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Decay of Solutions of Some Nonlinear Wave Equations*

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We study the large-time behaviour of solutions to the initial-value problem for the Korteweg–de Vries equation and for the regularized long-wave equation, with a dissipative term appended. Using energy estimates, a maximum principle, and a transformation of Cole-Hopf type, sharp rates of temporal decay of certain norms of the solution are obtained.

1. INTRODUCTION

When attempting to describe the propagation of small-amplitude long waves in nonlinear dispersive media, it is frequently necessary to take account of dissipative mechanisms to accurately reflect real situations. Oftentimes the mechanisms leading to the degradation of the wave are quite complex and not well understood. In such cases one may be forced to rely upon *ad hoc* models of dissipation (cf. Bona *et al.* [9]). Two equations that have gained some currency when the need to append dissipation to nonlinearity and dispersion arises in modelling unidirectional propagation of planar waves are

$$u_t + u_x + uu_x - \nu u_{xx} + u_{xxx} = 0 \tag{1.1}$$

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and

$$u_t + u_x + uu_x - \nu u_{xx} - u_{xxt} = 0, \tag{1.2}$$

where ν is a fixed, positive constant. Here $u = u(x, t)$ is a real-valued function of the two real variables x and t, which, in applications, are typically proportional to distance in the direction of propagation and to elapsed time, respectively. The dependent variable may represent a displacement of the underlying medium or a velocity, for example. Equation (1.1) is sometimes referred to as the Korteweg–de Vries–Burgers equation (KdVB equation) since it represents a marriage of the Korteweg–de Vries equation (KdV) equation (see Korteweg and de Vries [19]),

$$u_t + u_x + uu_x + u_{xxx} = 0 \tag{1.3}$$

and the Burgers equation,

$$u_t + uu_x = \nu u_{xx}. \tag{1.4}$$

Equation (1.2) is the so-called regularized long-wave equation (RLW equation; cf. Benjamin *et al.* [4]),

$$u_t + u_x + uu_x - u_{xxt} = 0, \tag{1.5}$$

with a Burgers-type dissipative term appended. Equations (1.3) and (1.5), and their dissipative counterparts (1.1) and (1.2), respectively, have been the subject of numerous investigations. The review articles of Benjamin [3], Bona [7], Jeffrey and Kakutani [14], Miura [23], Scott *et al.* [28], or the texts of Ablowitz and Segur [1], Lamb [20], or Whitham [29] can help the interested reader into the literature.

The *pure initial-value problem* for any of the above equations is to ask for a solution u defined for (x, t) in $\mathbb{R} \times \mathbb{R}^+$, having a specified initial configuration

$$u(x, 0) = f(x), \tag{1.6}$$

for x in $\mathbb{R}$. In case $u(x, t)$ represents the displacement of the medium from its equilibrium position at the location labelled by x at time t, then one thinks of f as representing a known initial displacement of the medium. A typical situation arises wherein the initial disturbance has sensibly finite extent, so corresponding to a datum f being drawn from some class of functions having limits at $\pm\infty$. The pure initial-value problem for either of Eqs. (1.1) or (1.2) is well-posed in Hadamard's classical sense. That is, corresponding to suitable functions f, there are unique functions $u(x, t)$ defined for $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ satisfying the differential equation there, and for which (1.6) holds. Moreover, if f is perturbed slightly within its function class, then the solution u changes only slightly in response. Precise statements of these results are provided in Section 2.

The present paper is concerned with the large-time behaviour of solutions to either (1.1) or (1.2). Experience with equations that incorporate dissipation, for example, the Burgers equation (1.4), leads one to conjecture that solutions of (1.1) or (1.2) should approach zero as t approaches infinity. Roughly speaking, the energy is continually degraded, and there is no mechanism to replenish this lost energy. Hence if $\|\cdot\|$ is some norm for real-valued functions of a real variable, then it is expected that

$$\|u(\cdot, t)\| \to 0 \tag{1.7}$$

as $t \to \infty$. Our concern herein will be to determine, for various natural choices of norm, if (1.7) is valid, and if so what can be said about the rate at which $\|u(\cdot, t)\|$ approaches zero.

It will be shown that (1.7) is valid for many choices of the norm, and in certain, important circumstances sharp rates of decay will be established. For example, we shall show that solutions u of (1.1) and (1.2) have the property that

$$\int_{-\infty}^{\infty} u^2(x, t)\, dx = O(t^{-1/2}), \tag{1.8}$$

as $t \to \infty$ and that this result cannot in general be improved. (In fact, in Theorem 5.5 we establish explicitly the limit as $t \to \infty$ of $t^{1/2} \int u^2(x, t)\, dx$ in terms of the initial data f.) It is worth contrasting this theorem with the analogous result for the equation

$$u_t + u_x + uu_x + \nu u + u_{xxx} = 0, \tag{1.9}$$

which features the KdV equation with a different dissipative term appended (cf. Knickerbocker and Newell [18]). Whilst the result (1.8) appears to be somewhat subtle, the decay rate for the L_2-norm of solutions of (1.9) is very easily derived. Assume that the solution u of (1.9) is smooth, and that u, u_x, and u_{xx} tend rapidly to zero as $|x| \to \infty$. Then if (1.9) is multiplied by u and the result integrated over $\mathbb{R}$ we obtain, by formal calculations that can be rigorously justified, the relationship

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} u^2(x, t)\, dx + \nu \int_{-\infty}^{\infty} u^2(x, t)\, dx = 0.$$

From this there follows at once the exact formula

$$\int_{-\infty}^{\infty} u^2(x, t)\, dx = e^{-2\nu t} \int_{-\infty}^{\infty} f^2(x)\, dx,$$

where, as before, $f(x) = u(x, 0)$. Thus in this case,

$$\int_{-\infty}^{\infty} u^2(x, t)\, dx = O(e^{-2\nu t}), \tag{1.10}$$

a result considerably different from that expressed in (1.8). One concludes from this that the rate of decay of a given norm is very much dependent on the particular dissipative term featured in the equation.

If a certain norm of a solution u of (1.1) or (1.2) tends to zero at a particular rate, then heuristic considerations would lead one to expect that the norm of the derivatives of u will tend to zero even faster. Thus it will be demonstrated that solutions u of (1.1) or (1.2) satisfy

$$\int_{-\infty}^{\infty} u_x^2(x, t)\, dx = O(t^{-3/2}), \tag{1.11}$$

and that this result is also sharp in general. Again, by contrast, it is readily established that a smooth solution u of (1.9) with appropriate spatial decay still has only the property

$$\int_{-\infty}^{\infty} u_x^2(x, t)\, dx = O(e^{-2\nu t})$$

as $t \to \infty$. Thus no enhanced decay for derivatives appears to obtain for solutions of (1.12).

It is worth noting that not all norms of solution of (1.1) or (1.2) tend to zero in the limit of unboundedly large time. Again assuming appropriate smoothness and spatial decay of a solution u of (1.1) or (1.2), one derives easily a conservation of mass result stating that

$$\int_{-\infty}^{\infty} u(x, t)\, dx = \int_{-\infty}^{\infty} f(x)\, dx, \tag{1.12}$$

for all $t \geqslant 0$. It follows that if the right-hand side of (1.12) is nonzero, then

$$\int_{-\infty}^{\infty} |u(x, t)|\, dx$$

does not tend to zero as $t \to \infty$.

The paper is organized as follows. In Section 2 we set the notation and present the aforementioned theory pertaining to the well-posedness of the initial-value problem for Eqs. (1.1) and (1.2). In Sections 3, 4, and 5 the large-time asymptotics of the solutions of (1.2) are considered in detail. Section 3 contains elementary results of boundedness and decay while

Section 4 focusses on solutions of the linear version of (1.2). The results for the linearized problem are interesting in their own right, suggestive of the situation that obtains for the nonlinear problem, and technically useful in Section 5 where more subtle results like (1.8) and (1.11) for the nonlinear equation are obtained. Section 6 is devoted to remarks concerning Eq. (1.1). It is indicated there that for the most part the same theorems hold for Eqs. (1.1) and (1.2). In the main, the proofs of the theorem are quite similar for both equations. Section 7 is a short conclusion summarizing the paper's accomplishments and indicating future lines of inquiry.

2. Notation and Preliminary Results

The notation in force throughout is briefly reviewed here, along with some prior results that bear on the subsequent theory.

If X is any Banach space its norm will generally be denoted $\| \ \|_X$, except for the special abbreviations mentioned below. The symbol $\mathbb{R}$ is reserved for the real line, and $\mathbb{R}^N$ is N-dimensional Euclidean space. If $\Omega \subset \mathbb{R}^N$ is a Lebesgue measurable set, $L_p(\Omega)$ is the Banach space of the pth-power integrable functions normed by

$$|f|_{L_p(\Omega)} = \left\{ \int_\Omega |f(x)|^p \, dx \right\}^{1/p}$$

If Ω is understood from the context, we shall write L_p for $L_p(\Omega)$ and $|f|_p$ for the norm of $f \in L_p(\Omega)$. This will very often be the case when $\Omega = \mathbb{R}$. The usual modification will be presumed if $p = \infty$.

If $f \in L_p(\Omega)$ and its distributional derivatives up to order k also lie in $L_p(\Omega)$, we write $f \in W_p^k(\Omega)$. This class of functions is a Banach space with norm

$$\|f\|_{W_p^k(\Omega)} = \sum_{|\alpha| \leqslant k} |\partial^\alpha f|_{L_p(\Omega)}, \tag{2.1}$$

where the usual multi-index notation is being employed. The case $p = 2$ will appear often and so deserves the special notation $H^k(\Omega)$. If Ω is understood, then H^k will stand for $H^k(\Omega)$, and the norm of f in $H^k(\Omega)$ will be abbreviated to

$$\|f\|_k = \|f\|_{H^k(\Omega)}.$$

Again, this shorthand will frequently be used in case $\Omega = \mathbb{R}$. We let $H^\infty(\Omega) = \bigcap_k H^k(\Omega)$, but do not bother to topologize this collection of functions. It is useful to remember that $H^\infty(\mathbb{R})$ is dense in $H^k(\mathbb{R})$, for any

k. Note that there are two shortened notations for L_2, namely $|f|_2$ and $\|f\|_0$. We will systematically prefer the second notation, and will usually drop the subscript 0 if no confusion results thereby. Thus an unadorned norm $\|\ \|$ is the norm in L_2.

If $f \in L_1(\mathbb{R})$, its Fourier transform is defined as

$$\hat{f}(y) = (\mathscr{F}f)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixy}\, dx. \tag{2.2}$$

If $f \in L_2(\mathbb{R})$, then $\mathscr{F}f \in L_2(\mathbb{R})$ and $\mathscr{F}$ extends by continuity to a Hilbert-space isomorphism of $L_2(\mathbb{R})$ onto itself. If k is a positive integer, and $f \in H^k(\mathbb{R})$, then

$$\|f\|_k^2 = \int_{-\infty}^{\infty} (1 + y^2 + \cdots + y^{2k}) |\hat{f}(y)|^2\, dy. \tag{2.3}$$

These elementary facts may be found, for example, in Yosida's text [30].

If I is a closed interval in $\mathbb{R}$, say $I = [a, b]$ where $a = -\infty$ or $b = +\infty$ is allowed, and X is a Banach space, let $C_{\mathrm{b}}(a, b; X) = C_{\mathrm{b}}(I; X)$ denote the bounded continuous mappings $u: I \to X$. This is again a Banach space with the norm

$$\|u\|_{C_{\mathrm{b}}(I;X)} = \sup_{t \in I} \|u(t)\|_X.$$

If I is bounded, the subscript b, for bounded, will be dropped. Similarly, if $1 \leqslant p \leqslant \infty$ and I and X are as above, $L_p(I; X)$ is the collection of measurable functions $u: I \to X$ such that

$$\|u\|_{L_p(I;X)} = \left\{ \int_I \|u(s)\|_X^p\, ds \right\}^{1/p} < \infty.$$

Logically prior to the developments in this paper are general results of existence, uniqueness, regularity, and continuous dependence for the initial-value problems under study. Fortunately there is an adequate theory of well-posedness for both the initial-value problems to be considered here. With a few minor extensions, this theory will serve our purposes very nicely.

THEOREM 2.1. *Let $f \in H^s$ where $s \geqslant 2$. Then there exists a unique function $u \in C_b(0, \infty; H^s)$ such that $u(\cdot, 0) = f(\cdot)$ and which solves* (1.1) *in $\mathbb{R} \times \mathbb{R}^+$. For each $T > 0$, $u(\cdot, T)$ lies in H^∞ and the mapping that associates to f in H^s the solution u of* (1.1) *with initial value f is continuous from H^s to $C(0, T; H^s)$. Moreover, if $f \in W_1^k$, then $u \in C(0, T; W_1^k)$ for any $T > 0$, and $u(\cdot, T) \in W_1^r$ for any $r > 0$.*

Remark. This result of existence, uniqueness, continuous dependence and smoothing is established in Bona and Smith [10]. The hypothesis $s \geqslant 2$ can be weakened to $s \geqslant 0$, but the corresponding conclusions are more complicated to state and are not needed herein. The H^s results do not depend on ν, and remain in the limit as ν tends to 0. The smoothing and the L_1 results need $\nu > 0$ and are not true in general in the limit as ν tends to zero.

THEOREM 2.2. *Let $f \in H^s$, where $s \geqslant 1$. Then there exists a unique function in $C_b(0, \infty; H^1)$ which also lies in the class $C(0, T; H^s)$ for each $T > 0$ and is such that u solves (1.2) in $\mathbb{R} \times \mathbb{R}^+$ with $u(\cdot, 0) = f(\cdot)$. For each $k > 0$, $\partial_t^k u$ also lies in $C(0, T; H^s)$ for each $T > 0$. The mapping that associates to f in H^s the solution u of (1.2) with initial value f is continuous from H^s to $C^k(0, T; H^s)$, for $k \geqslant 0$. If $f \in W_1^k$ then u and all of its temporal derivatives lie in $C(0, T; W_1^k)$, for all $T > 0$.*

Remark. Except for the L_1 theory this theorem is proved in Benjamin *et al.* [4]. The L_1-results may be obtained using the sort of argument favoured in the last-quoted reference, as we show presently. It is worth noting that despite the dissipation, there is no smoothing effect in the spatial variable. All the above results carry over in the limit as ν tends to zero, but with $\nu = 0$, $\partial_t^k u \in C(0, T; H^{s+1})$ for $k > 0$ and $T > 0$.

Proof. As mentioned above, only the L_1 results are new. Let u be a solution of (1.2) and let the Green's function $\frac{1}{2}e^{-|z|}$ for $1 - \partial_x^2$ on the whole line be denoted by $K(z)$. Evaluate Eq. (1.2) for u at the point (y, t), multiply the result by $K(x - y)$, and then integrate with respect to y over $\mathbb{R}$. After a little manipulation, there appears

$$\begin{aligned} u_t(x, t) = &- \nu u(x, t) + \nu \int_{-\infty}^{\infty} K(x-y)\, u(y, t)\, dy \\ &- \int_{-\infty}^{\infty} K'(x-y) \left[u(y, t) + \frac{1}{2} u^2(y, t) \right] dy, \end{aligned} \tag{2.4}$$

where $K'(z) = -\frac{1}{2}\mathrm{sgn}(z) e^{-|z|}$. Equation (2.4) may be viewed as an ordinary differential equation of the form

$$\dot{u} = -\nu u + g,$$

where the dot connotes differentiation with respect to the temporal variable. This interpretation of (2.4) allows one to deduce readily the equivalence of (2.4) and the formula

$$u(x, t) = e^{-\nu t} f(x) + \int_0^t e^{\nu(s-t)} \left\{ \nu \int_{-\infty}^{\infty} K(x-y)\, u(y, s)\, dy \right.$$

$$\left. - \int_{-\infty}^{\infty} K'(x-y) \left[u(y, s) + \frac{1}{2} u^2(y, s) \right] dy \right\} ds$$

$$= f_0(x, t) + A(u)(x, t) = B(u)(x, t) \tag{2.5}$$

for functions u which are reasonably smooth. It is easy to see that if T is sufficiently small, then B is a contraction mapping of a ball centered at zero in $C(0, T; H^1)$. Hence (2.5) possesses a solution in $C(0, T; H^1)$, at least for T small. It is immediate from (2.5) that if $f \in H^s$ for $s > 1$, then $u \in C(0, T; H^s)$. Moreover, the time interval T for which B is known to be contractive depends inversely on $\|f\|_1$. It follows that if $\|u(\cdot, t)\|_1$ is shown to be bounded on finite time intervals, then this contraction-mapping argument may be successfully iterated to produce a solution of (2.5) defined for all $t \geq 0$. More precisely, suppose u to be defined in $C(0, T_1; H^1)$ as a solution of (2.5). Then one can imagine using $u(\cdot, T_1)$ as initial data and using the corresponding reduction of (1.2) to (2.5) to extend u as a solution of (1.2) over the interval $[0, T_1 + \Delta T]$. The positive quantity ΔT depends inversely on $\|u(\cdot, T_1)\|_1$. One continues this procedure inductively and makes use of the *a priori* information about $\|u(\cdot, t)\|_1$ to conclude that u may be extended to any finite interval in a finite number of such steps.

It turns out (see formula (3.2)) that the H^1-norm of a solution of (1.2) is actually bounded for all time, and hence that the above argument is effective in producing global solutions of the initial-value problem for (1.2).

One way to establish the L_1 results in the statement of the theorem is to mimic the argument just outlined relative to the function class H^1. It turns out that using both the L_1 and H^1 norms simultaneously is effective. To this end, define

$$X_T = C(0, T; H^1) \cap C(0, T; L_1)$$

with the norm

$$|||u|||_T = \|u\|_{C(0, T; H^1)} + \|u\|_{C(0, T; L_1)}.$$

If $u, v \in X_T$, then

$$|A(u)(\cdot, t) - A(v)(\cdot, t)|_1$$

$$\leq t\{(|K|_1 + |K'|_1) + \sup_{0 \leq s \leq t} [1 + |u(\cdot, s)|_\infty + |v(\cdot, s)|_\infty]\}$$

$$\times \sup_{0 \leq s \leq t} |u(\cdot, t) - v(\cdot, t)|_1$$

$$\leq tC\|u - v\|_{C(0, t; L_1)}.$$

Hence, for any $T>0$,

$$\|A(u)-A(v)\|_{C(0,T;L_1)} \leqslant TC\|u-v\|_{C(0,T;L_1)}, \tag{2.6}$$

where

$$C = C(\|u\|_{C(0,T;H^1)}, \|v\|_{C(0,T;H^1)}). \tag{2.7}$$

Exactly the same estimate as (2.6) holds in $C(0, T; H^1)$, and the constant has the dependence depicted in (2.7). It follows that

$$|||A(u)-A(v)|||_T \leqslant TC|||u-v|||_T, \tag{2.8}$$

where $C = C(|||u|||_T, |||v|||_T)$, say. The argument in Benjamin *et al.* [4, Sect. 3] may now be seen to carry over intact to X_T. We conclude that B is contractive in a ball of radius R centered at zero in X_T provided R is large enough and T is small enough. The result will therefore be in hand if it can be shown that, for a solution u of (2.5), $|||u|||_T$ is uniformly bounded as T ranges over any bounded interval. Since

$$\|u\|_{C(0,T;H^1)} \leqslant \|f\|_1 \tag{2.9}$$

for all $T>0$ (see again (3.2)), it suffices to bound the L_1-norm of u on bounded time intervals. But from (2.5),

$$|u(\cdot, t)|_1 \leqslant |f|_1 + \int_0^t C|u(\cdot, s)|_1 \, ds, \tag{2.10}$$

where C is as in (2.7), except that in light of (2.9), C is now known to be bounded independently of t. Gronwall's lemma applies to (2.10) and gives the desired result.

The further spatial and temporal regularity of u is now considered. Notice that the mapping A defined in (2.5) carries $C(0, T; H^s) \cap C(0, T; W_1^s)$ into $C(0, T; H^{s+1}) \cap C(0, T; W_1^{s+1})$ for any $s \geqslant 0$. This fact follows easily from the properties of the kernel K and formula (2.5). A simple induction based on (2.5) shows that u is as smooth spatially as the initial datum f. From this and (2.4) it is then adduced that u_t lies in $C(0, T; H^s) \cap C(0, T; W_1^s)$. For $m>1$, another easy induction demonstrates that

$$\partial_t^m u = \nu \partial_t^{m-1} u - \nu \int_{-\infty}^{\infty} K(x-y)\, \partial_t^{m-1} u(y, t)\, dy$$
$$- \int_{-\infty}^{\infty} K'(x-y)\, \partial_t^{m-1} \left[u(y,t) + \frac{1}{2} u^2(y,t) \right] dy,$$

and consequently the final result in the statement of the theorem is seen to be valid.

Remark. We shall see later that if u is a solution of (1.1) or (1.2), then $|u(\cdot, t)|_1$ is in fact bounded independently of t.

3. Elementary Estimates for the RLW-Burgers Equation

In this section and the two sections following, interest will be focussed on the initial-value problem

$$u_t + u_x + uu_x - \nu u_{xx} - u_{xxt} = 0, \tag{3.1a}$$

with

$$u(x, 0) = f(x), \tag{3.1b}$$

for $x \in \mathbb{R}$ and $t \geqslant 0$, where ν is a fixed, positive constant.

If Eq. (3.1a) is multiplied by u, and the result integrated over $\mathbb{R}$ and over $[0, T]$, there appears after appropriate integrations by parts,

$$\int_{-\infty}^{\infty} [u^2(x, t) + u_x^2(x, t)]\, dx$$
$$= \int_{-\infty}^{\infty} [f(x)^2 + f'(x)^2]\, dx - 2\nu \int_0^t \int_{-\infty}^{\infty} u_x^2(x, s)\, dx\, ds. \tag{3.2}$$

It follows that $\|u(\cdot, t)\|_1$ is a monotone nonincreasing function of t, and that $u_x \in L_2(\mathbb{R} \times \mathbb{R}^+)$. In particular, for all $t \geqslant 0$,

$$\|u(\cdot, t)\|_1 \leqslant \|f\|_1 \tag{3.3}$$

and

$$\int_0^{\infty} \int_{-\infty}^{\infty} u_x^2(x, s)\, dx\, ds \leqslant \frac{1}{2\nu} \|f\|_1^2. \tag{3.4}$$

Lemma 3.1. *Let u be the solution of* (3.1) *corresponding to initial data $f \in H^1(\mathbb{R})$. Then,*

(a) *inequalities* (3.3) *and* (3.4) *hold,*

(b) $u_{xx} \in L_2(\mathbb{R} \times \mathbb{R}^+)$, $\|u_x(\cdot, t)\|$, $\|u_{xx}(\cdot, t)\| \to 0$, *as* $t \to \infty$, *and*

(c) $|u(\cdot, t)|_\infty \to 0$, *as* $t \to \infty$.

Proof. Part (a) has already been established using formula (3.2). A corollary of (3.3) is the following inequality: for any $x \in \mathbb{R}$ and $t \geqslant 0$,

$$u^2(x,t) = 2\int_{-\infty}^{x} (uu_x)\,dx \leqslant 2\|u(\cdot,t)\|\,\|u_x(\cdot,t)\|$$
$$\leqslant \|u(\cdot,t)\|^2 + \|u_x(\cdot,t)\|_2^2 \leqslant \|f\|_1^2. \tag{3.5}$$

In particular, u is bounded and

$$|u(\cdot,t)|_\infty^2 \leqslant 2\|u(\cdot,t)\|\,\|u_x(\cdot,t)\| \leqslant \|f\|_1^2. \tag{3.6}$$

To prove part (b), multiply Eq. (3.1a) by u_{xx} and integrate over $\mathbb{R}$ to obtain

$$\frac{d}{dt}\int_{-\infty}^{\infty} (u_x^2 + u_{xx}^2)\,dx = 2\int_{-\infty}^{\infty} (-\nu u_{xx}^2 + uu_x u_{xx})\,dx. \tag{3.7}$$

Bounding above the right-hand side of this last equation gives a revealing inequality, namely

$$2\int_{-\infty}^{\infty} (-\nu u_{xx}^2 + uu_x u_{xx})\,dx$$
$$\leqslant -2\nu\int_{-\infty}^{\infty} u_{xx}^2\,dx + \nu\int_{-\infty}^{\infty} u_{xx}^2\,dx + \frac{1}{\nu}\int_{-\infty}^{\infty} (u^2u_x^2)\,dx$$
$$\leqslant -\nu\int_{-\infty}^{\infty} u_{xx}^2\,dx + \frac{1}{\nu}|u(\cdot,t)|_\infty^2\,\|u_x(\cdot,t)\|^2$$
$$\leqslant -\nu\int_{-\infty}^{\infty} u_{xx}^2\,dx + \frac{\|f\|_1^2}{\nu}\|u_x(\cdot,t)\|^2.$$

Rearranging and integrating over $[0, t]$ gives

$$\int_{-\infty}^{\infty} (u_x^2 + u_{xx}^2)\,dx + \nu\int_0^t\int_{-\infty}^{\infty} u_{xx}^2\,dx\,dx$$
$$\leqslant \frac{\|f\|_1^2}{\nu}\int_0^t\int_{-\infty}^{\infty} u_x^2\,dx\,ds + \|f\|_2^2. \tag{3.8}$$

Since $u_x \in L_2(\mathbb{R}\times\mathbb{R}^+)$, from (3.4), the right-hand side of (3.8) is bounded, independently of $t \geqslant 0$. Thus $u_{xx} \in L_2(\mathbb{R}\times\mathbb{R}^+)$. As a consequence,

$$\int_{-\infty}^{\infty} (-\nu u_{xx}^2 + uu_x u_{xx})\,dx$$

lies in $L_1(\mathbb{R}^+)$. Hence, from (3.7), the integral

$$\int_{-\infty}^{\infty} (u_x^2 + u_{xx}^2)\,dx$$

approaches a limit as $t \to \infty$. Since this integral lies in $L_1(\mathbb{R}^+)$ as a function of t, its limit at ∞ must be zero.

Part (c) follows immediately upon noting, as in (3.5), that

$$|u(\cdot, t)|_\infty^2 \leqslant 2\|u(\cdot, t)\| \, \|u_x(\cdot, t)\| \leqslant 2\|f\|_1 \, \|u_x(\cdot, t)\|,$$

and remarking that the right-hand side of this inequality tends to zero as $t \to \infty$.

Since $\|u(\cdot, t)\|_1$ is monotone decreasing, it has a limit as $t \to \infty$. Moreover, $\|u_x(\cdot, t)\|$ tends to zero as $t \to \infty$. Hence $\|u(\cdot, t)\|$ has a limit as $t \to \infty$. Whether or not this limit is zero turns out to be important in understanding several aspects of the large-time asymptotics of u. The following lemma suggests, but does not establish, that this limit is zero. (In Section 5 it will be proved that this limit is always zero, and the optimal decay rate delineated.)

LEMMA 3.2. *Let u be the solution of* (3.1) *corresponding to initial data $f \in H^2(\mathbb{R})$. Then,*

(a) $u_t, u_{xt} \in L_2(\mathbb{R} \times \mathbb{R}^+)$,

(b) *for every real number a*

$$\int_{-\infty}^a u^2 \, dx \to 0, \qquad \textit{as} \quad t \to \infty,$$

and

(c) *if Ω is a bounded open set in $\mathbb{R}$, then $u \in L_2(\Omega \times \mathbb{R}^+)$.*

Proof. If Eq. (3.1a) is multiplied by u_t and then integrated over $\mathbb{R}$, and integration by parts is performed, we obtain

$$\begin{aligned}
\frac{\nu}{2} \frac{d}{dt} \int_{-\infty}^{\infty} u_x^2 \, dx &+ \int_{-\infty}^{\infty} (u_t^2 + u_{xt}^2) \, dx \\
&= -\int_{-\infty}^{\infty} u_t(u_x + u u_x) \, dx \\
&\leqslant \frac{1}{2} \int_{-\infty}^{\infty} u_t^2 \, dx + \frac{1}{2} \int_{-\infty}^{\infty} (1+u)^2 \, u_x^2 \, dx \\
&\leqslant \frac{1}{2} \int_{-\infty}^{\infty} u_t^2 \, dx + \frac{(1 + \|f\|_1)^2}{2} \int_{-\infty}^{\infty} u_x^2 \, dx.
\end{aligned}$$

Since $u_x \in L_2(\mathbb{R} \times \mathbb{R})$, part (a) follows.

For part (b), let $\varepsilon > 0$ be given and choose $T = T(\varepsilon)$ such that

$$\int_T^\infty \int_{-\infty}^\infty (\nu^2 u_x^2 + u_{xt}^2)\, dx\, dt \leqslant \frac{\varepsilon}{12}, \qquad |u(\cdot, t)|_\infty \leqslant \frac{1}{4}, \tag{3.9}$$

for all $t \geqslant T$. Then choose $N = N(\varepsilon) < 0$ such that

$$\int_{-\infty}^{N+1} [u^2(x, T) + u_x^2(x, T)]\, dx \leqslant \frac{\varepsilon}{2}. \tag{3.10}$$

Multiply (3.1a) by u as before, but integrate only over $(-\infty, M)$ to obtain that

$$0 = \frac{d}{dt} \int_{-\infty}^M (u^2 + u_x^2)\, dx + 2\nu \int_{-\infty}^M u_x^2\, dx + u^2(M, t) + \frac{2}{3} u^3(M, t)$$

$$- 2\nu u(M, t)\, u_x(M, t) - 2u(M, t)\, u_{xt}(M, t).$$

If we take $t \geqslant T$ in the last formula, so that $|u| \leqslant \frac{1}{4}$, and use Young's inequality, we may derive that

$$\frac{d}{dt} \int_{-\infty}^M (u^2 + u_x^2)\, dx + 2\nu \int_{-\infty}^M u_x^2\, dx + \frac{1}{2} u^2(M, t)$$

$$\leqslant 6[\nu^2 u_x^2(M, t) + u_{xt}^2(M, t)], \tag{3.11}$$

and this holds for any $M \in \mathbb{R}$ and $t \geqslant T$. Integrate (3.11) over the t-interval (T, S) to obtain

$$\int_{-\infty}^M [u^2(x, S) + u_x^2(x, S)]\, dx + \frac{1}{2} \int_T^S u^2(M, t)\, dt + 2\nu \int_T^S \int_{-\infty}^M u_x^2\, dx\, dt$$

$$\leqslant \int_{-\infty}^M [u^2(x, T) + u_x^2(x, T)]\, dx$$

$$+ 6 \int_T^S [\nu^2 u_x^2(M, t) + u_{xt}^2(M, t)]\, dt, \tag{3.12a}$$

for any $M \in \mathbb{R}$. Using (3.10), it is easily adduced that the right-hand side of (3.12a) is bounded above by

$$\frac{\varepsilon}{2} + 6 \int_T^S [\nu^2 u_x^2(M, t) + u_{xt}^2(M, t)]\, dt, \tag{3.12b}$$

provided $M \leqslant N+1$. Finally, integrate (3.12) with respect to M over the interval $(N, N+1)$, and use (3.9) to obtain

$$\int_{-\infty}^{N} [u^2(x, S) + u_x^2(x, S)]\, dx$$

$$\leqslant \frac{\varepsilon}{2} + 6 \int_T^{\infty} \int_{-\infty}^{\infty} [v^2 u_x^2(x, t) + u_{xt}^2(x, t)]\, dx\, dt \leqslant \varepsilon,$$

for any $S \geqslant T$. Since $\|u_x(\cdot, t)\| \to 0$, as $t \to \infty$, it is concluded from the last inequality that

$$\limsup_{t \to \infty} \int_{-\infty}^{N} u^2(x, t)\, dx \leqslant \varepsilon.$$

Moreover, $|u(\cdot, t)|_\infty \to 0$ as $t \to \infty$. Hence, for any $a \in \mathbb{R}$,

$$\limsup_{t \to \infty} \int_{-\infty}^{a} u^2(x, t)\, dx \leqslant \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, part (b) is established.

If (3.12a) is integrated with respect to M over the set Ω and the result considered in the limit as $S \to \infty$, then part (c) is readily deduced. This completes the proof of the lemma.

It is a corollary of the methods appearing in the last proof that for any $\alpha \in (0, 1)$ and any $\beta \in (1, \infty)$,

$$\int_{-\infty}^{\alpha t} u^2(x, t)\, dx + \int_{\beta t}^{\infty} u^2(x, t)\, dx \to 0, \tag{3.13}$$

as $t \to \infty$. If the initial data f is such that

$$\int_{-\infty}^{a} |x| [f^2(x) + f_x^2(x)]\, dx < \infty,$$

for some $a \in \mathbb{R}$ (and so for all $a \in \mathbb{R}$), then an argument like the one given in the last proof (multiply equation (3.1a) by xu and then integrate) ensures that

$$\int_{-\infty}^{a} |x| (u^2(x, t) + u_x^2(x, t))\, dx + \int_0^t \int_{-\infty}^{a} |x|\, u_x^2(x, s)\, dx\, ds$$

is bounded, with a bound depending upon a but not on t. This in turn implies that

$$\int_{-\infty}^{a} \int_{-\infty}^{M} [u^2(x,t) + u_x^2(x,t)]\, dx\, dM$$

has a bound which depends upon a, but not on t. Hence, for such data, the expression on the right-hand side of (3.12a) may be integrated with respect to M over $(-\infty, a)$, and the resulting quantity is seen to be bounded, independently of T and S. A consequence of these observations is that both u and $|x|^{1/2}u_x$ lie in $L_2((-\infty, a)\times \mathbb{R}^+)$, for any $a \in \mathbb{R}$. It will appear later (Theorem 5.5 and the remark thereafter) that if $u \in L_2(\mathbb{R}\times\mathbb{R}^+)$, then necessarily u has zero mass. That is,

$$\int_{-\infty}^{\infty} u(x,t)\, dx = \int_{-\infty}^{\infty} f(x)\, dx = 0.$$

Hence if f does not happen to have zero mass, then $u \notin L_2(\mathbb{R}\times\mathbb{R}^+)$ and so the fact that $u \in L_2((-\infty, a)\times\mathbb{R}^+)$ is a reflection of the expectation that most of the mass of the wave moves to the right as t increases.

In any case, $u_x \in L_2(\mathbb{R}\times\mathbb{R}^+)$, and so one expects that $\|u_x(\cdot, t)\|$ tends to zero at least at the rate $t^{-1/2}$. The following intermediate result verifies this presumption. The optimal rate $t^{-3/4}$ will be proved in Section 5, at the cost of some minor additional restrictions on f.

PROPOSITION 3.3. *Let u be the solution of* (3.1) *corresponding to initial data $f \in H^2(\mathbb{R})$. Then,*

$$\lim_{t\to\infty} t^{1/2} \int_{-\infty}^{\infty} [u_x^2(x,t) + u^4(x,t)]\, dx = 0$$

and

$$\lim_{t\to\infty} t^{1/2} \int_{t}^{\infty} \int_{-\infty}^{\infty} u_{xt}^2(x,s)\, dx\, ds = 0.$$

Proof. Multiply (3.1a) by $u_t + u_x$ and integrate the result over $\mathbb{R}$ to reach the expression

$$\int_{-\infty}^{\infty} (u_t+u_x)^2\, dx + \int_{-\infty}^{\infty} uu_x(u_t+u_x)\, dx + \frac{\nu}{2}\frac{d}{dt}\int_{-\infty}^{\infty} u_x^2\, dx$$
$$+ \int_{-\infty}^{\infty} u_{xx}u_{xt}\, dx + \int_{-\infty}^{\infty} u_{xt}^2\, dx = 0.$$

On the other hand, if (3.1a) is multiplied by u_{xt} and then integrated over $\mathbb{R}$, there appears

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} u_x^2\,dx+\int_{-\infty}^{\infty} uu_xu_{xt}\,dx-\nu\int_{-\infty}^{\infty} u_{xx}u_{xt}\,dx=0.$$

These latter two relations are combined to give

$$\begin{aligned}\frac{1}{2}\left(\nu+\frac{1}{\nu}\right)\frac{d}{dt}\int_{-\infty}^{\infty} u_x^2\,dx&+\int_{-\infty}^{\infty}(u_t+u_x)^2\,dx\\&+\int_{-\infty}^{\infty} uu_x(u_t+u_x)\,dx+\int_{-\infty}^{\infty} u_{xt}^2\,dx\\&+\frac{1}{\nu}\int_{-\infty}^{\infty} uu_xu_{xt}\,dx=0.\end{aligned}\tag{3.14}$$

Finally, (3.1a) is multiplied by u^3 and integrated over $\mathbb{R}$: the result is the relationship

$$\frac{1}{4}\frac{d}{dt}\int_{-\infty}^{\infty} u^4\,dx+3\nu\int_{-\infty}^{\infty} u^2u_x^2\,dx+3\int_{-\infty}^{\infty} u^2u_xu_{xt}\,dx=0.\tag{3.15}$$

Equation (3.15) is multiplied by a suitably large positive constant b and added to (3.14). Then Young's inequality yields

$$\begin{aligned}\frac{1}{2}\left(\nu+\frac{1}{\nu}\right)&\frac{d}{dt}\int_{-\infty}^{\infty} u_x^2\,dx+\frac{b}{4}\frac{d}{dt}\int_{-\infty}^{\infty} u^4\,dx\\&=-\int_{-\infty}^{\infty}(u_t+u_x)^2\,dx-\int_{-\infty}^{\infty} uu_x(u_t+u_x)\,dx\\&\quad-\int_{-\infty}^{\infty} u_{xt}^2\,dx-\frac{1}{\nu}\int_{-\infty}^{\infty} uu_xu_{xt}\,dx\\&\quad-3b\nu\int_{-\infty}^{\infty} u^2u_x^2\,dx-3b\int_{-\infty}^{\infty} u^2u_xu_{xt}\,dx\\&\leqslant\left(-\frac{3b\nu}{2}+\frac{1}{4}+\frac{1}{2\nu^2}\right)\int_{-\infty}^{\infty} u^2u_x^2\,dx\\&\quad+\left(\frac{3b}{2\nu}|u(\cdot,t)|_\infty^2-\frac{1}{2}\right)\int_{-\infty}^{\infty} u_{xt}^2\,dx.\end{aligned}\tag{3.16}$$

First choose b so large that

$$-\frac{3b\nu}{2}+\frac{1}{4}+\frac{1}{2\nu^2}<0,\tag{3.17a}$$

and then, choose T so that for $t \geqslant T$,

$$\frac{3b}{2\nu}|u(\cdot, t)|_\infty^2 - \frac{1}{2} < 0. \tag{3.17b}$$

For such a choice of b, and for $t \geqslant T$, it is assured that

$$\frac{d}{dt}\Gamma(t) \leqslant 0, \tag{3.18a}$$

where

$$\Gamma(t) = \frac{1}{2}\left(\nu + \frac{1}{\nu}\right)\int_{-\infty}^{\infty} u_x^2\, dx + \frac{b}{4}\int_{-\infty}^{\infty} u^4\, dx. \tag{3.18b}$$

On the other hand, $\Gamma(t) \in L_2(\mathbb{R}^+)$. For $\|u_x(\cdot, t)\|$ is bounded and u_x lies in $L_2(\mathbb{R} \times \mathbb{R}^+)$ whilst

$$\int_{-\infty}^{\infty} u^4(x, t)\, dx \leqslant |u(\cdot, t)|_\infty^2 \, \|u(\cdot, t)\|^2 \leqslant 2\|u_x(\cdot, t)\| \, \|u(\cdot, t)\|^3.$$

Since $\|u(\cdot, t)\|$ is bounded and $u_x \in L_2(\mathbb{R} \times \mathbb{R}^+)$, our assertion follows. Because Γ is eventually decreasing, it is evident that

$$\int_\tau^\infty \Gamma^2(s)\, ds \geqslant \int_\tau^t \Gamma^2(s)\, ds \geqslant (t - \tau)\, \Gamma^2(t),$$

for $t > \tau$, and τ large enough. In consequence,

$$\int_\tau^\infty \Gamma^2(s)\, ds \geqslant \limsup_{t \to \infty} t\Gamma^2(t),$$

for any τ large enough. As $\Gamma^2 \in L_1(\mathbb{R}^+)$, the left-hand side of the last inequality can be made as small as desired by choosing τ large enough. It is therefore established that

$$\lim_{t \to \infty} t^{1/2} \int_{-\infty}^{\infty} (u_x^2 + u^4)\, dx = 0,$$

as in the statement of the proposition.

Return now to (3.16) and choose T so large that for $t \geqslant T$,

$$\frac{3b}{2\nu}|u(\cdot, t)|_\infty^2 - \frac{1}{2} < -\frac{1}{4}$$

(see (3.17b)). For such t, (3.16) and the definition (3.18b) of Γ imply that

$$\frac{d\Gamma}{dt}(t) \leq -\frac{1}{4}\int_{-\infty}^{\infty} u_{xt}^2(x, t)\, dx.$$

Integrating this relation over the temporal interval $[t, \infty)$, where $t \geq T$, it appears that

$$-\Gamma(t) \leq -\frac{1}{4}\int_t^{\infty}\int_{-\infty}^{\infty} u_{xt}^2(x, t)\, dx,$$

whence

$$\int_t^{\infty}\int_{-\infty}^{\infty} u_{xt}^2(x, t)\, dx = o(t^{-1/2})$$

as $t \to \infty$. This concludes the proof of the proposition.

From Proposition 3.3 one may deduce a decay rate of the L_∞-norm of the solution u as follows. For any $x \in \mathbb{R}$,

$$|u(x, t)|^3 = 3\left|\int_{-\infty}^{x} u^2 u_x\, dx\right| \leq \frac{3}{2}\int_{-\infty}^{\infty} (u^4 + u_x^2)\, dx = o(t^{-1/2})$$

as $t \to \infty$. Thus

$$|u(\cdot, t)|_\infty = o(t^{-1/6})$$

as $t \to \infty$. The optimal rate $|u(\cdot, t)|_\infty = O(t^{-1/2})$ is adduced in Section 5, Corollary 5.2, where f is assumed to satisfy slightly stronger hypotheses.

4. The Linearized RLW-Burgers Equation

Since u tends to zero in various senses as $t \to +\infty$, it appears interesting to consider problem (3.1) linearized around the zero solution, namely

$$w_t + w_x - \nu w_{xx} - w_{xxt} = 0, \qquad (4.1a)$$

$$w(x, 0) = f(x). \qquad (4.1b)$$

Using information gleaned from problem (4.1) we hope to understand more precisely how u behaves for large t. Moreover, at certain points in what follows, the theory obtained for the linear problem will be used in the analysis of the solutions to the nonlinear equation (3.2).

As will become apparent, certain gross features of the asymptotic behaviour of a solution of (4.1) are determined by the number $\int f(x)\,dx$, and it is worth noting again that solutions of (4.1), or indeed of the full problem (3.1), have $\int u(x,t)\,dx$ independent of t.

Problem (4.1) can easily be solved by formally taking the Fourier transform (see (2.2)) of Eq. (4.1a) with respect to the spatial variable x: there results

$$\frac{d}{dt}\hat{w}(y,t)+iy\hat{w}(y,t)+vy^2\hat{w}(y,t)+y^2\frac{d}{dt}\hat{w}(y,t)=0 \tag{4.2}$$

for $(y,t)\subset\mathbb{R}\times\mathbb{R}^+$, and so

$$\hat{w}(y,t)=\exp\left(\frac{-vy^2t-iyt}{1+y^2}\right)\hat{w}(y,0). \tag{4.3}$$

It is clear from (4.3) that if $w(\cdot,0)=f\in H^k$, for some k, then the same is true of $w(\cdot,t)$, for all $t>0$. Indeed, this is clear from (4.3) and formula (2.3) for the H^k norm.

Throughout Section 4 it will be supposed that w is the solution of (4.1) corresponding to the initial data f, as given in (4.3).

LEMMA 4.1. *If $f\in H^1\cap L_1$, then*

(a) $\lim_{t\to\infty} t^{1/2}\int_{-\infty}^{\infty} w^2(x,t)\,dx=(8v\pi)^{-1/2}\,[\int_{-\infty}^{\infty} w(x,0)\,dx]^2$, *and*

(b) $\lim_{t\to\infty} t^{3/2}\int_{-\infty}^{\infty} w_x^2(x,t)\,dx=(128v^3\pi)^{-1/2}\,[\int_{-\infty}^{\infty} w(x,0)\,dx]^2$.

Proof. (a) By Parseval's theorem,

$$\int_{-\infty}^{\infty} w^2\,dx=\int_{-\infty}^{\infty}|\hat{w}|^2\,dy=\int_{-\infty}^{\infty}\exp\left(\frac{-2vy^2t}{1+y^2}\right)|\hat{w}(y,0)|^2\,dy. \tag{4.4}$$

Let $\varepsilon>0$ be given and choose $\delta=\delta(\varepsilon)\in(0,1)$ such that

$$\big|\,|\hat{w}(y,0)|^2-|\hat{w}(0,0)|^2\big|\leqslant\varepsilon, \tag{4.5}$$

for all $|y|\leqslant\delta$. The continuity of $\hat{w}(y,0)$ follows since $w(x,0)\in L_1$. The integral in (4.4) may be written in the form

$$\begin{aligned}\int_{-\infty}^{\infty} w^2\,dx&=\int_{-\delta}^{\delta}\exp\left(\frac{-2vy^2t}{1+y^2}\right)|\hat{w}(0,0)|^2\,dy\\&\quad+\int_{-\delta}^{\delta}\exp\left(\frac{-2vy^2t}{1+y^2}\right)(|\hat{w}(y,0)|^2-|\hat{w}(0,0)|^2)\,dy\\&\quad+\int_{|y|>\delta}\exp\left(\frac{-2vy^2t}{1+y^2}\right)|\hat{w}(y,0)|^2\,dy.\end{aligned} \tag{4.6}$$

The second term on the right side of (4.6) may be bounded in magnitude by

$$\varepsilon \int_{-\delta}^{\delta} \exp\left(\frac{-2\nu y^2 t}{1+y^2}\right) dy < \varepsilon \int_{-\delta}^{\delta} \exp(-\nu y^2 t)\, dy$$
$$< \frac{\varepsilon}{\sqrt{\nu t}} \int_{-\infty}^{\infty} e^{-y^2}\, dy = \varepsilon \left(\frac{\pi}{\nu t}\right)^{1/2}, \tag{4.7}$$

on account of (4.5) and the fact that $\delta < 1$. The final term on the right-hand side of (4.6) may be bounded by

$$\exp(-\nu\delta^2 t) \int_{|y|>\delta} |\hat{w}(y,0)|^2\, dy \leqslant \exp(-\nu\delta^2 t) \|w(\cdot,0)\|^2. \tag{4.8}$$

The use of (4.7) and (4.8) in (4.6) leads to the conclusion

$$\lim_{t\to\infty} t^{1/2} \int_{-\infty}^{\infty} w^2\, dx = O(\varepsilon) + \lim_{t\to\infty} t^{1/2} |\hat{w}(0,0)|^2 \int_{-\delta}^{\delta} \exp\left(\frac{-2\nu y^2 t}{1+y^2}\right) dy$$
$$= O(\varepsilon) + \left(\frac{\pi}{2\nu}\right)^{1/2} |\hat{w}(0,0)|^2,$$

as $\delta \downarrow 0$. Upon letting ε tend to zero, result (a) follows.

(b) Parseval's theorem gives

$$\int_{-\infty}^{\infty} w_x^2\, dx = \int_{-\infty}^{\infty} y^2 |\hat{w}(y,t)|^2\, dy$$
$$= \int_{-\infty}^{\infty} y^2 \exp\left(\frac{-2\nu y^2 t}{1+y^2}\right) |\hat{w}(y,0)|^2\, dy. \tag{4.9}$$

If we argue as in the proof of part (a) from (4.9), we readily obtain that

$$\lim_{t\to\infty} t^{3/2} \int_{-\infty}^{\infty} w_x^2\, dx = O(\varepsilon) + \lim_{t\to\infty} t^{3/2} |\hat{w}(0,0)|^2 \int_{-\delta}^{\delta} y^2$$
$$\times \exp\left(\frac{-2\nu y^2 t}{1+y^2}\right) dy$$
$$= O(\varepsilon) + \left(\frac{\pi}{32\nu^3}\right)^{1/2} |\hat{w}(0,0)|^2,$$

as $\delta \downarrow 0$. The result stated in part (b) now follows and the proof of the lemma is complete.

In view of the last results, one is naturally led to inquire what happens in case the total mass,

$$\int_{-\infty}^{\infty} w(x, 0)\, dx,$$

of the disturbance is zero. If it is presumed that

$$\int_{-\infty}^{\infty} |x|\, |w(x, 0)|\, dx < \infty, \tag{4.10}$$

then $d\hat{w}(y, 0)/dy$ is the Fourier transform of an L_1 function, and therefore is a uniformly continuous function of y. Because $\hat{w}$ is C^1 in its first variable and since $\hat{w}(0, 0) = 0$, Taylor's theorem implies that

$$\left| \hat{w}(y, 0) - y \frac{d}{dy} \hat{w}(0, 0) \right| = o(|y|)$$

as $y \to 0$. In consequence, if one computes as in (4.6)–(4.8), there appears

$$\lim_{t \to \infty} t^{3/2} \int_{-\infty}^{\infty} w^2\, dx = \lim_{t \to \infty} t^{3/2} \left| \frac{d\hat{w}}{dy}(0, 0) \right|^2 \int_{-\delta}^{\delta} y^2 \exp\left(\frac{-2\nu y^2 t}{1 + y^2} \right) dy$$

$$= \frac{1}{8\nu(2\pi\nu)^{1/2}} \left(\int_{-\infty}^{\infty} x w(x, 0)\, dx \right)^2. \tag{4.11}$$

Similarly, it is found that

$$\lim_{t \to \infty} t^{5/2} \int_{-\infty}^{\infty} w_x^2\, dx$$

exists, and is a constant depending on ν times the square of the first moment about the origin of the initial data, as in (4.11). The generalization of this result to the case where the total mass of the disturbance is zero and the first few moments about the origin of the initial data are also zero is straightforward; we content ourselves with a statement covering the general situation.

LEMMA 4.2. *Let $f \in H^r$, where $r \geqslant 0$ and let k be a nonnegative integer such that*

$$\int_{-\infty}^{\infty} |x|^j\, |f(x)|\, dx < \infty,$$

for $0 \leqslant j \leqslant k$. *Suppose in addition that*

$$\int_{-\infty}^{\infty} x^j f(x)\, dx = 0,$$

for $0 \leqslant j \leqslant k-1$. *Then,*

$$\lim_{t \to \infty} t^{k+i+1/2} \int_{-\infty}^{\infty} [\partial_x^i w(x, t)]^2\, dx = \frac{1 \cdot 3 \cdot 5 \cdots (2(k+i)-1)}{(8\nu\pi)^{1/2}\, (4\nu)^{k+i}} \left[\int_{-\infty}^{\infty} x^k f(x)\, dx \right]^2, \tag{4.12}$$

for $0 \leqslant i \leqslant r$.

An immediate consequence of the last lemma is the following corollary.

COROLLARY 4.3. *Let* f *satisfy the hypotheses of Lemma* 4.2. *Then,*

$$|\partial_x^j w(\cdot, t)|_\infty = O(t^{-(k+j+1)/2}) \tag{4.13}$$

as $t \to \infty$, *for* $0 \leqslant j < r$.

Proof. This follows immediately from inequality (3.5) and the results of the last lemma.

Remark. The decay rates appearing in (4.13) for the maximum value of w or one of its spatial derivatives turn out to be sharp. For example, suppose $k = 0$ and $r = 1$, and that

$$\int_{-\infty}^{\infty} f(x)\, dx \neq 0.$$

Corollary 4.3 indicates that $|w(\cdot, t)|_\infty = O(t^{-1/2})$, as $t \to \infty$. Suppose in fact that

$$\liminf_{t \to \infty} t^{1/2} |w(\cdot, t)|_\infty = 0.$$

Then,

$$\liminf_{t \to \infty} t^{1/2} \int_{-\infty}^{\infty} w^2(x, t)\, dx \leqslant (\liminf_{t \to \infty} t^{1/2} |w(\cdot, t)|_\infty) \left(\limsup_{t \to \infty} \int_{-\infty}^{\infty} |w(x, t)|\, dx \right).$$

Borrowing a result from the later part of this section, the L_1-norm of w is bounded independently of t. Hence

$$\liminf_{t\to\infty} t^{1/2}\int_{-\infty}^{\infty} w^2(x, t)\, dx = 0,$$

which contradicts Lemma 4.1.

The last remark shows a feature of various types of evolution equations, that an understanding of $|w(\cdot, t)|_1$ is crucial for establishing optimal order decay rates (cf. Schonbek [26, 27] and the references contained therein). For the linearized equation (4.1), the next result has the desired information. Similar issues regarding the nonlinear equation will be addressed in the next section.

THEOREM 4.4. *If $f \in L_1 \cap H^1$, then w has the following properties.*

(a) $\sup_{0\leqslant t<\infty}\int_{-\infty}^{\infty} |w(x, t)|\, dx < \infty$.

(b) *If $v > 1$ and $f \not\equiv 0$, then $f(x) \geqslant 0$ ($f(x) \leqslant 0$), for all $x \in \mathbb{R}$, implies that $w(x, t) > 0$ ($w(x, t) < 0$) for all (x, t) with $t > 0$.*

(c) *If $v \geqslant 1$ and for some $t \geqslant 0$, $w(\tilde{x}, t) = \sup_x w(x, t)$, then $w_t(\tilde{x}, t) < 0$, and if for some $t \geqslant 0$, $w(\tilde{y}, t) = \inf_y w(y, t)$, then $w_t(\tilde{y}, t) > 0$.*

Remark. Part (b) of the above result is a maximum principle rather like those appearing in the theory of parabolic equations. In effect, (b) states that if $v > 1$, the dissipative term $-vw_{xx}$ dominates the dispersive term w_{xxt}. This observation appears to have been made first by Lucier [21, 22]. A similar result holds for the nonlinear equations, as Lucier points out. This aspect will appear in the next section.

Proof. The proof of (a) follows by calculation. For (b), multiply (4.1a) by $K(x-y)$, where $K(z) = \frac{1}{2}e^{-|z|}$ is the Green's function for $1 - \partial_x^2$. After integration with respect to y over $\mathbb{R}$, there results

$$\begin{aligned} w_t(x, t) = & -vw(x, t) + \frac{1}{2}(v-1)\int_x^{\infty} e^{x-y} w(y, t)\, dy \\ & + \frac{1}{2}(v+1)\int_{-\infty}^{x} e^{y-x} w(y, t)\, dy. \qquad (4.14) \end{aligned}$$

If for some $t \geqslant 0$, $w(x, t) \geqslant 0$ for all x, and $w(x_0, t) = 0$ for some x_0, then (4.14) makes clear that $w_t(x_0, t) > 0$ since $v > 1$ and $w(\cdot, t) \not\equiv 0$ (cf. (4.3)). As w and w_t are continuous functions of (x, t), it follows that $w(x, t) > 0$ if $t > 0$, provided only that $f \geqslant 0$ and f is not identically zero. Similar considerations apply if $f \leqslant 0$. (For $v = 1$, the same argument shows only that $w(x, t) \geqslant 0$, for $t \geqslant 0$.)

For part (c), suppose that for some $t \geqslant 0$, there is an $\tilde{x} = \tilde{x}(t)$ such that $w(\tilde{x}, t) = \sup_x w(x, t)$. Then since $w \not\equiv w(\tilde{x}, t)$ and $v > 1$, if (4.14) is evaluated at $x = \tilde{x}$, there follows

$$\begin{aligned} w_t(\tilde{x}, t) &< -vw(\tilde{x}, t) + \frac{1}{2}(v-1)\int_{\tilde{x}}^{\infty} e^{\tilde{x}-y} w(\tilde{x}, t)\, dy \\ &\quad + \frac{1}{2}(v+1)\int_{-\infty}^{\tilde{x}} e^{y-\tilde{x}} w(\tilde{x}, t)\, dy \\ &= -vw(\tilde{x}, t) + \frac{1}{2}(v-1)\, w(\tilde{x}, t) \\ &\quad + \frac{1}{2}(v+1)\, w(\tilde{x}, t) = 0. \end{aligned}$$

Similary, $w_t(\tilde{y}, t) > 0$ if $w(\tilde{y}, t) = \inf_y w(y, t)$. The theorem is established.

Since solutions of (4.1) have

$$\int_{-\infty}^{\infty} w(x, t)\, dx = \int_{-\infty}^{\infty} f(x)\, dx,$$

it follows that if $v \geqslant 1$ and $f \geqslant 0$, then

$$\int_{-\infty}^{\infty} |w(x, t)|\, dx = \int_{-\infty}^{\infty} w(x, t)\, dx = \int_{-\infty}^{\infty} f(x)\, dx,$$

for all $t \geqslant 0$. In general, if we write $f = f_+ - f_-$ where $f_+, f_- \geqslant 0$, $f_+, f_- \in L_1 \cap H^1$, and let w, w_+, and w_- be the solution of (3.1) corresponding to f, f_+, and f_-, respectively, then $w_+, w_- \geqslant 0$ and $w = w_+ - w_-$. Hence,

$$\begin{aligned} \int_{-\infty}^{\infty} |w(x, t)|\, dx &= \int_{-\infty}^{\infty} w_+(x, t)\, dx + \int_{-\infty}^{\infty} w_-(x, t)\, dx \\ &= \int_{-\infty}^{\infty} f_+(x)\, dx + \int_{-\infty}^{\infty} f_-(x)\, dx, \end{aligned}$$

and so for $v \geqslant 1$, the conclusion of part (a) in Theorem 4.4 follows easily.

5. Further Results for the RLW-Burgers Equation

It is our goal here to demonstrate that solutions u of the nonlinear problem (3.1) have many of the same gross asymptotic properties as solutions w of the linearized problem (4.1). The following theorem shows that various properties of u are equivalent; in particular, a time-independent bound for $|u(\cdot, t)|_1$ is equivalent to one for $t^{1/4}\|u(\cdot, t)\|$. We shall prove in Theorem 5.5 that condition (c) below holds for any $\nu > 0$. The proof depends on a transformation of Cole–Hopf type which allows us to replace (3.1) by an equation similar to the heat equation. A much simpler proof of the equivalent condition (a) is possible when $\nu > 1$, and this is given in Theorem 5.4. The proof depends on maximum principles and is analogous to that for Theorem 4.4.

THEOREM 5.1. *Let $f \in L_1 \cap H^2$ and u the solution of* (3.1) *corresponding to the initial data f. Then the following properties of u are equivalent*:

(a) $\sup_{0 \leqslant t < \infty} |u(\cdot, t)|_1 < \infty$,

(b) $\sup_{0 \leqslant t < \infty} |\hat{u}(\cdot, t)|_\infty < \infty$,

(c) $\sup_{0 \leqslant t < \infty} t^{1/2} \int_{-\infty}^{\infty} u^2(x, t)\, dx < \infty$.

Proof. (a) $\Rightarrow$ (b). This is clear since

$$|\hat{u}(y, t)| \leqslant \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |u(x, t)|\, dx \leqslant \frac{1}{\sqrt{2\pi}} |u(\cdot, t)|_1.$$

(b) $\Rightarrow$ (c). The use of (3.2) and Parseval's theorem gives

$$\begin{aligned}
\frac{d}{dt} t \int_{-\infty}^{\infty} (u^2 + u_x^2)\, dx
&= \int_{-\infty}^{\infty} (u^2 + u_x^2)\, dx - 2\nu t \int_{-\infty}^{\infty} u_x^2\, dx \\
&= \int_{-\infty}^{\infty} |\hat{u}(y, t)|^2\, dy - 2\nu t \int_{-\infty}^{\infty} y^2 |\hat{u}(y, t)|^2\, dy + \int_{-\infty}^{\infty} u_x^2\, dx \\
&\leqslant \int_{|y| < (2\nu t)^{-1/2}} |\hat{u}(y, t)|^2\, dy + \int_{-\infty}^{\infty} u_x^2\, dx \\
&\leqslant C t^{-1/2} + \int_{-\infty}^{\infty} u_x^2\, dx,
\end{aligned}$$

where C is a constant provided by (b). This inequality, when integrated with respect to t over $[0, T]$, yields (c) since $u_x \in L_2(\mathbb{R} \times \mathbb{R}^+)$.

(c) $\Rightarrow$ (a). Applying the Fourier transform in the spatial variable x to (3.1a) gives the relation

$$(1 + y^2) \frac{d}{dt} \hat{u}(y, t) + (\nu y^2 + iy)\, \hat{u}(y, t) = -\frac{iy}{2} \widehat{u^2}(y, t),$$

for $(y, t) \in \mathbb{R} \times \mathbb{R}^+$. Viewing this as an ordinary differential equation for $\hat{u}$, we obtain $\hat{u}$ implicitly in the form

$$\hat{u}(y, t) = \exp\left(\frac{-vy^2t - iyt}{1+y^2}\right)\hat{u}(y, 0) - \frac{iy}{2(1+y^2)} \int_0^t \exp\left[\left(\frac{vy^2 + iy}{1+y^2}\right)(s-t)\right]\widehat{u^2}(y, s)\, ds. \tag{5.1}$$

The first term on the right-hand side of (5.1) is just the Fourier transform $\hat{w}$ of the solution w of the linearized problem (4.1) with initial data $f = u(\cdot, 0)$. Part (a) of Theorem 4.4 implies that $|w(\cdot, t)|_1$ is bounded, independently of $t \geqslant 0$. The elementary theory of the Fourier transform allows us to represent the second term on the right-hand side of (5.1) as the Fourier transform of the function

$$g(x, t) = \frac{-i}{2\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} \tau(x-y, s-t)\, u^2(y, s)\, dy\, ds,$$

where

$$\hat{\tau}(y, r) = \frac{y}{1+y^2} \exp\left[\left(\frac{vy^2 + iy}{1+y^2}\right) r\right].$$

To establish (a) it thus suffices to show that $|g(\cdot, t)|_1$ is bounded independently of t, and this will follow if it can be shown that

$$h(t) = \int_0^t |\tau(\cdot, s-t)|_1 \, \|u(\cdot, s)\|^2 \, ds \tag{5.2}$$

is bounded. A calculation shows that

$$|\tau(\cdot, -r)|_1 \leqslant \text{constant}\; r^{-1/2}, \tag{5.3}$$

for $r \geqslant 0$, where the constant is independent of r. Using this bound and (c) leads to

$$h(t) \leqslant C \int_0^t \frac{ds}{\sqrt{t-s}\sqrt{s}} = C \int_0^1 \frac{dz}{\sqrt{z(1-z)}},$$

where C connotes a constant. The theorem is proved.

The following corollary improves upon Proposition 3.3, bringing the rates of decay for certain norms into line with those obtained for the linear problem in Lemma 4.1.

COROLLARY 5.2. *Let u be as in Theorem* 5.1 *and suppose u satisfies one of the equivalent conditions* (a), (b), *or* (c) *appearing in the statement of that theorem. Then it follows that*

(a) $\sup_{0\leqslant t<\infty} t^{1/2}|u(\cdot,t)|_\infty<\infty$, *and*

(b) $\sup_{0\leqslant t<\infty} t^{3/2}\int_{-\infty}^{\infty}(u^4+u_x^2)\,dx<\infty$.

Proof. (a) Suppose (b) holds. Then,

$$\begin{aligned}|u(\cdot,t)|_\infty^3&\leqslant 3\int_{-\infty}^{\infty}|u^2u_x|\,dx\\&\leqslant 3|u(\cdot,t)|_4^2\,\|u_x(\cdot,t)\|=O(t^{-3/2}),\end{aligned}$$

as $t\to\infty$. Hence (a) holds.

(b) Multiply Eq. (3.1a) by u_{xx} and integrate over $\mathbb{R}$ to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}(u_x^2+u_{xx}^2)\,dx&=-\nu\int_{-\infty}^{\infty}u_{xx}^2\,dx+\int_{-\infty}^{\infty}uu_xu_{xx}\,dx\\&\leqslant-\frac{\nu}{2}\int_{-\infty}^{\infty}u_{xx}^2\,dx+\frac{1}{2\nu}\int_{-\infty}^{\infty}u^2u_x^2\,dx.\end{aligned}\tag{5.4}$$

If (5.4) is added to formula (3.16), there appears

$$\frac{dH}{dt}\leqslant-A\int_{-\infty}^{\infty}u_{xx}^2\,dx-B\int_{-\infty}^{\infty}u^2u_x^2\,dx,\tag{5.5}$$

for $t\geqslant T$, where

$$\begin{aligned}H(t)=\frac{1}{2}\left(\nu+\frac{1}{\nu}+1\right)&\int_{-\infty}^{\infty}u_x^2\,dx\\&+\frac{b}{4}\int_{-\infty}^{\infty}u^4\,dx+\frac{1}{2}\int_{-\infty}^{\infty}u_{xx}^2\,dx\end{aligned}$$

and A, B, b, and T are suitably chosen positive constants. It follows from (5.5) that

$$\begin{aligned}\frac{d}{dt}(t^2H(t))\leqslant t\left(\nu+\frac{1}{\nu}+1\right)&\int_{-\infty}^{\infty}u_x^2\,dx+\frac{bt}{2}\int_{-\infty}^{\infty}u^4\,dx\\&-\frac{At^2}{2}\int_{-\infty}^{\infty}u_{xx}^2\,dx-Bt^2\int_{-\infty}^{\infty}u^2u_x^2\,dx,\end{aligned}\tag{5.6}$$

provided $t \geqslant \tilde{T} = \max(T, 2/A)$. Now use Parseval's theorem and part (b) of Theorem 5.1 to conclude that

$$\begin{aligned} t\left(v+\frac{1}{v}+1\right)\int_{-\infty}^{\infty} u_x^2\,dx - \frac{At^2}{2}\int_{-\infty}^{\infty} u_{xx}^2\,dx &\leqslant t\left(v+\frac{1}{v}+1\right)\int_{|y|\leqslant \alpha/\sqrt{t}} y^2|\hat{u}(y,t)|^2\,dy \\ &\leqslant \frac{\text{constant}}{t^{1/2}}, \end{aligned} \tag{5.7}$$

where $\alpha^2 = 2(v + v^{-1} + 1)/A$. If we set $v = u^2$, then

$$\begin{aligned} \frac{bt}{2}\int_{-\infty}^{\infty} u^4\,dx - Bt^2\int_{-\infty}^{\infty} u^2u_x^2\,dx &= \frac{bt}{2}\int_{-\infty}^{\infty} v^2\,dx - \frac{Bt^2}{4}\int_{-\infty}^{\infty} v_x^2\,dx \\ &\leqslant \frac{bt}{2}\int_{|y|\leqslant \beta/\sqrt{t}} |\hat{v}(y,t)|^2\,dy \leqslant \frac{\text{constant}}{t^{1/2}}, \end{aligned} \tag{5.8}$$

with $\beta^2 = 2b/B$, where Parseval's theorem and part (c) of Theorem 5.1 have been used. (Since $v = u^2$,

$$|\hat{v}(\cdot,t)|_\infty \leqslant \frac{1}{\sqrt{2\pi}}|v(\cdot,t)|_1 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u^2(x,t)\,dx \leqslant \frac{\text{constant}}{t^{1/2}}.)$$

Using (5.7) and (5.8) in (5.6) gives

$$\frac{d}{dt}(t^2H(t)) \leqslant \frac{\text{constant}}{t^{1/2}},$$

whence $H(t) \leqslant Ct^{-3/2}$, as desired. The corollary is thus proved.

Inequality (3.5), Corollary 5.2(b), and the arguments after Corollary 4.3 ensure that

$$\liminf_{t\to\infty} t^{1/2}|u(\cdot,t)|_\infty = 0$$

if and only if

$$\liminf_{t\to\infty} t^{1/4}\|u(\cdot,t)\| = 0.$$

We shall show in Theorem 5.5(b) that the latter relation holds if and only if the initial data f has zero mass

$$\int_{-\infty}^{\infty} f(x)\,dx = 0.$$

Theorems 5.1 and 5.2 yield the estimate

$$\sup_{t \geqslant 0} t^{j+1/2} \int_{-\infty}^{\infty} |\partial_x^j u(x,t)|^2\,dx < \infty$$

for $j = 0$ and 1, and it is natural to expect similar estimates for $j = 2, 3, \ldots$. This is shown in the following corollary which is stated without proof. (The proof follows lines that are by now familiar.)

COROLLARY 5.3. *Let $f \in L_1 \cap H^{k+1}$, $k \geqslant 1$, and let u denote the solution of* (3.1) *corresponding to the initial data f. Suppose u satisfies the conditions of Theorem* 5.1. *Then*

$$\sup_{t \geqslant 0} t^{j+1/2} \int_{-\infty}^{\infty} [\partial_x^j u(x,t)]^2\,dx < \infty \qquad \textit{for} \quad 0 \leqslant j \leqslant k$$

and

$$\sup_{t \geqslant 0} t^{(j+1)/2} |\partial_x^j u(\cdot, t)|_\infty < \infty \qquad \textit{for} \quad 0 \leqslant j \leqslant k-1.$$

The rest of this section is devoted to showing that one of the conditions (and therefore all of them) in Theorem 5.1 does in fact hold. First, we present an easy proof based on the work of Lucier [21, 22] that condition (a) of Theorem 5.1 holds in case $\nu > 1$. We also show that solutions of (3.1) eventually satisfy maximum and minimum principles, just as for the linear problem (see Theorem 4.4). Afterwards, the main result of this section is stated and proved, namely that condition (c) of Theorem 5.1 obtains for solutions of (3.1) provided only that the data is suitably restricted and $\nu > 0$. In fact, even more precise information about the temporal asymptotics is obtained, as will be apparent presently.

THEOREM 5.4. *Let $\nu > 1$, $f \in L_1 \cap H^2$, and u the solution of* (3.1) *corresponding to the initial data f. Then,*

(a) $\sup_{0 \leqslant t < \infty} |u(\cdot, t)|_1 < \infty$, *and*

(b) *if T is such that $|u(\cdot, t)|_\infty \leqslant 2(\nu - 1)$ for all $t \geqslant T$, $|u(\cdot, t)|_1$ is non-increasing for $t \geqslant T$. If $u(x, s) \geqslant 0$ ($\leqslant 0$) for all x and some $s > T$, then $u(x, t) \geqslant 0$ ($\leqslant 0$) for all x and for all $t \geqslant s$. Moreover, if for some $t \geqslant T$, $u(x, t)$ takes its maximum value at $\tilde{x}$, then $u_t(\tilde{x}, t) < 0$, and if $u(x, t)$ takes its minimum value at $\tilde{y}$, then $u_t(\tilde{y}, t) > 0$.*

Proof. Theorem 2.1 assures that $|u(\cdot, t)|_1$ is bounded on bounded temporal intervals. Since $|u(\cdot, t)|_\infty \to 0$ as $t \to \infty$, the hypothesis of part (b) above is satisfied for T large enough. If (b) holds, then for $t \geqslant T$, $|u(\cdot, t)|_1$ is nonincreasing, and so (a) holds.

Following the calculations that led to (4.14), but applied to (3.1) instead of the linearized equation (4.1) leads to the formula

$$u_t(x, t) = -\nu u(x, t) + \frac{1}{2}\int_x^\infty e^{x-y}\left[\nu u(y, t) - u(y, t) - \frac{1}{2}u^2(y, t)\right] dy$$
$$+ \frac{1}{2}\int_{-\infty}^x e^{y-x}\left[\nu u(y, t) + u(y, t) + \frac{1}{2}u^2(y, t)\right] dy, \tag{5.9}$$

which holds for any (x, t) in $\mathbb{R} \times \mathbb{R}^+$. Since $\nu > 1$, Lemma 3.1 assures there is a $T > 0$ such that $|u(\cdot, t)|_\infty \leqslant 2(\nu - 1)$, for $t \geqslant T$. Equation (5.9) implies that

$$\frac{\partial}{\partial t}\left[e^{\nu t} u(x, t)\right] = \frac{1}{2} e^{\nu t}\left[\int_x^\infty e^{x-y}\left(\nu u - u - \frac{1}{2}u^2\right) dy\right.$$
$$\left. + \int_{-\infty}^\infty e^{y-x}\left(\nu u + u + \frac{1}{2}u^2\right) dy\right],$$

and so, if $t \geqslant s \geqslant T$,

$$e^{\nu t}|u(x, t)| \leqslant e^{\nu s}|u(x, s)| + \int_s^t e^{\nu r} A(x, r)\, dr. \tag{5.10a}$$

Here, we have implicitly defined

$$A(x, s) = \frac{1}{2}\int_x^\infty e^{x-y}\left|\nu u - u - \frac{1}{2}u^2\right| dy$$
$$+ \frac{1}{2}\int_{-\infty}^x e^{y-x}\left|\nu u + u + \frac{1}{2}u^2\right| dy. \tag{5.10b}$$

A change in the order of integration shows that

$$\int_{-\infty}^\infty A(x, s)\, dx = \frac{1}{2}\int_{-\infty}^\infty \left[\left|\nu u(x, s) - u(x, s) - \frac{1}{2}u^2(x, s)\right|\right.$$
$$\left. + \left|\nu u(x, s) + u(x, s) + \frac{1}{2}u^2(x, s)\right|\right] dx$$
$$= \frac{1}{2}\int_{-\infty}^\infty |u(x, s)|\left[\left|\nu - 1 - \frac{1}{2}u(x, s)\right|\right.$$
$$\left. + \left|\nu + 1 + \frac{1}{2}u(x, s)\right|\right] dx.$$

Since $|u(x, s)| \leqslant 2(\nu - 1)$,

$$|\nu - 1 - \tfrac{1}{2}u(x, s)| + |\nu + 1 + \tfrac{1}{2}u(x, s)| = 2\nu.$$

Hence if (5.10a) is integrated with respect to x over $\mathbb{R}$, there obtains

$$e^{\nu t} \int_{-\infty}^{\infty} |u(x, t)| \, dx \leqslant e^{\nu s} \int_{-\infty}^{\infty} |u(x, s)| \, dx + \nu \int_s^t e^{\nu r} \int_{-\infty}^{\infty} |u(x, r)| \, dx \, dr, \tag{5.11}$$

holding whenever $t \geqslant s \geqslant T$. If we set

$$H(t) = \int_s^t e^{\nu r} \int_{-\infty}^{\infty} |u(x, r)| \, dx \, dr,$$

then (5.11) may be written compactly as

$$H'(t) \leqslant H'(s) + \nu H(t), \tag{5.12}$$

for $t \geqslant s \geqslant T$. Inequality (5.12) yields immediately that

$$\nu e^{-\nu t} H(t) \leqslant H'(s)(e^{-\nu s} - e^{-\nu t}) = [1 - e^{-\nu(t-s)}] \int_{-\infty}^{\infty} |u(x, s)| \, dx.$$

If the last inequality is used in (5.11), we find that

$$\begin{aligned} \int_{-\infty}^{\infty} |u(x, t)| \, dx &\leqslant e^{-\nu(t-s)} \int_{-\infty}^{\infty} |u(x, s)| \, dx + [1 - e^{-\nu(t-s)}] \int_{-\infty}^{\infty} |u(x, s)| \, dx \\ &= \int_{-\infty}^{\infty} |u(x, s)| \, dx, \end{aligned}$$

for any $t \geqslant T$. Thus the L_1-norm of u is nonincreasing for $t \geqslant T$.

The rest of part (b) follows just as in the proof of Theorem 4.4, once the restriction $|u(x, t)| \leqslant 2(\nu - 1)$, holding for $t \geqslant T$, is taken into account. Accepting this remark as valid, the theorem is established.

THEOREM 5.5. *Let $\nu > 0$, $f \in L_1 \cap H^2$, and u the solution of* (3.1) *corresponding to the initial data f. Then it follows that*

$$\text{(a)} \quad \sup_{t \geqslant 0} t^{1/2} \int_{-\infty}^{\infty} u^2(x, t) \, dx < \infty,$$

and in fact

$$\text{(b)}\quad \lim_{t\to\infty} t^{1/2}\int_{-\infty}^{\infty} u^2(x,t)\,dx$$
$$=\frac{4\nu^2(v_+-1)^2}{2\pi\sqrt{\nu}}\int_{-\infty}^{\infty}\frac{e^{-2u^2}}{(1+[(v_+-1)/\sqrt{\pi}]\int_u^\infty e^{-s^2}\,ds)^2}\,du,$$

where $v_+=\exp(-(1/2\nu)\int_{-\infty}^{\infty} f(x)\,dx)$.

The proof of Theorem 5.5 will depend on an initial lemma which shall appear shortly. It will be assumed throughout the remainder of this section that the hypotheses of Theorem 5.5 hold.

Set $\tilde{u}(x,t)=\frac{1}{2}u(x+t,t)$ so that

$$\tilde{u}_t+2\tilde{u}\tilde{u}_x-\nu\tilde{u}_{xx}+\tilde{u}_{xxx}-\tilde{u}_{xxt}=0. \tag{5.13}$$

Since $f\in L_1$, it follows from Theorem 2.2 that $\tilde{u}_t$, $\tilde{u}\in C(0,T;L_1)$ for each $T>0$. We define

$$U(x,t)=\int_{-\infty}^{x}\tilde{u}(y,t)\,dy.$$

Then because of Theorem 2.2, U is uniformly bounded on $\mathbb{R}\times[0,T]$, for any $T>0$, and satisfies

$$U_t(x,t)+(U_x(x,t))^2-\nu U_{xx}(x,t)-\tfrac{1}{2}u_{xt}(x+t,t)=0. \tag{5.14}$$

Following the classical Cole–Hopf idea, set $U=-\nu\log v$ so that

$$v_t-\nu v_{xx}=Fv, \tag{5.15}$$

where $F(x,t)=-u_{xt}(x+t,t)/2\nu$. Note that, for any $t\geqslant 0$, $v(x,t)=\exp(-\nu^{-1}U(x,t))$, and so

$$\lim_{x\to-\infty} v(x,t)=1, \tag{5.16a}$$

$$\begin{aligned}\lim_{x\to\infty} v(x,t)&=\exp\left(-\nu^{-1}\int_{-\infty}^{\infty}\frac{1}{2}u(y+t,t)\,dy\right)\\&=\exp\left(-(2\nu)^{-1}\int_{-\infty}^{\infty}u(y,t)\,dy\right)\\&=\exp\left(-(2\nu)^{-1}\int_{-\infty}^{\infty}u(y,0)\,dy\right),\end{aligned} \tag{5.16b}$$

since the total mass is independent of time. Note also that since U is uniformly bounded on $\mathbb{R}\times[0, T]$ for any $T>0$, it follows that

$$0<\inf_{\substack{x\in\mathbb{R}\\ 0\leqslant t\leqslant T}} v(x,t)\leqslant \sup_{\substack{x\in\mathbb{R}\\ 0\leqslant t\leqslant T}} v(x,t)<\infty. \tag{5.17}$$

Our first task will be to demonstrate that the last-displayed inequalities hold uniformly in T.

LEMMA 5.6. *Let* $v(x,t)=\exp(-\nu^{-1}U(x,t))$, *where*

$$U(x,t)=\frac{1}{2}\int_{-\infty}^{x} u(y+t,t)\,dy.$$

Then

$$0<\inf_{t\geqslant 0}\inf_{x\in\mathbb{R}} v(x,t)\leqslant\sup_{t\geqslant 0}\sup_{x\in\mathbb{R}} v(x,t)<\infty.$$

Proof. Let

$$G(z,\tau)=(4\pi\nu\tau)^{-1/2}\exp\left(-\frac{z^2}{4\nu\tau}\right) \tag{5.18}$$

be the Green's function for the heat equation. Then if $0\leqslant T\leqslant t$, the solution v of (5.15) satisfies the integral equation

$$\begin{aligned} v(x,t)&=\int_T^t\int_{-\infty}^{\infty} G(x-y,t-s)\,v(y,s)\,F(y,s)\,dy\,ds\\ &\quad+\int_{-\infty}^{\infty} G(x-y,t-T)\,v(y,T)\,dy\\ &=R(x,t)+S(x,t), \end{aligned} \tag{5.19a}$$

say. Fix $t>T$ and observe that

$$\begin{aligned} |R(x,t)|&\leqslant \sup_{T\leqslant s\leqslant t}|v(\cdot,s)|_\infty\int_T^t\int_{-\infty}^{\infty}|F(y,s)|\,G(x-y,t-s)\,dy\,ds\\ &\leqslant \sup_{T\leqslant s\leqslant t}|v(\cdot,s)|_\infty\int_T^t\|F(\cdot,s)\|\,\|G(\cdot,t-s)\|\,ds. \end{aligned}$$

One determines readily that

$$\|G(\cdot,t-s)\|\leqslant C(t-s)^{-1/4},$$

where C is a constant. By absorbing ν^{-1} into the constant C, we have

$$|R(\cdot, t)|_\infty \leqslant C \sup_{T \leqslant s \leqslant t} |v(\cdot, s)|_\infty \int_T^t \frac{\|u_{xt}(\cdot, s)\|}{(t-s)^{1/4}}\, ds. \tag{5.19b}$$

We also have

$$|S(\cdot, t)|_\infty \leqslant |G(\cdot, t-T)|_1 \, |v(\cdot, T)|_\infty = |v(\cdot, T)|_\infty .$$

It thus follows from (5.19a) that

$$\sup_{T \leqslant s \leqslant t} |v(\cdot, s)|_\infty \leqslant |v(\cdot, T)|_\infty + C \sup_{T \leqslant s \leqslant t} |v(\cdot, s)|_\infty \sup_{T \leqslant s \leqslant t} K(s, T), \tag{5.20}$$

where

$$K(s, T) = \int_T^s \frac{\|u_{xt}(\cdot, \tau)\|}{(s-\tau)^{1/4}}\, d\tau. \tag{5.21}$$

The rest of the proof of Lemma 5.6 is broken into two parts, (a) and (b).

(a) Our first goal is to show the final inequality in the statement of the lemma, and this will necessitate consideration of the term involving K in (5.20). In estimating $K(s, T)$, first consider the case where $s \in [T, 2T]$. Then, it transpires that

$$\begin{aligned} K(s, T) &\leqslant \left\{ \int_T^s \frac{1}{(s-\tau)^{1/2}}\, d\tau \right\}^{1/2} \left\{ \int_T^s \|u_{xt}(\cdot, \tau)\|^2\, d\tau \right\}^{1/2} \\ &\leqslant \sqrt{2}\, (s-T)^{1/4} \left\{ \int_T^\infty \|u_{xt}(\cdot, \tau)\|^2\, d\tau \right\}^{1/2} \\ &\leqslant \left\{ 2T^{1/2} \int_T^\infty \|u_{xt}(\cdot, \tau)\|^2\, d\tau \right\}^{1/2}, \end{aligned} \tag{5.22}$$

and this tends to zero as $T \to \infty$ by Proposition 3.3.

We now consider the case that $s > 2T$. Using the inequality in (5.22) yields

$$\begin{aligned} K(s, T) &= \int_T^{s/2} \frac{\|u_{xt}(\cdot, \tau)\|}{(s-\tau)^{1/4}}\, d\tau + \int_{s/2}^s \frac{\|u_{xt}(\cdot, \tau)\|}{(s-\tau)^{1/4}}\, d\tau \\ &\leqslant \int_T^{s/2} \frac{\|u_{xt}(\cdot, \tau)\|}{(s-\tau)^{1/4}}\, d\tau + \left\{ 2s^{1/2} \int_s^\infty \|u_{xt}(\cdot, \tau)\|^2\, d\tau \right\}^{1/2} \end{aligned} \tag{5.23}$$

The second term on the right-hand side of the last inequality tends to zero as $T \to \infty$, since $s > T$, as already remarked. The other term on the right can be bounded above as follows:

$$\int_T^{s/2} \frac{\|u_{xt}(\cdot,\tau)\|}{(s-\tau)^{1/4}}\,d\tau \leqslant \left\{10 s^{-1/4} \int_T^s \tau^{3/4} \|u_{xt}(\cdot,\tau)\|^2\,d\tau\right\}^{1/2}. \tag{5.24}$$

Formulae (3.16) and (3.17) derived in the proof of Proposition 3.3 give

$$\|u_{xt}(\cdot,\tau)\|^2 \leqslant -\frac{d}{d\tau}\left[A\|u_x(\cdot,\tau)\|^2 + B|u(\cdot,\tau)|_4^4\right] \tag{5.25}$$

for certain positive constants A and B. Using the latter relation in (5.24) leads to the inequality

$$\begin{aligned}
&\int_T^s \tau^{3/4}\|u_{xt}(\cdot,\tau)\|^2\,d\tau \\
&\quad\leqslant -\int_T^s \tau^{3/4}\frac{d}{d\tau}\left[A\|u_x(\cdot,\tau)\|^2 + B|u(\cdot,\tau)|_4^4\right] d\tau \\
&\quad\leqslant T^{3/4}\left[A\|u_x(\cdot,T)\|^2 + B|u(\cdot,T)|_4^4\right] \\
&\qquad + \frac{3}{4}\int_T^s \frac{1}{\tau^{1/4}}\left[A\|u_x(\cdot,\tau)\|^2 + B|u(\cdot,\tau)|_4^4\right] d\tau \\
&\quad\leqslant T^{3/4}\left[A\|u_x(\cdot,T)\|^2 + B|u(\cdot,T)|_4^4\right] \\
&\qquad + \frac{3}{4}\left\{\int_T^s \frac{d\tau}{\tau^{1/2}}\right\}^{1/2}\left\{C\int_T^s \left[\|u_x(\cdot,\tau)\|^4 + |u(\cdot,\tau)|_4^8\right] d\tau\right\}^{1/2} \\
&\quad\leqslant T^{3/4}\left[A\|u_x(\cdot,T)\|^2 + B|u(\cdot,T)|_4^4\right] \\
&\qquad + C s^{1/4}\left\{\int_T^\infty \|u_x(\cdot,\tau)\|^2\,d\tau\right\}^{1/2},
\end{aligned} \tag{5.26}$$

where C denotes various positive constants. To achieve the last inequality we used the fact that $\|(u(\cdot,\tau)\|$ and $\|u_x(\cdot,\tau)\|$ are bounded and the elementary inequality

$$\begin{aligned}
|u(\cdot,\tau)|_4^8 &= \left(\int_{-\infty}^{\infty} u(x,\tau)^4\,dx\right)^2 \leqslant \left(|u(\cdot,\tau)|_\infty^2\,\|u(\cdot,\tau)\|^2\right)^2 \\
&\leqslant 4\|u(\cdot,\tau)\|^6\,\|u_x(\cdot,\tau)\|^2 \leqslant C\|u_x(\cdot,\tau)\|^2.
\end{aligned} \tag{5.27}$$

The use of (5.26) with (5.23) and (5.24) gives, for $s \geqslant 2T$,

$$K(s, T) \leqslant \left\{2s^{1/2} \int_s^{\infty} \|u_{xt}(\cdot, \tau)\|^2 \, d\tau\right\}^{1/2} + C\left\{\int_T^{\infty} \|u_x(\cdot, \tau)\|^2 \, d\tau\right\}^{1/4} + C\{T^{1/2}\|u_x(\cdot, T)\|^2 + T^{1/2}|u(\cdot, T)|_4^4\}^{1/2}. \tag{5.28}$$

The use of Proposition 3.3 and the fact that $u_x \in L_2(\mathbb{R} \times \mathbb{R}^+)$ coupled with (5.28) show that

$$\lim_{T \to \infty} \sup_{t \geqslant T} K(t, T) = 0. \tag{5.29}$$

Reference to (5.20) shows therefore that for T sufficiently large, and any $t \geqslant T$,

$$\sup_{T \leqslant s \leqslant t} |v(\cdot, s)|_\infty \leqslant |v(\cdot, T)|_\infty + \tfrac{1}{2} \sup_{T \leqslant s \leqslant t} |v(\cdot, s)|_\infty,$$

or equivalently,

$$\sup_{T \leqslant s \leqslant t} |v(\cdot, s)|_\infty \leqslant 2|v(\cdot, T)|_\infty.$$

It follows that

$$\sup_{t \geqslant 0} |v(\cdot, t)|_\infty < \infty. \tag{5.30}$$

Thus the right-hand inequality in (5.17) is seen to hold if T is taken to be ∞.

(b) We now prove the first inequality in the statement of the lemma. From Eq. (5.19a), $v(x, t) - S(x, t) = R(x, t)$, so that

$$|v(\cdot, t) - S(\cdot, t)|_\infty = |R(\cdot, t)|_\infty.$$

Hence (5.19b) and the definition (5.21) of K imply that

$$\sup_{T \leqslant s \leqslant t} |v(\cdot, s) - S(\cdot, s)|_\infty \leqslant C \sup_{T \leqslant s \leqslant t} |v(\cdot, s)|_\infty \sup_{T \leqslant s \leqslant t} K(s, T) \leqslant C_1 \sup_{T \leqslant s \leqslant t} K(s, T),$$

since v is uniformly bounded by (a). It follows from (5.29) that

$$\lim_{T \to \infty} \sup_{s \geqslant T} |v(\cdot, s) - S(\cdot, s)|_\infty = 0, \tag{5.31}$$

whence

$$\liminf_{t\to\infty} \inf_{x\in R} v(x,t) = \liminf_{t\to\infty} \inf_{x\in R} S(x,t). \tag{5.32}$$

By its definition in (5.19a), one discerns that

$$S(x,t) = [4\pi\nu(t-T)]^{-1/2} \int_{-\infty}^{\infty} v(y,T) \exp\left(-\frac{(x-y)^2}{4\nu(t-T)}\right) dy. \tag{5.33}$$

Define k by

$$k = \min\{v(-\infty, T), v(\infty, T)\} = \min\left\{1, \exp\left(-\frac{1}{2\nu}\int_{-\infty}^{\infty} u(x,0)\,dx\right)\right\}. \tag{5.34}$$

Let $\varepsilon > 0$ be given and choose $M = M(\varepsilon)$ so large that $v(y,T) \geqslant k - \varepsilon$ provided $|y| \geqslant M$. Then, since v is uniformly bounded and M is fixed,

$$\begin{aligned} S(x,t) &= [4\pi\nu(t-T)]^{-1/2} \int_{-\infty}^{\infty} [v(y,T) - k + \varepsilon] \\ &\quad \times \exp\left(-\frac{(x-y)^2}{4\nu(t-T)}\right) dy + k - \varepsilon \\ &\geqslant k - \varepsilon - [4\pi\nu(t-T)]^{-1/2} \int_{-M}^{M} |v(y,T) - k + \varepsilon|\,dy \\ &\geqslant k - \varepsilon - C(t-T)^{-1/2}. \end{aligned}$$

Hence, it follows that

$$\liminf_{t\to\infty} \inf_{x\in R} S(x,t) \geqslant k - \varepsilon,$$

and because $\varepsilon > 0$ was arbitrary, it is concluded that

$$\liminf_{t\to\infty} \inf_{x\in R} S(x,t) \geqslant k.$$

(In fact, the reverse inequality also holds since $S(x,t) \to v(\pm\infty, T)$ as $x \to \pm\infty$.) The use of this estimate with (5.31) implies that

$$\liminf_{t\to\infty} \inf_{x\in R} v(x,t) \geqslant k. \tag{5.35}$$

This concludes the proof of the lemma.

By definition,

$$U_x(x, t) = \tilde{u}(x, t) = -\nu \frac{v_x(x, t)}{v(x, t)}.$$

Because v is bounded away from zero according to Lemma 5.6, questions concerning evanescence of $\|u(\cdot, t)\| = 2\|\tilde{u}(\cdot, t)\|$, as $t \to \infty$, are equivalent to the same questions posed about $\|v_x(\cdot, t)\|$, as $t \to \infty$. Advantage will be taken of this observation in the next stage of the proof.

Proof of Theorem 5.5. Part (a). For the moment, the right-hand side of (5.15) is renamed g, so that

$$v_t = \nu v_{xx} + g, \tag{5.36}$$

where $g(x, t) = -v(x, t)\, u_{xt}(x + t, t)/2\nu$. Let $T > 0$ be fixed and, as in (5.19a), define S to be the bounded solution of

$$\begin{aligned} S_t &= \nu S_{xx}, \qquad && (x, t) \in \mathbb{R} \times (T, \infty), \\ S(x, T) &= v(x, T), && x \in \mathbb{R}. \end{aligned} \tag{5.37}$$

Let $\chi = S_x$, so that

$$\begin{aligned} \chi_t &= \nu \chi_{xx}, \qquad && (x, t) \in \mathbb{R} \times (T, \infty), \\ \chi(x, T) &= v_x(x, T), && x \in \mathbb{R}. \end{aligned} \tag{5.38}$$

Arguing exactly as in the proof of Lemma 4.1, part (a), it is inferred that

$$\lim_{t \to \infty} t^{1/2} \int_{-\infty}^{\infty} S_x^2(x, t)\, dx = (8\nu\pi)^{-1/2} \left[\int_{-\infty}^{\infty} S_x(x, T)\, dx \right]^2. \tag{5.39}$$

Thus in particular $t^{1/4}\|S_x(\cdot, t)\|$ is bounded as $t \to \infty$.

We examine now the dependent variable $y \equiv v_x - S_x = R_x$ (see (5.19a)); our intention is to show that

$$t^{1/4}\|y(\cdot, t)\| \to 0 \qquad \text{as} \quad t \to \infty. \tag{5.40}$$

The use of (5.40) with (5.39) will then prove part (a) of the theorem. We break the proof of (5.40) into two pieces: (i) $t^{\mu}\|y(\cdot, t)\| \to 0$ as $t \to \infty$ for any μ in $(0, \frac{1}{4})$, and then (ii) the proof of (5.40).

(i) A short computation shows that

$$\begin{aligned} y_t &= \nu y_{xx} + g_x, \qquad && (x, t) \in \mathbb{R} \times (T, \infty), \\ y(x, T) &= 0, && x \in \mathbb{R}. \end{aligned}$$

In Fourier transformed variables, this equation has the form

$$\frac{d}{dt}\hat{y}(\xi, t)+\nu\xi^2\hat{y}(\xi, t)=i\xi\hat{g},$$
$$\hat{y}(\xi, T)=0.$$

The latter initial-value problem may be solved to obtain

$$\hat{y}(\xi, t)=i\xi\int_T^t e^{\nu\xi^2(s-t)}\hat{g}(\xi, s)\,ds,$$

whence

$$\begin{aligned}\|y(\cdot, t)\|=\|\hat{y}(\cdot, t)\| &\leqslant \int_T^t \|\xi e^{\nu\xi^2(s-t)}\hat{g}(\xi, s)\|\,ds\\ &=\int_T^t\left\{\int_{-\infty}^{\infty}\xi^2e^{2\nu\xi^2(s-t)}|\hat{g}(\xi, s)|^2\,d\xi\right\}^{1/2}ds.\end{aligned}$$

Elementary considerations show that $\max_{x\geqslant 0} xe^{-\alpha x}=1/\alpha e$, and it is therefore concluded that

$$\xi^2e^{2\nu\xi^2(s-t)}\leqslant\frac{1}{2\nu e(t-s)}.$$

Thus we arrive at the estimate

$$\begin{aligned}\|y(\cdot, t)\| &\leqslant C_1\int_T^t (t-s)^{-1/2}\,\|g(\cdot, s)\|\,ds\\ &\leqslant C_2\int_T^t (t-s)^{-1/2}\,\|u_{xt}(\cdot, s)\|\,ds,\end{aligned}\tag{5.41}$$

where C_1 and C_2 are constants. This estimate is extended as follows:

$$\begin{aligned}&\int_T^t (t-s)^{-1/2}\,\|u_{xt}(\cdot, s)\|\,ds\\ &\quad\leqslant\int_T^{t/2}(t-s)^{-1/2}\,\|u_{xt}(\cdot, s)\|\,ds+\int_{t/2}^t (t-s)^{-1/2}\,\|u_{xt}(\cdot, s)\|\,ds\\ &\quad\leqslant\left(\frac{2}{t}\right)^{1/4}\int_T^{t/2}\frac{\|u_{xt}(\cdot, s)\|}{(t-s)^{1/4}}\,ds\\ &\qquad+\int_{t/2}^t (t-s)^{-1/2}\,\|u_{xt}(\cdot, s)\|\,ds.\end{aligned}\tag{5.42}$$

As in (5.24) and (5.26),

$$\left(\frac{2}{t}\right)^{1/4} \int_T^{t/2} \frac{\|u_{xt}(\cdot, s)\|}{(t-s)^{1/4}}\, ds$$
$$\leqslant \frac{C}{t^{1/4}} \left\{ T^{1/2}[\|u_x(\cdot, t)\|^2 + |u(\cdot, T)|_4^4] + \left(\int_T^\infty \|u_x(\cdot, s)\|^2\, ds\right)^{1/2} \right\}^{1/2},$$

where C denotes an absolute constant. The use of Proposition 3.3 then gives

$$\limsup_{t \to \infty} t^{1/4} \left[\left(\frac{2}{t}\right)^{1/4} \int_T^{t/2} \|u_{xt}(\cdot, s)\|\, ds\right] = 0. \tag{5.43}$$

We now turn to the second term on the right-hand side of (5.42). By Young's inequality, for any $\varepsilon > 0$,

$$\int_{t/2}^t (t-s)^{-1/2} \|u_{xt}(\cdot, s)\|\, ds \leqslant \varepsilon \int_{t/2}^t (t-s)^{-p/2} s^{-\alpha p}\, dx + \frac{1}{\varepsilon} \int_{t/2}^t s^{\alpha q} \|u_{xt}(\cdot, s)\|^q\, ds, \tag{5.44}$$

where $p^{-1} + q^{-1} = 1$ and $\alpha > 0$. Here $p \in (1, 2)$ and in due course it will be taken to be near 2. If the constant α is chosen as

$$\alpha = \frac{5-2p}{4p}, \qquad \text{so} \quad \frac{p}{2} + \alpha p = \frac{5}{4},$$

then

$$\varepsilon \int_{t/2}^t (t-s)^{-p/2} s^{-\alpha p}\, ds = \frac{\varepsilon}{t^{1/4}} \int_{1/2}^1 (1-x)^{-p/2} x^{-\alpha p}\, dx \leqslant \frac{C\varepsilon}{t^{1/4}}. \tag{5.45}$$

Since $q > 2$, $\|u_{xt}(\cdot, s)\|^q \leqslant C\|u_{xt}(\cdot, s)\|^2$ because $\|u_{xt}(\cdot, s)\|$ is a bounded function of s. Indeed, the Schwartz inequality applied to (3.14) ensures that $\|u_{xt}(\cdot, s)\| = o(s^{-1/4})$, as $s \to +\infty$. As for the second term on the right-hand side of (5.44), proceed as follows:

$$\frac{1}{\varepsilon} \int_{t/2}^t s^{\alpha q} \|u_{xt}(\cdot, s)\|^q\, ds \leqslant \frac{Ct^{\alpha q}}{\varepsilon} \int_{t/2}^t \|u_{xt}(\cdot, s)\|^2\, ds \leqslant \frac{C}{\varepsilon} t^{\alpha q} \int_{t/2}^\infty \|u_{xt}(\cdot, s)\|^2\, ds. \tag{5.46a}$$

As shown in Proposition 3.3, the last integral is $o(t^{-1/2})$, as $t \to \infty$, and so it transpires that

$$\frac{1}{\varepsilon}\int_{t/2}^{t} s^{\alpha q}\|u_{xt}(\cdot, s)\|^q\, ds = \frac{C}{\varepsilon}\, o(t^{-\mu}), \tag{5.46b}$$

where $\mu = \frac{1}{2} - \alpha q$. Note that

$$\frac{1}{2} - \alpha q = \frac{4p-7}{4(p-1)} < \frac{1}{4}, \tag{5.47}$$

and that μ can be made as close as desired to $\frac{1}{4}$ by taking p near 2. Combining (5.41)–(5.46), it is demonstrated that

$$\limsup_{t \to \infty} t^{\mu}\|y(\cdot, t)\| = 0, \tag{5.48}$$

where $\mu = \frac{1}{2} - \alpha q = (4p-7)/4(p-1)$ and p is arbitrary in $(1, 2)$. Putting (5.39) and (5.48) together gives

$$\lim_{t \to \infty} t^{\mu}\|v_x(\cdot, t)\| = 0, \tag{5.49}$$

and so by our earlier remarks,

$$\lim_{t \to \infty} t^{\mu}\|u(\cdot, t)\| = 0. \tag{5.50}$$

(ii) We now use (5.50) to prove (5.40). To begin, note that because of (3.5)

$$\int_{-\infty}^{\infty} u^4(x, t)\, dx \leqslant 2\|u(\cdot, t)\|^3\, \|u_x(\cdot, t)\| \leqslant C\,\frac{\|u_x(\cdot, t)\|}{t^{3\mu}},$$

for all $t \geqslant 0$, where μ is as above. Since $\|u_x(\cdot, t)\|$ lies in $L_2(\mathbb{R}^+)$, it follows that if $\frac{19}{10} < p < 2$, then $|u(\cdot, t)|_4^4 \in L_1(\mathbb{R}^+)$. Returning to the proof of Proposition 3.3, if

$$\Gamma(t) = \frac{1}{2}\left(v + \frac{1}{v}\right)\int_{-\infty}^{\infty} u_x^2(x, t)\, dx + \frac{b}{4}\int_{-\infty}^{\infty} u^4(x, t)\, dx$$

as before, then we still have from (3.18a) that

$$\frac{d}{dt}\Gamma(t) \leqslant 0$$

for sufficiently large t. However, now it is known that $\Gamma \in L_1(\mathbb{R}^+)$. In consequence, it is deduced that

$$\Gamma(t) = o(t^{-1}),$$

as $t \to \infty$. Thus,

$$\int_{-\infty}^{\infty} u_x^2(x, t)\, dx, \qquad \int_{-\infty}^{\infty} u^4(x, t)\, dx = o(t^{-1}),$$

as $t \to \infty$. With this improved estimate it is deduced as in the proof of Proposition 3.3 that

$$\int_t^{\infty} \int_{-\infty}^{\infty} u_{xt}^2(x, s)\, dx\, ds = o(t^{-1}),$$

as $t \to \infty$. The last relation, when used in the inequality (5.46a) gives

$$\frac{1}{\varepsilon} \int_{t/2}^{t} s^{\alpha q} \|u_{xt}(\cdot, s)\|^q\, ds \leqslant \frac{C}{\varepsilon}\, t^{\alpha q - 1}.$$

But $\alpha q - 1 = (9 - 6p)/4(p - 1)$, and if $p > \frac{8}{5}$ this means that $\alpha q - 1 < -\frac{1}{4}$. It follows from (5.41)–(5.45) and the last remark that

$$\lim_{t \to \infty} t^{1/4} \|y(\cdot, t)\| = \lim_{t \to \infty} t^{1/4} \|R_x(\cdot, t)\| = 0. \tag{5.51}$$

Now $v_x = y + S_x$, and so to study the asymptotic behaviour of $t^{1/4}\|v_x(\cdot, t)\|$, it suffices to consider the behaviour of $t^{1/4}\|S_x(\cdot, t)\|$. In (5.39) we saw that

$$\lim_{t \to \infty} t^{1/2} \int_{-\infty}^{\infty} S_x^2(x, t)\, dx = (8 \nu \pi)^{-1/2} \left[\int_{-\infty}^{\infty} S_x(x, T)\, dx \right]^2. \tag{5.52}$$

But $S_x(\cdot, T) = v_x(\cdot, T)$, and so

$$\begin{aligned} &\lim_{t \to \infty} t^{1/2} \int_{-\infty}^{\infty} S_x^2(x, t)\, dx \\ &\quad = (8 \nu \pi)^{-1/2} \left(\int_{-\infty}^{\infty} v_x(x, T)\, dx \right)^2 \\ &\quad = (8 \nu \pi)^{-1/2} \left[v(x, T) \big|_{-\infty}^{\infty} \right]^2 \\ &\quad = (8 \nu \pi)^{-1/2} \left[\exp\left(-\frac{1}{2\nu} \int_{-\infty}^{\infty} u(x, 0)\, dx \right) - 1 \right]^2, \end{aligned}$$

because of (5.16).

We are now in sight of our objective. Recall that $\tilde{u} = -\nu v_x/v$, so

$$\begin{aligned}
&\limsup_{t\to\infty} t^{1/2}\int_{-\infty}^{\infty} \tilde{u}^2(x,t)\,dx \\
&\quad= \limsup_{t\to\infty} t^{1/2}\nu^2\int_{-\infty}^{\infty} \frac{v_x^2(x,t)}{v^2(x,t)}\,dx \\
&\quad\leqslant \frac{\nu^2}{k^2}\lim_{t\to\infty} t^{1/2}\int_{-\infty}^{\infty} v_x^2(x,t)\,dx \\
&\quad= \frac{\nu^2}{k^2(8\nu\pi)^{1/2}}\left[\exp\left(-\frac{1}{2\nu}\int_{-\infty}^{\infty} u(x,0)\,dx\right)-1\right]^2,
\end{aligned}$$

where k is defined in (5.34). It follows from the definition of $\tilde{u}$ that

$$\begin{aligned}
&\limsup_{t\to\infty} t^{1/2}\int_{-\infty}^{\infty} u^2(x,t)\,dx \\
&\quad\leqslant \frac{4\nu^2}{k^2(8\nu\pi)^{1/2}}\left[\exp\left(\frac{-1}{2\nu}\int_{-\infty}^{\infty} u(x,0)\,dx\right)-1\right]^2.
\end{aligned}$$

In particular,

$$t^{1/4}\|u(\cdot,t)\| = O(1),$$

as $t\to\infty$. The proof of (a) is complete.

Part (b). We return to the formula

$$\begin{aligned}
\lim_{t\to\infty} t^{1/2}\int_{-\infty}^{\infty} u^2(x,t)\,dx &= \lim_{t\to\infty} 4t^{1/2}\int_{-\infty}^{\infty} \tilde{u}^2(x,t)\,dx \\
&= \lim_{t\to\infty} 4t^{1/2}\nu^2\int_{-\infty}^{\infty} \frac{v_x^2(x,t)}{v^2(x,t)}\,dx.
\end{aligned}$$

Recall from (5.19a) that $v = R + S$, while (5.35) gives

$$\liminf_{t\to\infty}\inf_{x\in R} v(x,t) \geqslant k > 0$$

and (5.51) gives

$$\lim_{t\to\infty} t^{1/2}\int_{-\infty}^{\infty} R_x^2\,dx = 0.$$

It follows from these estimates that

$$\lim_{t\to\infty} t^{1/2}\int_{-\infty}^{\infty} u^2\,dx = \lim_{t\to\infty} 4t^{1/2}\nu^2\int_{-\infty}^{\infty} \frac{S_x^2}{v^2}\,dx.$$

Equation (5.31) implies that $|R(\cdot, t)|_\infty \to 0$ as $t \to \infty$, and so

$$\lim_{t \to \infty} t^{1/2} \int_{-\infty}^{\infty} u^2 \, dx = \lim_{t \to \infty} 4t^{1/2} \nu^2 \int_{-\infty}^{\infty} \frac{S_x^2}{S^2} \, dx. \tag{5.53}$$

Recall from (5.19a) that

$$S_t = \nu S_{xx} \qquad \text{for} \quad (x, t) \in \mathbb{R} \times (T, \infty),$$

and

$$S(x, T) = v(x, T),$$

where

$$v(x, T) = \exp\left(-\frac{1}{\nu} \int_{-\infty}^{x} \tilde{u}(y, T) \, dy \right).$$

Set

$$v_+ = \exp\left(-\frac{1}{\nu} \int_{-\infty}^{\infty} \tilde{u}(y, T) \, dy \right) = \exp\left(-\frac{1}{2\nu} \int_{-\infty}^{\infty} u(y, 0) \, dy \right)$$

and define

$$Z(x) = \begin{cases} 1 & \text{for} \quad x < 0, \text{ and} \\ v_+ & \text{for} \quad x > 0. \end{cases}$$

Set $S(x, t) = A(x, t) + C(x, t) + 1$, where

$$A_t = \nu A_{xx} \qquad \text{for} \quad (t, x) \in (T, \infty) \times \mathbb{R},$$

$$A(x, T) = v(x, T) - Z(x),$$

and

$$C_t = \nu C_{xx} \qquad \text{for} \quad (x, t) \in \mathbb{R} \times (T, \infty),$$

$$C(x, T) = Z(x) - 1.$$

Note that

$$1 + C(x, t) \geqslant \min(v_+, 1) \tag{5.54a}$$

and that

$$\sup_{T \leqslant t < \infty} t^{1/2} \int_{-\infty}^{\infty} C_x^2 \, dx < \infty. \tag{5.54b}$$

Since $A(\pm\infty, T)=0$, it follows easily that

$$|A(\cdot, t)|_{\infty} \to 0 \quad \text{as} \quad t \to \infty \tag{5.55a}$$

and

$$t^{1/2}\int_{-\infty}^{\infty} A_x^2\,dx \to 0 \quad \text{as} \quad t \to \infty. \tag{5.55b}$$

A bit of algebra shows that

$$t^{1/2}\int_{-\infty}^{\infty}\left[\frac{S_x^2}{S^2}-\frac{C_x^2}{(1+C)^2}\right]dx$$
$$= t^{1/2}\int_{-\infty}^{\infty}\frac{1}{S(1+C)}\left(\frac{S_x}{S}+\frac{C_x}{1+C}\right)\{(1+C)A_x - AC_x\}\,dx. \tag{5.56}$$

Manipulation of Eqs. (5.53)–(5.56) yields

$$\lim_{t\to\infty} t^{1/2}\int_{-\infty}^{\infty} u^2\,dx = \lim_{t\to\infty} 4\nu^2 t^{1/2}\int_{-\infty}^{\infty}\frac{C_x^2}{(1+C)^2}\,dx. \tag{5.57}$$

The fundamental solution for the heat equation allows us to write an explicit formula for C, namely

$$C(x, t)=\frac{v_+ - 1}{\sqrt{4\pi\nu}\sqrt{t-T}}\int_0^{\infty}\exp\left(-\frac{(x-y)^2}{4\nu(t-T)}\right)dy,$$

or, equivalently,

$$C(x, t)=\frac{v_+ - 1}{\sqrt{\pi}}\int_{-x/\sqrt{4\nu(t-T)}}^{\infty} e^{-s^2}\,ds,$$

and consequently

$$C_x(x, t)=\frac{v_+ - 1}{\sqrt{4\nu\pi(t-T)}}\exp\left(\frac{-x^2}{4\nu(t-T)}\right).$$

The use of these formulae and a change of variables gives

$$\nu^{1/2}t^{1/2}\int_{-\infty}^{\infty}\frac{C_x^2}{(1+C)^2}\,dx$$
$$=\frac{\nu^2(v_+ - 1)^2}{2\pi\sqrt{\nu}}\sqrt{\frac{t}{t-T}}\int_{-\infty}^{\infty}\frac{e^{-2u^2}}{(1+((v_+ - 1)/\sqrt{\pi})\int_u^{\infty} e^{-s^2}\,ds)^2}\,du.$$

If we use this formula in (5.57), then the proof of (b) is complete.

Remarks. If $\int_{-\infty}^{\infty} f = 0$, $xf(x) \in L_1$, and $f \in H^2$, then one can show (cf. Lemma 4.2) that

$$\limsup_{t \to \infty} t^{3/2} \int_{-\infty}^{\infty} u^2(x, t)\, dx < \infty.$$

It is interesting to compare the estimate from Theorem 5.5(b) for the asymptotic behaviour of $t^{1/2}\|u(\cdot, t)\|^2$ with that for the linear equation in Lemma 4.1. The ratio J of the "nonlinear rate" to the "linear rate" is given explicitly by

$$J(X) = \sqrt{\frac{2}{\pi}} \left(\frac{e^X - 1}{X}\right)^2 \int_{-\infty}^{\infty} \times \frac{e^{-2u^2}}{(1 + ((e^X - 1)/\sqrt{\pi}) \int_u^{\infty} e^{-s^2}\, ds)^2}\, du,$$

where $X = -(1/2\nu) \int_{-\infty}^{\infty} f(x)\, dx$. Note that $J(0) = 1$ and that J is an even function of X. One can easily prove the following two results:

(i) $J(X) < 1$ for small $X > 0$; more precisely

$$J(X) \simeq 1 + \left(\frac{7}{12} - \frac{3}{\pi} \tan^{-1} \frac{1}{\sqrt{2}}\right) X^2 \simeq 1 - 0.0044 X^2$$

and

(ii) $J(X) \to 0$ as $X \to \infty$.

The obvious conjecture is that

$$J(X) < 1 \qquad \text{for all} \quad X > 0, \tag{5.58}$$

but we have not pursued this point. If such a result held (and we know it does for small and large X by (i), (ii)), then the presence of the nonlinear term uu_x has the effect of making the solution smaller (as measured by $t^{1/4}\|u(\cdot, t)\|$) than the corresponding solution to the linear equation.

It follows from Theorem 5.5(b) and the first remark above that $u \in L_2(\mathbb{R} \times \mathbb{R}^+)$ if and only if $\int_{-\infty}^{\infty} f(x)\, dx = 0$.

6. Properties of Solutions of the KdV-Burgers Equation

In this section consideration is given to the initial-value problem

$$u_t + u_x + uu_x - \nu u_{xx} + u_{xxx} = 0, \tag{6.1a}$$

$$u(x, 0) = f(x), \tag{6.1b}$$

for $t \geqslant 0$, $x \in R$. The detailed regularity properties of solutions of the KdV equation and the RLW equation differ somewhat (see Section 2 and the references mentioned there). Nonetheless, the theory developed in Sections 3, 4, and 5 for solutions of Eq. (3.1) goes over to solutions of (6.1) with only two reservations. One difference is that the estimates for u_{xt} (cf. Lemmas 3.1, 3.2, and Proposition 3.3) must be replaced by analogous ones for u_{xx}—this is expected since (3.1) has a term u_{xxt} while (6.1) has u_{xxx}. The other difference is the absence of an analogue to Theorem 4.4(b)–(c). Such a result is unlikely to hold.

The proofs are largely similar to those presented in the context of (3.1), and consequently are not presented here.

7. Discussion

The foregoing theory provides sharp rates of decay for solutions of the KdV-Burgers equation and the RLW-Burgers equation. There are several important conclusions emerging from our analysis. First, the considerable difference in linearized dispersion relations evidenced in the two equations in view here does not change the algebraic rate of temporal decay of the standard norms of solutions corresponding to initial data that is smooth and tends to zero appropriately as the spatial variable becomes unbounded. Indeed, we observe in both cases that the asymptotic rates of decay are the same as those exhibited by solutions of the one-dimensional heat equation posed on the infinite line with similar initial data. However, while the decay rates exhibit the same algebraic power of t, the nonlinear term is seen to play a role in the value of the constant of proportionality that appears in the sharp asymptotics of norms of solutions. We attribute this phenomenon to the nonlinearity's propensity to cascade energy into higher wavenumbers, which are damped more rapidly than smaller wavenumbers by the Burgers-type dissipative term that we have considered.

More general equations than those in view here arise in modelling unidirectional wave motion in nonlinear, dispersive, dissipative media. A couple of classes of models that have been suggested in a number of contexts (cf. Albert *et al.* [2], Benjamin *et al.* [4], Biler [5. 6], Bona [7, 8], Dix [11], Felland [12], Kakutani and Matsuuchi [17], Saut [25]) are the following,

$$u_t + F(u)_x + Mu - Lu_x = 0, \tag{7.1a}$$

or

$$u_t + F(u)_x + Mu + Lu_t = 0, \tag{7.1b}$$

where $u = u(x, t)$ is as before a real-valued function of the two real variables x and t, F is a smooth, real-valued function of a real variable which may be supposed to have $F(0) = 0$ without loss of generality, and the operators L and M are Fourier multiplier operators given by the formulae

$$\widehat{Lv}(k) = \alpha(k)\, \hat{v}(k)$$

and

$$\widehat{Mv}(k) = \beta(k)\, \hat{v}(k),$$

where α and β are nonnegative functions that model the effects of dispersion and dissipation, respectively. Some preliminary decay results are available for this class of equations (cf. Biler [5, 6], Dix [11], Felland [12]). However, detailed results such as those obtained here for the relatively simple, local, KdV-Burgers and RLW-Burgers equations have so far proved to be elusive. Certainly the analysis that comes to the fore in Sections 5 and 6 is not applicable because of the lack of the analogue of the Cole–Hopf transformation.

We conjecture that similar results do obtain in the context of the class of Eqs. (7.1), and that one will again observe that the optimal decay rate is that of the linearized equation corresponding to (7.1) in which the nonlinear term is simply dropped. Moreover, we also expect that the analogue of (5.58) will obtain, pointing to the conclusion that nonlinear effects will generally result in a cascade of energy into higher wavenumbers. Information concerning such systems would surely enhance our understanding of the interaction between nonlinearity, dispersion, and dissipation in wave motion.

References

1. M. J. Ablowitz and H. Segur, "Solitons and the Inverse Scattering Transform," SIAM, Philadelphia, 1981.
2. J. P. Albert, J. L. Bona, and D. Henry, Sufficient conditions for stability of solitary-wave solutions of model equations for long waves, *Phys. D* **24** (1987), 343.
3. T. B. Benjamin, Lectures on nonlinear wave motion, *in* "Lectures in Applied Mathematics," (A. Newell, Ed.), Vol. 15, pp. 3–47, American Mathematical Society, Providence, RI, 1974.
4. T. B. Benjamin, J. L. Bona, and J. J. Mahony, Model equations for long waves in nonlinear dispersive systems, *Philos. Trans. Roy. Soc. London Ser. A* **272** (1972), 47.
5. P. Biler, Asymptotic behavior in time of some equations generalizing Korteweg–de Vries–Burgers, *Bull. Polish Acad. Sci.* **32** (1984), 275.
6. P. Biler, Large-time behaviour of periodic solutions of two dispersive equations of Korteweg-de Vries–Burgers type, *Bull. Polish Acad. Sci.* **32** (1984), 401.
7. J. L. Bona, Model equations for waves in nonlinear dispersive systems, *in* "Proceedings of the International Congress of Mathematicians, Helsinki, 1978," pp. 887–894, Academia Scientarim Fennica, Hungary, 1980.

8. J. L. Bona, On solitary waves and their role in the evolution of long waves, *in* "Applications of Nonlinear Analysis in the Physical Sciences" (H. Amann, N. Bazely, and K. Kirchgässner, Eds.), pp. 193–205. Pitmann, London, 1981.
9. J. L. Bona, W. G. Pritchard, and L. R. Scott, Solitary-wave interaction, *Phys. Fluids* **13** No. 3 (1980), 438.
10. J. L. Bona and R. Smith, The initial-value problem for the Korteweg–de Vries equation, *Philos. Trans. Roy. Soc. London Ser. A* **278** (1975), 555.
11. D. Dix, "Decay of Solutions to the Benjamin–Ono–Burgers equation," Ph.D. thesis, University of Chicago, 1988.
12. M. Felland, "Decay Rates for Nonlinear, Dispersive, Dissipative wave Equations," Ph.D. thesis, University of Chicago, 1989.
13. H. Grad and P. N. Hu, Unified shock profile in a plasma, *Phys. Fluids* **10** (1967), 2596.
14. A. Jeffrey and T. Kakutani, Weak nonlinear dispersive waves; a discussion centred around the Korteweg–de Vries equations, *SIAM Rev.* **14** (1972), 582.
15. R. S. Johnson, A nonlinear equation incorporating damping and dispersion, *J. Fluid. Mech.* **42** (1970), 49.
16. R. S. Johnson, Shallow water waves on a viscous fluid—The undular bore, *Phys. Fluids* **15** (1978), 1693.
17. T. Kakutani and K. Matsuuchi, Effects of viscosity on long gravity waves, *J. Phys. Soc. Japan* **39** (1975), 237.
18. C. J. Knickerbocker and A. C. Newell, Shelves and the Korteweg–de Vries equation, *J. Fluid Mech.* **98** (1980), 803.
19. D. J. Korteweg and G. de Vries, On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves, *Philos. Mag.* (5) **39** (1895), 422.
20. G. L. Lamb, "Elements of Soliton Theory," Wiley, New York, 1980.
21. B. J. Lucier, "Dispersive Approximation for Hyperbolic Conservation Laws," Ph.D. thesis, University of Chicago, 1981.
22. B. J. Lucier, On Sobolev regularizations of hyperbolic conservation laws, *Comm. Partial Differential Equations* **10** (1985), 1.
23. R. M. Miura, The Korteweg–de Vries equation: A survey of results, *SIAM Rev.* **18** (1976), 412.
24. D. H. Peregrine, Calculations of the development of an undular bore, *J. Fluid Mech.* **25** (1966), 321.
25. J.-C. Saut, Sur quelques généralisations de l'equation de Korteweg–de Vries, *J. Math. Pures Appl.* (9) **58** (1979), 21.
26. M. E. Schonbek, Decay of solutions to parabolic conservation laws, *Comm. Partial Differential Equations* **7** (1980), 449.
27. M. E. Schonbek, L^2 decay for weak solutions of the Navier–Stokes equations, *Arch. Rational Mech. Anal.* **88** (1985), 209.
28. A. C. Scott, F. Y. F. Chu, and D. W. McLaughlin, The soliton: A new concept in applied science. *Proc. IEEE* **61** (1973), 1443.
29. G. B. Whitham, "Linear and Nonlinear Waves," Wiley, New York, 1974.
30. K. Yosida, "Functional Analysis," Springer-Verlag, Berlin, 1974.