on compact subsets of } \bar{\mathbf{R}}^+\times[0,T].$$

By a different argument, which makes use of the fact that $H^1(0,M)$ is compactly embedded in $L_2(0,M)$ for any $M>0$ (cf. [8, Lemma 7]) it may also be presumed that

$$(6.10)\qquad \partial_t^k u_n\to\partial_t^k u \quad \text{as } n\to+\infty, \text{ almost everywhere in } \bar{\mathbf{R}}^+\times[0,T].$$

By passing to a further subsequence, if necessary, it may be supposed as well that, as $n\to+\infty$,

$$\begin{aligned}
& u_n\partial_x u_n\to w && \text{weak-star in } L^\infty\big(0,T;H^2(\mathbf{R}^+)\big),\\
& \partial_x u_n\to v && \text{weak-star in } L^\infty\big(0,T;H^2(\mathbf{R}^+)\big),\\
& \partial_x^3 u_n\to V && \text{weak-star in } L^\infty\big(0,T;L^2(\mathbf{R}^+)\big).
\end{aligned}$$

Because of (6.9), $u_n\to u$ and $u_n^2\to u^2$ in $\mathfrak{D}'(\mathbf{R}^+\times(0,T))$. Hence the identifications $w=\frac{1}{2}\partial_x u^2$, $v=\partial_x u$, $V=\partial_x^3 u$ follow. Moreover, $\partial_t\partial_x^2 u_n$ is bounded in $L^\infty(0,T;H^{-1}(\mathbf{R}^+))$, so $\varepsilon_n\partial_t\partial_x^2 u_n\to 0$ strongly in this space, as $n\to+\infty$.

The reader will now appreciate that there is in hand enough information to pass to the limit $n\to+\infty$ in the regularized equation and conclude that, at least in the distributional sense, u satisfies the KdV equation,

$$u_t+u_x+uu_x+u_{xxx}=0,$$

in $\mathbf{R}^+\times(0,T)$. Moreover, as $u_\varepsilon(x,0)\equiv f(x)$ and $u_\varepsilon(0,t)=g_\varepsilon(t)$ for $0<\varepsilon\leq\delta$, it follows from (6.9), for example, that

$$u(x,0)=f(x) \quad \text{for } x\in\mathbf{R}^+,$$

and

$$u(0,t)=g(t) \quad \text{for } t\in[0,T].$$

Thus u does indeed provide a solution of (6.1) on $\overline{\mathbf{R}}^+\times[0,T]$. Moreover, by the lower-semicontinuity of the norm, relative to weak-star convergence, (6.7) implies that

$$\|u^{(j)}(\cdot,t)\|_3\leq b,$$

for $0\leq j<k$, and

$$\|u^{(k)}(\cdot,t)\|_1\leq b,$$

where $b=b(\|f\|_{3k+1},|g|_{k+1,T})$ is the constant obtained earlier from Theorem 5.3.

Notice that, if $k=1$, then $u_t\in L^\infty(0,T;H^1(\mathbf{R}^+))$ and u_x, $uu_x\in L^\infty(0,T;H^2(\mathbf{R}^+))$. Hence, from the differential equation, $u_{xxx}\in L^\infty(0,T;H^1(\mathbf{R}^+))$, whence $u\in L^\infty(0,T;H^4(\mathbf{R}^+))$. If $k>1$, this type of simple argument may be continued inductively. The outcome is that

$$\partial_t^j u\in L^\infty\big(0,T;H^{3(k-j)+1}(\mathbf{R}^+)\big), \tag{6.11}$$

for $0\leq j\leq k$.

Finally, (6.11) and standard interpolation results ([19, Chap. 1, Thm. 3.1]) yield the following additional smoothness results:

$$\partial_t^j u\in C\big(0,T;H^{3(k-j)-1/2}(\mathbf{R}^+)\big), \tag{6.12}$$

for $0\leq j<k$.

In particular, if $k>1$, certainly $u\in C(0,T;H^4(\mathbf{R}^+))$. Therefore, u_t, u_x, uu_x, and u_{xxx} all lie in $C(0,T;H^1(\mathbf{R}^+))$. As this latter space is embedded in $C_b(\overline{\mathbf{R}}^+\times[0,T])$, it follows that, after possible modification on a set of measure zero, all the derivatives in the differential equation are continuous, and bounded, functions. Consequently, if $k>1$, u is a classical solution of the quarter-plane problem for KdV.

The proof of the proposition is now completed. □

Remark. Because the solution u obtained in Proposition 6.4 lies within the realm of the uniqueness theorem 6.1, the entire family $\{u_\varepsilon\}_{0<\varepsilon\leq\delta}$ is inferred to converge to u, in the various senses appearing in the proof. This is because we actually prove that any sequence $\{\varepsilon_n\}_1^\infty$ in $(0,\delta]$, with $\varepsilon_n\to0$, as $n\to+\infty$, has a subsequence such that the corresponding functions $\{u_n\}$ converge to a solution of (6.1), which by uniqueness must be u.

The last proposition gives very nearly the result stated in Theorem 6.2. The only essential difference is that f and g are assumed to be infinitely differentiable. Using Lemma 6.3, this added assumption is shown to be unnecessary.

Proof of Theorem 6.2. Suppose now that $f\in H^{3k+1}(\mathbf{R}^+)$ and $g\in H^{k+1}_{\text{loc}}(\mathbf{R}^+)$ are fixed, and that f and g satisfy the first $k+1$ compatibility conditions, as in the statement of the theorem. Fix $T>0$. By Lemma 6.3, there exist sequences $\{f_N\}_1^\infty \subseteq H^\infty(\mathbf{R}^+)$ and $\{g_N\}_1^\infty \subseteq C^\infty(\mathbf{R}^+)$ such that

$$(6.13)\qquad \begin{aligned} f_N &\to f \quad \text{in } H^{3k+1}(\mathbf{R}^+),\\ g_N &\to g \quad \text{in } H^{k+1}(0,T), \end{aligned}$$

as $N\to+\infty$. And, for each $N>0$, f_N and g_N satisfy the same $k+1$ compatibility conditions satisfied by f and g. The last proposition thus applies, and it is concluded that there is a solution u_N of (6.1), on $\bar{\mathbf{R}}^+\times[0,T]$, corresponding to the data f_N and g_N. Moreover, $\partial_t^j u_N \in L^\infty(0,T;H^{3(k-j)+1}(\mathbf{R}^+))$, for $0\le j\le k$, and if

$$b_N = b\left(\|f_N\|_{3k+1}, |g_N|_{k+1,T}\right),$$

then for $0\le j<k$,

$$\left\|\partial_t^j u_N\right\|_{L^\infty(0,T;H^3(\mathbf{R}^+))} \le b_N, \qquad \left\|\partial_t^k u_N\right\|_{L^\infty(0,T;H^1(\mathbf{R}^+))} \le b_N.$$

Because of (6.13) and the fact that b is bounded as its arguments vary over a bounded set, there is a constant B, independent of N, such that

$$(6.14\text{a})\qquad \left\|\partial_t^j u_N\right\|_{L^\infty(0,T;H^3(\mathbf{R}^+))} \le B,$$

for $0\le j<k$, and

$$(6.14\text{b})\qquad \left\|\partial_t^k u_N\right\|_{L^\infty(0,T;H^1(\mathbf{R}^+))} \le B.$$

In consequence of the bounds expressed in (6.14), the arguments of Proposition 6.4 may be repeated without essential change (the extra smoothness available during the proof of the proposition was not used, nor was the regularizing term $-\varepsilon u_{xxt}$). It is concluded therefore that $\{u_N\}_1^\infty$ converges to a function u_T, say, in the various ways already detailed in the proof of Proposition 6.4. As before, u_T provides a solution of (6.1) corresponding to the data f and g, on $\bar{\mathbf{R}}^+\times[0,T]$.

The above argument applies for any fixed $T>0$. Define a function U on $\mathbf{R}^+\times\mathbf{R}^+$ by,

$$U(x,t)=u_T(x,t),$$

provided that $t<T$. This is well defined because of the uniqueness result. It is clear that U provides the solution whose existence was contemplated in the statement of Theorem 6.2. The fact that U is a classical solution of the problem (6.1), if $k>1$, follows exactly as in the proof of Proposition 6.4. The theorem is thus established. □

It is perhaps worth comment that Theorem 6.2 also holds if $k=0$. This result subsists on the ε-independent $H^1(\mathbf{R}^+)$-bound established in Corollary 3.6. The proof of existence of these weaker solutions, while a little more delicate than the proof of Theorem 6.2, fits more or less directly into the framework exposed in the proof of Proposition 6.3. (The extra ingredients may be found, for example, in [8, App. A].) For this reason, we content ourselves with a statement of this further consequence.

THEOREM 6.5. *Let $f\in H^1(\mathbf{R}^+)$ and $g\in H^1_{\text{loc}}(\mathbf{R}^+)$, and suppose $f(0)=g(0)$. Then there exists a solution u in $L^\infty_{\text{loc}}(\mathbf{R}^+;H^1(\mathbf{R}^+))$ of problem* (6.1) *corresponding to the data f and g.*

Remarks. By a solution we mean as usual a solution in the sense of distributions. In this case the uniqueness result does not apply.

Note that, for any $T>0$, $u_t\in L^\infty(0,T;H^{-2}(\mathbf{R}^+))$, from the equation. Hence $u\in C(0,T;H^{-1/2}(\mathbf{R}^+))$ (cf. again [19, Chap. 1]), so the initial-value is taken on in a weak, but meaningful way. Note as well that $L^\infty(0,T;H^1(\mathbf{R}^+))\subseteq L^\infty(0,T;C_b(\mathbf{R}^+))$. Hence for almost every t in $[0,T]$, $u(x,t)$ is continuous in x at $x=0$. Thus the boundary-values are also obtained in a meaningful way.

7. Conclusion. The quarter-plane problem (1.3) is argued to be a natural configuration in which to use the KdV equation for the prediction of wave propagation in a uniform channel. The general idea behind the use of this form of initial- and boundary-value problem for testing the appurtenance of the KdV equation may be appreciated by reference to Fig. 1. With the liquid initially at rest ($f\equiv 0$), a wavemaker located at one end of the channel is activated. The passage of the waves down the channel is recorded by probes, the recording nearest the wavemaker being construed as the boundary data $g(t)$. Note that if the waves are in the regime to which, formally, KdV applies, then they are expected to be smooth, and so g will lie in $\mathcal{D}(0,T)$, for some $T>0$. In consequence, the data so determined will satisfy the compatibility conditions, expressed for example below (6.5), to all orders. Hence the theory developed herein is applicable.

Our theory demonstrates that problem (1.3) has unique smooth solutions corresponding to such smooth and compatible data. This is a step in the direction of a satisfactory mathematical analysis of the situation envisaged in Fig. 1. Another important step, which has not been treated here, is a result of continuous dependence of the solutions on variations of the data. Also, in considering comparisons of the model's predictions with laboratory-scale experiments, some compensation for dissipative effects must be included (cf. [10]). Less important, but still of some mathematical interest, is a possible improvement of the regularity theory to bring this aspect into line with the theory for the pure initial-value problem (cf. [8] or [16]). We have shown that if $f\in H^{3k+1}(\mathbf{R}^+)$ and $g\in H^{k+1}_{\mathrm{loc}}(\mathbf{R}^+)$ satisfy the appropriate compatibility conditions at $(x,t)=(0,0)$, then the quarter-plane problem has a solution in $L^\infty_{\mathrm{loc}}(\mathbf{R}^+;H^{3k+1}(\mathbf{R}^+))$. Whereas, we confidently expect the solutions to lie in $C(\mathbf{R}^+;H^{3k+1}(\mathbf{R}^+))$. In fact, this latter point seems to be related to a sharp version of continuous dependence of solutions on the data.

It deserves emphasis that a satisfactory numerical scheme for the configuration in view here is essential to effect any quantitative comparisons of laboratory data with predictions of the model. Especial care must be exercised here. First, control of the high-frequency end of the Fourier spectrum must be assured. Otherwise an untenable error may be created near $x=0$, due to the large negative phase and group velocity associated to such components (cf. [4, §2]). Secondly, the integration will in fact take place on a bounded spatial domain, forcing the imposition of additional boundary conditions. This in turn will lead to consideration of an initial- and two-point-boundary-value problem for the KdV equation, and to consideration of the relation of such a problem to the situation studied here. The difficulties seem numerous enough to warrant insisting on a scheme having rigorously derived error bounds. Thus far, such schemes seem to be available only for the periodic initial-value problem (cf. [1], [2], [29] and [30]).

Finally, it is worth remarking that the methods embodied in this paper might yield a comparison theorem between the quarter-plane problem (1.3) for KdV and the analogous quarter-plane problem for (1.4) studied in [5], and used in the comparisons with experimental data reported in [10]. Such a program of comparison of model equations has been carried out for the associated pure initial-value problems in [11], using the general line pursued herein. Thus there is some cause for hope that a similar result is obtained in the present context.

Acknowledgments. The first author wishes to record stimulating and useful conversations and correspondence with M. Heard, W. G. Pritchard, L. R. Scott and R. Smith concerning the problem studied herein. Both authors gratefully acknowledge support from the National Science Foundation during part of this collaboration. Both authors were also supported by the U. S. Army Research Office as visiting members of the Mathematics Research Center, University of Wisconsin-Madison, during part of this collaboration.

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