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THE KORTEWEG-DE VRIES EQUATION IN A QUARTER PLANE, CONTINUOUS DEPENDENCE RESULTS*

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Abstract. Considered herein is an initial- and boundary-value problem that arises in modeling the propagation of small-amplitude, long waves generated by a wavemaker at one end of a homogeneous stretch of nonlinear, dispersive media. The principle accomplishment is to show that the solutions to this problem depend continuously in strong norms on both the initial and the boundary data.

1. Introduction. This paper is a continuation of an earlier one (Bona and Winther 1983) in which an initial- and boundary-value problem for the Korteweg-de Vries equation was analysed. This classical model appears in the study of small amplitude, long wave propagation in an impressive variety of physical situations. It was argued in this previous work that the problem

$$u_t + u_x + uu_x + u_{xxx} = 0, \quad \text{for } x, t \geq 0 \tag{1.1a}$$

with

$$u(x,0) = f(x), \quad \text{for } x \geq 0, \quad u(0,t) = g(t), \quad \text{for } t \geq 0, \tag{1.1b}$$

is especially interesting and apropriate as regards the use of this equation in situations where a wavetrain is created at one end of and travels into an undistured patch of the medium of propagation. A common example to which the Korteweg-de Vries equation might be expected to apply arises in a flume with a wavemaker afixed at one end which, when appropriately oscillated, generates unidirectional, small amplitude, long waves that travel down the channel (cf. Bona, Pritchard and Scott 1981, Hammack and Segur 1974 and Zabusky and Galvin 1971).

The problem posed in (1.1) has been investigated by Bona and Heard, as well as in the present authors' earlier paper. The work of Bona and Heard provides existence of relatively weak solutions corresponding to weak assumptions on the initial and boundary data f and

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g in (1.1b). In our earlier effort, we were able to provide arbitrarily smooth solutions corresponding to smooth initial and boundary data, provided that certain compatibility conditions between these data obtain.

The principle accomplishment of the present paper is a result of continuous dependence of solutions on the data in spaces that are as restrictive as the solutions allow. Continuous dependence results are important, but somewhat rare for nonlinear evolution equations in the rather strong topologies considered here. As a biproduct, we are able to bring the existence theory into line with that available for the pure initial value problem

$$\begin{aligned} &u_t + u_x + uu_x + u_{xxx} = 0, \qquad \text{for } x \in \mathbb{R}, \quad t \geq 0, \quad \text{with} \\ &u(x,0) = f(x), \qquad \text{for } x \in \mathbb{R} \end{aligned} \tag{1.2}$$

(cf. Bona and Smith 1975, Bona and Scott 1976, Kato 1975 and 1983, and Saut and Temam 1976). Another consequence of our theory is some results of local smoothing of solutions of (1.2) which are analogous to those obtained by Kato (1983) for solutions of (1.2). We are also able to show that the local smoothing inherent in (1.1) depends continuously upon variations in the initial and boundary data, as did Bona and Saut (1988) for (1.2).

The paper is organized as follows. Section 2 sets out notation, reviews the central theorem from our previous paper, and provides a precise statement of the main results to which attention is given here. In Section 3 bounds are obtained on the difference between two solutions of problem (1.1) in terms of the difference between the corresponding initial and boundary data. Such results are obtained by energy methods that were already exploited in our earlier work on this problem. There follows in Section 4 a technical lemma that provides a special approximation scheme for the data in (1.1b). With the bounds and the approximation scheme in hand, the proof of our main result is readily deduced in Section 5. The theory of local smoothing is established in Section 6.

2. Notation and statement of the main results. In this section, the notation in force throughout is reviewed and the main theorem given precise enunciation.

With a few exceptions noted below, the notation utilized will be that which is currently standard in the theory of partial differential equations (cf. Lions and Magenes, 1968). In general, if X is a Banach space the norm on X will be denoted $\|\cdot\|_x$. However, in the special case wherein $X = H^k(\mathbb{R}^+)$, the Sobolev space of real-valued, square integrable functions defined on the half line $\mathbb{R}^+$ whose first k derivatives are also square integrable, we shall write

$$\|f\|_k \quad \text{for} \quad \|f\|_{H^k(\mathbb{R}^+)}$$

if $f \in H^k(\mathbb{R}^+)$. Similarly, if $g \in H^k(0,T)$, we write

$$|g|_{k,T} \quad \text{for} \quad \|g\|_{H^k(0,T)} \,.$$

If $k = 0$, the subscript k will be omitted altogether, so that

$$\|f\| = \|f\|_{L_2(\mathbb{R}^+)} \quad \text{and} \quad |g|_T = |g|_{0,T} \,.$$

In Section 4 use will be made of the closed linear subspace $H_0^m(\mathbb{R}^+)$ of $H^m(\mathbb{R}^+)$. The space $H_0^m(\mathbb{R}^+)$ is the closure in the norm $\|\cdot\|_m$ of the subspace $C_0^\infty(\mathbb{R}^+)$, and thus an element $f \in H_0^m(\mathbb{R}^+)$ has the property that $f(0) = f'(0) = \cdots = \partial_x^{m-1} f(0) = 0$. Of course, the space $H_0^m(\mathbb{R}^+)$ inherits its topology from $H^m(\mathbb{R}^+)$.

A central role in our theory will be played by the following, non-standard collections of functions. For non-negative integers k, define X_k as follows:

$$X_k = \{(f,g) \in H^{3k+1}(\mathbb{R}^+) \times H^{k+1}_{\text{loc}}(\mathbb{R}^+) : \phi_f^{(j)}(0) = g^{(j)}(0) \text{ for } 0 \le j \le k\}. \tag{2.1}$$

In this definition $g^{(j)}(t)$ is shorthand for

$$\frac{d^j}{dt^j} g\Big|_t \tag{2.2a}$$

and $\phi^{(j)} = \phi_f^{(j)}$ is defined recursively by

$$\begin{cases} \phi^{(0)} = f \quad \text{and} \\ \phi^{(j+1)} = -\big[\phi_x^{(j)} + \phi_{xxx}^{(j)} + \frac{1}{2}\big(\sum_{i=0}^{j} \binom{j}{i} \phi^{(i)}\phi^{(j-i)}\big)_x\big]. \end{cases} \tag{2.2b}$$

(The abbreviation $F^{(j)}$ for the j^{th} derivative of F with respect to t will be used throughout, even when F is a function of both x and t.) The class of functions X_k is given the topology induced by $H^{3k+1}(\mathbb{R}^+) \times H^k_{\text{loc}}(\mathbb{R}^+)$. Note that the traces to which reference is made in (2.1) all make sense because H^1-functions defined on open subsets of $\mathbb{R}$ have a realization as bounded, continuous functions. Consequently, X_k is a closed subset of $H^{3k+1}(\mathbb{R}^+) \times H^{k+1}_{\text{loc}}(\mathbb{R}^+)$.

A word of explanation is warranted concerning the relevance of these spaces to our endeavors. Suppose that there is to hand a solution u of (1.1) taking the initial and boundary values f and g, respectively, smoothly in the closed quarter plane $\{(x,t) : x \ge 0 \text{ and } t \ge 0\}$. It follows that

$$\begin{aligned} g'(0) = \lim_{t\to 0} g'(t) &= \lim_{t\to 0} -\big[u_x(0,t) + u(0,t)u_x(0,t) + u_{xxx}(0,t)\big] \\ &= -[f'(0) + f(0)f'(0) + f'''(0)], \end{aligned}$$

or, in the notation introduced above,

$$g^{(1)}(0) = \phi_f^{(1)}(0).$$

Inductively, one determines that g and f must satisfy the further relations

$$g^{(j)}(0) = \phi_f^{(j)}(0) \tag{2.3}$$

for all j such that both sides of (2.3) are defined. This observation leads naturally to the introduction of the function classes X_k.

In the precursor to the present work (Bona and Winther 1983) the following result was proved.

Theorem 2.1. *Assume that $(f,g) \in X_k$ for some integer $k \ge 1$. There then exists a unique solution u of (1.1) in $L^\infty_{\text{loc}}(\mathbb{R}^+; H^{3k+1}(\mathbb{R}^+))$ corresponding to the data f and g. Furthermore, there is a constant c, depending continuously on T, $\|f\|_{3k+1}$ and $|g|_{k+1,T}$ such that*

$$\|u\|_{L^\infty(0,T;H^{3k+1}(\mathbb{R}^+))} \le c\,.$$

The purpose of this paper is to extend Theorem 2.1 in several ways. The major advance consists in showing that the initial- and boundary-value problem (1.1) is well posed in Hadamard's classical sense. That is, the mapping that assigns to suitable pairs of functions (f, g) the unique solution u of (1.1) corresponding to specifying f as initial data and g as boundary data is continuous as a mapping between their respective function classes. As is well understood, this sort of property of a system is crucial to its perspicuity as a model for physical phenomena that do not feature any catastrophic change of type. Moreover, a good continous dependence theory is a strong indicator that satisfactory numerical approximations can be devised for the system in question, and is sometimes a useful tool in analyzing numerical approximation schemes. As a minor consequence of the analysis, it is also deduced that the solutions whose existence is guaranteed by Theorem 2.1 actually lie in the stronger spaces $C(\mathbb{R}^+;\ H^{3k+1}(\mathbb{R}^+))$ and, for each $T > 0$, in $L_2(0,T;\ H^{3k+2}_{\text{loc}}(\mathbb{R}^+))$. Here is the precise result in view.

Theorem 2.2. *Assume that the initial and boundary data (f, g) lies in X_k for an integer $k \geq 1$. There exists a unique solution u of (1.1) in $C(\mathbb{R}^+; H^{3k+1}(\mathbb{R}^+)) \cap L_{2,\text{loc}}(\mathbb{R}^+; H^{3k+2}_{\text{loc}}(\mathbb{R}^+))$ corresponding to the data f and g. Furthermore, the map $(f, g) \mapsto u$ is continuous as a map from X_k into $C(\mathbb{R}^+;\ H^{3k+1}(\mathbb{R}^+)) \cap L_{2,\text{loc}}(\mathbb{R}^+;\ H^{3k+2}_{\text{loc}}(\mathbb{R}^+))$.*

Remarks. The continuous dependence result means that if $\{u_n\}_{n=1}^{\infty}$ is a sequence of solutions to (1.1) corresponding to the sequence $\{(f_n, g_n)\}_{n=1}^{\infty}$ of auxiliary data in X_k, and

$$\|f_n - f\|_{3k+1} + |g_n - g|_{k+1,T} \to 0$$

as $n \to \infty$, for all $T > 0$, then $(f, g) \in X_k$ and if u is the solution of (1.1) corresponding to the data (f, g), it follows that

$$\|u_n - u\|_{L_2(0,T;\, H^{3k+2}(0,R))} + \|u_n - u\|_{C(0,T;\, H^{3k+1}(\mathbb{R}^+))} \to 0$$

as $n \to \infty$, for all positive R and T.

Since u satisfies the KdV equation and lies in $C(\mathbb{R}^+;\ H^{3k+1}(\mathbb{R}^+)) \cap L_2(0,T;\ H^{3k+2}_{\text{loc}}(\mathbb{R}^+))$, it follows immediately that $u_t \in C(\mathbb{R}^+;\ H^{3k-2}(\mathbb{R}^+)) \cap L_2(0,T;\ H^{3k-1}_{\text{loc}}(\mathbb{R}^+))$. A simple inductive argument yields regularity of higher-order, temporal derivatives. The outcome of this argument is summarized in the next theorem.

Theorem 2.3. *Let $(f, g) \in X_k$ where $k \geq 1$ and let u be the solution of (1.1) corresponding to the initial and boundary data f and g, respectively. Then u lies in*

$$Y = \bigcap_{j=0}^{k} \left\{ C^j(\mathbb{R}^+;\ H^{3(k-j)+1}(\mathbb{R}^+)) \cap L_{2,\text{loc}}(\mathbb{R}^+;\ H^{3(k-j)+2}_{\text{loc}}(\mathbb{R}^+)) \right\}.$$

Moreover, the correspondence $(f, g) \mapsto u$ is continuous from X_k into Y.

3. A priori bounds. The purpose of this section is to derive bounds for the difference between two solutions u_1 and u_2 of the initial- and boundary-value problem (1.1). These bounds, which are derived via energy-type arguments, may be expressed in terms of corresponding differences in the initial and the boundary data for the two solutions. Such results comprise an essential tool in the proof of Theorem 2.2 given in Sections 5 and 6; they are collected together in Lemma 3.3 below.

Throughout this section we shall assume that (f_1, g_1) and (f_2, g_2) are two sets of data for the problem (1.1) which lie in X_1. By Theorem 1.1 the corresponding solutions u_1 and u_2 of (1.1) will be elements of $L^\infty_{\text{loc}}(\mathbb{R}^+;\, H^4(\mathbb{R}^+))$. If the data (f_1, g_1) and (f_2, g_2) happen to lie in X_{k+1} where $k \geq 1$, then the solutions u_1 and u_2 are in $L^\infty_{\text{loc}}(\mathbb{R}^+;\, H^{3k+4}(\mathbb{R}^+))$, and, moreover, there exist constants $C_{k,T}$ which depend continuously on T, $\|f_i\|_{3k+1}$ and $|g_i|_{k+1,T}$, $i = 1, 2$, such that

$$\|u_i\|_{L^\infty(0,T;H^{3k+1}(\mathbb{R}^+))}\,, \quad |\partial_x^{3k+1} u_i(0,\cdot)|_{0,T} \leq C_{k,T}\,, \tag{3.1}$$

for $i = 1, 2$. Whilst not stated explicitly in the paper of Bona and Winther (1983), (3.1) is a consequence of the arguments that lead to what we have here called Theorem 2.1 (see particularly formula (6.7) in the proof of Proposition 6.4 in the last-quoted reference).

To begin the derivation of the desired bounds, introduce the notation

$$\Delta f = f_1 - f_2, \quad \Delta g = g_1 - g_2, \quad \text{and} \quad w = u_1 - u_2.$$

From (1.1) it follows that the function w satisfies the variable-coefficient, initial- and boundary-value problem

$$\left.\begin{aligned} & w_t + w_x + \frac{1}{2}((u_1 + u_2)w)_x + w_{xxx} = 0, && \text{for } (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ & w(0,t) = \Delta g(t), && \text{for } t \in \mathbb{R}^+, \\ & w(x,0) = \Delta f(x), && \text{for } x \in \mathbb{R}^+, \end{aligned}\right\} \tag{3.2}$$

If j is a non-negative integer, remember that $u_i^{(j)} = \partial_t^j u_i$, for $i = 1, 2$, and $w^{(j)} = \partial_t^j w$. Then $w^{(k)}$ satisfies the partial differential equation

$$w_t^{(k)} + w_x^{(k)} + (u_1 w^{(k)} + u_2^{(k)} w)_x + w_{xxx}^{(k)} = F^{(k)}\,, \tag{3.3}$$

where

$$F^{(k)} = -\frac{1}{2} \sum_{j=1}^{k-1} \binom{k}{j} \left([u_1^{(k-j)} + u_2^{(k-j)}] w^{(j)}\right)_x. \tag{3.4}$$

The first result is a bound for the $L_2(\mathbb{R}^+)$- and $H^1(\mathbb{R}^+)$-norms of the difference w.

Lemma 3.1. *Assume that $(f_i, g_i) \in X_1$ for $i = 1, 2$. Then, for any $T > 0$ there is a constant C_T depending continuously on T, $\|f_i\|_4$ and $|g_i|_{2,T}$, $i = 1, 2$, such that*

$$\|w(\cdot,t)\|^2 + \int_0^t w_x^2(0,s)\, ds \leq C_T\{\|\Delta f\|^2 + |\Delta g|_{1,T}^2\} \tag{3.5}$$

and

$$\|w_x(\cdot,t)\|^2 + \int_0^t w_{xx}^2(0,s)\, ds \leq C_T\{\|\Delta f\|_1^2 + |\Delta g|_{1,T}^2\} \tag{3.6}$$

for $0 \leq t \leq T$.

Proof: As mentioned above, these assumptions imply that $u_i \in L^\infty(0,T;\, H^4(\mathbb{R}^+))$, for $i = 1, 2$. Membership in this function class suffices to justify all the calculations below. Also, from (3.1) it follows that there is a constant C_T such that

$$\|u_i\|_{L^\infty(0,T;H^4(\mathbb{R}^+))} \leq C_T \tag{3.7}$$

for $i = 1, 2$. Here, and below, C_T will denote different constants possessing the same properties as the constant C_T specified in the statement of the lemma. To establish (3.5) let $y = w - r$ where $r(x,t) = \Delta g(t)e^{-x}$. Then y satisfies the initial- and boundary-value problem

$$\begin{aligned} &y_t + y_x + \frac{1}{2}\big((u_1+u_2)y\big)_x + y_{xxx} = h \\ &y(0,t) = 0, \quad y(x,0) = \Delta f(x) - r(x,0), \end{aligned} \tag{3.8}$$

where $h = -\big(r_t + r_x + \frac{1}{2}((u_1+u_2)r)_x + r_{xxx}\big)$. Upon multiplying (3.8) by 2y and integrating over $\mathbb{R}^+ \times (0,t)$, there appears after integrations by parts the relation

$$\|y(\cdot,t)\|^2 + \int_0^t y_x^2(0,s)\,ds + \int_0^t\int_0^\infty \big((u_1+u_2)y\big)_x y\,dx\,ds = \|f - r(\cdot,0)\|^2 + 2\int_0^t\int_0^\infty (hy)\,dx\,ds. \tag{3.9}$$

Further integrations by parts shows

$$\int_0^\infty \big((u_1+u_2)y\big)_x y\,dx = -\frac{1}{2}\int_0^\infty (u_1+u_2)_x y^2\,dx,$$

and so from (3.7) and the definition of r we obtain that

$$\|(u_1+u_2)_x\|_{L^\infty(\mathbb{R}^+\times(0,T))} \le C_T\,,$$

$$\|h\|_{L^2(\mathbb{R}^+\times(0,T))} \le C_T|\Delta g|_{1,T}\,,$$

and

$$\|r\|_{L^\infty(0,T;L^2(\mathbb{R}^+))} \le C_T|\Delta g|_{1,T}\,.$$

It thus follows from (3.9) and Gronwall's Lemma that for $0 \le t \le T$

$$\|y(\cdot,t)\|^2 + \int_0^t y_x^2(0,s)\,ds \le C_T\big\{\|\Delta f\|^2 + |\Delta g|_{1,T}^2\big\}\,,$$

and this implies (3.5).

To demonstrate (3.6), multiply (3.2) by $-2w_{xx}$ and integrate over $\mathbb{R}^+ \times (0,t)$ to obtain the relation

$$\begin{aligned} &\|w(\cdot,t)\|^2 + 2\int_0^t g_t(s)w_x(0,s)\,ds + \int_0^t \big(w_x^2(0,s) + w_{xx}^2(0,s)\big)\,ds \\ &= \|w_x(\cdot,0)\|^2 + \int_0^t\int_0^\infty \big((u_1+u_2)w\big)_x w_{xx}\,dx\,ds. \end{aligned} \tag{3.10}$$

Integration by parts gives the expression

$$\begin{aligned} \int_0^\infty \big((u_1+u_2)w\big)_x w_{xx}\,dx = &-(u_1+u_2)_x(0,s)\Delta g(s)w_x(0,s) - \frac{1}{2}(u_1+u_2)(0,s)w_x^2(0,s) \\ &- \int_0^\infty (u_1+u_2)_{xx} w w_x\,dx - \frac{3}{2}\int_0^\infty (u_1+u_2)_x w_x^2\,dx, \end{aligned}$$

from which one adduces using (3.10) along with (3.5) and (3.7) that

$$\|w_x(\cdot,t)\|^2 + \int_0^t w_{xx}^2(0,s)\,ds \le C_T\big\{\|\Delta f\|_1^2 + |\Delta g\|_{1,T}^2 + \int_0^t \|w_x(\cdot,s)\|^2\,ds\big\}$$

for $0 \le t \le T$. Hence, (3.6) follows from Gronwall's Lemma and the proof of Lemma 3.1 is thus completed. ∎

In order to prove continuous dependence results in higher-order Sobolev spaces, we will obtain bounds similar to the bounds derived in Lemma 3.1 in higher-order Sobolev norms. Such results can be derived from the estimates of the functions $w^{(k)} = \partial_t^k w$ given next.

Lemma 3.2. *Assume that the data* $(f_i, g_i) \in X_{k+1}$ *where* $k \geq 1$ *is an integer. For any* $T > 0$ *there are constants* $C_{k,T}$ *depending continuously on* T, $\|f_i\|_{3k+1}$ *and* $|g_i|_{k+1,T}$ *for* $i = 1, 2$, *such that*

$$\|w^{(k)}(\cdot,t)\|^2 + \int_0^t \left(w_x^{(k)}(0,s)\right)^2 ds \leq C_{k,T}\{\|\Delta f\|_{3k}^2 + |\Delta g|_{k+1,T}^2\} \tag{3.11}$$

and

$$\begin{aligned}&\|w_x^{(k)}(\cdot,t)\|^2 + \int_0^t \left(w_{xx}^{(k)}(0,s)\right)^2 ds \leq \\ &C_{k,T}\{\|\Delta f\|_{3k+1}^2 + |\Delta g|_{k+1,T}^2 + \|w\|_{L^\infty(\mathbb{R}^+\times(0,T))}^2 \cdot \|u_2^{(k)}\|_{L^2(0,T;H^2(\mathbb{R}^+))}^2\}\end{aligned} \tag{3.12}$$

for $0 \leq t \leq T$.

Proof: Note that since $(f_i, g_i) \in X_{k+1}$, Theorem 2.1 implies that $u_i \in L^\infty(0,T;H^{3k+4}(\mathbb{R}^+))$. Membership in the latter function space is sufficient to justify the calculations below.

Define a family of constants $C_{0,T}$ which depends continuously on T, $\|f_i\|_4$ and $\|g_i\|_{2,T}$ such that the estimates (3.11) and (3.12) hold for $k = 0$. This is possible on account of Lemma 3.1. The present lemma is proved by induction on k. Assume that the estimates (3.11) and (3.12) regarding $w^{(j)}$ hold for $0 \leq j < k$. To establish (3.11) for $j = k$, it is convenient to write

$$y = w^{(k)} - r^{(k)},$$

where

$$r^{(k)} = \Delta g^{(k)} e^{-x}$$

and $\Delta g^{(k)}$ denotes $\partial_t^k \Delta g$. From (3.3) and the definition of y it follows that

$$\begin{aligned}&y_t + y_x + (u_1 y)_x + y_{xxx} = h, \\ &y(0,t) = 0, \quad y(x,0) = w^{(k)}(x,0) - r^{(k)}(x,0),\end{aligned} \tag{3.13}$$

where

$$h = F^{(k)} - (u_2^{(k)} w)_x - \left(r_t^{(k)} + r_x^{(k)} + (u_1 r^{(k)})_x + r_{xxx}^{(k)}\right)$$

and $F^{(k)}$ is given in (3.4). After multiplying (3.13) by $2y$ and integrating over $\mathbb{R}^+ \times (0,t)$, we obtain

$$\|y(\cdot,t)\|^2 + \int_0^t y_x^2(0,s)\,ds + 2\int_0^t\int_0^\infty (u_1 y)_x y\,dx\,ds = \|y(\cdot,0)\|^2 + 2\int_0^t\int_0^\infty hy\,dx\,ds. \tag{3.14}$$

Note that

$$\int_0^\infty (u_1 y)_x y\,dx = \frac{1}{2}\int_0^\infty (u_1)_x y^2\,dx$$

and that, from (3.1),

$$\|(u_1)_x\|_{L^\infty(\mathbb{R}^+\times(0,T))} \leq C_{1,T}.$$

(Here, and below, $C_{k,T}$ denotes various constants having the same properties as those specified in the statement of the lemma). Also, from (3.1) and the induction hypothesis,

$$\|h\|_{L^2(\mathbb{R}^+\times(0,T))} \leq C_{k,T}\{\|\Delta f\|_{3k-2} + |\Delta g|_{k+1,T}\}.$$

Finally, from the definition of $r^{(k)}$ one sees immediately that

$$\|y(\cdot,0)\| \le C_{k,T}\{\|\Delta f\|_{3k} + |\Delta g|_{k+1,T}\}.$$

Using the preceding facts in (3.14) and applying Gronwall's Lemma, it is confirmed that

$$\|y(\cdot,t)\|^2 + \int_0^t y_x^2(0,s)\,ds \le C_{k,t}\{\|\Delta f\|_{3k}^2 + |\Delta g|_{k+1,T}^2\}$$

for $0 \le t \le T$. This latter relation implies (3.11).

For the derivation of (3.12), multiply (3.3) by $-2w_{xx}^{(k)}$ and integrate over $\mathbb{R}^+ \times (0,t)$. After suitable integrations by parts, there appears

$$\begin{aligned}\|w_x^{(k)}(\cdot,t)\|^2 &+ 2\int_0^t g^{(k+1)}(s)w_x^{(k)}(0,s)\,ds + \int_0^t \big((w_x^{(k)}(0,s))^2 + (w_{xx}^{(k)}(0,s))^2\big)\,ds \\ =&\|w_x^{(k)}(\cdot,0)\|^2 + 2\int_0^t\int_0^\infty (u_1 w^{(k)} + u_2^{(k)}w)_x w_{xx}^{(k)}\,dx\,ds - 2\int_0^t\int_0^\infty F^{(k)}w_{xx}^{(k)}\,dx\,ds.\end{aligned} \tag{3.15}$$

First observe that

$$\int_0^\infty F^{(k)}w_{xx}^{(k)}\,dx = -F^{(k)}(0,s)w_x^{(k)}(0,s) - \int_0^\infty (F^{(k)})_x w_x^{(k)}\,dx$$

and that (3.1) and the induction hypothesis imply

$$|F^{(k)}(0,\cdot)|_T \le C_{k,T}\{\|\Delta f\|_{3k-3} + |\Delta g|_{k,T}\}.$$

Note also that (3.3), (3.11) and the induction hypothesis entails the inequality

$$\|w^{(j)}\|_{L^\infty(0,T;H^3(\mathbb{R}^+))} \le C_{k,T}\{\|\Delta f\|_{3k} + |\Delta g|_{k+1,T}\}$$

for $0 \le j \le k-1$. This in turn implies that

$$\|F_x^{(k)}\|_{L^2(\mathbb{R}^+\times(0,T))} \le C_{k,T}\{\|\Delta f\|_{3k} + |\Delta g|_{k+1,T}\}.$$

It follows from the estimates above that

$$\int_0^t\int_0^\infty F^{(k)}w_{xx}\,dx\,ds \le \int_0^t \|w_x^{(k)}(\cdot,s)\|^2\,ds + C_{k,T}\{\|\Delta f\|_{3k}^2 + |\Delta g|_{k+1,T}^2\}. \tag{3.16}$$

Next, integration by parts gives

$$\begin{aligned}\int_0^\infty (u_1 w^{(k)})_x w_{xx}^{(k)}\,dx &= \int_0^\infty \{\tfrac{1}{2}u_1\big((w_x^{(k)})^2\big)_x + (u_1)_x w^{(k)}w_{xx}^{(k)}\}\,dx \\ &= -\frac{1}{2}u_1(0,s)(w_x^{(k)}(0,s))^2 + (u_1)_x(0,s)w^{(k)}(0,s)w_x^{(k)}(0,s) \\ &\quad - \frac{3}{2}\int_0^\infty (u_1)_x(w_x^{(k)})^2\,dx - \int_0^\infty (u_1)_{xx}w^{(k)}w_x^{(k)}\,dx.\end{aligned}$$

Therefore, by (3.1) and (3.11) it follows that

$$\int_0^t \int_0^\infty (u_1 w^{(k)})_x w_{xx}^{(k)}\, dx\, ds \le C_{k,T}\big\{\|\Delta f\|_{3k}^2 + |\Delta g|_{k+1,T}^2 + \int_0^t \|w_x^{(k)}\|^2\, ds\big\} \tag{3.17}$$

for $0 \le t \le T$. Finally, consider the term $\int_0^t \int_0^\infty (u_2^{(k)} w)_x w_{xx}^{(k)}\, dx\, ds$ in (3.15). As before, integrate by parts to obtain

$$\begin{aligned}\int_0^\infty (u_2^{(k)} w)_x w_{xx}^{(k)}\, dx = &- (u_2^{(k)} w)_x(0,s) w_x^{(k)}(0,s) \\ &- \int_0^\infty \big\{(u_2^{(k)})_{xx} w + 2(u_2^{(k)})_x w_x + u_2^{(k)} w_{xx}\big\} w_x^{(k)}\, dx.\end{aligned}$$

Then (3.1), (3.11) and the induction hypothesis again give

$$\begin{aligned}&\int_0^t \int_0^\infty (u_2^{(k)} w)_x w_{xx}^{(k)}\, dx\, ds \le \\ &C_{k,T}\big\{\|\Delta f\|_{3k}^2 + |\Delta g|_{k+1,T}^2 + \int_0^t \|w_x^{(k)}\|^2\, ds\big\} + \int_0^t \int_0^\infty (u_2^{(k)})_{xx} w w_x^{(k)}\, dx\, ds\end{aligned} \tag{3.18}$$

for $0 \le t \le T$. To gain control of the last term on the right-hand side of (3.18), write

$$\begin{aligned}\int_0^t \int_0^\infty (u_2^{(k)})_{xx} w w_x^{(k)}\, dx\, ds &\le \int_0^t \int_0^\infty \big\{\big[(u_2^{(k)})_{xx} w\big]^2 + (w_x^{(k)})^2\, dx\, ds \\ \le \|w\|_{L^\infty(\mathbf{R}^+ \times (0,T))}^2 \cdot \|u_2^{(k)}\|_{L^2(0,T;H^2(\mathbf{R}^+))}^2 &+ \int_0^t \int_0^\infty (w_x^{(k)})^2\, dx\, ds\,.\end{aligned} \tag{3.19}$$

The estimate (3.12) is now a consequence of (3.15), (3.16), (3.17), (3.18), (3.19) and Gronwall's Lemma.

This completes the induction argument and hence the proof of the Lemma. ∎

An inductive use of (3.2) and (3.3), combined with the estimates derived in Lemma 3.2 immediately gives the following estimates for $\|w(\cdot,t)\|_{3k}$ and $\|w(\cdot,t)\|_{3k+1}$, $|\partial_x^{3k+1} w(0,\cdot)|_T$, and $|\partial_x^{3k+2} w(0,\cdot)|_T$.

Lemma 3.3. *Assume that the data (f_i, g_i) lies in X_{k+1}, for $i = 1,2$, where $k \ge 1$ is an integer. For any $T > 0$ there is a constant $C_{k,T}$ depending continuously on T, $\|f_i\|_{3k+1}$ and $|g_i|_{k+1,T}$ for $i = 1,2$, such that for $0 \le t \le T$,*

$$\|w(\cdot,t)\|_{3k}^2 + |\partial_x^{3k+1} w(0,\cdot)|_T^2 \le C_{k,T}\big\{\|\Delta f\|_{3k}^2 + |\Delta g|_{k+1,T}^2\big\} \tag{3.20}$$

and

$$\begin{aligned}&\|w(\cdot,t)\|_{3k+1}^2 + |\partial_x^{3k+2} w(0,\cdot)|_T^2 \le \\ &C_{k,T}\big\{\|\Delta f\|_{3k+1}^2 + |\Delta g|_{k+1,T}^2 + \|w\|_{L^\infty(\mathbf{R}^+ \times (0,T))}^2 \cdot \|u_2\|_{L^2(0,T;H^{3k+2}(\mathbf{R}^+))}^2\big\}\,.\end{aligned} \tag{3.21}$$

If the last results are specialized to the case wherein $f_2 = g_2 \equiv 0$, so that $u_2 \equiv 0$, a useful improvement on the bound (3.1) is obtained. This improvement is recorded here for use in the smoothing theory worked out in Section 6.

Lemma 3.4. *Let $(f, g) \in X_{k+1}$ where $k \geq 1$ and let u be the solution of (1.1) corresponding to the initial data f and boundary data g. Then for any $T > 0$ there is a constant $C_{k,T}$ depending continuously upon T, $\|f\|_{3k+1}$, and $|g|_{k+1,T}$ such that*

$$\|u(\cdot,t)\|_{3k}^2 + |\partial_x^{3k+1}u(0,\cdot)|_T^2 \leq C_{k,T}\{\|f\|_{3k}^2 + |g|_{k+1,T}^2\} \tag{3.22}$$

and

$$\|u(\cdot,t)\|_{3k+1}^2 + |\partial_x^{3k+2}u(0,\cdot)|_T^2 \leq C_{k,T}\{\|f\|_{3k+1}^2 + |g|_{k+1,T}^2\}. \tag{3.23}$$

4. Smooth and compatible approximation of the data. To use the result of Lemma 3.3 in the estimation of the difference of two solutions of (1.1) in the norm of $L^\infty(0,T;\, H^{3k+1}(\mathbb{R}))$, the present theory requires that the corresponding initial and boundary data lie in the space X_{k+1}. However, our goal is to establish that the map

$$(f, g) \mapsto u$$

is continuous from X_k into $C(\mathbb{R}^+;\, H^{k+1}(\mathbb{R}^+))$. To prove continuous dependence with respect to the norms advertised in Theorem 2.2, the natural strategy is to approximate elements of X_k by elements of X_{k+n} where $n \geq 1$. The existence of smooth and compatible approximations of the data will be established in this section. In Sections 5 and 6 these approximations are used, together with the results from Section 3 to prove the desired continuous dependence theorem.

The main accomplishments of this section are collected in the following proposition.

Proposition 4.1. *Let there be given $(f, g) \in X_k$ where $k \geq 1$. For any integer $n \geq 0$ and $\epsilon \in (0, 1]$ there exist functions*

$$(f_\epsilon, g_\epsilon) \in X_{k+n} \cap (H^\infty(\mathbb{R}^+) \times H^\infty_{\text{loc}}(\mathbb{R}^+))$$

such that for any $T > 0$,

(i) $\|f_\epsilon - f\|_{3(k-j)+1},\ |g_\epsilon - g|_{k+1-j,T} = o(\epsilon^j)$ *for $k \geq j \geq 0$, and*

(ii) $\|f_\epsilon\|_{3(k+j)+1}\,,\ |g_\epsilon|_{k+j+1,T} \leq c\epsilon^{-j}$ *for $j \geq 0$*

as $\epsilon \downarrow 0$, where the constant c depends only on $\|f\|_{3k+1}$, $|g|_{k+1,T+1}$, j, n and T. Furthermore, the convergence in (i) depends upon j, n and T, but is uniform on compact subsets of $H^{3k+1}(\mathbb{R}^+) \times H^{k+1}(0, T+1)$. Finally, for any fixed $\epsilon \in (0, 1]$, the map $(f, g) \mapsto (f_\epsilon, g_\epsilon)$ is continuous from X_k into $X_{k+n} \cap (H^\infty(\mathbb{R}^+) \times H^\infty_{\text{loc}}(\mathbb{R}^+))$.

The desired approximations in Proposition 4.1 will be constructed via two operators, a cut-off operator K_ϵ and a smoothing operator J_ϵ. Before presenting the proof of this proposition, these operators and some of their relevant properties are delineated.

For the rest of this section, let θ denote a fixed $C^\infty(\mathbb{R})$-function with the properties

$$\left.\begin{array}{ll} \theta(x) \equiv 0 & \text{for } x \leq 1, \\ \theta(x) \equiv 1 & \text{for } x \geq 2, \\ 0 \leq \theta(x) \leq 1 & \text{for all } x \in \mathbb{R}. \end{array}\right\} \tag{4.1}$$

For each $\epsilon \in (0, 1]$ and $f \in L^2(\mathbb{R}^+)$ define $K_\epsilon f \in L^2(\mathbb{R}^+)$ by $(K_\epsilon f)(x) = \theta(x/\epsilon)f(x)$. The operator K_ϵ maps $H^m(\mathbb{R}^+)$ continuously into $H_0^m(\mathbb{R}^+)$ for any $m \geq 0$.

The following lemma exposes the most important property of the family of operators $\{K_\epsilon : 0 < \epsilon \leq 1\}$ for the present purposes.

Lemma 4.2. *Let m and k be integers such that $0 \le k \le m$. Then, as $\epsilon \to 0$,*

$$\|K_\epsilon f - f\|_k = o(\epsilon^{m-k})$$

for all $f \in H_0^m(\mathbb{R}^+)$. Furthermore, the convergence is uniform on compact subsets of $H_0^m(\mathbb{R}^+)$.

Proof: If $m = 0$, the result is easy, so take $m \ge 1$ and assume that $f \in H_0^m(\mathbb{R}^+)$. We first prove that

$$\|K_\epsilon f - f\| = o(\epsilon^m) \tag{4.2}$$

as $\epsilon \downarrow 0$. Since $f \in H_0^m(\mathbb{R}^+)$ it follows from Taylor's theorem with remainder that

$$f(x) = \frac{1}{(m-1)!}\int_0^x (x-\xi)^{m-1} f^{(m)}(\xi)\,d\xi\,,$$

and consequently

$$\begin{aligned}
\|K_\epsilon f - f\|^2 &= \int_0^{2\epsilon} (\theta(x/\epsilon) - 1)^2 |f(x)|^2\,dx \\
&\le \Big(\frac{1}{(m-1)!}\Big)^2 \int_0^{2\epsilon} \Big(\int_0^{2\epsilon} |x-\xi|^{m-1} |f^{(m)}(\xi)|\,d\xi\Big)^2\,dx \\
&\le \Big(\frac{1}{(m-1)!}\Big)^2 \int_0^{2\epsilon}\int_0^{2\epsilon} |x-\xi|^{2m-2}\,d\xi\,dx \int_0^{2\epsilon} |f^{(m)}(\xi)|^2\,d\xi \\
&\le C_m \epsilon^{2m} \int_0^{2\epsilon} |f^{(m)}(\xi)|^2\,d\xi\,.
\end{aligned}$$

Since $\int_0^{2\epsilon} |f^{(m)}(\xi)|^2\,d\xi \to 0$ as $\epsilon \to 0$, uniformly on compact subsets of $H_0^m(\mathbb{R}^+)$, (4.2) follows.

Note particularly that (4.2) implies the desired result for $k = 0$. Also, by interpolation, the desired result for $0 < k \le m$ will follow from the estimate

$$\|(d/dx)^m (K_\epsilon f - f)\| = o(1) \tag{4.3}$$

as $\epsilon \to 0$. To demonstrate the validity of (4.3), define operators $K_\epsilon^{(j)}$ for each integer $j \ge 1$ and $\epsilon \in (0,1]$ by the formula

$$(K_\epsilon^{(j)} f)(x) = \epsilon^{-j}\theta^{(j)}(x/\epsilon) f(x)\,,$$

where $\theta^{(j)}(x) = (d/dx)^j \theta(x)$. Note that

$$\Big(\frac{d}{dx}\Big)^m (K_\epsilon f - f) = K_\epsilon f^{(m)} - f^{(m)} + \sum_{j=1}^m \binom{m}{j} K_\epsilon^{(j)} f^{(m-j)}\,.$$

Because (4.2) is valid, the inequality (4.3) follows from the estimate

$$\|K_\epsilon^{(j)} \tilde{f}\| = o(1) \tag{4.4}$$

as $\epsilon \downarrow 0$ for $\tilde{f} \in H_0^j(\mathbb{R}^+)$ and $j \ge 1$, a fact which is established next.

From the definition of the operator $K_\epsilon^{(j)}$ it appears that for any $\tilde{f} \in H_0^j(\mathbb{R}^+)$

$$\|K_\epsilon^{(j)}\tilde{f}\|^2 = \epsilon^{-2j}\int_\epsilon^{2\epsilon} |\theta^{(j)}(x/\epsilon)\tilde{f}(x)|^2\,dx$$
$$\leq \Big(\frac{1}{(j-1)!}\max_y \theta^{(j)}(y)\Big)^2 \epsilon^{-2j}\int_0^{2\epsilon}\Big(\int_\epsilon^{2\epsilon}|x-\xi|^{j-1}|\tilde{f}^{(j)}(\xi)|\,d\xi\Big)^2\,dx\,.$$

By the Cauchy-Schwartz inequality it therefore transpires that

$$\|K_\epsilon^{(j)}\tilde{f}\|^2 \leq c_j\int_0^{2\epsilon}|\tilde{f}^{(j)}(\xi)|^2\,d\xi \to 0$$

as $\epsilon \downarrow 0$, where the convergence is uniform on compact subsets of $H_0^j(\mathbb{R}^+)$. This establishes (4.4) and hence completes the proof of the Lemma. ∎

In addition to the cut-off operators studied above, we shall also use a family of smoothing operators J_ϵ. To define these operators, let Φ be a fixed C^∞-function on $\mathbb{R}$ with the properties

$$\text{support}\,\Phi \subset [-1/2, 1/2], \qquad \Phi \geq 0, \qquad \text{and} \quad \int_{-\infty}^{\infty}\Phi(x)\,dx = 1.$$

For each $\epsilon \in (0,1]$ and $f \in L^2(\mathbb{R}^+)$ define $J_\epsilon f \in H^\infty(\mathbb{R}^+)$ by

$$(J_\epsilon f)(x) = \int_0^\infty \Phi\Big(\frac{x-\xi}{\epsilon}\Big)f(\xi)\,d\xi\,.$$

Note that the integration is only taken over $\mathbb{R}^+$. However, if we let $\bar{f}$ be the extension of f to all of $\mathbb{R}$ defined by $\bar{f}(x) \equiv 0$ for $x < 0$, then the formula

$$J_\epsilon f(x) = \int_{-\infty}^\infty \Phi\Big(\frac{x-\xi}{\epsilon}\Big)\bar{f}(\xi)\,d\xi$$

defines $J_\epsilon f(x)$ for all real x. The map $f \mapsto \bar{f}$ is continuous as a map from $H_0^m(\mathbb{R}^+)$ into $H^m(\mathbb{R})$. Hence, by an argument completely analogous to the proof of Lemma 5 in Bona and Smith (1975), the following result may be established.

Lemma 4.3. *Let $f \in H_0^m(\mathbb{R}^+)$ for some integer $m \geq 0$. Then, as $\epsilon \to 0$, the functions $J_\epsilon f$ have the following properties:*

$$\|J_\epsilon f - f\|_{m-1} = o(\epsilon^j), \qquad \text{for } m \geq j \geq 0, \tag{4.5a}$$
$$\|J_\epsilon f\|_{m+j} \leq c\epsilon^{-j}, \qquad \text{for } j \geq 0, \tag{4.5b}$$

where the constant c depends only on $\|f\|_m$ and j. Furthermore, the convergence in (4.5a) is uniform on compact subsets of $H_0^m(\mathbb{R}^+)$.

Before proving Proposition 4.1, it is worth noting that Lemma 4.2 and Lemma 4.3 both have analogous versions in $H_{0,\mathrm{loc}}^m(\mathbb{R}^+)$. By using precisely the same arguments that come to the fore in the proofs of these previous lemmas, the following lemma may be validated.

Lemma 4.4. *Let $g \in H^m_{0,\mathrm{loc}}(\mathbb{R}^+)$ for some integer $m \geq 0$. Then, as $\epsilon \to 0$, the functions $K_\epsilon g$ and $J_\epsilon g$ have the properties that for any $T > 0$*

$$|J_\epsilon g - g|_{m-j,T}\,,\quad |K_\epsilon g - g|_{m-j,T} = o(\epsilon^j), \qquad \text{for } m \geq j \geq 0, \tag{4.6a}$$

$$|J_\epsilon g|_{m+j,T} \leq c\epsilon^{-j}\,, \qquad \text{for } j \geq 0, \tag{4.6b}$$

where the constant c depends only on $|g|_{m,T+1}$ and j. Furthermore, the convergence in (4.6a) is uniform on compact subsets of $H^m_0(0, T+1)$.

We are now in a position to prove the proposition.

Proof of Proposition 4.1: Let $(f, g) \in X_k$ be given, where $k \geq 1$. Let $\psi(x) = 1 - \theta(x)$, where θ is a fixed, smooth function with the defining properties (4.1). Then $\psi \in C^\infty(\mathbb{R}^+)$ and

$$\psi(x) = 0, \qquad \text{for } x \geq 2, \quad \psi(0) = 1,$$

and

$$(\frac{d}{dx})^j \psi(0) = 0 \qquad \text{for } j \geq 1.$$

Consider first the given initial function $f \in H^{3k+1}(\mathbb{R}^+)$. Let

$$p_f(x) = \psi(x) \sum_{j=0}^{3k} f^{(j)}(0) \frac{x^j}{j!}\,,$$

and let $f_0 = f - p_f$. Define

$$f_\epsilon(x) = f_{0,\epsilon}(x) + p_f(x),$$

where $f_{0,\epsilon} = J_\delta K_\delta f_0$ with $\delta = \epsilon^{1/3}$. Clearly, one has

$$f_\epsilon - f = f_{0,\epsilon} - f_0 = (J_\delta - I)K_\delta f_0 + (K_\delta - I) f_0,$$

and so Lemma 4.2, Lemma 4.3, and the definition of the operators K_ϵ and J_ϵ therefore imply that

$$\|f_\epsilon - f\|_{3(k-j)+1} = o(\epsilon^j), \qquad \text{for } 0 \leq j \leq k, \tag{4.7a}$$

$$\|f_\epsilon\|_{3(k+j)+1} = O(\epsilon^{-j}), \qquad \text{for } j \geq 0, \tag{4.7b}$$

$$\phi_\epsilon^{(j)}(0) = \phi^{(j)}(0), \qquad \text{for } 0 \leq j \leq k, \tag{4.7c}$$

$$(\frac{d}{dx})^j f_\epsilon(0) = 0, \qquad \text{for } j > 3k, \tag{4.7d}$$

where $\phi^{(j)}$ and $\phi_\epsilon^{(j)}$ is shorthand for the functions $\phi_f^{(j)}$ and $\phi_{f_\epsilon}^{(j)}$, associated as in (2.2b) to the functions f and f_ϵ, respectively. Furthermore, the bound (4.7b) holds uniformly on bounded sets of $H^{3k+1}(\mathbb{R}^+)$ and the convergence (4.7a) is uniform on compact sets in $H^{3k+1}(\mathbb{R}^+)$.

To define the approximations g_ϵ of g, first apply a procedure similar to that used to construct the family $\{f_\epsilon\}$. Define the function

$$p_g(t) = \psi(t) \sum_{j=0}^{k} g^{(j)}(0) (\frac{t^j}{j!})\,,$$

where, as before $g^{(j)} = (d/dt)^j g$. Furthermore, let

$$g_0 = g - p_g$$

and let

$$\tilde{g}_\epsilon(t) = g_{0,\epsilon}(t) + p_g(t)$$

where $g_{0,\epsilon} = J_\epsilon K_\epsilon g_0$. Finally, define $g_\epsilon(t)$ by

$$g_\epsilon(t) = \tilde{g}_\epsilon(t) + h_\epsilon(t),$$

where

$$h_\epsilon(t) = \psi(t/\epsilon) \sum_{j=k+1}^{k+n} \phi_\epsilon^{(j)}(0) \frac{t^j}{j!} .$$

It follows from the construction of f_ϵ and g_ϵ that they are both smooth functions satisfying the relation

$$\phi_\epsilon^{(j)}(0) = g_\epsilon^{(j)}(0) \quad \text{for } j = 0, 1, \cdots, k+n,$$

and consequently $(f_\epsilon, g_\epsilon) \in X_{k+n} \cap (H^\infty(\mathbb{R}^+) \times H^\infty_{\text{loc}}(\mathbb{R}^+))$. Observe also that for any $T > 0$, Lemma 4.4 implies

$$|\tilde{g}_\epsilon - g|_{k+1-j,T} = o(\epsilon^j), \quad \text{for } 0 \le j \le k+1, \tag{4.8a}$$

$$|\tilde{g}_\epsilon|_{k+1+j,T} = O(\epsilon^{-j}), \quad \text{for } j \ge 0, \tag{4.8b}$$

where the bound (4.8b) holds uniformly on bounded sets of $H^{k+1}(0, T+1)$ and the convergence (4.8a) is uniform on compact sets in this space. The desired properties of the functions g_ϵ will therefore follow from the estimates

$$|h_\epsilon|_{k+1-j,T} \le c\epsilon^{j+1/2} \tag{4.9}$$

for $j \le k+1$. Here the constant c is to depend only on $\|f\|_{3k+1}$, j and n.

The validity of (4.9) is now established. Note first that a direct calculation shows

$$|h_\epsilon|_{k+1-j,T} \le \tilde{c}\epsilon^{j+1/2}$$

for $j \le k+1$, where the constant $\tilde{c}$ depends on $\phi_\epsilon^{(i)}(0)$ for integers i in the interval $[k+1, n]$ and on j. Furthermore, for any $i \ge 0$, $\phi_\epsilon^{(i)}(0)$ can be bounded independently of ϵ by constants only depending on $\|f\|_{3k+1}$ and j because of (4.7a) and (4.7d). Hence, (4.9) follows and the desired estimates (i) and (ii) are now consequences of (4.7), (4.8) and (4.9).

The continuous dependence result that refers to the mapping $(f, g) \to (f_\epsilon, g_\epsilon)$ is a consequence of the continuity properties of the operators J_ϵ and K_ϵ and the fact that the map

$$f(0), f'(0), \cdots, f^{(3k)}(0) \to h_\epsilon$$

is continuous from $\mathbb{R}^{3k+1}$ into $H^\infty(\mathbb{R}^+)$. ∎

Finally, we shall derive a bound for the solution of the equation (1.1) with data (f_ϵ, g_ϵ) generated by Proposition 4.1. This result will be used in Sections 5 and 6 to bound the norm of u_2 appearing on the right-hand side of the estimate (3.21). The result is obtained via energy arguments similar to those used in Section 3.

Lemma 4.5. *Assume that $(f,g) \in X_k$, where $k \geq 1$ is an integer, and let $(f_\epsilon, g_\epsilon) \in X_{k+2}$ for $\epsilon \in (0,1]$ be approximations to (f,g) whose existence is guaranteed by Proposition 4.1 with $n = 2$. If u_ϵ denotes the solution of (1.1) with data (f_ϵ, g_ϵ), then for any $T > 0$*

$$\|u_\epsilon\|_{L^\infty(0,T;H^{3k+2}(\mathbb{R}^+))} \leq C_{k,T}\epsilon^{-1/2}.$$

Here the constant $C_{k,T}$ depends only on T, $\|f\|_{3k+1}$ and $|g|_{k+1,T+1}$.

Proof: First observe from (3.1) and Proposition 4.1 that

$$\|u_\epsilon\|_{L^\infty(0,T;H^{3k+1}(\mathbb{R}^+))} \leq C_{k,T}, \tag{4.10}$$

where here, and below, $C_{k,T}$ denotes constants with the same properties as those specified in the statement of the Lemma. Thus, the desired estimate will follow from the validity of an inequality of the form

$$\|u_\epsilon\|_{L^\infty(0,T;H^{3k+3}(\mathbb{R}^+))} \leq C_{k,T}\epsilon^{-1}. \tag{4.11}$$

Furthermore, because of the equation (1.1), (4.11) will follow from

$$\|\partial_t^{k+1} u_\epsilon\|_{L^\infty(0,T;L^2(\mathbb{R}^+))} \leq C_{k,T}\epsilon^{-1}. \tag{4.12}$$

If $u^{(j)} = u_\epsilon^{(j)}$ connotes $\partial_t^j u_\epsilon$, then for $j \geq 1$ the function $u^{(j)}$ satisfies the equation

$$u_t^{(j)} + u_x^{(j)} + (uu^{(j)})_x + u_{xxx}^{(j)} = F^{(j)}(u), \tag{4.13}$$

where $u = u_\epsilon$ and

$$F^{(j)}(u) = -\frac{1}{2}\partial_x \sum_{i=1}^{j-1} \binom{j}{i} u^{(i)} u^{(j-i)}.$$

For any non-negative integer j, let $r^{(j)}(x,t) = g_\epsilon^{(j)}(t)e^{-x}$ and define

$$y = u^{(k+1)} - r^{(k+1)}.$$

Then y satisfies the equation

$$y_t + y_x + (uy)_x + y_{xxx} = h$$

with initial and boundary values given by

$$y(0,t) = 0, \quad y(x,0) = u^{(k+1)}(x,0) - r^{(k+1)}(x,0),$$

where

$$h = -\left(r_t^{(k+1)} + r_x^{(k+1)} + (ur^{(k+1)})_x + r_{xxx}^{(k+1)} + F^{(k+1)}\right).$$

After a little manipulation, we obtain

$$\|y(\cdot,0)\|^2 + \int_0^t y_x^2(0,s)\,ds + 2\int_0^t \int_0^\infty (uy)_x y\,dx\,ds = \|y(\cdot,0)\|^2 + \int_0^t \int_0^\infty hy\,dx\,ds. \tag{4.14}$$

Note that

$$\int_0^t \int_0^\infty (uy)_x y \, dx \, ds = \frac{1}{2} \int_0^t \int_0^\infty u_x y^2 \, dx \, ds,$$

and thus by using (4.10) it follows that

$$\int_0^t \int_0^\infty (uy)_x y \, dx \le C_{k,T} \int_0^t \|y(\cdot, s)\|^2 \, ds$$

for $0 \le t \le T$. From the properties of (f_ϵ, g_ϵ) and the definition of y it also follows that

$$\|y(\cdot, 0)\| \le C_{k,T} \epsilon^{-2/3}$$

and

$$\|h\|_{L^2(\mathbb{R}^+ \times (0,T))} \le C_{k,T} \epsilon^{-1} .$$

Hence, (4.14) and Gronwall's Lemma imply that

$$\|y\|_{L^\infty(0,T; L^2(\mathbb{R}^+))} \le C_{k,T} \epsilon^{-1} .$$

This in turn gives (4.12) and so completes the proof. ∎

5. Proof of the main result. The purpose of this section is to begin the proof of Theorem 2.2 using the auxiliary results derived in Sections 3 and 4. Two propositions will be established, the first, Proposition 5.1, states that the solution u of (1.1) corresponding to data $(f, g) \in X_k$ is in $C(\mathbb{R}^+; H^{3k+1}(\mathbb{R}^+))$, while Proposition 5.2 gives the continuous dependence result in this space.

Proposition 5.1. *Assume that $(f, g) \in X_k$ for some $k \ge 1$. There exists a unique solution u of (1.1) in $C(\mathbb{R}^+; H^{3k+1}(\mathbb{R}^+))$ corresponding to the data f and g.*

Proof: Since it is already known from Theorem 2.1 that there exists a unique solution u of (1.1) in $L^\infty(\mathbb{R}^+; H^{3k+1}(\mathbb{R}^+))$ corresponding to the data f and g, it is only necessary to show that this solution actually lies in $C(\mathbb{R}^+; H^{3k+1}(\mathbb{R}^+))$.

Fix a positive value of T, let a sequence of approximations $\{(f_\epsilon, g_\epsilon)\}_{\epsilon \in (0,1]} \subset X_{k+2}$ to the data (f, g) be constructed for which the properties delineated in Proposition 4.1 hold, and let $\{u_\epsilon\}_{\epsilon \in (0,1]}$ denote the corresponding family of solutions of (1.1). From Theorem 2.1 we have that

$$u_\epsilon \in L^\infty(0, T;\, H^{3k+7}(\mathbb{R}^+)) \quad \text{and} \quad \partial_t u_\epsilon \in L^\infty(0, T;\, H^{3k+4}(\mathbb{R}^+)).$$

Hence for all $\epsilon \in (0, 1]$, u_ϵ certainly lies in $C(0, T;\, H^{3k+1}(\mathbb{R}^+))$. It will now be argued that $\{u_\epsilon\}$ is Cauchy in $C(0, T;\, H^{3k+1}(\mathbb{R}^+))$. Suppose that $0 < \delta < \epsilon \le 1$. From Lemma 3.3 and Proposition 4.1 there follows the existence of a constant $C_{k,T}$ depending continuously on $\|f\|_{3k+1}$, $|g|_{k+1,T+1}$ and T such that for $0 \le t \le T$

$$\begin{aligned} \|u_\epsilon(\cdot, t) - u_\delta(\cdot, t)\|_{3k+1} \le C_{k,T} \big\{ &\|f_\epsilon - f_\delta\|_{3k+1} + |g_\epsilon - g_\delta|_{k+1,T} \\ &+ \|u_\epsilon - u_\delta\|_{L^\infty(\mathbb{R}^+ \times (0,T))} \cdot \|u_\epsilon\|_{L^2(0,T; H^{3k+2}(\mathbb{R}^+))} \big\} . \end{aligned} \tag{5.1}$$

From Lemma 3.1 and Proposition 4.1 we have

$$\|u_\epsilon - u_\delta\|_{L^\infty(\mathbb{R}^+ \times (0,T))} \le \|u_\epsilon - u_\delta\|_{L^\infty(0,T;\, H^1(\mathbb{R}^+))} \le C_{k,T} \epsilon^k .$$

Also, from Lemma 4.5, it appears that

$$\|u_\epsilon\|_{L^\infty(0,T;\, H^{3k+2}(\mathbb{R}^+))} \le C_{k,T}\epsilon^{-1/2}.$$

Together with the fact that

$$\|f_\epsilon - f_\delta\|_{3k+1}, \quad |g_\epsilon - g_\delta|_{k+1,T} \to 0, \quad \text{as } \epsilon \to 0,$$

the last three inequalities entail that $\{u_\epsilon\}$ is a Cauchy sequence in $C(0,T;\, H^{3k+1}(\mathbb{R}^+))$. Hence, as $\epsilon \to 0$, $\{u_\epsilon\}$ converges to a function $\bar{u} \in C(0,T;\, H^{3k+1}(\mathbb{R}^+))$. By continuity it certainly follows that $\bar{u}$ satisfies the differential equation (1.1) in the sense of distributions on $\mathbb{R}^+ \times (0,T)$. Furthermore,

$$\|\bar{u}(\cdot,0) - f\|_{3k+1} \le \|\bar{u}(\cdot,0) - u_\epsilon(\cdot,0)\|_{3k+1} + \|f_\epsilon - f\|_{3k+1} \to 0,$$

as $\epsilon \downarrow 0$, and

$$|\bar{u}(0,\cdot) - g|_{k+1,T} \le |\bar{u}(0,\cdot) - u_\epsilon(0,\cdot)|_{k+1,T} + |g_\epsilon - g|_{k+1,T} \to 0$$

as $\epsilon \to 0$. Hence $\bar{u}$ is a solution of (1.1) with initial and boundary data f and g, respectively. From the uniqueness result of Theorem 2.1 it is therefore implied that

$$u = \bar{u} \in C(0,T;\, H^{3k+1}(\mathbb{R}^+)). \qquad \blacksquare$$

Proposition 5.2. *The map $(f,g) \mapsto u$ is continuous from X_k into $C(\mathbb{R}^+;\, H^{3k+1}(\mathbb{R}^+))$, where X_k is considered as a closed subset of the Fréchet space $H^{3k+1}(\mathbb{R}^+) \times H^{k+1}_{\text{loc}}(\mathbb{R}^+)$.*

Proof: Let $\{(f_n, g_n)\}_{n=1}^\infty$ be a sequence in X_k that converges to $(f,g) \in X_k$, which is to say that for any $T > 0$

$$\|f_n - f\|_{3k+1} + |g_n - g|_{k+1,T} \to 0 \quad \text{as } n \to \infty.$$

Let u_n and u be the solutions of (1.1) corresponding to the data (f_n, g_n) and (f,g), respectively, $n = 1, 2, \cdots$. Furthermore, let $T > 0$ be fixed but arbitrary for the rest of this proof. By Proposition 5.1, it is known that

$$u_n,\ u \in C(0,T;\, H^{3k+1}(\mathbb{R}^+))$$

for all $n \ge 1$. The goal here is to show that

$$\|u_n - u\|_{C(0,T;H^{3k+1}(\mathbb{R}^+))} \to 0$$

as $n \to \infty$. In order that this be established, define, for all $n \ge 1$ and $\epsilon \in (0,1]$ approximations $(f_{n,\epsilon}, g_{n,\epsilon}) \in X_{k+2}$ of (f_n, g_n) for which the properties stated in Proposition 4.1 hold. Let also (f_ϵ, g_ϵ) be similar approximations of (f,g) and let $u_{n,\epsilon}$ and u_ϵ be the solutions of (1.1) corresponding to the data $(f_{n,\epsilon}, g_{n,\epsilon})$ and (f_ϵ, g_ϵ), respectively, $n = 1, 2, \cdots$.

The difference $u_n - u$ will be estimated by considering separately each term in the identity

$$u_n - u = (u_n - u_{n,\epsilon}) + (u_{n,\epsilon} - u_\epsilon) + (u_\epsilon - u).$$

From the proof of Proposition 5.1 we derive immediately that

$$\|u_\epsilon - u\|_{C(0,T;H^{3k+1}(\mathbb{R}^+))} \to 0 \tag{5.2}$$

as $\epsilon \to 0$. Consider next the difference $u_n - u_{n,\epsilon}$. Again it follows from Proposition 5.1 that

$$\|u_{n,\epsilon} - u_n\|_{C(0,T;H^{3k+1}(\mathbb{R}^+))} \to 0 \tag{5.3}$$

as $\epsilon \to 0$ for all fixed $n \geq 1$. Furthermore, by Proposition 4.1,

$$\|f_{n,\epsilon} - f_n\|_{3k+1}\,, \quad |g_{n,\epsilon} - g_n|_{k+1,T} \to 0$$

as $\epsilon \to 0$, uniformly in n. Also, Lemmas 3.1, 3.3 and 4.5, and Proposition 4.1 imply that

$$\|u_{n,\epsilon} - u_n\|_{L^\infty(\mathbb{R}^+\times(0,T))}\|u_{n,\epsilon}\|_{L^\infty(0,T;H^{3k+2}(\mathbb{R}^+))} \leq C_{k,T}\epsilon^{k-1/2}\,,$$

where $C_{k,T}$ is independent of n. Hence (cf. (5.1)), the convergence in (5.3) is uniform in n. Let $\delta > 0$ be arbitrary. From (5.2) and the uniform convergence in (5.3), it is concluded that there exists an $\epsilon_1 \in (0,1]$ such that

$$\|u_{n,\epsilon} - u_n\|_{C(0,T;H^{3k+1}(\mathbb{R}^+))} + \|u_\epsilon - u\|_{C(0,T;H^{3k+1}(\mathbb{R}^+))} \leq \delta/2 \tag{5.4}$$

for all $\epsilon \in (0,\epsilon_1]$ and all $n \geq 1$. For the rest of this proof let $\epsilon \in (0,\epsilon_1)$ be fixed. From Lemma 3.3 we have

$$\begin{aligned}\|u_{n,\epsilon} - u_\epsilon\|_{C(0,T;H^{3k+1}(\mathbb{R}^+))} \leq C_{k,T}\big\{&\|f_{n,\epsilon} - f_\epsilon\|_{3k+1} + |g_{n,\epsilon} - g_\epsilon|_{k+1,T} \\ &+ \|u_{n,\epsilon} - u_\epsilon\|_{L^\infty(\mathbb{R}^+\times(0,T))}\cdot\|u_\epsilon\|_{L^\infty(0,T;H^{3k+2}(\mathbb{R}^+))}\big\}\,,\end{aligned}$$

where the constant $C_{k,T}$ is independent of n. From the continuity of the map $(f,g) \mapsto (f_\epsilon, g_\epsilon)$ in $H^{3k+1}(\mathbb{R}^+) \times H^{k+1}_{\text{loc}}(\mathbb{R}^+)$ (see Proposition 4.1) it is readily concluded that

$$\|f_{n,\epsilon} - f_\epsilon\|_{3k+1} + |g_{n,\epsilon} - g_\epsilon|_{k+1,T} \to 0$$

as $n \to \infty$. Also by Lemma 3.1 and Proposition 4.1

$$\|u_{n,\epsilon} - u_\epsilon\|_{L^\infty(\mathbb{R}^+\times(0,T))} \to 0$$

as $n \to \infty$. Hence, there exists a positive integer $N = N(\delta)$ such that

$$\|u_{n,\epsilon} - u_\epsilon\|_{C(0,T;H^{3k+1}(\mathbb{R}^+))} \leq \delta/2$$

provided $n \geq N$.
Together with (5.4), this implies that u_n converges to u in $C(0,T;\, H^{3k+1}(\mathbb{R}^+))$. Since $T > 0$ was arbitrary, this completes the proof of Proposition 5.2. ∎

6. Smoothing of solutions. In this final section of the paper the proof of Theorem 2.2 will be completed by showing that the solutions of (1.1) that have been the focus of attention thus far possess smoothness beyond what has already been established. These results, enunciated explicitly as Proposition 6.1, are similar to those first discovered by Kato (1983) and Bona and Saut (1988) for the pure initial-value problem (1.2). Indeed, our proof follows the lines laid out in Section 6 in Kato (1983) and in Theorem 9 in Bona and Saut (1988). (Further work on the pure initial-value problem (1.2) along the lines of Kato's work may be found in Ponce (1988) where the initial-value problem for the Benjamin-Ono equation is the principle point of departure). There is an important difference arising in the proof of smoothing in the case where the underlying spatial domain is the half line, namely the boundary conditions at $x = 0$ that appear upon integration by parts. These present difficulties not arising in the previous theory.

The starting point is a one-parameter class $\{p_R\}_{R>0}$ of real-valued functions defined on $\mathbb{R}^+$ which are at least piecewise twice continuously differentiable and such that

$$\begin{cases} p_R(0) = 0, \\ p_R,\ p'_R,\ p''_R \text{ are bounded on } \mathbb{R}^+, \\ p'_R(x) \to 0 \text{ as } x \to \infty, \\ p'_R(x) = 1 \quad \text{for} \quad 0 \le x \le R, \\ \text{and there is a positive } \mu \text{ for which} \\ p'_R(x) - \mu|p''_R(x)| \ge 0 \ \text{ for all } x \in \mathbb{R}^+ . \end{cases} \tag{6.1}$$

Note that the last condition in (6.1) entails that p'_R is everywhere non-negative. The construction of functions p_R satisfying (6.1) presents no difficulty. Indeed, the choice

$$p_R(x) = \int_0^x \theta(y)\, dy$$

where

$$\theta(y) = \begin{cases} 1 & \text{for } 0 \le y \le R, \text{ and} \\ 1 - \tanh(y - R) & \text{for } R \le y \end{cases}$$

satisfies all of the properties described in (6.1) with $\mu = \frac{1}{2}$, say. The family $\{p_R\}$ will figure prominently in the proofs of the last proposition.

Proposition 6.1. *Let $(f, g) \in X_k$ where $k \ge 1$ and let $u \in C(\mathbb{R}^+;\ H^{3k+1}(\mathbb{R}^+))$ be the solution of (1.1) corresponding to initial and boundary data f and g, respectively. Then for every positive T,*

$$u \in L_2(0, T;\ H^{3k+2}_{\text{loc}}(\mathbb{R}^+)) . \tag{6.2}$$

Moreover, the map that assigns to $(f, g) \in X_k$ the unique solution of (1.1) in $C(0, T;\ H^{3k+1}(\mathbb{R}^+))$ is continuous from X_k into $L_2(0, T;\ H^{3k+2}_{\text{loc}}(\mathbb{R}^+))$.

Proof: To prove (6.2) it suffices to show that

$$\int_0^T \int_0^R |\partial_x^{3k+2} u(x, t)|^2\, dx\, dt$$

is bounded for all finite, positive values of R and T. To prove the continuous dependence result, it suffices to show that if $\{(f_n, g_n)\}_{n=1}^{\infty}$ is a sequence from X_k converging to (f, g) there, and if $\{u_n\}_{n=1}^{\infty}$ and u are the associated solutions of (1.1) in $C(\mathbb{R}^+;\ H^{3k+1}(\mathbb{R}^+))$, then for each value of R and T,

$$\int_0^T \int_0^R |\partial_x^{3k+2}(u_n - u)|^2\, dx\, dt \to 0 \tag{6.3}$$

as $n \to \infty$. As in Section 5, we begin by regularizing the data. Let $\{(f_{n,\epsilon}, g_{n,\epsilon})\}_{0<\epsilon\leq 1}$ and $\{(f_\epsilon, g_\epsilon)\}_{0<\epsilon\leq 1}$ be families of approximate data in X_{k+2} as defined in Proposition 4.1 relative to (f_n, g_n) and (f, g), respectively, and let $\{u_{n,\epsilon}\}_{0<\epsilon\leq 1}$ and $\{u_\epsilon\}_{0<\epsilon\leq 1}$ be solutions of (1.1) associated to this data, $n = 1, 2, \cdots$. Let W stand for either $u_{n,\epsilon} - u_{n,\delta}$ or $u_\epsilon - u_\delta$ where $0 < \delta < \epsilon \leq 1$ and let U stand for $u_{n,\epsilon}$ in the first case and for u_ϵ in the second case. Then the functions W and U lie in $C(\mathbb{R}^+; H^{3k+7}(\mathbb{R}^+))$ and they satisfy the initial- and boundary-value problem

$$\left.\begin{aligned} &W_t + W_x + WW_x + (UW)_x + W_{xxx} = 0 && \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}^+ \text{ with} \\ &W(x,0) = f_{n,\epsilon} - f_{n,\delta} && \text{for } x \in \mathbb{R}^+, \text{ and} \\ &W(0,t) = g_{n,\epsilon} - g_{n,\delta} && \text{for } t \in \mathbb{R}^+. \end{aligned}\right\} \tag{6.4}$$

If W is standing for $u_\epsilon - u_\delta$ then the n's are absent in (6.4).

Differentiate equation (6.4) $3k+1$ times with respect to x, multiply the result by $p_R \partial_x^{3k+1} W$, and integrate with respect to the spatial variable over $\mathbb{R}^+$. After integrations by parts, we come to the relationship

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^\infty p_R(x)|\partial_x^{3k+1}W(x,t)|^2\,dx - \frac{1}{2}\int_0^\infty p_R'(x)|\partial_x^{3k+1}W(x,t)|^2\,dx \\ &+\frac{1}{2}\int_0^\infty p_R(x)[\partial_x^{3k+1}W\partial_x^{3k+2}(W^2)]\,dx + \int_0^\infty p_R(x)[\partial_x^{3k+1}W\partial_x^{3k+2}(UW)]\,dx \\ &+\frac{3}{2}\int_0^\infty p_R'(x)|\partial_x^{3k+2}W(x,t)|^2\,dx + \int_0^\infty p_R''(x)[\partial_x^{3k+1}W\partial_x^{3k+2}W]\,dx \\ &\quad - \partial_x^{3k+1}W(0,t)\partial_x^{3k+2}W(0,t) = 0. \end{aligned}$$

Rearranging this formula and integrating with respect to the temporal variable over $[0,T]$ leads to

$$\begin{aligned} &\frac{3}{2}\int_0^T\int_0^\infty p_R'(x)|\partial_x^{3k+2}W|^2\,dx\,dt = -\frac{1}{2}\int_0^\infty p_R(x)|\partial_x^{3k+1}W(x,T)|^2\,dx \\ &+\frac{1}{2}\int_0^\infty p_R(x)|\partial_x^{3k+1}W(x,0)|^2\,dx + \frac{1}{2}\int_0^T\int_0^\infty p_R'(x)|\partial_x^{3k+1}W(x,t)|^2\,dx\,dt \\ &-\frac{1}{2}\int_0^T\int_0^\infty p_R(x)[\partial_x^{3k+1}W\partial_x^{3k+2}(W^2)]\,dx\,dt - \int_0^T\int_0^\infty p_R(x)[\partial_x^{3k+1}W\partial_x^{3k+2}(UW)]\,dx\,dt \\ &-\int_0^T\int_0^\infty p_R''(x)[\partial_x^{3k+1}W\partial_x^{3k+2}W]\,dx\,dt + \int_0^T \partial_x^{3k+1}W(0,t)\partial_x^{3k+2}W(0,t)\,dt. \end{aligned} \tag{6.5}$$

Estimating the penultimate term on the right-hand side of (6.5) in a straightforward way, we come to the inequality

$$
\begin{aligned}
\frac{3}{2}\int_0^T\int_0^R |\partial_x^{3k+2}W|^2\,dx\,dt &\le \frac{3}{2}\int_0^T\int_0^\infty [p_R'(x)-\mu|p_R''|]|\partial_x^{3k+2}W|^2\,dx\,dt\\
&\le \frac{1}{2}\int_0^\infty p_R(x)\{|\partial_x^{3k+1}W(x,0)|^2-|\partial_x^{3k+1}W(x,T)|^2\}\,dx\\
&+\frac{1}{2}\int_0^T\int_0^\infty \{p_R'(x)+\frac{1}{3\mu}|p_R''(x)|\}|\partial_x^{3k+1}W(x,t)|^2\,dx\,dt\\
&-\frac{1}{2}\int_0^T\int_0^\infty p_R(x)[\partial_x^{3k+1}W\partial_x^{3k+2}(W^2)]\,dx\,dt \qquad (6.6)\\
&-\int_0^T\int_0^\infty p_R(x)[\partial_x^{3k+1}W\partial_x^{3k+2}(UW)]\,dx\,dt\\
&+\int_0^T \partial_x^{3k+1}W(0,t)\partial_x^{3k+2}W(0,t)\,dt\,,
\end{aligned}
$$

where use has been made of the last property of the weight p_R specified in (6.1) to deduce the left-hand inequality.

The goal now is to establish a bound on the right-hand side of formula (6.6) that is independent of n, and which tends to zero as ϵ tends to zero. Supposing for the moment that such a result has been provided, the remainder of the proof is then straightforward. For it first transpires that the family $\{u_\epsilon\}_{0<\epsilon\le 1}$ is Cauchy in the space $L_2(0,T;\ H^{3k+2}(0,R))$, where R, $T>0$ are arbitrary. It follows that $u_\epsilon \to \bar{u}$ in $L_{2,\mathrm{loc}}(\mathbb{R}^+;\ H^{3k+2}(\mathbb{R}^+))$ as $\epsilon\to 0$. However, we already know from the results in Section 5 that as $\epsilon\to 0$, $u_\epsilon \to u$, the unique solution of (1.1) with data (f,g), in $C(\mathbb{R}^+;\ H^{3k+1}(\mathbb{R}^+))$. In consequence, $u=\bar{u}$ and it is thus assured that $u\in L_2(0,T;\ H^{3k+2}_{\mathrm{loc}}(\mathbb{R}^+))$ for all $T>0$. Similarly, $u_n\in L_2(0,T;\ H^{3k+2}_{\mathrm{loc}}(\mathbb{R}^+))$ for all n and $u_{n,\epsilon}\to u_n$ in this space. Moreover, for fixed, positive R and T,

$$
\begin{aligned}
\Big[\int_0^T\int_0^R |\partial_x^{3k+2}(u_n-u)|^2\,dx\,dt\Big]^{1/2} &\le \Big[\int_0^T\int_0^R |\partial_x^{3k+2}(u_n-u_{n,\epsilon})|^2\,dx\,dt\Big]^{1/2}\\
+\Big[\int_0^T\int_0^R |\partial_x^{3k+2}(u_{n,\epsilon}-u_\epsilon)|^2\,dx\,dt\Big]^{1/2} &+ \Big[\int_0^T\int_0^R |\partial_x^{3k+2}(u_\epsilon-u)|^2\,dx\,dt\Big]^{1/2}.
\end{aligned}
$$

Under the present assumption, the first and third terms on the right-hand side of the last formula can be made as small as we like simply by choosing ϵ small enough, and this holds uniformly in n. Thus given $\gamma>0$, there is an $\epsilon_0>0$ such that for all ϵ in $(0,\epsilon_0]$,

$$
\Big[\int_0^T\int_0^R |\partial_x^{3k+2}(u_n-u)|^2\,dx\,dt\Big]^{1/2} \le \gamma + \Big[\int_0^T\int_0^R |\partial_x^{3k+2}(u_{n,\epsilon}-u_\epsilon)^2\,dx\,dt\Big]^{1/2}. \qquad (6.7)
$$

By the continuity of the regularization defined in Proposition 4.1, $(f_{n,\epsilon},g_{n,\epsilon})\to(f_\epsilon,g_\epsilon)$ as $n\to\infty$ in X_{k+2}, and so Proposition 5.2 implies $u_{n,\epsilon}\to u_\epsilon$ as $n\to\infty$ in $C(0,T;\ H^{3k+7}(\mathbb{R}^+))$. Hence it follows from (6.7) that if ϵ is fixed and less than ϵ_0, then

$$
\overline{\lim_{n\to\infty}}\Big[\int_0^T\int_0^R |\partial_x^{3k+2}(u_n-u)^2|\,dx\,dt\Big]^{1/2} \le \gamma\,,
$$

and since $\gamma>0$ was arbitrary, it follows that $u_n\to u$ in $L_2(0,T;H^{3k+2}(0,R))$, as desired.

Attention is thus focussed on proving the bound whose validity was just argued to imply all the conclusions specified in the statement of the Proposition. To this end, it is useful to notice the following properties of W and U which follow readily from Lemma 3.1, Lemma 3.3, Lemma 3.4 and Proposition 4.1:

$$\|W(\cdot,t)\|_{3k+1} = o(1) \qquad \text{as } \epsilon \to 0, \tag{6.8a}$$

$$\|W(\cdot,t)\|_{1} = o(\epsilon^k) \qquad \text{as } \epsilon \to 0, \tag{6.8b}$$

$$|\partial_x^{3k+1} W(0,\cdot)|_T = o(1) \qquad \text{as } \epsilon \to 0, \tag{6.8c}$$

$$|\partial_x^{3k+2} W(0,\cdot)|_T = o(1) \qquad \text{as } \epsilon \to 0, \tag{6.8d}$$

$$\|U(\cdot,t)\|_{3k+1} = O(1) \qquad \text{as } \epsilon \to 0, \quad \text{and} \tag{6.8e}$$

$$\|U(\cdot,t)\|_{3k+2} = O(\epsilon^{-1/2}) \qquad \text{as } \epsilon \to 0, \tag{6.8f}$$

uniformly with regard to n and t in $[0,T]$. Indeed, several of these relations were already commented upon and used in Section 5.

Turning now to consideration of the right-hand side of (6.6), since p_R, p'_R and p''_R are all bounded functions, the first two terms are $o(1)$ as $\epsilon \to 0$, uniformly in n by (6.8a). The final term on the right side of (6.6) is also $o(1)$ as $\epsilon \to 0$ because of (6.8c) and (6.8d). Using Leibniz' rule, and with one integration by parts, the term that is cubic in W is written as

$$\int_0^T \int_0^\infty [(3k+\frac{3}{2})p_R(x)W_x - \frac{1}{2}p'_R(x)W][\partial_x^{3k+1}W]^2\,dx\,dt$$
$$+\frac{1}{2}\sum_{j=2}^{3k}\binom{3k+2}{j}\int_0^T\int_0^R p_R(x)\left[\partial_x^{3k+1}W\partial_x^j W\partial_x^{3k+2-j}W\right]dx\,dt\,.$$

An examination of the various summands appearing in this expression and reference to (6.8) shows each one to be $o(1)$ as $\epsilon \to 0$, uniformly in n. Using (6.8a) and (6.8e), a similar conclusion is readily reached regarding the integral explicitly involving U, except that the term

$$\int_0^T\int_0^\infty p_R(x)\left[W\partial_x^{3k+1}W\partial_x^{3k+2}U\right]dx\,dt$$

requires the use of (6.8f) and (6.8b).

The crucial bound being established, the proof of the Proposition is now complete. ∎

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