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NONLOCAL MODELS FOR NONLINEAR, DISPERSIVE WAVES*

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Considered herein are model equations for the unidirectional propagation of small-amplitude, nonlinear, dispersive, long waves such as those governed by the classical Korteweg–de Vries equation. Of especial interest are physical situations in which the linear dispersion relation is not appropriately approximated by a polynomial, so that the operator modelling dispersive effects is nonlocal. Particular cases in view here are the Benjamin–Ono equation and the intermediate long-wave equation which arise in internal-wave theory, and Smith's equation which governs certain types of continental-shelf waves.

The initial-value problem for these equations is shown to be globally well posed in the classical sense, including continuous dependence upon the initial data and, in certain cases upon the modelling of nonlinear and dispersive effects. Whilst the results are stated for the specific equations listed above, the techniques utilized are seen to have a considerable range of generality as regards application to nonlinear, dispersive evolution equations. Particularly worthy of note is our theorem implying that solutions of the intermediate long-wave equation converge strongly to solutions of the Korteweg–de Vries equation, or to solutions of the Benjamin–Ono equation, in appropriate asymptotic limits.

1. Introduction

The classical Korteweg–de Vries equation [1] was originally derived in 1895 as an approximate model for planar, uni-directional, irrotational waves propagating on the surface of shallow water. This model equation's range of application has broadened considerably in the last twenty-five years, and now includes many physical situations that feature wave motion wherein a balance is struck between the weak effects of nonlinearity and dispersion (cf. refs. [2, 3]). Not all such situations lead to the Korteweg–de Vries equation, however. For example, waves in certain crystalline lattices have a cubic rather than a quadratic nonlinearity such as appears in the Korteweg–de Vries equation. More commonly, the Korteweg–de Vries equation does not appear when the linearized dispersion relation $P(\xi)$ for the full system of equations cannot be approximated adequately near $\xi = 0$ by a quadratic expression of the form $1 - \xi^2$. In this case the dispersion operator L that appears in the model equation is usually nonlocal. It is to this latter situation that the present paper is devoted.

The linearized dispersion relation for wave equations is now explained in more detail. If the full equations of motion for some wave phenomenon in a medium that is homogeneous in the direction of the waves' propagation are linearized around an appropriate rest state and plane-wave solutions of the linearization are sought, there will in general be determined a dispersion relation $\omega = Q(\xi)$ that is implied in order that $e^{i(\xi x - \omega t)}$ be a solution of the linearized equation. Here x is

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proportional to the distance in the direction of propagation, t is proportional to the elapsed time, and the wavenumber ξ and frequency ω are constants. A general motion of the linearized system is then presumed to be realized as a superposition of plane waves of the form

$$v(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} w(\xi)\,\mathrm{e}^{\mathrm{i}(\xi x - \omega t)}\,\mathrm{d}\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} w(\xi)\,\mathrm{e}^{\mathrm{i}\xi(x - P(\xi)t)}\,\mathrm{d}\xi,$$

where $P(\xi) = Q(\xi)/\xi$ and the weight $w(\xi)$ describes the relative amount of the wave's energy that is present in wavenumber ξ. The associated dispersion operator L is defined by $\widehat{Lv}(\xi) = P(\xi)\,\hat{v}(\xi)$, where a circumflex adorning a function connotes that function's Fourier transform. This operator arises from the observation that the wave motion represented by v formally satisfies the equation $v_t + Lv_x = 0$. (Here, and below, subscripts denote partial differentiation.)

A telling example is the situation that arises when one considers two-dimensional surface water waves. In this case it transpires that $P(\xi) = [\tanh(\xi)/\xi]^{1/2}$ in suitably normalized variables. If a long-wave approximation is taken wherein only values of the wave number ξ near to 0 are considered important, then it may be appropriate to approximate $P(\xi)$ by a simpler expression $\tilde{P}(\xi)$ that is valid near $\xi = 0$. For the case of surface water waves, a natural choice of $\tilde{P}(\xi)$ is $1 - \frac{1}{6}\xi^2$, which approximates $P(\xi)$ correctly up to order ξ^4 and corresponds to the dispersion operator $\tilde{L}v = v + \frac{1}{6}v_{xx}$.

In many situations where uni-directional wave motion arises, the dispersion relation $P(\xi)$ does not admit a good polynomial approximation near the origin. Important examples include certain waves in stratified fluids (the Benjamin–Ono equation [4, 5] and the intermediate long-wave equation [6, 7]), continental-shelf waves (Smith's equation, see ref. [8]), and waves in rotating flows (Pritchard [9] and Leibovich [10]). Assuming that nonlinear effects still arise as they do for the Korteweg–de Vries equation, such situations invariably lead to model equations of the form

$$u_t + uu_x - Lu_x = 0, \tag{1.1}$$

where the dependent variable u depends upon x and t as defined above and L is an approximation to, or the full dispersion operator for the system in question. Eq. (1.1) is written relative to a frame of reference that moves to the right with the phase velocity of infinitesimal waves of extreme length, a quantity that typically becomes equal to one in suitable schemes for nondimensionalizing the independent and dependent variables.

Our purpose is to investigate several specific equations of the form (1.1). Of especial interest will be the pure initial-value problem for the Benjamin–Ono equation, the intermediate long-wave equation, and Smith's equation, in which $u(x,t)$ is specified for all real x at some fixed time, say $t = 0$, and attention is given to proving that a unique solution of (1.1) exists for all x and all non-negative t which has the specified values at $t = 0$. In addition, a theory of continuous dependence of solutions on the data is established, as well as results concerning the continuous dependence of solutions on the dispersion relation.

The theory for existence of smooth solutions of equations like (1.1) with given, smooth, initial data is derived by parabolic regularization. That is, a small amount of dissipation is introduced into the model, solutions obtained for this perturbed equation, and then the limiting form of these solutions as the artificial dissipation tends to zero is sought as the solution to the original problem. This technique proves to be effective in the present context, as it was earlier in other, similar situations (cf. refs. [11–13]), because of various a priori estimates that obtain for these equations. The continuous dependence of solutions on initial data follows directly from the strong convergence of the dissipated solutions to the solution of (1.1), as noted already in the context of the Korteweg–de Vries equation in ref. [14] (see also ref. [15]). The results pertaining to Smith's equation for continental-shelf waves and those of continuous dependence of solution upon the dispersion relation rely upon further a priori estimates, in this case for the difference between solutions to different model equations.

Although there has been enormous scientific activity centered around the equations studied herein, there has been comparatively little written concerning rigorous theory for the initial-value problems that form the backbone of most previous studies. The principle general results are in the early paper of Saut [12] mentioned before, which is in many ways a direct ancestor of the present paper. Later, results pertaining particularly to the Benjamin–Ono equation were stated and used by Bennett et al. [16] in a study of the stability of the solitary-wave solutions of this equation discovered by Benjamin [4]. Their theorem is established in detail here. Very recently, Iorio [17] has also written on the initial-value problem for the Benjamin–Ono equation, obtaining very interesting results, some of which overlap with those presented here. He does not deal with other than the Benjamin–Ono equation, however, nor does he address the issue of the solutions' continuous dependence upon the initial data and upon the symbol of the dispersion relation. Also worth noting is the manuscript of Ponce [18] dealing with smoothing associated to solving the Benjamin–Ono equation.

In addition to providing some advance in our fundamental knowledge concerning the initial-value problem for models for nonlinear, dispersive waves that possess a nonlocal dispersion relation, another point of our analysis is worth noting. It will be shown that if the initial data are fixed, then solutions of the intermediate long-wave equation converge, uniformly on bounded time intervals, to the solution with the same initial data of the Korteweg–de Vries equation or of the Benjamin–Ono equation, respectively, as the relative depth of the two-fluid system that the intermediate long-wave equation models approaches the limiting value 0 or $+\infty$, respectively. This result is already suggested by Joseph's analysis in ref. [6] of the solitary-wave solutions of the intermediate long-wave equation, and was conjectured explicitly in ref. [19]. Indeed, detailed analysis by Santini, Ablowitz, and Fokas [20] of the Benjamin–Ono limit of the intermediate long-wave equation provided the first clue to a formulation of the inverse scattering transform for nonlinear, dispersive equations posed in two spatial dimensions.

It deserves remark that a good deal of the analysis contained herein applies equally well to certain systems of evolution equations for nonlinear, dispersive wave motion. However, to keep this article to a reasonable size, the discussion of systems such as that proposed by Liu et al. [21] as a model for internal waves in a three-layer, stratified medium will be given in a subsequent publication.

A good deal of the theory developed here appears first in the theses of Abdelouhab [22] and Felland [23]. Their work was supervised by Saut and Bona, respectively, and the present paper owes its existence to subsequent, cooperative efforts among the four authors.

The paper is organized as follows. The particular versions of eq. (1.1) studied here are recounted in section 2, along with some elementary properties of solutions of the equation. Section 3 is concerned with the invariant functionals for the Benjamin–Ono equation, and the a priori information that may be deduced therefrom. Similar information for the intermediate long-wave equation is presented in section 4. In section 5 the Cauchy problem for the Benjamin–Ono equation is solved, as regards existence, uniqueness, and continuous dependence on the initial data, for data that decay appropriately to zero at infinity. Similar results are available via the same methods for the intermediate long-wave equation, as is remarked in section 6. In section 7 Smith's equation is considered and the associated Cauchy problem resolved by considering it as a perturbation of the Benjamin–Ono equation. Section 8 is devoted to proving that the Korteweg–de Vries equation and the Benjamin–Ono equation arise as singular limits of the intermediate long-wave equation. Finally, section 9 deals with the periodic initial-value problem for the foregoing equations, in which the initial data are supposed to be periodic in the spatial variable x and a solution is sought that preserves this property.

The notations employed throughout are the standard ones used in the modern theory of nonlinear partial differential equations. For most of our symbolism, we may safely rely upon Lion's text [24] to guide the reader. There is one abbreviation followed here that should be explained: If X is any Banach space of real-valued functions defined on some subset of Euclidean space, its usual norm will be denoted $\| \ \|_X$. In the special case where $X = H^s(\mathbb{R})$, $s \geq 0$, is the Sobolev class of $L^2(\mathbb{R})$ functions having derivatives of all orders up to s which also lie in $L^2(\mathbb{R})$, the norm will be denoted simply $\| \ \|_s$. The same applies to the negative-norm spaces, $H^{-s}(\mathbb{R})$. (If $s \geq 0$ is not an integer, the norm of a function f in H^s is defined in terms of the Fourier transform $\hat{f}$ of f in any of the usual ways, as for example in Lions and Magenes [25]). Thus the $L^2(\mathbb{R})$ norm is denoted $\| \ \|_0$. Whilst the spaces $H^s(\mathbb{R})$, $s \in \mathbb{R}$, are all Hilbert spaces, the only inner product that intervenes in our analysis is that of $L^2(\mathbb{R})$, which we therefore write unadorned as $(\ , \ )$.

2. Some nonlocal, nonlinear dispersive equations

As explained in section 1 the equations to be treated here can all be written in the form

$$u_t + uu_x - L(u_x) = 0, \tag{2.1}$$

where $u = u(x, t)$, $t \in \mathbb{R}_+$, $x \in \mathbb{R}$, the dispersion operator L is defined by

$$\widehat{Lu}(\xi) = p(\xi)\, \hat{u}(\xi), \tag{2.2}$$

and the symbol $p(\xi)$ characterizes the linearized dispersion relation of the model equation (2.1). In this paper, interest will be focused on the Cauchy problem for equations of type (2.1) in which the dispersion relation p has one of the following special forms:

$$p(\xi) = 2\pi|\xi| \quad \text{(Benjamin–Ono equation)}, \tag{2.3}$$

$$p_\delta(\xi) = 2\pi\xi \coth(2\pi\delta\xi) - \frac{1}{\delta} \quad (\delta > 0)$$
$$\text{(intermediate long-wave equation)}, \tag{2.4}$$

$$p_s(\xi) = 2\pi\left(\sqrt{\xi^2 + 1} - 1\right) \quad \text{(Smith equation)}. \tag{2.5}$$

The Benjamin–Ono and the intermediate long-wave equation can be written in the alternative forms

$$u_t + uu_x + H(u_{xx}) = 0$$
$$\text{(Benjamin–Ono equation)}, \tag{2.6}$$

where H denotes the Hilbert transform defined by the principle-value integral

$$Hu(x) = \frac{1}{\pi} \mathrm{PV} \int_{-\infty}^{\infty} \frac{u(y)}{x - y}\, \mathrm{d}y,$$

and

$$u_t + uu_x + \frac{1}{\delta} u_x + T(u_{xx}) = 0$$
$$\text{(intermediate long-wave equation)}, \tag{2.7}$$

where T is defined by the principle-value convolution

$$Tu(x) = -\frac{1}{2\delta} \mathrm{PV} \int_{-\infty}^{\infty} \coth\left(\frac{\pi(x - y)}{2\delta}\right) u(y)\, \mathrm{d}y,$$

respectively. The positive parameter δ characterizes the depth of the lighter fluid layer in a two-fluid system in which the light fluid rests upon a heavier fluid (cf. ref. [7]).

It is obvious that the operator L defined by (2.2) commutes with differentiation, and that, if the dispersion relation $p(\xi)$ is real-valued, as will always be the case here, then L is self-adjoint on its domain in $L^2(\mathbb{R})$. Under these circumstances (2.1) admits the classical invariants

$$I_{-1}(u) = \int_{-\infty}^{\infty} u \, \mathrm{d}x, \quad M_{-1}(u) = 1,$$

$$I_0(u) = \int_{-\infty}^{\infty} \tfrac{1}{2}u^2 \, \mathrm{d}x, \quad M_0(u) = u,$$

$$I_1(u) = -\int_{-\infty}^{\infty} \left[\tfrac{1}{3}u^3 - uL(u)\right] \mathrm{d}x, \quad M_1(u) = -u^2 + 2L(u).$$

Here $M_i(u)$ is the gradient of the corresponding functional $I_i(u)$, $i = -1, 0, 1$. Recall that if J is a smooth functional defined on a suitable subspace of $L^2(\mathbb{R})$, its gradient G is characterized by

$$(G(u), v) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} J(u + \varepsilon v)|_{\varepsilon=0},$$

where the inner product is that of $L^2(\mathbb{R})$. By definition,

$$\frac{\mathrm{d}}{\mathrm{d}t} I_i(u) = (u_t, M_i(u)),$$

for $i = -1, 0, 1$. Moreover, provided u is sufficiently smooth and decays to zero at infinity along with the first few of its partial derivatives, it is obvious from the self-adjointness of L that

$$(uu_x - Lu_x, M_i(u)) = 0,$$

for $i = -1, 0, 1$. Hence it follows that if u is a suitably restricted solution of (2.1), then

$$\frac{\mathrm{d}}{\mathrm{d}t} I_i(u) = 0.$$

If L possesses further properties, one can exhibit other nontrivial invariants of eq. (2.1). For instance, if L satisfies

$$\begin{aligned} &L(u)\,L(v_x) + L(u_x)\,L(v) - \lambda L(u_x L(v) + v_x L(u)) \\ &\quad = \alpha[u_x F(v) + v_x F(u)] + \beta[uG(v_x) + vG(u_x)] + \gamma W(uv)_x, \end{aligned} \tag{2.8}$$

where α, λ, β and γ are constants and F, G, W are self-adjoint operators that commute with L and differentiation, then it is easily seen that

$$\begin{aligned} I_2(u) = \int_{-\infty}^{\infty} \Big(&\tfrac{1}{4}u^4 - \tfrac{3}{2}u^2 L(u_x) \\ &- \frac{3}{2(1+2\lambda)} \times [\alpha u F(u) - 2\beta u G(u) + \gamma u W(u)] \\ &+ \tfrac{3}{2} L(u)^2 \Big) \mathrm{d}x \end{aligned}$$

and

$$\begin{aligned} M_2(u) = u^3 &- 3\left[uL(u) + L(\tfrac{1}{2}u^2)\right] \\ &- \frac{3}{1+2\lambda}[\alpha F(u) - 2\beta G(u) + \gamma W(u)] \\ &+ 3L^2(u) \end{aligned}$$

satisfy the relations

$$\int_{-\infty}^{\infty} [uu_x - L(u_x)]\, M_2(u)\, \mathrm{d}x = 0$$

and

$$\int_{-\infty}^{\infty} u_t M_2(u)\, \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t} I_2(u)$$

for every suitable function u. Here, the notation $L^2 u$ stands for $L(Lu)$. Relation (2.8) appears to be central to establishing the existence of infinitely many invariants for eq. (2.1), at least for the equations in view here.

For later purposes, it is worth recalling the definition of the Poisson bracket of two functionals F_1 and F_2, namely

$$[F_1, F_2] = \int_{-\infty}^{\infty} \operatorname{grad} F_1(u)\, \partial_x \operatorname{grad} F_2(u)\, \mathrm{d}x. \tag{2.9}$$

Using the notation, eq. (2.1) may be written in the Hamiltonian form

$$u_t = \tfrac{1}{2}[I, I_1], \tag{2.10}$$

where $I(u) = u$.

3. Invariants for the Benjamin–Ono equation

The Benjamin–Ono equation possesses infinitely many integral invariants of the sort discussed in section 2 (cf. refs. [26, 27]). Reproduced below are the first six invariants, along with their gradients, in the form given by Case [26]. The general form of these invariants is also presented, and some simple properties deduced therefrom which will be useful for the study of the Cauchy problem associated to the Benjamin–Ono equation.

3.1. The first six invariants

$$I_{-1}(u) = \int_{-\infty}^{\infty} u \, \mathrm{d}x, \qquad I_0(u) = \tfrac{1}{2}\int_{-\infty}^{\infty} u^2 \, \mathrm{d}x,$$

$$I_1(u) = -\int_{-\infty}^{\infty} \left[\tfrac{1}{3}u^3 + uH(u_x)\right] \mathrm{d}x,$$

$$I_2(u) = \int_{-\infty}^{\infty} \left[\tfrac{1}{4}u^4 + \tfrac{3}{2}u^2H(u_x) + 2u_x^2\right] \mathrm{d}x,$$

$$I_3(u) = \int_{-\infty}^{\infty} \Big\{-\tfrac{1}{5}u^5 - \left[\tfrac{4}{3}u^3H(u_x) + u^2H(uu_x)\right] - \left[2uH(u_x)^2 + 6uu_x^2\right] + 4uH(u_{xxx})\Big\} \, \mathrm{d}x,$$

$$I_4(u) = \int_{-\infty}^{\infty} \Big\{\tfrac{1}{6}u^6 + \left[\tfrac{5}{4}u^4H(u_x) + \tfrac{5}{3}u^3H(uu_x)\right] + \tfrac{5}{2}\left[5u^2u_x^2 + u^2H(u_x)^2 + 2uH(u_x)H(uu_x)\right] - 10\left[u_x^2H(u_x) + 2uu_{xx}H(u_x)\right] + 8u_{xx}^2\Big\} \, \mathrm{d}x,$$

$$M_{-1}(u) = 1, \qquad M_0(u) = u,$$

$$M_1(u) = -u^2 - 2H(u_x),$$

$$M_2(u) = u^3 + 3\left[uH(u_x) + H(uu_x)\right] - 4u_{xx},$$

$$M_3(u) = -u^4 - 4\left[u^2H(u_x) + uH(uu_x) + H(u^2u_x)\right] - \left[2H(u_x)^2 + 4H(uH(u_x))_x\right] + \left(6u_x^2 + 12uu_{xx}\right) + 8H(u_{xxx}),$$

$$M_4(u) = u^5 + 5\left[u^3H(u_x) + u^2H(uu_x) + uH(u^2u_x) + H(u^3u_x)\right] + \left[-25uu_x^2 - 25u^2u_{xx} + 5uH(u_x)^2 + 5H(u^2H(u_x))_x + 5H(u_x)H(uu_x) + 5H(uH(uu_x))_x + 5uH(uH(u_x))_x\right] + \left[-40H(u_xu_{xx}) - 20(u_xH(u_x))_x - 20uH(u_{xxx}) - 20H(uu_{xxx})\right] + 16u_{xxxx}.$$

3.2. The general form of an invariant of the Benjamin–Ono equation

It is easily seen from the induction formulas of Case [26] that the known hierarchy of polynomial invariants of the Benjamin–Ono equation can be written in the form $I_n(u)$, $n = 0, 1, \ldots$, with

$$I_n(u) = \int_{-\infty}^{\infty} (-1)^n \frac{1}{n+2} u^{n+2} \, \mathrm{d}x + \sum_{m=1}^{n-1} \int_{-\infty}^{\infty} P_{n+2-m,m}(u) \, \mathrm{d}x + i(n)\, c_n \int_{-\infty}^{\infty} u \frac{\partial^n}{\partial x^n} A_n(u) \, \mathrm{d}x, \tag{3.1}$$

where c_n is a positive constant and

$$\begin{aligned} A_n(u) &= u && \text{if } n \text{ is even},\\ &= H(u) && \text{if } n \text{ is odd},\\ i(n) &= (-1)^p && \text{if } n = 2p,\\ &= (-1)^{p+1} && \text{if } n = 2p+1. \end{aligned}$$

The polynomial $P_{j,k}(u)$ denotes the sum of all terms which are homogeneous of degree j in u and which involve exactly k derivatives in x. With this notation, the gradient of $I_n(u)$ has the form

$$M_n(u) = (-1)^n u^{n+1} + \sum_{m=1}^{n-1} Q_{n+1-m,m}(u) + 2i(n)\, c_n \frac{\partial^n}{\partial x^n} A_n(u),$$

where $Q_{j,k}(u)$ consists of terms homogeneous in

u of degree j which have exactly k x-derivatives in total.

3.3. Some properties of the invariants $I_n(u)$, $n \geq 0$

3.3.1. The $I_n(u)$ are in involution when u is a solution of the Benjamin–Ono equation (see ref. [28]). That is, in the notation introduced in section 2 (see (2.9)),

$$[I_n, I_m] = 0,$$

for all n and m. Using the classical properties of the Hilbert transform [29, 30]

(H1) $\displaystyle\int_{-\infty}^{\infty} uH(v)\,\mathrm{d}x = -\int_{-\infty}^{\infty} H(u)\,v\,\mathrm{d}x,$

(H2) $\displaystyle\int_{-\infty}^{\infty} H(u)\,H(v)\,\mathrm{d}x = \int_{-\infty}^{\infty} uv\,\mathrm{d}x,$

(H3) $H[uH(v) + vH(u)] = H(u)\,H(v) - uv,$

all of which hold for arbitrary functions $u, v \in L^2(\mathbb{R})$, it is easy to prove the following result.

Proposition 3.3.1. Let $u = u(x,t)$ be a C^∞-function all of whose partial derivatives lie in $L_2(\mathbb{R})$. Then it follows that

$$\int_{-\infty}^{\infty} [uu_x + H(u_{xx})]\,M_n(u)\,\mathrm{d}x = 0,$$

$$\int_{-\infty}^{\infty} u_t M_n(u)\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t} I_n(u),$$

for $n = 0, 1, \ldots, 4$. Hence if u is a smooth solution of the Benjamin–Ono equation, $I_n(u)$ is independent of t, for $n = 0, \ldots, 4$.

Remark. In fact, proposition 3.3.1 holds for all $n = 0, 1, \ldots,$ but we shall only need the first few invariants in our analysis, and for these the proof of the proposition is just a calculation involving the formulae in section 3.1 in conjunction with (H1)–(H3).

3.3.2. The invariance of $I_n(u)$ leads to bounds on the $H^{n/2}(\mathbb{R})$ norm of u. By the Parseval identity, the term $i(n)\,c_n \int_{-\infty}^{\infty} u[\partial^n A_n(u)/\partial x^n]\,\mathrm{d}x$ in $I_n(u)$ is equivalent to the $\|u\|_{n/2}^2$ for smooth functions u. Therefore, one is tempted to write

$$I_n(u) = c\|u\|_{n/2}^2 + R_n(u),$$

where c is a positive constant. The remainder term $R_n(u)$ may be bounded advantageously, as the following result shows.

Lemma 3.3.2. For all $\eta > 0$, there exists a constant $c = c(\eta) > 0$ such that

$$|R_n(u)| \leq \eta\|u\|_{n/2}^2 + c(\eta)\|u\|_0^{2+2n}$$

for all $u \in H^{n/2}(\mathbb{R})$.

Proof. From (3.1) it follows that

$$R_n(u) = (-1)^n \int_{-\infty}^{\infty} \frac{1}{n+2} u^{n+2}\,\mathrm{d}x + \sum_{m=1}^{n-1} \int_{-\infty}^{\infty} P_{n+2-m,m}(u)\,\mathrm{d}x.$$

Consider the first integral in the above expression for $R_n(u)$ and use the embedding $H^{n/(4+2n)}(\mathbb{R}) \subset L^{n+2}(\mathbb{R})$ and an interpolation inequality to obtain that

$$\left|\int_{-\infty}^{\infty} u^{n+2}\,\mathrm{d}x\right| \leq \|u\|_{L^{n+2}}^{n+2} \leq c\|u\|_{n/(4+2n)}^{n+2} \leq c\|u\|_{n/2}\|u\|_0^{n+1}. \tag{3.2}$$

Then, for any $\eta > 0$, Young's inequality therefore implies the existence of a constant $c = c(\eta)$ such that

$$\left|\int_{-\infty}^{\infty} u^{n+2}\,\mathrm{d}x\right| \leq \eta\|u\|_{n/2}^2 + c(\eta)\|u\|_0^{2+2n}. \tag{3.3}$$

We turn now to consideration of the other $n-1$ summands making up $R_n(u)$. The terms comprising $P_{n+2-m,m}(u)$ are homogeneous of degree

$n+2-m$ in u and involve m x-derivatives. Moreover, the Hilbert transform H is a bounded operator from $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$, for any $p>1$ and from $H^s(\mathbb{R})$ into $H^s(\mathbb{R})$, for all s (cf. ref. [31]). Performing several integrations by parts, we can therefore bound $P_{n+2-m,m}(u)$ above by estimating terms of the type

$$\int_{-\infty}^{\infty}\left(\prod_{i=1}^{n+2-m} u_{(\alpha_i)}\right)\mathrm{d}x,$$

where

$$u_{(\alpha_i)} = \partial_x^{\alpha_i}u, \quad \alpha_i \in \{0,\dots,N(m)\},$$
$$N(m) = \tfrac{1}{2}m \quad \text{if } m \text{ is even}, \qquad = \tfrac{1}{2}(m+1) \quad \text{if } m \text{ is odd},$$

and

$$\sum_{i=1}^{n+2-m} \alpha_i = m.$$

Provided $m \le n-1$, an application of Hölder's inequality shows that

$$\left|\int_{-\infty}^{\infty}\left(\prod_{i=1}^{n+2-m} u_{(\alpha_i)}\right)\mathrm{d}x\right| \le \prod_{i=1}^{n+2-m}\|u_{(\alpha_i)}\|_{L^{\gamma_i}} \le c\prod_{i=1}^{n+2-m}\|u_{(\alpha_i)}\|_{s_i} \le c\prod_{i=1}^{n+2-m}\|u\|_{s_i+\alpha_i},$$

where the γ_i will be chosen presently so that $s_i = (\gamma_i - 2)/2\gamma_i$, $\gamma_i > 2$, and $\sum_{i=1}^{n+2-m} 1/\gamma_i = 1$. The constant c arises from the embedding of $H^{s_i}(\mathbb{R})$ into $L^{\gamma_i}(\mathbb{R})$, which has been utilized for $1 \le i \le n+2-m$. Define $\lambda_i = s_i + \alpha_i = \frac{1}{2} - 1/\gamma_i + \alpha_i$. By the definition of α_i it is easily seen that one can choose γ_i such that $\lambda_i < n/2$, for $i = 1,2,\dots, n+2-m$. Since $\lambda_i = (1-\theta)n/2$, with $\theta = (n-2\lambda_i)/n$, a standard interpolation inequality thus yields

$$\left|\int_{-\infty}^{\infty}\left(\prod_{i=1}^{n+2-m} u_{(\alpha_i)}\right)\mathrm{d}x\right| \le c\prod_{i=1}^{n+2-m}\|u\|_{n/2}^{2\lambda_i/n}\|u\|_0^{1-2\lambda_i/n}$$
$$= c\|u\|_{n/2}^{(2/n)\sum_{i=1}^{n+2-m}\lambda_i}\|u\|_0^{n+2-m-(2/n)\sum_{i=1}^{n+2-m}\lambda_i}$$
$$= c\|u\|_{n/2}^{1+m/n}\|u\|_0^{n+1-m-m/n}.$$

Since $1 \le m \le n-1$, we have $1 + m/n < 2$, and therefore by Young's inequality, for any $\eta > 0$ there is a $c = c(\eta)$ such that

$$\left|\int_{-\infty}^{\infty}\left(\prod_{i=1}^{n+2-m} u_{(\alpha_i)}\right)\mathrm{d}x\right| \le \eta\|u\|_{n/2}^2 + c(\eta)\|u\|_0^{2+2n},$$

for $m = 1,2,\dots,n-1$. The last inequality combined with (3.3) suffice to establish lemma 3.3.2. □

Lemma 3.3.3. In the above notation, there is a $c > 0$ such that

$$\int_{-\infty}^{\infty}(-1)^j u_{(2j)}M_n(u)\,\mathrm{d}x = c\|u\|_{n/2+j}^2 + \tilde{R}_{n,j}(u).$$

Moreover, for every $\eta > 0$, there exists a constant $c = c(\eta) > 0$ such that

$$|\tilde{R}_{n,j}(u)| \le \eta\|u\|_{n/2+j}^2 + c(\eta)\|u\|_0^{2+2(n+2j)}.$$

Proof. The proof is similar to the proof of lemma 3.3.2 since $\tilde{R}_{n,j}(u)$ is a sum of homogeneous terms of degree $n+2-m$ in u, with $m+2j$ x-derivatives ($m \le n-1$). We are therefore reduced to estimating terms of the form

$$\int_{-\infty}^{\infty} P_{n+2-m,m+2j}(u)\,\mathrm{d}x,$$

and for this we can apply the reasoning in the proof of lemma 3.3.2 once it is noted that $P_{n+2-m,m+2j} = P_{(n+2j)+2-(m+2j),m+2j}$. The result then follows immediately. □

4. Invariants for the intermediate long-wave equation

Write the intermediate long-wave equation in the form

$$u_t + uu_x + \frac{1}{\delta} u_x + T(u_{xx}) = 0, \tag{4.1}$$

where

$$Tu(x) = -\frac{1}{2\delta} \mathrm{PV} \int_{-\infty}^{\infty} \coth\left(\frac{\pi(x-y)}{2\delta}\right) u(y)\,\mathrm{d}x. \tag{4.2}$$

This equation possesses an infinite sequence of invariants which are in involution (cf. refs. [19], [28] or [32]) the first few of which are written below along with their gradients.

$$I_{-1}(u) = \int_{-\infty}^{\infty} u\,\mathrm{d}x, \quad M_{-1}(u) = 1,$$

$$I_0(u) = \int_{-\infty}^{\infty} \tfrac{1}{2}u^2\,\mathrm{d}x, \quad M_0(u) = u,$$

$$I_1(u) = -\int_{-\infty}^{\infty}\left(\tfrac{1}{3}u^3 - uT(u_x) + \frac{1}{\delta}u^2\right)\mathrm{d}x,$$

$$M_1(u) = -u^2 - 2T(u_x) - \frac{2}{\delta}u,$$

$$I_2(u) = \int_{-\infty}^{\infty}\left(\tfrac{1}{4}u^4 + \tfrac{3}{2}u^2T(u_x) + \tfrac{1}{2}u_x^2 + \tfrac{3}{2}[T(u)_x]^2 + \frac{1}{\delta}\left[\tfrac{3}{2}u^3 + \tfrac{9}{2}uT(u_x)\right] + \frac{3}{2\delta^2}u^2\right)\mathrm{d}x,$$

$$M_2(u) = u^3 + 3[uT(u_x) + T(uu_x)] - u_{xx} + 3T^2(u_{xx}) + \frac{3}{\delta}\left[\tfrac{3}{2}u^2 + 3T(u_x)\right] + \frac{3}{\delta^2}u.$$

The counterparts of lemmas 3.3.2 and 3.3.3 for the Benjamin–Ono equation are valid in the present context. To see this straightforwardly, it suffices to note the following elementary result.

Lemma 4.1. For every $\delta > 0$ and for all $\xi \in \mathbb{R}$,

$$-\frac{1}{\delta} + 2\pi|\xi| \le 2\pi\xi\coth(2\pi\delta\xi) \le \frac{1}{\delta} + 2\pi|\xi|.$$

Proof. This is easily adduced from the classical series expansion

$$\coth(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 + n^2\pi^2},$$

which is valid for $z \in \mathbb{C}$, $z \neq in\pi$ for any integer n. □

It follows from lemma 4.1 that the operators $H\partial_x^2$ and $T\partial_x^2 + (1/\delta)\partial_x$ differ by a pseudo-differential operator of order 0, which is bounded in all the Sobolev spaces $H^s(\mathbb{R})$. Therefore, one can write

$$I_n(u) = c\|u\|_{n/2}^2 + d\sum_{k=1}^{n} \frac{1}{\delta^k}\|u\|_{(n-k)/2}^2 + S_n(u) + \sum_{k=1}^{n}\frac{1}{\delta^k}S_{n-k}(u), \tag{4.3}$$

where d is a constant and for every $\eta > 0$, there is a $c = c(\eta)$ such that the remainder S_{n-k} satisfies

$$|S_{n-k}(u)| \le \eta\|u\|_{(n-k)/2}^2 + c(\eta)\|u\|_0^{2+2(n-k)}, \tag{4.4}$$

for $k = 0, 1, \ldots, n$. The proof of (4.4) is similar to the proof of lemma 3.3.2.

On the other hand, the multipliers $M_n(u)$ have the property that

$$\int_{-\infty}^{\infty}(-1)^j u_{(2j)} M_n(u)\,\mathrm{d}x = c\|u\|_{n/2+j}^2 + d\sum_{k=1}^{n}\frac{1}{\delta^k}\|u\|_{(n-k)/2+j}^2 + \tilde{S}_{n,j}(u) + \sum_{k=1}^{n}\frac{1}{\delta^k}\tilde{S}_{n-k,j}(u), \tag{4.5}$$

where for every $\eta > 0$ there is a $c = c(\eta) > 0$ for which the $\tilde{S}_{n-k,j}(u)$ satisfy

$$|\tilde{S}_{n-k,j}(u)| \le \eta\|u\|_{(n-k)/2+j}^2 + c(\eta)\|u\|_0^{2+2(n-k+2j)}. \tag{4.6}$$

The proof of (4.6) is similar to that of lemma 3.3.3.

5. The Cauchy problem for the Benjamin–Ono equation

5.1. Review of known results on the Cauchy problem for nonlocal, nonlinear dispersive equations

The initial-value problems under study in the present paper are particular cases of initial-value problems for general, nonlocal, nonlinear, dispersive equations of the type

$$u_t + f(u)_x - Lu_x = 0, \quad \text{for } x \in \mathbb{R},\ t \geq 0,$$
$$u(x,0) = u_0(x), \quad \text{for } x \in \mathbb{R}, \tag{5.1}$$

which were studied in ref. [12] under the assumption that $f(u)$ is a polynomial in u of degree d and L is defined by

$$\widehat{Lu}(\xi) = p(\xi)\,\hat{u}(\xi),$$

where p satisfies

$$p \in L^\infty_{\text{loc}}(\mathbb{R}),\ p \geq 0 \text{ a.e., } p \text{ real}, \tag{5.2}$$

and

$$\text{there exist constants } \lambda, \mu, 0 \leq \lambda \leq \mu, R, c_1, c_2 > 0,$$
$$\text{such that } c_1|\xi|^\lambda \leq p(\xi) \leq c_2|\xi|^\mu \quad \text{for a.e. } |\xi| \geq R. \tag{5.3}$$

In this situation, the following theorem of global existence of relatively weak solutions of (5.1) was established in ref. [12].

Theorem 5.1.1. Suppose that $d < 2\lambda + 1$. Let $u_0 \in D(L^{1/2}) = \{v \in L^2(\mathbb{R}):\ L^{1/2}v \in L^2(\mathbb{R})\}$. Then there exists a solution u of (5.1) with initial value u_0 which lies in the space $L^\infty(\mathbb{R}_+, D(L^{1/2}))$.

Moreover, under very weak assumptions on the symbol p of L, one can easily obtain the following result on local existence of smooth solutions (see again ref. [12]).

Theorem 5.1.2. Let $s \in \mathbb{R}$, $s > \frac{3}{2}$ and $u_0 \in H^s(\mathbb{R})$. Then there exists $T_* = T_*(\|u_0\|_s) > 0$, such that (5.1) possesses a unique solution $u \in L^\infty(0, T; H^s(\mathbb{R}))$ for all $T < T_*$. Moreover $T_*(\|u_0\|_s)$ depends only on $\|u_0\|_{3/2+\eta}$, for η small enough.

We shall now complete the preceding results for the Benjamin–Ono, the intermediate-long-wave, and the Smith equations, and prove in particular the existence of global smooth solutions. In this section, we deal with the Benjamin–Ono equation.

5.2. Global solutions for the Benjamin–Ono equation in the spaces $H^1(\mathbb{R})$ and $H^{3/2}(\mathbb{R})$

Theorem 5.2.1. Let $u_0 \in H^{n/2}(\mathbb{R})$, for $n = 2$ or $n = 3$. There exists a solution u of (2.6) such that $u \in L^\infty(\mathbb{R}_+;\ H^{n/2}(\mathbb{R}))$.

Proof. The strategy is to use the parabolic regularization (as in refs. [11, 12])

$$u^\varepsilon_t + u^\varepsilon u^\varepsilon_x + H(u^\varepsilon_{xx}) + \varepsilon(u^\varepsilon_{xxxx} + u^\varepsilon) = 0,$$
$$u^\varepsilon(x,0) = u_{0\varepsilon}, \tag{5.4}$$

where $u_{0\varepsilon} \in H^\infty(\mathbb{R}) = \bigcap_{k>0} H^k(\mathbb{R})$ is such that $u_{0\varepsilon} \to u_0$ in $H^{n/2}(\mathbb{R})$. For $\varepsilon > 0$ fixed, the classical theory of parabolic equations shows that (5.4) possesses a unique solution u^ε in the class $C^\infty(\mathbb{R}_+;\ H^\infty(\mathbb{R}))$.

Using lemmas 3.3.2 and 3.3.3 we shall obtain a priori estimates on u^ε which will lead to the required solution u of (2.6) by a standard limiting procedure. First take the scalar product of eq. (5.4) with $M_n(u^\varepsilon)$. On account of proposition 3.3.1, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} I_n(u^\varepsilon) + \varepsilon \int_{-\infty}^{\infty} \left[(u^\varepsilon_{xxxx} + u^\varepsilon) M_n(u^\varepsilon) \right] \mathrm{d}x = 0. \tag{5.5}$$

Next integrate (5.5) over the temporal interval

$[0, t]$. The case $n = 0$ yields

$$\tfrac{1}{2}\|u^\varepsilon(\bullet, t)\|_0^2 + \varepsilon\int_0^t\int_{-\infty}^{\infty}\left[u^\varepsilon(x,s)^2 + u_{xx}^\varepsilon(x,s)^2\right]\mathrm{d}x\,\mathrm{d}s = \tfrac{1}{2}\|u_{0\varepsilon}\|_0^2. \tag{5.6}$$

It follows that for any $t > 0$,

$$\|u^\varepsilon\|_{L^\infty(\mathbb{R}_+;\, L^2(\mathbb{R}))} \le \|u_{0\varepsilon}\|_0,$$
$$\varepsilon\int_0^t \|u^\varepsilon(\bullet, s)\|_0^2\,\mathrm{d}s \le \|u_{0\varepsilon}\|_0^2. \tag{5.7}$$

For $n > 0$, integrate (5.5) over the temporal interval $[0, T]$. Using lemmas 3.3.2 and 3.3.3, it follows that for any $T > 0$, there is a positive constant c such that

$$\begin{aligned}\|u^\varepsilon(\bullet, T)\|_{n/2}^2 &+ c\varepsilon\int_0^T\left[\|u^\varepsilon(\bullet, t)\|_{n/2+2}^2 + \|u^\varepsilon(\bullet, t)\|_{n/2}^2\right]\mathrm{d}t \\ &\le c\left(\|u_{0\varepsilon}\|_{n/2}^2 + \|u_{0\varepsilon}\|_0^{2+2n}\right) \\ &\quad + c\varepsilon\int_0^T\left[\|u^\varepsilon(\bullet, t)\|_0^{2+2(n+4)} + \|u^\varepsilon(\bullet, t)\|_0^{2+2n}\right]\mathrm{d}t.\end{aligned} \tag{5.8}$$

Because of the bounds already available in (5.7) it is easy to show that

$$\begin{aligned}\varepsilon\int_0^T&\left[\|u^\varepsilon(\bullet, t)\|_0^{2+2(n+4)} + \|u^\varepsilon(\bullet, t)\|_0^{2+2n}\right]\mathrm{d}t \\ &\le \left(\|u^\varepsilon\|_{L^\infty(\mathbb{R}_+;\, L^2(\mathbb{R}))}^{2(n+4)} + \|u^\varepsilon\|_{L^\infty(\mathbb{R}_+;\, L^2(\mathbb{R}))}^{2n}\right) \times \varepsilon\int_0^T \|u^\varepsilon(\bullet, t)\|_0^2\,\mathrm{d}t \\ &\le \|u_{0\varepsilon}\|_0^{2(n+5)} + \|u_{0\varepsilon}\|_0^{2(n+1)},\end{aligned}$$

and so (5.8) gives

$$\|u^\varepsilon(\bullet, T)\|_{n/2}^2 + c\varepsilon\int_0^T \|u^\varepsilon(\bullet, t)\|_{n/2+2}^2\,\mathrm{d}t \le c\left(\|u_{0\varepsilon}\|_0^{2(n+5)} + \|u_{0\varepsilon}\|_0^{2(n+1)} + \|u_{0\varepsilon}\|_{n/2}^2\right). \tag{5.9}$$

Thus it transpires that

u^ε is bounded, independently of ε,

$$\text{in } L^\infty\left(\mathbb{R}_+;\, H^{n/2}(\mathbb{R})\right), \tag{5.10}$$

and

$\sqrt{\varepsilon}\,u^\varepsilon$ is bounded, independently of ε,

$$\text{in } L^2\left(\mathbb{R}_+;\, H^{n/2+2}(\mathbb{R})\right).$$

One can now easily pass to the limit by extracting a subsequence, still denoted u^ε, such that

$$u^\varepsilon \to u \text{ in } L^\infty\left(\mathbb{R}_+;\, H^{n/2}(\mathbb{R})\right) \quad \text{weak*}, \tag{5.11}$$
$$u_t^\varepsilon \to u_t \text{ in } L^2\left(\mathbb{R}_+;\, H^{-1}(\mathbb{R})\right) \quad \text{weakly}. \tag{5.12}$$

Using classical compactness arguments, one deduces from (5.11) and (5.12) that the limiting function u is indeed a solution of (2.6). □

5.3. Global existence of smooth solutions for the Cauchy problem associated with the Benjamin–Ono equation

This section is devoted to existence results for solutions of the Benjamin–Ono equation in classes where uniqueness holds.

Theorem 5.3.1. Let $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. Corresponding to the initial data u_0, there exists a unique solution u of (2.6) such that

$$u \in C^k\left(\mathbb{R}_+;\, H^{s-2k}(\mathbb{R})\right)$$
for all $k \in \mathbb{N}$ with $s - 2k \ge -1$.

Moreover, for every fixed $T > 0$, let $\mathfrak{A}_T$ be the mapping which associates to u_0 the solution u on the interval $[0, T]$. Then $\mathfrak{A}_T$ is continuous from $H^s(\mathbb{R})$ into $C^k(0, T;\, H^{s-2k}(\mathbb{R}))$, for the same range of k.

Corollary 5.3.2. Let $u_0 \in H^\infty(\mathbb{R})$. Then corresponding to initial data u_0, there exists a unique solution u of (2.6) satisfying $u \in C^\infty(\mathbb{R}_+;\, H^\infty(\mathbb{R}))$.

Proof of theorem 5.3.1. The proof of theorem 5.3.1 consists of several steps; namely

(i) proving existence of solutions in $C_w(\mathbb{R}_+; H^s(\mathbb{R}))$ ($C_w(\mathbb{R}_+; H^s(\mathbb{R}))$ is the space of continuous functions from $\mathbb{R}_+$ with values in the space $H^s(\mathbb{R})$ when the latter space is equipped with its weak topology);
(ii) showing the strong continuity of these solutions with respect to the temporal variable, and
(iii) establishing the continuous dependence of the solution on the initial data.

The uniqueness assertion in theorem 5.3.1 follows by a straightforward energy estimate and Gronwall's lemma, since in the considered classes one has $u_x \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ (cf. ref. [12]).

(i) Existence of solutions in $C_w(\mathbb{R}_+; H^s(\mathbb{R}))$

Theorem 5.3.3. Let $u_0 \in H^s(\mathbb{R})$, where $s > \frac{3}{2}$. Then for any $T > 0$, there exists a unique solution u of (2.6) such that $u \in C_w(0, T; H^s(\mathbb{R}))$.

Proof of theorem 5.3.3. Again we consider the parabolic regularization (5.4) with initial data $u_{0\varepsilon}$ in $H^\infty(\mathbb{R})$ such that

$$u_{0\varepsilon} \to u_0 \text{ in } H^s(\mathbb{R}) \text{ as } \varepsilon \downarrow 0. \tag{5.13}$$

Lemma 5.3.4. For any $T > 0$ there exists a constant $c = c(T, \|u_{0\varepsilon}\|_{3/2})$ such that

$$\|u^\varepsilon\|_{L^\infty(0,T;\, H^s(\mathbb{R}))} \le c\big(T, \|u_{0\varepsilon}\|_{3/2}\big) \|u_{0\varepsilon}\|_s, \tag{5.14}$$

$$\sqrt{\varepsilon}\, \|u^\varepsilon\|_{L^2(0,T;\, H^{s+2}(\mathbb{R}))} \le c\big(T, \|u_{0\varepsilon}\|_{3/2}\big) \|u_{0\varepsilon}\|_s. \tag{5.15}$$

Proof of lemma 5.3.4. The techniques of Saut and Temam [13] are used. Define the derivation operator D^s by

$$D^s u(x) = \int_{-\infty}^{\infty} |\xi|^s \hat{u}(\xi)\, \mathrm{e}^{2\pi i x \xi}\, \mathrm{d}\xi. \tag{5.16}$$

Apply the operator D^s to the regularized equation (5.4) and take the $L^2(\mathbb{R})$ scalar product of the result with $D^s u^\varepsilon$. Using the fact that H is skew-adjoint, we obtain

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|D^s u^\varepsilon\|_0^2 + \big(D^s(u^\varepsilon u_x^\varepsilon), D^s u^\varepsilon\big) + \varepsilon\big(\|D^s u_{xx}^\varepsilon\|_0^2 + \|D^s u^\varepsilon\|_0^2\big) = 0. \tag{5.17}$$

To estimate the second term in (5.17), write

$$\big(D^s(u^\varepsilon u_x^\varepsilon), D^s u^\varepsilon\big) = \big(D^s(u^\varepsilon u_x^\varepsilon) - u^\varepsilon D^s u_x^\varepsilon, D^s u^\varepsilon\big) + \big(u^\varepsilon D^s u_x^\varepsilon, D^s u^\varepsilon\big) \tag{5.18}$$

and apply the Hölder inequality to obtain

$$\big|\big(D^s(u^\varepsilon u_x^\varepsilon) - u^\varepsilon D^s u_x^\varepsilon, D^s u^\varepsilon\big)\big| \le \|D^s(u^\varepsilon u_x^\varepsilon) - u^\varepsilon D^s u_x^\varepsilon\|_0 \|D^s u^\varepsilon\|_0. \tag{5.19}$$

To estimate the right-hand side of (5.19), we shall use the following commutation lemma, the proof of which may be found in ref. [13].

Lemma 5.3.5. Let $s \ge \gamma + 1 > \frac{3}{2}$ and let $u, v \in H^s(\mathbb{R})$. Then there is a constant $c = c(\gamma, s)$ such that

$$\|D^s(uv) - u D^s v\|_0 \le c(\gamma, s)\big(\|u\|_s \|v\|_\gamma + \|u\|_{\gamma+1} \|v\|_{s-1}\big).$$

Moreover, $c(\gamma, s) = c'(s)/\sqrt{\gamma - \frac{1}{2}}$.

Since $s > \frac{3}{2}$ there is an $\eta > 0$ such that $s > \frac{3}{2} + \eta$. Apply lemma 5.3.5. with $u = u^\varepsilon$, $v = u_x^\varepsilon$, and $\gamma = \frac{1}{2} + \eta$ to obtain that

$$\|D^s(u^\varepsilon u_x^\varepsilon) - u^\varepsilon D^s u_x^\varepsilon\|_0 \le \frac{c}{\sqrt{\eta}} \|u^\varepsilon\|_{3/2+\eta} \|u^\varepsilon\|_s. \tag{5.20}$$

On the other hand,

$$\big(u^\varepsilon D^s u_x^\varepsilon, D^s u^\varepsilon\big) = -\tfrac{1}{2}\big(u_x^\varepsilon D^s u^\varepsilon, D^s u^\varepsilon\big),$$

and since (cf. ref. [12], lemma 6), for all $\eta > 0$,

$$\|u_x\|_{L^\infty} \le \frac{c}{\sqrt{\eta}} \|u\|_{3/2+\eta},$$

one derives the inequality

$$|(u^\varepsilon D^s u_x^\varepsilon, D^s u^\varepsilon)| \le \frac{c}{\sqrt{\eta}} \|u^\varepsilon\|_{3/2+\eta} \|D^s u^\varepsilon\|_0^2, \tag{5.21}$$

which holds for all $\eta > 0$.

Formulas (5.18), (5.20) and (5.21) give

$$|(D^s(u^\varepsilon u_x^\varepsilon), D^s u^\varepsilon)| \le \frac{c}{\sqrt{\eta}} \|u^\varepsilon\|_{3/2+\eta} \|u^\varepsilon\|_s^2, \tag{5.22}$$

and this holds for any $\eta > 0$ such that $\frac{3}{2} + \eta < s$. Since $H^{3/2+\eta}(\mathbb{R}) = [H^s(\mathbb{R}), H^{3/2}(\mathbb{R})]_\theta$, with $\theta = 1 - 2\eta/(2s-3) = 1 - 2\gamma n$, we have the interpolation inequality

$$\|u\|_{3/2+\eta} \le c \|u\|_s^{2\gamma\eta} \|u\|_{3/2}^{1-2\gamma\eta}, \tag{5.23}$$

where c is a constant independent of η. On the other hand, it was established in the proof of theorem 5.2.1 that

$$\|u^\varepsilon\|_{L^\infty(\mathbb{R}_+;\, H^{3/2}(\mathbb{R}))} \le c a_3(\|u_{0\varepsilon}\|_{3/2}). \tag{5.24}$$

Combining (5.6), (5.17), (5.22), (5.23) and (5.24) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^\varepsilon\|_s^2 + \varepsilon\|u^\varepsilon\|_{s+2}^2 \le \varphi\left(\frac{1}{\sqrt{\eta}}\|u^\varepsilon\|_s^{2+2\gamma\eta}\right) \tag{5.25}$$

for all $\eta > 0$ such that $3/2 + \eta < s$, where $\varphi = [c a_3(\|u_{0\varepsilon}\|_{3/2})]^{1-2\gamma\eta}$.

Integrating (5.25) over the temporal interval $[0, t]$ leads to

$$\|u^\varepsilon(\bullet, t)\|_s^2 + 2\varepsilon\int_0^t \|u^\varepsilon(\bullet, \tau)\|_{s+2}^2 \,\mathrm{d}\tau \le y(t), \tag{5.26}$$

where y is the solution of the differential equation

$$y'(t) = \varphi\frac{1}{\sqrt{\eta}} y(t)^{1+\gamma\eta}, \quad y(0) = \|u_{0\varepsilon}\|_s^2, \tag{5.27}$$

on its maximal interval of existence $[0, T(\eta))$. Here, the constant γ is $1/(2s-3)$. Eq. (5.27) is easily integrated and one finds that

$$y(t) = \left(\|u_{0\varepsilon}\|_s^{-2\gamma\eta} - \gamma\sqrt{\eta}\,\varphi t\right)^{-1/\gamma\eta}, \tag{5.28}$$

whence

$$T(\eta) = \frac{1}{\gamma\varphi\sqrt{\eta}}\|u_{0\varepsilon}\|_s^{-2\gamma\eta} \to +\infty \quad \text{as } \eta \to 0.$$

For any fixed $T > 0$, we can choose $\eta > 0$ so small that $T < \frac{1}{2}T(\eta)$. Then it follows that for $0 \le t \le T$,

$$y(t) \le c(T; \|u_{0\varepsilon}\|_{3/2}) \|u_{0\varepsilon}\|_s^2,$$

and lemma 5.3.4 is proved. □

Lemma 5.3.4 implies that u^ε (respectively, $\sqrt{\varepsilon}\, u^\varepsilon$) belongs to a bounded subset of $L^\infty(0, T; H^s(\mathbb{R}))$ (respectively, $L^2(0, T; H^{s+2}(\mathbb{R}))$. By a standard limiting argument, it is inferred that a subsequence of u^ε converges in $L^\infty(0, T; H^s(\mathbb{R}))$ weak* to a solution u of (2.6) such that for all $T > 0$, $u \in L^\infty(0, T; H^s(\mathbb{R}))$. Moreover, using (2.6), it is apparent that for all $T > 0$, $u_t \in L^\infty(0, T; H^{s-2}(\mathbb{R}))$, and by classical results of Lions and Strauss (see ref. [24]) we arrive at the desired conclusion, namely that for all positive T, $u \in C_w(0, T; H^s(\mathbb{R}))$.

Note that the conclusion just deduced implies that if $u_0 \in H^\infty$, then the unique solution u of (2.6) corresponding to the initial data u_0 lies in $C^\infty(\mathbb{R}_+; H^\infty(\mathbb{R}))$.

(ii) Strong continuity

Here it will be proved that the solution u obtained in (i) lies in $C^k(\mathbb{R}_+; H^{s-2k}(\mathbb{R}))$ for $k \in \mathbb{N}$ such that $s - 2k \ge -1$. We shall follow the technique used by Bona and Smith [14] in the context of the KdV equation, whereby for a particular sequence $\{u_{0\varepsilon}\}_{\varepsilon>0}$ of regularizations of the initial data u_0, the corresponding sequence of solutions $\{u^\varepsilon\}_{\varepsilon>0}$ is shown to be Cauchy in the space $C^k(0,T;\,H^{s-2k}(\mathbb{R}))$ for all $T>0$ and all $k\in\mathbb{N}$ such that $s-2k\geq -1$.

In fact, the same regularizing sequence used in ref. [14] suffices for our purposes. Let $\varphi = C^\infty(\mathbb{R})$ be such that

(1) $0\leq\varphi(\xi)\leq 1$, for $\xi\in\mathbb{R}$, (2) $\varphi(0)=1$,

(3) $\dfrac{\mathrm{d}^k}{\mathrm{d}\xi^k}\psi(0)=0$, for $k\in\mathbb{N}$,

where $\psi(\xi)=1-\varphi(\xi)$, and

(4) $\varphi(\xi)$ tends exponentially to 0 as $|\xi|\to\infty$.

Then, for $\xi\in\mathbb{R}$, define

$$\hat{u}_{0\varepsilon}(\xi)=\varphi(\varepsilon^{1/6}\xi)\,\hat{u}_0(\xi). \tag{5.29}$$

The following proposition was proved in ref. [14].

Proposition 5.3.6. Let $s>0$ and $u_0\in H^s(\mathbb{R})$. Then

$$u_{0\varepsilon}\in H^\infty(\mathbb{R}) \quad\text{and}\quad \|u_0-u_{0\varepsilon}\|_s\to 0 \quad \text{as } \varepsilon\downarrow 0. \tag{5.30}$$

For any $r\geq 0$,

$$\|u_{0\varepsilon}\|_{s+r}\leq c\varepsilon^{-r/6}\|u_0\|_s, \tag{5.31}$$

$$\|u_0-u_{0\varepsilon}\|_{s-r}\leq c\varepsilon^{r/6}\|u_0\|_s, \tag{5.32}$$

and

$$\|u_0-u_{0\varepsilon}\|_s\leq c\|u_0\|_s. \tag{5.33}$$

Moreover, if $u_0^n\to u_0$ in $H^s(\mathbb{R})$ as $n\to+\infty$, then

$$\|u_{0\varepsilon}^n-u_0^n\|_s\to 0 \quad \text{uniformly in } n, \text{ as } \varepsilon\to 0. \tag{5.34}$$

The constants occurring in (5.31), (5.32), and (5.33) depend only on r and the choice of φ, and are independent of ε, provided ε is restricted to a bounded subset of $\mathbb{R}_+$.

We are now ready to prove our central approximation lemma as regards solutions of the Benjamin–Ono equation.

Lemma 5.3.7. Let $T>0$ and $u_0\in H^s$, where $s>\frac{3}{2}$, be given. For $0<\alpha\leq\varepsilon$, let u_α and u_ε denote the solution of (2.6) corresponding to the initial data $u_{0\alpha}$ and $u_{0\varepsilon}$ respectively. There exists a constant $c=c(T,\|u_{0\varepsilon}\|_s)$ such that for ε sufficiently small,

$$\sup_{0\leq t\leq T}\|u^\alpha(\bullet,t)-u^\varepsilon(\bullet,t)\|_s \leq c\varepsilon^{\gamma(s)}+c\left(\|u_0-u_{0\varepsilon}\|_s+\|u_0-u_{0\alpha}\|_s\right), \tag{5.35}$$

where

$$\gamma(s)=\frac{\nu s}{6(\nu+1)}, \tag{5.36}$$

and ν is any nonnegative number such that $\nu<s-3/2$.

Proof of lemma 5.3.7. Set $u=u^\varepsilon$, $v=u^\alpha$, and $w=u-v$, so that w satisfies the initial-value problem

$$w_t+H(w_{xx})=-\left(uw-\tfrac{1}{2}w^2\right)_x, \tag{5.37}$$

$$w(\bullet,0)=u_{0\varepsilon}-u_{0\alpha}. \tag{5.38}$$

Of course u and v both obey the inequality (5.14). We first claim that for $0<\alpha\leq\varepsilon$,

$$\sup_{0\leq t\leq T}\|w(\bullet,t)\|_0\leq c(T,\|u_0\|_s)\,\varepsilon^{s/6}. \tag{5.39}$$

To prove (5.39), take the $L^2(\mathbb{R})$ scalar product of (5.37) with w to obtain, after suitable integrations by parts,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_0^2=-\frac{1}{2}\int_{-\infty}^{\infty}u_xw^2\,\mathrm{d}x. \tag{5.40}$$

Because of (5.14),

$$\left|\int_{-\infty}^{\infty}u_xw^2\,\mathrm{d}x\right|\leq c(T,\|u_0\|_s)\|w\|_0^2. \tag{5.41}$$

Integrating (5.40), over the temporal interval [0, t], and using (5.41) leads to the inequality

$$\|w(\bullet, t)\|_0^2 \le \|w(\bullet, 0)\|_0^2 + c(T, \|u_0\|_s) \int_0^t \|w(\bullet, s)\|_0^2 \, ds, \tag{5.42}$$

and Gronwall's lemma immediately implies that

$$\|w(\bullet, t)\|_0^2 \le \|w(\bullet, 0)\|_0^2 e^{ct}. \tag{5.43}$$

On the other hand, by the triangle inequality,

$$\|w(\bullet, 0)\|_0 = \|u_{0\varepsilon} - u_{0\alpha}\|_0 \le \|u_0 - u_{0\varepsilon}\|_0 + \|u_0 - u_{0\alpha}\|_0,$$

so that

$$\|w(\bullet, 0)\|_0 \le c(\|u_0\|_s)(\varepsilon^{s/6} + \alpha^{s/6}) \tag{5.44}$$

by (5.32). Since $\alpha \le \varepsilon$, we reach the inequality

$$\|w(\bullet, t)\|_0^2 \le c\varepsilon^{s/3} e^{cT} \tag{5.45}$$

for $0 \le t \le T$, which completes the proof of (5.39). Note that the constant c in (5.39) does not depend on ε for ε sufficiently small (e.g. $\varepsilon \le 1$).

The proof of lemma 5.3.7 continues by establishing estimates like (5.39) for higher Sobolev norms. To this end, apply the operator D^s to (5.37) and take the $L^2(\mathbb{R})$ scalar product with $D^s w$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|D^s w\|_0^2 = -(D^s(uw)_x, D^s w) + (D^s(ww_x), D^s w). \tag{5.46}$$

Since $w = u - v$, we may write

$$|-(D^s(uw_x), D^s w) + (D^s(ww_x), D^s_w)| = |(D^s(vw_x), D^s w)|.$$

As $s > \frac{3}{2}$, lemma 5.3.5 may be applied, with $\gamma = s - 1$ and taken in conjunction with (5.14) and (5.31) to yield the inequality

$$|(D^s(vw_x), D^s w)| \le c\|v\|_s \|w\|_s^2 \le c\|v\|_{L^\infty(0,T;\, H^s(\mathbb{R}))} \|w\|_s^2 \le c(T; \|u_0\|_s)\|w\|_s^2. \tag{5.47}$$

It remains to estimate the term $(D^s(u_x w), D^s w)$. For this purpose, lemma 5.3.5 is reformulated as follows.

Lemma 5.3.8. For any real numbers s, γ_1, γ_2 such that $s > \frac{3}{2}$, $s - 1 \ge \gamma_i > \frac{1}{2}$, $i = 1, 2$, there exists a constant $c = c(s, \gamma_1, \gamma_2) > 0$ such that

$$\|D^s(uv)\|_0 \le c(\|u\|_s \|v\|_{\gamma_1} + \|u\|_{\gamma_2 + 1} \|v\|_{s-1}) + \|uD^s v\|_0 \tag{5.48}$$

for all $u, v \in H^s(\mathbb{R})$.

Choosing γ_1 and γ_2 in the range stipulated in lemma 5.3.8, formula (5.48) gives immediately the bound

$$|(D^s(u_x w), D^s w)| \le c(\|u\|_{s+1}\|w\|_{\gamma_1} + \|u\|_{\gamma_2+2}\|w\|_{s-1})\|w\|_s + \|u_x D^s w\|_0 \|w\|_s \tag{5.49}$$

on the remaining term. Because $s > \frac{3}{2}$, it transpires that

$$\|u_x D^s w\|_0 \le \|u_x\|_{L^\infty} \|w\|_s \le c\|u\|_s \|w\|_s \le c(T; \|u_0\|_s)\|w\|_s. \tag{5.50}$$

It remains to estimate $\|u\|_{s+1} \|w\|_{\gamma_1}$ and $\|u\|_{\gamma_2+2}\|w\|_{s-1}$ for $\frac{1}{2} < \gamma_i \le s - 1$. First consider the combination $\|u\|_{s+1} \|w\|_\gamma$, where $\frac{1}{2} < \gamma \le s - 1$. Because of (5.14) and (5.31),

$$\|u(\bullet, t)\|_{s+1} \le \|u\|_{L^\infty(0,T;\, H^{s+1}(\mathbb{R}))} \le c\|u_{0\varepsilon}\|_{s+1} \le c(T; \|u_0\|_s)\varepsilon^{-1/6}. \tag{5.51}$$

By interpolation, we may bound $\|w\|_\gamma$ in terms of $\|w\|_s$ and $\|w\|_0$ as follows:

$$\|w\|_\gamma \le c\|w\|_s^{\gamma/s}\,\|w\|_0^{1-\gamma/s}. \tag{5.52}$$

From (5.52), (5.51), and (5.39) it follows that

$$\|u\|_{s+1}\|w\|_\gamma\,\|w\|_s \le c(T;\|u_0\|_s)\,\varepsilon^{\beta_0}\|w\|_s^{1+\gamma/s}, \tag{5.53}$$

where $\beta_0 = \frac{1}{6}(s-\gamma-1)$. Using Young's inequality, it is adduced that

$$\|u\|_{s+1}\|w\|_\gamma\,\|w\|_s \le c(T;\|u_0\|_s)\,\varepsilon^{[2s/(s-\gamma)]\beta_0} + \|w\|_s^2. \tag{5.54}$$

Write $\gamma = s-1-\nu$, so that $0 \le \nu < s-3/2$. Then

$$\frac{2s}{s-\gamma}\beta_0 = \frac{\nu s}{3(\nu+1)} = \lambda_0, \tag{5.55}$$

say, and so (5.54) leads to

$$\|u\|_{s+1}\|w\|_\gamma\,\|w\|_s \le c\varepsilon^{\lambda_0} + \|w\|_s^2. \tag{5.56}$$

Consider now the other combination appearing on the right-hand side of (5.49), namely $\|u\|_{\gamma+2}\|w\|_{s-1}$, where $\frac{1}{2} < \gamma \le s-1$. First, we have

$$\begin{aligned}\|u(\bullet,t)\|_{\gamma+2} &\le \|u\|_{L^\infty(0,T;\,H^{\gamma+2}(\mathbb{R}))}\\ &\le c\|u_{0\varepsilon}\|_{\gamma+2}\\ &\le c(T;\|u_0\|_s)\,\varepsilon^{(s-2-\gamma)/6}.\end{aligned} \tag{5.57}$$

Using successively a standard interpolation inequality and (5.39) gives the inequality

$$\begin{aligned}\|w\|_{s-1} &\le c\|w\|_s^{1-1/s}\|w\|_0^{1/s}\\ &\le c(T;\|u_0\|_s)\,\varepsilon^{1/6}\|w\|_s^{1-1/s},\end{aligned} \tag{5.58}$$

and combining this with (5.57) and Young's inequality yields

$$\begin{aligned}&\|u\|_{\gamma+2}\|w\|_{s-1}\,\|w\|_s\\ &\quad\le c(T;\|u_0\|_s)\,\varepsilon^{[s-(1+\gamma)]s/3} + \|w\|_s^2.\end{aligned}$$

Writing γ as in (5.55), we obtain

$$\|u\|_{\gamma+2}\|w\|_{s-1}\,\|w\|_s \le c(T;\|u_0\|_s)\,\varepsilon^{\nu s/3} + \|w\|_s^2. \tag{5.59}$$

Finally, from (5.59), (5.56), (5.49), (5.47), and (5.46), we conclude that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^s w\|_0^2 \le c\|w\|_s^2 + c(\varepsilon^{\lambda_0} + \varepsilon^{\nu s/3}),$$

where $\lambda_0 = \nu s/3(\nu+1) = 2\gamma(s)$ (see (5.36)) and $0 \le \nu < s-3/2$. Gronwall's lemma then leads to the inequality

$$\|w(t)\|_s^2 \le c(T;\|u_0\|_s)\left[\varepsilon^{\lambda_0} + \|w(\bullet,0)\|_s^2\right]. \tag{5.60}$$

This concludes the proof of lemma 5.3.7. □

It is now easy to deduce that $\{u^\varepsilon\}_{\varepsilon>0}$ is Cauchy in $C(0,T;\,H^s(\mathbb{R}))$. This is obviously equivalent to showing that

for any $\eta > 0$, there exists $\varepsilon_0 > 0$ such that

for all α and ε with $0 < \alpha \le \varepsilon \le \varepsilon_0$

$$\|u^\varepsilon - u^\alpha\|_{C(0,T;\,H^s(\mathbb{R}))} \le \eta. \tag{5.61}$$

But (5.61) is a direct consequence of (5.30) and the conclusion of lemma 5.3.7.

On the other hand, to show that $\{u^\varepsilon\}_{\varepsilon>0}$ is a Cauchy sequence in $C^k(0,T;\,H^{s-2k}(\mathbb{R}))$, for $k \in \mathbb{N}^*$ with $s-2k \ge -1$, it suffices to show that $\{\partial_t^k u^\varepsilon\}_{\varepsilon>0}$ is Cauchy in $C(0,T;\,H^{s-2k}(\mathbb{R}))$. To this purpose, notice that

$$w_t = -(uw)_x + ww_x - H(w_{xx})$$

and hence that

$$\|w_t\|_{s-2} \le c\left(\|u\|_{s-1}\,\|w\|_{s-1} + \|w\|_{s-1}^2 + \|w\|_s\right). \tag{5.62}$$

The fact that $\{u_t^\varepsilon\}_{\varepsilon>0}$ is Cauchy in $C(0,T;\,H^{s-2}(\mathbb{R}))$ is now a direct consequence of (5.62), (5.14) and lemma 5.3.7.

By iterating this procedure, one shows that $\{\partial_t^k u^\varepsilon\}_{\varepsilon>0}$ is a Cauchy sequence in $C(0,T; H^{s-2k}(\mathbb{R}))$ for $k \in \mathbb{N}$ such that $s-2k \geq -1$, so concluding part (ii) of the proof of theorem 5.3.1.

(iii) Continuous dependence with respect to the initial data.

We begin by proving the assertion that $\mathfrak{A}_T$ is continuous from $H^s(\mathbb{R})$ into $C(0,T; H^s(\mathbb{R}))$ if $s > \frac{3}{2}$. This amounts to proving that

if $\{u_0^n\}_{n\in\mathbb{N}}$ is a sequence in $H^s(\mathbb{R})$ such that

$u_0^n \to u_0$ in $H^s(\mathbb{R})$, then

$$\sup_{0\leq t\leq T} \|\mathfrak{A}_t(u_0^n) - \mathfrak{A}_t(u_0)\|_s \to 0 \text{ as } n\to\infty. \tag{5.63}$$

With an obvious notation, the triangle inequality assures that

$$\|u_n - u\|_s \leq \|u^n - u^{n\varepsilon}\|_s + \|u^{n\varepsilon} - u^\varepsilon\|_s + \|u^\varepsilon - u\|_s. \tag{5.64}$$

Letting α tend to 0 in the estimate derived in lemma 5.3.7 gives, for $s > \frac{3}{2}$,

$$\sup_{0\leq t\leq T} \|u - u^\varepsilon\|_s \leq c\varepsilon^{\gamma(s)} + c\|u_0 - u_{0\varepsilon}\|_s, \tag{5.65}$$

where $\gamma(s)$ is defined in (5.36) and the constants are independent of ε sufficiently small. Therefore, as $\varepsilon \downarrow 0$,

$$\sup_{0\leq t\leq T} \|u - u^\varepsilon\|_s \to 0,$$

$$\sup_{0\leq t\leq T} \|u^n - u^{n\varepsilon}\|_s \to 0, \tag{5.66}$$

and the last convergence is uniform in n because of proposition 5.3.6. Hence, for an arbitrary $\omega > 0$, there exists an $\varepsilon_0 > 0$ such that for all ε with $0 < \varepsilon \leq \varepsilon_0$, one has

$$\sup_{0\leq t\leq T} \|u^\varepsilon - u\|_s \leq \tfrac{1}{3}\omega,$$

$$\sup_{0\leq t\leq T} \|u^n - u^{n\varepsilon}\|_s \leq \tfrac{1}{3}\omega, \tag{5.67}$$

for all n. In order to prove (5.63) it is therefore sufficient to show that for any fixed ε with $0<\varepsilon\leq\varepsilon_0$ the following holds:

$$\sup_{0\leq t\leq T} \|u^{n\varepsilon} - u^\varepsilon\|_s \to 0, \quad \text{as } n\to\infty. \tag{5.68}$$

Set $w = u^{n\varepsilon} - u^\varepsilon$, so that w satisfies the initial-value problem

$$w_t + H(w_{xx}) = -(u^{n\varepsilon}w)_x + ww_x, \tag{5.69}$$

$$w(\bullet,0) = u_{0\varepsilon}^n - u_{0\varepsilon}. \tag{5.70}$$

As before, for $s \geq 0$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^2 w\|_0^2 = -(D^s(u^{n\varepsilon}w)_x, D^s w) + (D^s(ww_x), D^s w). \tag{5.71}$$

The terms on the right-hand side of (5.71) are estimated in a fashion that is by now quite familiar (see the proof of lemma 5.3.7), namely

$$|(D^s(u^{n\varepsilon}w)_x, D^s w)| \leq c(\|w\|_s^2 + \varepsilon^{\lambda_0}),$$

$$|(D^s(ww_x), D^s w)| \leq c\|w\|_s^2,$$

where the constant $c = c(T, \|u_{0\varepsilon}^n\|_s, \|u_{0\varepsilon}\|_s)$. Integrate (5.71) with respect to t and use Gronwall's lemma and the last bounds to get

$$\sup_{0\leq t\leq T} \|w(\bullet,t)\|_s^2 \leq c'\|w(\bullet,0)\|_s^2\,\mathrm{e}^{ct}, \tag{5.72}$$

where c' depends upon the fixed value of ε. Since $\|w(\bullet,0)\|_s \leq c\|u_0^n - u_0\|_s \to 0$ as $n\to+\infty$, we obtain (5.68). Thus there exists $N_0 = N_0(\varepsilon) \geq 0$ such that for any $n \geq N_0$,

$$\sup_{0\leq t\leq T} \|u^{n\varepsilon} - u^\varepsilon\|_s \leq \tfrac{1}{3}\omega. \tag{5.73}$$

The assertion (5.63) is a consequence of (5.64), (5.67), (5.73).

Now, from eq. (5.69), it is adduced, for $s \geq 2$, that

$$\|w_t\|_{s-2} \leq c(\|u\|_{s-1}\|w\|_s^2 + \|w\|_s^2),$$

which shows that $\|u_t^{n\epsilon} - u_t^\epsilon\|_s \to 0$ uniformly on $[0, T]$, as $n \to \infty$, and this proves our assertion for $k = 1$. By iteration we obtain the assertion for any $k \in \mathbb{N}$, such that $s - 2k \geq -1$.

The proof of theorem 5.3.1 is now complete. □

So far we have not established that the solution u obtained in theorem 5.3.1 is uniformly bounded for $t \in \mathbb{R}_+$. Such a result is the next order of business, and will be obtained in the spaces $H^s(\mathbb{R})$ for the special cases where $s = n/2$ with n an integer larger than 3.

Theorem 5.3.9. Let $u_0 \in H^{n/2}(\mathbb{R})$, $n \in \mathbb{N} \setminus \{0, 1, 2, 3\}$. Then the corresponding solution u of the Benjamin–Ono equation (2.6) with initial data u_0 satisfies $u \in C_b^k(\mathbb{R}_+, H^{n/2-2k}(\mathbb{R}))$, $k \in \mathbb{N}$, $n/2 - 2k \geq -1$. (Here $C_b^k(\mathbb{R}_+; X)$ stands for the space of functions u: $[0, \infty] \to X$ whose t-derivatives up to order k exist and are continuous and bounded with values in X).

Proof. Let $\{u_{0\epsilon}\}_{\epsilon > 0}$ be a sequence in $H^\infty(\mathbb{R})$ such that $u_{0\epsilon} \to u_0$ in $H^{n/2}(\mathbb{R})$ and consider the associated sequence of regularized problems

$$u_t^\epsilon + u^\epsilon u_x^\epsilon + H(u_{xx}^\epsilon) = 0, \tag{5.74}$$

$$u^\epsilon(\bullet, 0) = u_{0\epsilon}. \tag{5.75}$$

It results from corollary 5.3.2 that there exists a unique solution u^ϵ of (5.74)–(5.75) satisfying $u^\epsilon \in C^\infty(\mathbb{R}_+; H^\infty(\mathbb{R}))$. To obtain estimates in the norm $L^\infty(\mathbb{R}_+; H^{n/2}(\mathbb{R}))$ that are independent of ϵ, we shall make use of the invariants of the Benjamin–Ono equation. By proposition 3.3.1,

$$\frac{\mathrm{d}}{\mathrm{d}t} I_n(u^\epsilon) = 0. \tag{5.76}$$

This relation and lemma 3.3.2 gives

$$\|u^\epsilon(t)\|_{n/2}^2 \leq c\left(\|u_{0\epsilon}\|_{n/2}^{2+2n} + \|u_{0\epsilon}\|_{n/2}^2\right) \tag{5.77}$$

and it therefore transpires that

$$\|u^\epsilon\|_{L^\infty(\mathbb{R}_+;\, H^{n/2}(\mathbb{R}))} \leq c\left(\|u_{0\epsilon}\|_{n/2}^{n+1} + \|u_{0\epsilon}\|_{n/2}\right),$$

which proves that $u \in L^\infty(\mathbb{R}_+; H^{n/2}(\mathbb{R}))$. By using eq. (2.6) one easily proves recursively that in fact $u \in C_b^k(\mathbb{R}_+; H^{(n/2)-2k}(\mathbb{R}))$, for $k \in \mathbb{N}$ with $n/2 - 2k \geq -1$. □

Remark. Uniform bounds in $L^\infty(\mathbb{R}_+; H^s(\mathbb{R}))$ for arbitrary $s \geq 2$ can probably be obtained for solutions emanating from initial data in $H^s(\mathbb{R})$ by using the nonlinear interpolation techniques of Bona and Scott [15].

6. The Cauchy problem for the intermediate long-wave equation

Since the invariants of the intermediate long-wave equation (4.1) have properties similar to those of the Benjamin–Ono equation (cf. section 4), one obtains for the Cauchy problem exactly the same results. That is, the corresponding analog of theorem 5.2.1, theorem 5.3.1, corollary 5.3.2, theorem 5.3.3, and theorem 5.3.9 are still valid for eq. (4.1). The proofs parallel in detail those given for the Benjamin–Ono equation, and consequently are omitted. Here is a precise statement of the results to which allusion was just made.

Theorem 6.1. Let $u_0 \in H^s(\mathbb{R})$ be given initial data for the intermediate long-wave equation (4.1). If $u_0 \in H^{n/2}(\mathbb{R})$, for $n = 2$ or $n = 3$, then there exists a solution u of (4.1) with initial value u_0 such that $u \in L^\infty(\mathbb{R}_+; H^{n/2}(\mathbb{R})$, If $u_0 \in H^s(\mathbb{R})$ for $s > \frac{3}{2}$, then there exists a unique solution u of (4.1) with initial data u_0 such that, for each $T > 0$, $u \in C^k(0, T; H^{s-2k}(\mathbb{R}))$ for all k such that $s - 2k > -\frac{3}{2}$. Moreover, the mapping that associates to u_0 the unique solution u of (4.1) with initial value u_0 is continuous from $H^s(\mathbb{R})$ into $C^k(0, T; H^{s-2k}(\mathbb{R}))$, for all $T > 0$ and all k such that $s - 2k > -\frac{3}{2}$. If $s = n/2$, where n is an integer larger than 3, then $u \in C_b^k(\mathbb{R}_+; H^{s-2k}(\mathbb{R}))$ for all k such that $s - 2k > -\frac{3}{2}$.

7. The Cauchy problem for the Smith equation

In this section, attention is given to the Cauchy problem for Smith's equation

$$u_t + uu_x - L_s(u_x) = 0, \qquad u(\bullet, 0) = u_0 \tag{7.1}$$

where

$$\widehat{L_s u}(\xi) = p_s(\xi)\, \hat{u}(\xi),$$
$$p_s(\xi) = 2\pi\left(\sqrt{\xi^2+1} - 1\right). \tag{7.2}$$

Since only three invariants are known for Smith's equation, the initial-value problem cannot be treated in the same way that proved to be effective for the Benjamin–Ono equation and the intermediate long-wave equation. Instead Smith's equation will be viewed as a perturbation of the Benjamin–Ono equation. Specifically, write (7.1) as

$$u_t + uu_x + H(u_{xx}) + K(u_x) = 0, \tag{7.3}$$

where K is defined by

$$\widehat{Ku}(\xi) = q_s(\xi)\, \hat{u}(\xi), \tag{7.4}$$

with

$$q_s(\xi) = 2\pi|\xi| - p_s(\xi). \tag{7.5}$$

Obviously, $|q_s(\xi)|$ is bounded since it behaves as $2\pi + \pi/|\xi|$ as $|\xi| \to +\infty$, and thus K is a pseudo-differential operator of order 0 and is therefore a bounded operator on all the Sobolev spaces $H^s(\mathbb{R})$.

Our principal result is the counterpoint of theorem 5.2.1.

Theorem 7.1. Let $T > 0$ and $u_0 \in H^{n/2}(\mathbb{R})$, where $n = 2$ or $n = 3$. Then there exists a solution u of (7.1) such that for any $T > 0$, $u \in L^\infty(0, T; H^{n/2}(\mathbb{R}))$.

Proof. Consider as before the parabolic regularization

$$u_t^\varepsilon + u^\varepsilon u_x^\varepsilon + H(u_{xx}^\varepsilon) + K(u_x^\varepsilon)$$
$$+ \varepsilon(u_{xxxx}^\varepsilon + u^\varepsilon) = 0, \tag{7.6}$$
$$u^\varepsilon(x, 0) = u_{0\varepsilon}, \tag{7.7}$$

where $u_{0\varepsilon} \in H^\infty(\mathbb{R})$, $u_{0\varepsilon} \to u$ in $H^{n/2}(\mathbb{R})$.

It is sufficient to prove that u^ε remains in a bounded subset of $L^\infty(0, T; H^{n/2}(\mathbb{R}))$, independently of ε, as $\varepsilon \to 0$. This will be the aim of the subsequent lemmas, which distinguish the case $n = 2$ from the case $n = 3$.

Lemma 7.2. ($n = 2$). There exists $c > 0$ which is independent of ε such that

$$\|u^\varepsilon\|_{L^\infty(0,T;\, H^1(\mathbb{R}))} \le c, \tag{7.8}$$

$$\sqrt{\varepsilon}\, \|u^\varepsilon\|_{L^2(0,T;\, H^3(\mathbb{R}))} \le c. \tag{7.9}$$

Lemma 7.3. ($n = 3$). There exists $c > 0$ which is independent of ε such that

$$\|u^\varepsilon\|_{L^\infty(0,T;\, H^{3/2}(\mathbb{R}))} \le c, \tag{7.10}$$

$$\sqrt{\varepsilon}\, \|u^\varepsilon\|_{L^2(0,T;\, H^{7/2}(\mathbb{R}))} \le c. \tag{7.11}$$

Proof of lemma 7.2. Take the $L^2(\mathbb{R})$ scalar product of (7.6) with

$$M_2(u^\varepsilon) = (u^\varepsilon)^3 + 3[u^\varepsilon H(u_x^\varepsilon) + H(u^\varepsilon u_x^\varepsilon)] - 4u_{xx}^\varepsilon$$

to come to the relation

$$\frac{\mathrm{d}}{\mathrm{d}t} I_2(u^\varepsilon) + \varepsilon \int_{-\infty}^{\infty} (u_{xxxx}^\varepsilon + u^\varepsilon) M_2(u^\varepsilon)\, \mathrm{d}x$$
$$= -\int_{-\infty}^{\infty} K(u_x^\varepsilon)\, M_2(u^\varepsilon)\, \mathrm{d}x. \tag{7.12}$$

From lemma 3.3.2 and lemma 3.3.3 it suffices to estimate the right-hand side of (7.12). This will be accomplished in the following lemma.

Lemma 7.4. For any η in the range $(0, \frac{1}{2})$,

$$\left|\int_{-\infty}^{\infty} K(u_x^\varepsilon)\, M_2(u^\varepsilon)\,\mathrm{d}x\right| \le c\left(\frac{1}{\sqrt{\eta}}\|u^\varepsilon\|_1^{2+2\eta}\right).$$

Proof of lemma 7.4. There are four terms to estimate. First of all, $\int_{-\infty}^{\infty} K(u_x^\varepsilon)\, u_{xx}^\varepsilon\,\mathrm{d}x = 0$ since the symbol $q_s(\xi)$ is real and so K is self-adjoint. Also, we see that

$$\left|\int_{-\infty}^{\infty} (u^\varepsilon)^3 K(u_x^\varepsilon)\,\mathrm{d}x\right| \le \|u^\varepsilon\|_{L^\infty}^2 \|(u_x^\varepsilon)\|_0\, \|u^\varepsilon\|_0. \tag{7.13}$$

But $|u^\varepsilon|_{L^\infty(0,T;\, H^{1/2}(\mathbb{R}))}$ is bounded independently of ε by the result of theorem 5.1.1, and so therefore is $\|u^\varepsilon\|_{L^\infty(0,T;\, L^2(\mathbb{R}))}$ (here $T > 0$ is arbitrary). Since K is a bounded operator on $L^2(\mathbb{R})$, we obtain

$$\left|\int_{-\infty}^{\infty} (u^\varepsilon)^3 K(u_x^\varepsilon)\,\mathrm{d}x\right| \le c\|u_x^\varepsilon\|^2. \tag{7.14}$$

Let us now estimate $\int u^\varepsilon H(u_x^\varepsilon)\, K(u_x^\varepsilon)\,\mathrm{d}x$. The Hölder inequality yields

$$\begin{aligned}\left|\int_{-\infty}^{\infty} u^\varepsilon H(u_x^\varepsilon)\, K(u_x^\varepsilon)\,\mathrm{d}x\right| &\le \|u^\varepsilon\|_{L^\infty}\|H(u_x^\varepsilon)\|_0\, \|K(u_x^\varepsilon)\|_0 \\ &\le c\|u^\varepsilon\|_{L^\infty}\, \|u_x^\varepsilon\|_0^2 \end{aligned}\tag{7.15}$$

since H and K are bounded operators on $L^2(\mathbb{R})$. But for any $\eta > 0$ and $u \in H^{1/2+\eta}(\mathbb{R})$,

$$\|u\|_{L^\infty} \le \frac{\sqrt{2}}{\sqrt{\eta}}\|u\|_{1/2}^{1-2\eta}\|u\|_1^{2\eta}, \tag{7.16}$$

where the inequality

$$\|u\|_{L^\infty} \le \sqrt{\frac{2}{\eta}}\, \|u\|_{1/2+\eta}$$

valid for $0 < \eta < \frac{1}{2}$ has been employed again. Since $\{u^\varepsilon\}_{\varepsilon>0}$ is bounded in $L^\infty(0, T;\, H^{1/2}(\mathbb{R}))$ independently of ε, we have successively

$$\left|\int_{-\infty}^{\infty} (u^\varepsilon)^3 K(u_x^\varepsilon)\, H(u_x^\varepsilon)\,\mathrm{d}x\right| \le \frac{c}{\sqrt{\eta}}\|u^\varepsilon\|_1^{2+2\eta} \tag{7.17}$$

and so

$$\begin{aligned}&\left|\int_{-\infty}^{\infty} H(u^\varepsilon u_x^\varepsilon)\, K(u_x^\varepsilon)\,\mathrm{d}x\right| \\ &\quad\le c\|u^\varepsilon\|_{L^\infty}\, \|u_x^\varepsilon\|_0^2 \le \frac{c}{\sqrt{\eta}}\|u^\varepsilon\|_1^{2+2\eta}.\end{aligned} \tag{7.18}$$

Thus the proof of lemma 7.4 is achieved. □

Finally, (7.12) and lemmas 3.3.2, 3.3.3, and 7.4 lead to the inequality

$$\begin{aligned}&\|u^\varepsilon(t)\|_1^2 + \varepsilon\int_0^t \|u^\varepsilon(s)\|_3^2\,\mathrm{d}s \\ &\quad\le [a_2(\|u_{0\varepsilon}\|_1)]^2 \\ &\qquad + c\int_0^t\left(\frac{1}{\sqrt{\eta}}\|u^\varepsilon(s)\|_1^{2+2\eta} + \|u^\varepsilon(s)\|_1^2\right)\mathrm{d}s,\end{aligned} \tag{7.19}$$

which is valid at least for $0 < \eta < \frac{1}{2}$. Hence, it follows that

$$\|u^\varepsilon(t)\|_1^2 + \varepsilon\int_0^t \|u^\varepsilon(s)\|_3^2\,\mathrm{d}s \le y(t), \tag{7.20}$$

where y is the solution of the differential equation

$$\begin{aligned}y'(t) &= c\left(\frac{1}{\sqrt{\eta}}y(t)^{1+\eta} + y(t)\right), \\ y(0) &= c_* = [a_2(\|u_{0\varepsilon}\|_1)]^2,\end{aligned} \tag{7.21}$$

on its maximal interval of existence $[0, T(\eta)]$. Eq. (7.21) can be integrated and one finds

$$y(t) = \left[\left(c_*^{-\eta} + \frac{1}{\sqrt{\eta}}\right)\mathrm{e}^{-c\eta t} - \frac{1}{\sqrt{\eta}}\right]^{-1/\eta},$$

and therefore that

$$T(\eta)=\frac{1}{c}\left(\frac{1}{\eta}\log\left(1+c_*^{-\eta}\sqrt{\eta}\right)\right),$$

so that $\lim_{\eta\to 0}T(\eta)=+\infty$. Then, for any fixed $T>0$, there exists $c>0$ such that (7.8) and (7.9) hold, which proves lemma 7.2. □

Proof of lemma 7.3. Taking the scalar product of (7.6) with $M_3(u^\varepsilon)$ (cf. section 3.1) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}I_3(u^\varepsilon)+\varepsilon\int_{-\infty}^{\infty}(u^\varepsilon_{xxxx}+u^\varepsilon)M_3(u^\varepsilon)\,\mathrm{d}x=-\int_{-\infty}^{\infty}K(u^\varepsilon_x)\,M_3(u^\varepsilon)\,\mathrm{d}x. \qquad (7.22)$$

By lemma 3.3.3, it suffices to estimate the right-hand side of (7.22).

Lemma 7.5. There exists $c>0$, independent of ε, such that

$$\left|\int_{-\infty}^{\infty}K(u^\varepsilon_x)\,M_3\,\mathrm{d}x\right|\le c\|u^\varepsilon\|^2_{3/2}. \qquad (7.23)$$

Proof of lemma 7.5. Let u stand temporarily for u^ε. Using the formula for M_3 given in section 3.1, we compute as follows:

$$\begin{aligned}\int_{-\infty}^{\infty}K(u_x)\,M_3(u)\,\mathrm{d}x&=-\int_{-\infty}^{\infty}u^4K(u_x)\,\mathrm{d}x\\&+\int_{-\infty}^{\infty}\big[-4u^2H(u_x)\,K(u_x)-4H(u^2u_x)\,K(u_x)\\&-4uH(uu_x)\,K(u_x)\big]\,\mathrm{d}x\\&+\int_{-\infty}^{\infty}\big[-2H(u_x)^2K(u_x)\\&-4H(u_xH(u_x))\,K(u_x)\\&-4H(uH(u_{xx}))\,K(u_x)\\&+6u_x^2K(u_x)+12uu_{xx}K(u_x)\big]\,\mathrm{d}x\\&+8\int_{-\infty}^{\infty}H(u_{xxx})\,K(u_x)\,\mathrm{d}x.\end{aligned} \qquad (7.24)$$

The terms homogeneous of degree 5 and 4 in u on the right-hand side of (7.24) are easily bounded above, using in particular (7.8), by $c\|u^\varepsilon\|^2_{3/2}$ where c is a suitably large constant.

Consider now the terms that are homogeneous of degree 3. First, by using the embedding of $H^{1/4}(\mathbb{R})$ into $L^4(\mathbb{R})$, the bound implicit in (7.8), and the fact that K has order 0, it is deduced that

$$\begin{aligned}&\left|\int_{-\infty}^{\infty}u_x^2K(u_x)\,\mathrm{d}x\right|\\&\le\|u_x\|^2_{L^4}\|K(u_x)\|_0\le c\|u_x\|^2_{L^4}\|u_x\|_0\\&\le c\|u_x\|^2_{L^4}\le c\|u\|^2_{5/4}\le c\|u\|^2_{3/2}.\end{aligned} \qquad (7.25)$$

As H is also of order 0, a similar calculation suffices to bound the two integrals $\int_{-\infty}^{\infty}(Hu_x)^2Ku_x\,\mathrm{d}x$ and $\int_{-\infty}^{\infty}H(u_xHu_x)\,Ku_x\,\mathrm{d}x$.

By Parseval's identity,

$$\begin{aligned}&\left|\int_{-\infty}^{\infty}uu_{xx}Ku_x\,\mathrm{d}x\right|=\left|\int_{-\infty}^{\infty}\hat{u}_{xx}(\xi)\widehat{uKu_x}(\xi)\,\mathrm{d}\xi\right|\\&\le c\int_{-\infty}^{\infty}(1+|\xi|)^{3/2}|\hat{u}(\xi)|\\&\quad\times(1+|\xi|)^{1/2}|\widehat{uKu_x}(\xi)|\,\mathrm{d}\xi\\&\le c\|u\|_{3/2}\|uKu_x\|_{1/2}.\end{aligned} \qquad (7.26)$$

To estimate the term $\|uKu_x\|_{/2}$, use is made of the following lemma due to Grisvard [33].

Lemma 7.6. Let $a,b,c\in\mathbb{R}$ be such that $a\ge c$, $b\ge c$, $a+b\ge 0$, and $a+b-c>n/2$. Then the correspondence $(f,g)\to fg$ is a continuous bilinear form from $H^a(\mathbb{R}^n)\times H^b(\mathbb{R}^n)$ into $H^c(\mathbb{R}^n)$.

The lemma is applied with $a=1$, $b=1/2$, and $c=1/2$ to obtain that

$$\left|\int_{-\infty}^{\infty}uu_{xx}Ku_x\,\mathrm{d}x\right|\le c\|u\|^2_{3/2}\|u\|_1\le c\|u\|^2_{3/2}, \qquad (7.27)$$

again using (7.8). The term $\int_{-\infty}^{\infty}H(uHu_{xx})\,Ku_x\,\mathrm{d}x$ is handled in the same way.

To finish the proof of lemma 7.4, notice finally that

$$\int_{-\infty}^{\infty} H(u_{xxx})\, K u_x \,\mathrm{d}x = 0,$$

since H is skew-adjoint and K is self-adjoint. □

Lemma 7.3 results immediately from lemma 7.5, lemmas 3.3.2, 3.3.3 and (7.22).

The proof of theorem 7.1 is now complete. □

We now state a result analogous to theorem 5.3.1.

Theorem 7.7. Let $u_0 \in H^s(\mathbb{R})$ where $s > \frac{3}{2}$. Then there exists a unique solution u of (7.1) such that $u \in C^k(\mathbb{R}_+; H^{s-2k}(\mathbb{R}))$, for all $k \in \mathbb{N}$ such that $s - 2k \geq -1$. Moreover, for any $T > 0$, the mapping $\mathfrak{A}_T$: $u_0 \to u$ is continuous from $H^s(\mathbb{R})$ into $C^k(0, T; H^{s-2k}(\mathbb{R}))$, for the same values of k.

Proof. The proof is nearly identical to that of theorem 5.3.1. In fact the term $K(u_x)$ in (7.1) does not give any problem since the symbol of K is real and of order 0, so that

$$\int_{-\infty}^{\infty} D^s(K(u_x))\, D^s u \,\mathrm{d}x = 0,$$

for any $s \geq 0$, and K is a bounded operator on every space $H^s(\mathbb{R})$, for $s \in \mathbb{R}$. □

Corollary 7.8. Let $u_0 \in H^\infty(\mathbb{R})$. Then there exists a unique solution of (7.1) such that $u \in C^\infty(\mathbb{R}_+; H^\infty(\mathbb{R}))$.

8. The Korteweg–de Vries and Benjamin–Ono equations as limits of the intermediate long-wave equation

In this section attention is given to two potentially singular limits of the intermediate long-wave equation, namely those in which the positive parameter δ tends to zero or infinity. Recall that δ characterizes the relative depths of two homogeneous fluid layers, the deviation of the interface between which is governed approximately by the intermediate long-wave equation. If we write the equation in the form

$$u_t^\delta + u^\delta u_x^\delta + \frac{1}{\delta} u_x^\delta + T(u_{xx}^\delta) = 0 \tag{8.1}$$

as in (2.7), to emphasize the dependence of $u = u^\delta$ on δ, then there is a wealth of formal results (see refs. [7, 19, 20, 34, 35]) pointing to the conclusions that (8.1) reduces to the Korteweg–de Vries equation as $\delta \to 0$ and to the Benjamin–Ono equation as $\delta \to \infty$. Our aim here is to establish a rigorous theory of these convergences. Previous writing along these lines dealt with the convergence of special, travelling-wave solutions (the solitary waves) or else examined the formal convergence of the associated inverse-scattering transforms (see refs. [7, 19, 35–40]). In particular, even the formal analysis regarding the limit $\delta \to 0$ is misleading as is pointed out later in this section.

8.1. The Benjamin–Ono limit

We deal first with the limit $\delta \to \infty$. For the purposes at hand, write eq. (8.1) as a perturbation of the Benjamin–Ono equation in the following way:

$$u_t^\delta + u^\delta u_x^\delta + H(u_{xx}^\delta) + K_\delta(u_x^\delta) = 0, \tag{8.2}$$

where K_δ is defined by

$$\widehat{K_\delta u}(\xi) = q_\delta(\xi)\, \hat{u}(\xi), \tag{8.3}$$

and

$$q_\delta(\xi) = 2\pi|\xi| - 2\pi\xi \coth(2\pi\varepsilon\xi) + \frac{1}{\delta}. \tag{8.4}$$

By lemma 4.1, for all $\xi \in \mathbb{R}$,

$$0 \leq q_\delta(\xi) \leq \frac{2}{\delta}, \tag{8.5}$$

which proves that K has order 0 and acts as a bounded operator on all the Sobolev spaces $H^s(\mathbb{R})$.

Theorem 8.1.1. Let $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, and let u^δ be the solution of (8.1) with initial data u_0. Then, for any $T > 0$, $u^\delta \to u$ as $\delta \to \infty$ in $C^k(0, T; H^{s-2k}(\mathbb{R}))$ for all $k \in \mathbb{N}$ for which $s - 2k \geq 0$, where u is the solution of the Benjamin–Ono equation corresponding to the initial data u_0.

Proof. To begin with, notice that there exists a constant C, independent of $\delta \geq 1$ such that

$$\left\| \frac{\partial^k}{\partial t^k} u^\delta \right\|_{L^\infty(0,T;\, H^{s-2k}(\mathbb{R}))} \leq C$$

for all integers k such that $s - 2k \geq 0$ and for any $T > 0$. This result is easily inferred from (8.2) because the operator K_δ is bounded on every space $H^s(\mathbb{R})$, independently of $\delta \geq 1$, and commutes with the operator D^s. Indeed, it suffices to follow the proof of lemma 5.3.4 using the above-mentioned facts to conclude first that for any $T > 0$, there is a constant $c = c(T, \|u_0\|_{3/2})$ such that u^δ is bounded in $C(0, T; H^s(\mathbb{R}))$, by $c\|u_0\|_s$ independently of $\delta \geq 1$. Then a straightforward inductive argument as given near (5.62) shows that the kth temporal derivative is bounded in $C(0, T; H^{s-2k}(\mathbb{R}))$ for all k such that $s - 2k \geq 0$. □

Lemma 8.1.2. For any $s > \frac{3}{2}$ and $T > 0$, the one-parameter family $\{u^\delta\}_{\delta \geq 1}$ is Cauchy in $C(0, T; H^s(\mathbb{R}))$ as $\delta \to +\infty$.

To prepare for the proof of lemma 8.1.2, we introduce the following notation. For $\alpha > 0$ and $\varepsilon > 0$, let $u^{\alpha\varepsilon}$ be the solution of the regularized equation

$$u_t^{\alpha\varepsilon} + u^{\alpha\varepsilon} u_x^{\alpha\varepsilon} + H(u_{xx}^{\alpha\varepsilon}) - K_\alpha(u_x^{\alpha\varepsilon}) = 0,$$
$$u^{\alpha\varepsilon}(x, 0) = u_{0\varepsilon}(x), \tag{8.6}$$

where $u_{0\varepsilon}$ is the regularization of u_0 given by (5.29). For $\gamma, \delta > 0$, set $u = u^{\gamma\varepsilon}$, $v = u^{\delta\varepsilon}$, $w = u - v$ and note that w then satisfies the initial-value problem

$$w_t + H(w_{xx}) + K_\delta(w_x)$$
$$= -u_x w - v w_x - (K_\delta - K_\gamma) u_x,$$
$$w(x, 0) = 0. \tag{8.7}$$

It will be temporarily convenient to use the notation $T_{\delta,\gamma}(u) = K_\delta(u) - K_\gamma(u)$.

Lemma 8.1.3. There exists $C = C(T; \|u_0\|_s) > 0$, which is independent of $\gamma, \delta \geq 1$ and $\varepsilon \in (0, 1]$ such that

$$\sup_{0 \leq t \leq T} \|w(\bullet, t)\|_s^2 \leq C\left[\left(\frac{1}{\gamma} + \frac{1}{\delta}\right)^2 \varepsilon^{-s/3}\right]. \tag{8.8}$$

Proof of lemma 8.1.3. We first take the scalar product in $L^2(\mathbb{R})$ of (8.7) with w and use the usual estimates to obtain the inequality

$$\frac{1}{2}\frac{d}{dt}\|w(\bullet, t)\|_0^2$$
$$\leq \left[\tfrac{1}{2}\|v_x(\bullet, t)\|_{L^\infty} + \|u_x(\bullet, t)\|_{L^\infty} + 1\right] \|w(\bullet, t)\|_0^2$$
$$+ \tfrac{1}{2}\|T_{\delta,\gamma}(u_x)(\bullet, t)\|_0^2. \tag{8.9}$$

It follows easily from its definition that

$$\|T_{\delta,\gamma}(u_x)\|_0^2 \leq 4\left(\frac{1}{\delta} + \frac{1}{\gamma}\right)^2 \|u\|_1^2.$$

Since the family $\{u^\delta\}_{\delta \geq 1}$ is bounded in $L^\infty(0, T; H^s(\mathbb{R}))$, $s > \frac{3}{2}$, there exists a constant M such that

$$\frac{d}{dt}\|w\|_0^2 \leq M\left[\|w\|_0^2 + \left(\frac{1}{\delta} + \frac{1}{\gamma}\right)^2\right] \tag{8.10}$$

and by Gronwall's lemma we get

$$\sup_{0 \leq t \leq T} \|w(\bullet, t)\|_0^2$$
$$\leq MT\left(\frac{1}{\delta} + \frac{1}{\gamma}\right)^2 e^{MT} = C\left(\frac{1}{\delta} + \frac{1}{\gamma}\right)^2. \tag{8.11}$$

Next, apply D^s to the first equation in (8.7) and take the scalar product with $D^s w$ to come to the inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^s w\|_0^2 \le |(D^s(vw_x), D^s w)| + |(D^s(u_x w), D^s w)| + |(D^s(T_{\delta,\gamma}(u_x)), D^s w)|. \tag{8.12}$$

Using lemma 5.3.5 in a familiar fashion, and (8.8), we may conclude that

$$|(D^s(vw_x), D^s w)| \le C\|w\|_s^2, \tag{8.13}$$

where C is independent of $t \in [0, T]$ and of $\delta \ge 1$. By using in addition an analogue of (5.14), (5.31), and an interpolation inequality, one deduces

$$\begin{aligned}|(D^s(u_x w), D^s w)| &\le C(\|u\|_{s+1}\|w\|_{s-1}\|w\|_s) + \|u_x\|_{L^\infty}\|w\|_s^2 \\ &\le C\varepsilon^{-1/6}\|w\|_0^{1/2}\|w\|_s^{2-1/s} + C\|w\|_s^2 \\ &\le C\varepsilon^{-s/3}\|w\|_0^2 + C\|w\|_s^2.\end{aligned} \tag{8.14}$$

It remains only to majorize suitably the last term in (8.12); the following estimate suffices:

$$\begin{aligned}|(D^s(T_{\delta,\gamma}(u_x)), D^s w)| &\le \|D^s T_{\delta,\gamma}(u_x)\|_0 \|D^s w\|_0 \\ &\le \tfrac{1}{2}\|T_{\delta,\gamma}(u)\|_{s+1}^2 + \tfrac{1}{2}\|w\|_s^2 \\ &\le 2\left(\frac{1}{\delta}+\frac{1}{\gamma}\right)^2 \|u\|_{s+1}^2 + \tfrac{1}{2}\|w\|_s^2 \\ &\le C\left(\frac{1}{\delta}+\frac{1}{\gamma}\right)^2 \varepsilon^{-1/3}\|u_0\|_s^2 + \tfrac{1}{2}\|w\|_s^2.\end{aligned} \tag{8.15}$$

Finally (8.10) through (8.15) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_s^2 \le C\|w\|_s^2 + C\varepsilon^{-s/3}\left(\frac{1}{\delta}+\frac{1}{\gamma}\right)^2. \tag{8.16}$$

The inequality (8.8) is now a direct consequence of (8.16) and Gronwall's lemma. □

Notice that lemma 8.1.3 implies that for all γ and δ such that $1 \le \delta \le \gamma$, there exists a constant $C = C(T, \|u_0\|_s)$ not dependent on δ and γ such that, for any $\varepsilon \in (0, 1]$,

$$\sup_{0\le t\le T} \|u^{\delta\varepsilon}(\bullet, t) - u^{\gamma\varepsilon}(\bullet, t)\|_s^2 \le \frac{\varepsilon^{-s/3}}{\delta^2} C(T; \|u_0\|_s). \tag{8.17}$$

Proof of lemma 8.1.2. First, write

$$\|u^\delta - u^\gamma\|_s \le \|u^\delta - u^{\delta\varepsilon}\|_s + \|u^{\delta\varepsilon} - u^{\gamma\varepsilon}\|_s + \|u^{\gamma\varepsilon} - u^\gamma\|_s. \tag{8.18}$$

The techniques developed in section 5 (see lemma 5.3.7) for the Benjamin–Ono equation, when applied to the intermediate long-wave equation imply that there exist constants $C = C(T, \|u_0\|_s)$ and $\lambda = \lambda(s) > 0$ such that for all $\varepsilon \in (0, 1]$

$$\sup_{0\le t\le T} \|u^\delta(\bullet, t) - u^{\delta\varepsilon}(\bullet, t)\|_s \le (\varepsilon^{\lambda(s)} + \|u_{0\varepsilon} - u_0\|_s) C(T; \|u_0\|_s), \tag{8.19}$$

$$\sup_{0\le t\le T} \|u^\gamma(\bullet, t) - u^{\gamma\varepsilon}(\bullet, t)\|_s \le (\varepsilon^{\lambda(s)} + \|u_{0\varepsilon} - u_0\|_s) C(T; \|u_0\|_s). \tag{8.20}$$

Then, taking (8.17) into account, it is seen that there exists a new constant $C = C(T; \|u_0\|_s)$ which again does not depend on $\varepsilon \in (0, 1]$ and on γ and δ with $1 \le \delta \le \gamma$, such that

$$\|u^\delta(\bullet, t) - u^\gamma(\bullet, t)\|_s \le \left(\varepsilon^{\lambda(s)} + \frac{\varepsilon^{-s/6}}{\delta} + \|u_{0\varepsilon} - u_0\|_s\right) C(T; \|u_0\|_s). \tag{8.21}$$

Now it is propitious to choose $\varepsilon = \delta^{-3/s}$, so that $\varepsilon^{-s/6}/\delta = \delta^{-1/2}$ and (8.21) thus yields

$$\|u^\delta(\bullet, t) - u^\gamma(\bullet, t)\|_s \le C(T; \|u_0\|_s)(\delta^{-3\lambda(s)/s} + \delta^{-1/2} + \|u_{0\delta^{-3/s}} - u_0\|_s), \tag{8.22}$$

holding for all $1 \le \delta \le \gamma$, so establishing lemma 8.1.2. □

Using (8.7) it is now a simple inductive argument to show that in fact $\{\partial_t^j u^\delta\}_{\delta>0}$ is a Cauchy family in the space $C(0,T; H^{s-2j}(\mathbb{R}))$ for all $T>0$, and all j such that $s-2j \ge 0$. Consequently, if $s-2k \ge 0$, there exists $u \in C^k(0,T; H^{s-2k}(\mathbb{R}))$ such that $u_\delta \to u$ in $C^k(0,T; H^{s-2k}(\mathbb{R}))$ as $\delta \to +\infty$, and since

$$\|K_\delta(u^\delta)\|_{s-1} \le \frac{2}{\delta}\|u^\delta\|_s,$$

it is readily seen that u is actually the solution of the Benjamin–Ono equation corresponding to the initial data u_0.

Corollary 8.1.4. If $u_0 \in H^\infty(\mathbb{R})$, then for any $T>0$, $u_\delta \to u$ in $C^\infty(0,T; H^\infty(\mathbb{R}))$ as $\delta \to +\infty$.

8.2. *The Korteweg–de Vries limit*

Our goal in this subsection is to compare solutions of the intermediate long-wave equation with those of the Korteweg–de Vries equation in the limit as δ tends to zero. To this end it is necessary to rescale the intermediate long-wave equation by letting

$$\tilde{u}(x,t) = \frac{3}{\delta}u\left(x, \frac{3}{\delta}t\right) \tag{8.23}$$

so that $\tilde{u}$ satisfies the equation

$$\tilde{u}_t + \tilde{u}\tilde{u}_x - \frac{3}{\delta}\mathfrak{L}_\delta \tilde{u}_{xx} = 0, \tag{8.24}$$

where the symbol p_δ of $\mathfrak{L}_\delta$ is given as in (2.4) by

$$p_\delta(\xi) = 2\pi\xi \coth(2\pi\delta\xi) - \frac{1}{\delta}. \tag{8.25}$$

The factor of 3 appearing in (8.23) is for later convenience; the tilde adorning u will be dropped henceforth. While the rescaling (8.23) is harmless for fixed, positive values of δ, it has a very substantial effect in the limit as δ tends to zero. However, it is only the solutions of (8.24) that tend to those of the Korteweg–de Vries equation as $\delta \to 0$. The analysis made in ref. [39] is misleading since a δ appears in the purported limit equation. This is corrected in ref. [40], where an artificial form of the intermediate long-wave equation is introduced for consideration that obscures the difference between the scaling that leads to the Benjamin–Ono equation as in subsection 8.1 and that leads to the Korteweg–de Vries equation in the present subsection.

We begin with the following lemma about the symbol p_δ in formula (8.25).

Lemma 8.2.1. For all $\xi \in \mathbb{R}$ and $\delta > 0$,

$$2\pi\xi \coth(2\pi\delta\xi) = \frac{1}{\delta} + \tfrac{4}{3}\pi^2\delta\xi^2 + \tfrac{1}{3}\xi^2 h(\xi,\delta), \tag{8.26}$$

where $h(\xi,\delta)$ is a bounded function which is $\mathcal{O}(\delta^3)$ as $\delta \downarrow 0$, uniformly for ξ in any bounded set in $\mathbb{R}$. Moreover, there is a constant C which is independent of ξ such that $|h(\delta,\xi)| \le C\delta$, for all $\xi \in \mathbb{R}$.

Proof. For any $\xi \in \mathbb{R}$ and $\delta > 0$, it is classical that

$$\begin{aligned} 2\pi\xi \coth(2\pi\delta\xi) &= \frac{1}{\delta} + 8\xi^2\delta \sum_{n=1}^{\infty} \frac{1}{n^2 + 4\delta^2\xi^2} \\ &= \frac{1}{\delta} + 8\xi^2\delta\left[\sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{n=1}^{\infty}\left(\frac{1}{n^2+4\delta^2\xi^2} - \frac{1}{n^2}\right)\right] \\ &= \frac{1}{\delta} + \tfrac{4}{3}\pi^2\xi^2\delta - \xi^2\sum_{n=1}^{\infty}\frac{32\delta^3\xi^2}{n^2(n^2+4\delta^2\xi^2)}. \end{aligned}$$

The result now follows immediately. □

It is concluded from lemma 8.2.1 that

$$\frac{3}{\delta}p_\delta(\xi) = 4\pi^2\xi^2 + \xi^2\frac{h(\xi,\delta)}{\delta}, \tag{8.27}$$

where h is as in (8.26), and so $h(\xi, \delta)/\delta$ is bounded, uniformly for $\xi \in \mathbb{R}$ and $\delta > 0$, and $h(\xi, \delta)/\delta = \mathcal{O}(\delta^2)$ as $\delta \downarrow 0$, uniformly for ξ in any bounded subset of $\mathbb{R}$.

In our theory comparing the Korteweg–de Vries equation and the intermediate long-wave equation, an important role is played by the δ-independent bounds obtained in the next proposition.

Lemma 8.2.2. Let $u_0 \in H^s(\mathbb{R})$ where $s \geq 2$. Then the solution $u = u^\delta$ of (8.24) satisfies the relations

$$\|u^\delta\|_{L^\infty(0,T;\, H^s(\mathbb{R}))} \leq C, \tag{8.28}$$

where T is any finite value and the constant $C = C(T, \|u_0\|_s)$ is independent of $\delta \in (0,1]$, say.

Remarks. Some of the steps in the proof of this lemma are quite similar to those already delineated in section 4. The difference is that here the concern is with δ-independent bounds of solutions of the equation rather than ε-independent bounds on approximate solutions of the equation.

The complete proof of lemma 8.2.2 involves the use of the sixth polynomial invariant for the intermediate long-wave equation. The derivation of this invariant is somewhat long and tedious, though it follows straightforwardly, in principle, from the recursion relaxation developed in Kodama, Satsuma and Ablowitz [39]. We content ourselves here with a simple description of the form of the sixth invariant, and a consequent inequality.

Proof. Because of the continuous dependence result expressed in theorem 7.7, for any fixed, positive value of δ, the solution u^δ of the initial-value problem for eq. (8.24) is the limit in $C(0, T;\, H^s(\mathbb{R}))$ of solutions u_ε^δ of the initial-value problem

$$v_t + v v_x - \frac{3}{\delta} \mathfrak{L}_\delta v_x = 0, \tag{8.29}$$

$$v(x,0) = u_{0\varepsilon}(x), \tag{8.30}$$

where $u_{0\varepsilon} \in H^\infty(\mathbb{R})$ and $u_{0\varepsilon} \to u_0$ in $H^s(\mathbb{R})$. Moreover, by corollary 7.8, $\partial_t^k u_\varepsilon^\delta \in H^\infty(\mathbb{R})$ for all $k \geq 0$. Since, for any fixed, positive value of δ, $u_\varepsilon^\delta \to u^\delta$ as $\varepsilon \downarrow 0$ in $C(0, T;\, H^s(\mathbb{R}))$, it suffices to obtain bounds on u_ε^δ which are independent of small values of ε and δ.

To simplify the notation, let u connote u_ε^δ in what follows. We proceed in several steps.

(i) Take the $L_2(\mathbb{R})$ inner product of a solution u of (8.29)–(8.30) with eq. (8.29) to obtain that

$$\|u(\bullet, t)\|_0 = \|u_{0\varepsilon}\|_0 \leq C\|u_0\|_0, \tag{8.31}$$

which holds for all $t \geq 0$, for all $\delta > 0$ and for $\varepsilon \in (0,1]$, say.

(ii) Take the $L_2(\mathbb{R})$ inner product of $-u^2 + (6/\delta)\mathfrak{L}_\delta u$ with (8.29) to obtain

$$\frac{1}{\delta}\int_{-\infty}^{\infty} u \mathfrak{L}_\delta u \,\mathrm{d}x = \int_{-\infty}^{\infty} \tfrac{1}{9} u^3 \,\mathrm{d}x + \int_{-\infty}^{\infty} \left(\frac{1}{\delta} u_{0\varepsilon} \mathfrak{L}_\delta u_{0\varepsilon} - \tfrac{1}{9} u_{0\varepsilon}^3 \right) \mathrm{d}x. \tag{8.32}$$

(iii) Use the next invariant of the intermediate long-wave equation to deduce that

$$I_2(u(\bullet, t)) = I_2(u_{0\varepsilon}), \tag{8.33}$$

where I_2 is given in section 4 (up to a trivial change in the scaling). Using (8.31) and (8.32) in (8.33), we ascertain that

$$J_2(u(\bullet, t)) = J_2(u_{0\varepsilon}), \tag{8.34}$$

where, for any function $w \in H^1(\mathbb{R})$,

$$J_2(w) = \int_{-\infty}^{\infty} \left(\frac{\delta^2}{36} w^4 - \frac{\delta}{2} (w^2 \mathfrak{L}_\delta w) + \tfrac{1}{2} w_x^2 + \tfrac{3}{2} (\mathfrak{L}_\delta w)^2 - \frac{3}{2\delta} (w \mathfrak{L}_\delta w) \right) \mathrm{d}x. \tag{8.35}$$

To obtain an $H^1(\mathbb{R})$ bound on u which is independent of ε and δ, it therefore suffices to bound the quantity

$$\frac{\delta}{2}\int_{-\infty}^{\infty}(u^2\mathfrak{L}_\delta u)\,\mathrm{d}x + \frac{3}{2\delta}\int_{-\infty}^{\infty}(u\mathfrak{L}_\delta u)\,\mathrm{d}x + J_2(u_{0\varepsilon}) \tag{8.36}$$

by a quantity that just depends upon $\|u_0\|_s$. By using (8.32), bounding (8.36) is seen to be equivalent to bounding

$$\frac{\delta}{2}\int_{-\infty}^{\infty}(u^2\mathfrak{L}_\delta u)\,\mathrm{d}x + \frac{1}{6}\int_{-\infty}^{\infty}u^3\,\mathrm{d}x + \frac{3}{2\delta}\int_{-\infty}^{\infty}(u_{0\varepsilon}\mathfrak{L}_\delta u_{0\varepsilon})\,\mathrm{d}x - \frac{1}{6}\int_{-\infty}^{\infty}u_{0\varepsilon}^3\,\mathrm{d}x + J_2(u_{0\varepsilon}). \tag{8.37}$$

Recall from (8.4) and (8.5) that $p_\delta(\xi) = 2\pi|\xi| + q_\delta(\xi)$ where $q_\delta(\xi)$ lies always in the interval $[0, 2/\delta]$. Hence Hölder and interpolation inequalities combined with (8.31) yield

$$\left|\frac{\delta}{2}\int_{-\infty}^{\infty}(u^2\mathfrak{L}_\delta u)\,\mathrm{d}x\right| \le \frac{\delta}{2}\|\mathfrak{L}_\delta u\|_0\,\|u\|_{L_4}^2(\mathbb{R}) \le c\frac{\delta}{2}\|\mathfrak{L}_\delta u\|_0\,\|u\|_{1/4}^2 \le c\|u\|_1^{3/2}\,\|u\|_0^{3/2} \le c\|u\|_1^{3/2} \le \eta\|u\|_1^2 + c(\eta), \tag{8.38}$$

where $\eta > 0$ is arbitrary and $c(\eta)$ depends only upon the choice of η. Similar arguments lead to

$$\left|\int_{-\infty}^{\infty}u^3\,\mathrm{d}x\right| \le \|u\|_{L_3(\mathbb{R})}^3 \le c\|u\|_{1/6}^3 \le c\|u\|_1^{1/2}\,\|u\|_0^{5/2} \le \eta\|u\|_1^2 + c(\eta). \tag{8.39}$$

Applying (8.39) at $t = 0$ gives

$$\left|\int_{-\infty}^{\infty}u_{0\varepsilon}^3\,\mathrm{d}x\right| \le c = c(\|u_0\|_1). \tag{8.40}$$

The term

$$\left|\frac{1}{\delta}\int_{-\infty}^{\infty}(u_{0\varepsilon}\mathfrak{L}_\delta u_{0\varepsilon})\,\mathrm{d}x\right|$$

is equal to

$$\left|\frac{1}{\delta}\int_{-\infty}^{\infty}p_\delta(\xi)\,|\hat{u}_{0\varepsilon}(\xi)|^2\,\mathrm{d}\xi\right|$$

by Parseval's theorem, and hence (8.27) shows this expression to be bounded by a constant dependent only on

$$\sup_{0<\delta\le 1}\ \sup_{\xi\in\mathbb{R}}\left|\frac{h(\xi,\delta)}{\delta}\right|$$

and on $\|u_0\|_1$. It remains to estimate $J_2(u_{0\varepsilon})$. Using an interpolation inequality and (8.27) one sees immediately that

$$\int_{-\infty}^{\infty}\left[\delta^2 u_{0\varepsilon}^4 + \tfrac{1}{2}(u_{0\varepsilon})_x^2\right]\mathrm{d}x \le c_1 = c_1(\|u_0\|_1) \tag{8.41}$$

and that

$$\int_{-\infty}^{\infty}\left[\frac{1}{\delta}u_{0\varepsilon}\mathfrak{L}_\delta u_{0\varepsilon} + \frac{\delta}{2}u_{0\varepsilon}^2\mathfrak{L}_\delta u_{0\varepsilon} + (\mathfrak{L}_\delta u_{0\varepsilon})^2\right]\mathrm{d}x \le c_2 = c_2(\|u_0\|_1), \tag{8.42}$$

where both c_1 and c_2 are independent of $\delta \in (0,1]$. Putting together the pieces (8.31) and (8.35)–(8.42) with η chosen sufficiently small, gives

$$\|u_\varepsilon^\delta(\bullet, t)\|_1 \le c = c(\|u_0\|_1), \tag{8.43}$$

which holds good for all $\varepsilon, \delta \in (0,1]$. Taking the limit as $\varepsilon \downarrow 0$ gives (8.28) for $s = 1$.

(iv) The sixth polynomial invariant I_4 for the intermediate long-wave equation is analogous to the fourth invariant I_2 above; it consists of the term $\int_{-\infty}^{\infty}u_{xx}^2\,\mathrm{d}x$ plus a host of other terms, all of which can be bounded exactly as in part (iii) above by constants that depend upon $\|u_0\|_2$, but which are independent of $\varepsilon, \delta \in (0,1]$. Because of this, (8.28) is seen to hold with $s = 2$.

(v) For $s \ge 2$ any real number, apply D^s to the intermediate long-wave equation, and take the

$L_2(\mathbb{R})$ scalar product of the results with $D^s u$ itself, where as before $u = u_\varepsilon^\delta$. After a little manipulation, this procedure leads to

$$\frac{1}{2}\frac{d}{dt}\|D^s u\|_0^2 + \left(D^s(uu_x), D^s u\right) = 0. \tag{8.44}$$

The quantity $|(D^s(uu_x), D^s u)|$ is majorized by $c\|u\|_s^2$, where $c = c(\|u\|_2)$ depends only upon the $H^2(\mathbb{R})$ norm of u. An application of Gronwall's lemma leads to the conclusion

$$\|u_\varepsilon^\delta(\bullet, t)\|_s \le c = c(\|u_0\|_s) \tag{8.45}$$

where c does not depend upon $\varepsilon, \delta \in (0,1]$. Taking the limit as $\varepsilon \downarrow 0$ gives (8.28) for arbitrary $s \ge 2$.

The proof of the lemma is now complete. □

Returning now to the regularized problem (8.29)–(8.30), suppose that $u_0 \in H^s(\mathbb{R})$ where $s \ge 2$ and let $u_{0\varepsilon}$ be the specific regularization of u_0 as defined in section 5. As above, let u_ε^δ denote the solution of (8.29)–(8.30) with initial data $u_{0\varepsilon}$ and let u_ε denote the solution of the initial-value problem for the Korteweg–de Vries equation

$$u_t + uu_x + u_{xxx} = 0, \quad u(x,0) = u_{0\varepsilon}(x). \tag{8.46}$$

Lemma 8.2.3. Let $\varepsilon > 0$ be fixed. Then for every $s \ge 2$, and $T \ge 0$,

$$\|u_\varepsilon^\delta - u_\varepsilon\|_{C(0,T;\, H^s(\mathbb{R}))} \to 0 \tag{8.47}$$

as $\delta \downarrow 0$.

Proof. For $0 < \gamma < \delta$, set $u = u_\varepsilon^\gamma$, $v = u_\varepsilon^\delta$, and $w = u - v$. Then w satisfies the equation

$$w_t + w_{xxx} + H_\delta w_x = -\left(H_\delta - H_\gamma\right) v_x - u_x w - v w_x \tag{8.48}$$

with $w(x,0) \equiv 0$, where the symbol h_δ of H_δ is

$$h_\delta(\xi) = -\frac{3\xi^2}{\delta} h(\xi, \delta) \tag{8.49}$$

and $h(\xi, \delta)$ is defined in lemma 8.2.1. A similar formula applies to the symbol h_γ of H_γ where γ replaces δ throughout. It is convenient to let $S_{\delta,\gamma} = H_\delta - H_\gamma$.

To derive (8.47), proceed inductively starting with $s = 0$. Multiply (8.48) by w and integrate over $\mathbb{R}$. After integration by parts, one reaches the relation

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\|w(\bullet, t)\|_0^2 \\ &\le C\left[\|v_x(\bullet, t)\|_{L_\infty(\mathbb{R})}^2 + \|u_x(\bullet, t)\|_{L_\infty(\mathbb{R})}^2\right] \\ &\quad \times \|w(\bullet, t)\|_0^2 \\ &\quad + \tfrac{1}{2}\|S_{\delta,\gamma}(u_x)(\bullet, t)\|_0^2.\end{aligned} \tag{8.50}$$

Because of (8.49),

$$\|S_{\delta,\gamma}(u_x)\|_0^2 \le C\int_{-\infty}^{\infty} \xi^6\left(\frac{h^2(\xi,\delta)}{\delta^2} + \frac{h^2(\xi,\gamma)}{\gamma^2}\right)|\hat{u}(\xi)|^2\, d\xi.$$

Using lemma 8.2.2 and Lebesgue's dominated convergence theorem, one readily obtains that

$$\|S_{\delta,\gamma}(\partial_x u_\varepsilon^\gamma)\|_0 \le c_\varepsilon(\delta), \tag{8.51}$$

where, for fixed $\varepsilon > 0$, $c_\varepsilon(\delta) \to 0$ as $\delta \downarrow 0$. Gronwall's lemma applied to (8.50) gives

$$\|w(\bullet, t)\|_{C(0,T;\, H^0(\mathbb{R}))} \le C_\varepsilon(\delta) \tag{8.52}$$

where again $C_\varepsilon(\delta) \to 0$ as $\delta \downarrow 0$ in light of (8.51).

Using the commutation lemma 5.3.5, overestimates of the form

$$\|w(\bullet, t)\|_{C(0,T;\, H^s(\mathbb{R}))} \le c_\varepsilon(\delta) \tag{8.53}$$

may be derived for values of $s \ge 2$, where again $c_\varepsilon(\delta) \to 0$ as $\delta \downarrow 0$. The details of the derivation of (8.53) follow lines that are by now familiar, and so they are left to the reader.

Inequality (8.53) implies $\{u_\varepsilon^\delta\}_{\delta > 0}$ to be Cauchy in $C(0, T;\, H^s(\mathbb{R}))$ for any $T > 0$ and $s \ge 2$. Hence $\{u_\varepsilon^\delta\}_{\delta > 0}$ converges to some function $v_\varepsilon \in C(0, T;\, H^s(\mathbb{R}))$. Another application of lemma

8.2.1 shows v_ε to be a solution to the initial-value problem (8.46), and thus it follows by uniqueness that $v_\varepsilon = u_\varepsilon$ and so $u_\varepsilon^\delta \to u_\varepsilon$ as $\delta \downarrow 0$ in $C(0,T; H^s(\mathbb{R}))$ for any $T>0$ and $s \geq 0$, as required. The proof of lemma 8.2.3 is complete. □

The last lemma and our previously developed theory for the intermediate long-wave equation put us in range of proving our final result of this section, namely convergence of the solutions of the intermediate long-wave equation as scaled in (8.24), to an associated solution of the Korteweg–de Vries equation as $\delta \to 0$. As before, fix $u_0 \in H^s(\mathbb{R})$, $s \geq 2$, and let $u_{0\varepsilon}$ be the smooth approximations to u_0 defined in lemma 5.3.7. Let u^ε denote the solution of the Korteweg–de Vries equation (8.46) with initial data $u_{0\varepsilon}$ and let u denote the solution with the initial data u_0. The theory developed in ref. [14] implies that for any $s \geq 2$ and $T>0$, $u_\varepsilon \to u$ in $C(0,T; H^s(\mathbb{R}))$ as $\varepsilon \to 0$. Let u_ε^δ denote the solution of the intermediate long-wave equation (8.24) with initial data $u_{0\varepsilon}$ and u^δ denote the solution with initial data u_0. It is also known from the theory developed in section 5 as applied to the intermediate long-wave equation in section 6 that $u_\varepsilon^\delta \to u^\delta$ in $C(0,T; H^s(\mathbb{R}))$ as $\varepsilon \to 0$. Moreover, in light of the δ-independent bounds expressed in lemma 8.2.2 that apply to solutions of the intermediate long-wave equation scaled as in (8.24), one easily discerns by tracing through the proofs that $u_\varepsilon^\delta \to u^\delta$ uniformly for $\delta \in (0,1]$, say.

Here is the final result regarding convergence of intermediate long-wave solutions to those of the Korteweg–de Vries equation.

Theorem 8.2.4. Let $u_0 \in H^s(\mathbb{R})$ with $s \geq 2$ and let u^δ denote the solution of (8.29) with initial data u_0. Then for any $T>0$, $u^\delta \to u$ in $C(0,T; H^s(\mathbb{R}))$ as $\delta \downarrow 0$ where u is the solution of the Korteweg–de Vries equation (8.46) with initial data u_0.

Proof. As described above, introduce the families of functions $\{u_\varepsilon^\delta\}$ and $\{u_\varepsilon\}$. Then, by the triangle inequality,

$$\begin{aligned}
&\|u^\delta(\bullet,t) - u(\bullet,t)\|_s \\
&\quad\leq \|u^\delta(\bullet,t) - u_\varepsilon^\delta(\bullet,t)\|_s \\
&\qquad + \|u_\varepsilon^\delta(\bullet,t) - u_\varepsilon(\bullet,t)\|_s + \|u^\varepsilon(\bullet,t) - u(\bullet,t)\|_s.
\end{aligned}$$

Let $\nu > 0$ be given. Choose $\varepsilon > 0$ such that

$$\begin{aligned}
&\sup_{0\leq t\leq T} \big[\|u_\varepsilon^\delta(\bullet,t) - u^\delta(\bullet,t)\|_s \\
&\qquad + \|u_\varepsilon(\bullet,t) - u(\bullet,t)\|_s\big] \leq \nu
\end{aligned}$$

for all $\delta \in (0,1]$. Such a choice is possible because of the uniform convergence of u_ε^δ to u^δ as $\varepsilon \downarrow 0$. With $\varepsilon > 0$ now fixed, then by lemma 8.2.3,

$$\begin{aligned}
&\limsup_{\delta\downarrow 0} \|u^\delta(\bullet,t) - u(\bullet,t)\|_{C(0,T;\,H^s(\mathbb{R}))} \\
&\quad\leq \nu + \limsup_{\delta\downarrow 0} \|u_\varepsilon - u_\varepsilon^\delta\|_{C(0,T;\,H^s(\mathbb{R}))} = \nu.
\end{aligned}$$

As $\nu > 0$ was arbitrary, it is concluded that $u^\delta \to u$ as $\delta \downarrow 0$ in $C(0,T; H^s(\mathbb{R}))$, as desired. □

9. Benjamin–Ono and intermediate long-wave equation. The periodic case

In this section we turn to consideration of the initial-value problem for the Benjamin–Ono and the intermediate long-wave equations with spatial periodicity imposed. Since the results and methods are very close to those used in the study of the Cauchy problem on the line, we shall just emphasize the differences in the definition of the pseudo-differential operators involved in the model equation.

9.1. The Benjamin–Ono equation

For the situation wherein the wave motion is supposed to have imposed upon it some spatial periodicity, say with period $2L$, the Benjamin–Ono equation for internal waves takes the form

$$u_t + uu_x + \mathfrak{H}u_{xx} = 0, \tag{9.1}$$

where $u = u(x, t)$ is a $2L$-periodic function of x and $\mathfrak{H}$ is defined by the Hilbert kernel as

$$\mathfrak{H}f(x) = -\frac{1}{2L}\mathrm{PV}\int_{-L}^{L}\cot\left(\frac{\pi(x-y)}{2L}\right)f(y)\,\mathrm{d}y. \tag{9.2}$$

The associated initial-value problem is simply to pose (9.1) subject to the initial condition $u(x,0) = u_0(x)$ where u_0 is a given $2L$-periodic function that satisfies appropriate smoothness conditions to be specified momentarily.

The operator $\mathfrak{H}$ has properties similar to properties (H1)–(H3) of the Hilbert transform H presented in section 3.3. In particular, the following identities hold for f and g in $L^2(-L, L)$:

$$\int_{-L}^{L} fg\,\mathrm{d}x = \int_{-L}^{L}\mathfrak{H}(f)\,\mathfrak{H}(g)\,\mathrm{d}x, \tag{9.3}$$

$$\int_{-L}^{L} f\,\mathfrak{H}(g)\,\mathrm{d}x = -\int_{-L}^{L}\mathfrak{H}(f)\,g\,\mathrm{d}x, \tag{9.4}$$

$$\mathfrak{H}(f\,\mathfrak{H}(g) + \mathfrak{H}(f)g) = \mathfrak{H}(f)\,\mathfrak{H}(g) - fg, \tag{9.5}$$

$$\mathfrak{H}(f)(x) = \mathrm{i}\sum_{k\in\mathbb{Z}}\mathrm{sgn}(k)\,f_k\,\mathrm{e}^{\mathrm{i}k\pi x/L}, \tag{9.6}$$

where f_k is the kth Fourier coefficient of f defined by the formula

$$f_k = \frac{1}{2L}\int_{-L}^{L} f(x)\,\mathrm{e}^{-\mathrm{i}k\pi x/L}\,\mathrm{d}x = c_k(f).$$

A brief indication of the proof of (9.6) is perhaps warranted. It relies on the following lemma in Tricomi's book [30]:

Lemma 9.1. For $k = 1, 2, \ldots,$

$$\cos(k\theta) = \frac{1}{2\pi}\mathrm{PV}\int_{-\pi}^{\pi}\cot\left[\tfrac{1}{2}(x-\theta)\right]\sin(kx)\,\mathrm{d}x, \tag{9.7}$$

and

$$\sin(k\theta) = \frac{1}{2\pi}\mathrm{PV}\int_{-\pi}^{\pi}\cot\left[\tfrac{1}{2}(x-\theta)\right]\cos(kx)\,\mathrm{d}x. \tag{9.8}$$

Because $\mathfrak{H}$ is defined as a convolution, we know that

$$c_k(\mathfrak{H}(f)) = -c_k\left(\cot\left(\frac{\pi x}{2L}\right)\right)f_k.$$

Since the cotangent is an odd function,

$$\begin{aligned} c_k\left(\cot\left(\frac{\pi x}{2L}\right)\right) &= -\frac{\mathrm{i}}{2L}\int_{-L}^{L}\cot\left(\frac{\pi x}{2L}\right)\sin\left(k\frac{\pi}{L}x\right)\mathrm{d}x \\ &= -\frac{\mathrm{i}}{2\pi}\int_{-\pi}^{\pi}\cot\left(\tfrac{1}{2}\tau\right)\sin(k\tau)\,\mathrm{d}\tau \\ &= -\mathrm{i} \quad \text{if } k = 1, 2, \ldots, \\ &= +\mathrm{i} \quad \text{if } k = -1, -2, \ldots \end{aligned}$$

(in the last equality (9.7) has been applied with $\theta = 0$). Thus for all k in $\mathbb{Z}^*$,

$$c_k(\mathfrak{H}(f)) = \mathrm{i}\,\mathrm{sgn}(k)\,f_k,$$

and (9.6) follows.

All the computations of section 3 remain valid for eq. (9.1) if one replaces all integrations over $\mathbb{R}$ by integrations over $(-L, L)$ and the zero boundary conditions at infinity by periodic conditions. We therefore obtain results analogous to those in section 4 for the periodic initial-value problem for (9.1) in the following function-analytic setting, where to simplify the exposition, the period is taken to be equal to one.

For real s, denote by $H^s(C)$ the Sobolev space of order s on the unit circle C. For $s \geq 0$, $H^s(C)$ can be characterized as being the space of periodic real functions

$$u(x) = \sum_{k\in\mathbb{Z}} u_k\,\mathrm{e}^{2\pi\mathrm{i}kx}$$

of period 1 such that

$$\|u\|_s = \left(\sum_{k\in\mathbb{Z}}(1+k^2)^s|u_k|^2\right)^{1/2} < \infty.$$

With this definition, the results of section 4 hold exactly as stated upon replacing all appearances of $H^s(\mathbb{R})$ by $H^s(C)$, where the operator D^s that intervenes in the previous analysis is now defined by

$$D^s u(x) = \sum_{k\in\mathbb{Z}} u_k |k|^s \mathrm{e}^{2\pi i k x}$$

for $u \in H^s(C)$.

9.2. The intermediate long-wave equation

In the periodic case, the intermediate long-wave equation takes the form (see ref. [41])

$$u_t + uu_x + \frac{1}{\delta}u_x + \mathfrak{T}(u_{xx}) = 0, \tag{9.9}$$

where $\mathfrak{T}$ is defined by

$$\mathfrak{T}(f)(x) = \mathrm{i} \sum_{k\in\mathbb{Z}^*} \coth\left(\frac{k\pi\delta}{L}\right) f_k \mathrm{e}^{ik\pi x/L}, \tag{9.10}$$

and f is any sufficiently smooth $2L$-periodic function.

As in the case of the Benjamin–Ono equation, all the results of sections 4, 5, and 8 are valid for the periodic intermediate long-wave equation. Here is a precise statement of the results in view for both the Benjamin–Ono equation and the intermediate long-wave equation.

Theorem 9.1. Let $u_0 \in H^s(C)$ be given initial data for the periodic Benjamin–Ono equation (9.1) (respectively, for the periodic intermediate-depth equation (9.9)). If $s = 1$ or $s = \frac{3}{2}$, then there exists a solution u of (9.1) (respectively, of (9.9)) with initial value u_0 such that $u \in L^\infty(\mathbb{R}_+; H^s(C))$. If $s > \frac{3}{2}$, then there exists a unique solution u of (9.1) (respectively, (9.9)) with initial data u_0 such that, for each $T > 0$, $u \in C^k(0, T; H^{s-2k}(C))$ for all k for which $s - 2k > -\frac{3}{2}$. Moreover, the mapping that associates to u_0 the unique solution u of (9.1) (respectively (9.9)) with initial value u_0 is continuous from $H^s(C)$ to $C^k(0, T; H^{s-2k}(C))$, for all $T > 0$ and all k for which $s - 2k > -\frac{3}{2}$. If $s = n/2$ where n is an integer larger than 3, then $u \in C_b^k(\mathbb{R}_+; H^{s-2k}(C))$ for k with $s - 2k > -\frac{3}{2}$.

10. Conclusion

It has been shown that several nonlocal model equations for nonlinear, dispersive waves which are of current interest are classically well posed. These include the Benjamin–Ono equation and the intermediate depth equation for internal waves in stratified fluid, and Smith's equation for continental shelf waves. The limiting behavior as the relative depth of the layers approaches 0 and ∞ of suitably scaled versions of the intermediate long-wave equation is that of the Korteweg–de Vries equation and the Benjamin–Ono equation, respectively.

It should be acknowledged that the semi-group techniques of Kato [42] could also be used in the present context, at least for the theories regarding smooth solutions ($s > \frac{3}{2}$). Indeed, this was carried out in ref. [23].

The general ideas that have come to the fore herein are applicable to a considerably broader range of problems. For one, certain interesting systems of nonlocal, nonlinear dispersive wave equations such as those described in ref. [21] are amenable to the sort of treatment given here to single equations, though the details are naturally somewhat more complicated. A separate account of some of this material especially related to internal-wave propagation is being prepared for publication. Within the realm of single equations, the results on local existence, global existence of weak solutions, and continuity of strong solutions with respect to perturbation in the initial data and perturbations of the model all have counterparts for a broad class of equations of the form (5.1).

In addition to providing a secure underpinning for general analytical and numerical investigations of model equations of the form (5.1), the present theory is needed as a tool in the stability theory for solitary-wave solutions of these equations (see ref. [43] and the references contained therein).

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