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DECAY OF SOLUTIONS TO NONLINEAR, DISPERSIVE WAVE EQUATIONS

Jerry L. Bona
Department of Mathematics and Applied Research Laboratory
The Pennsylvania State University, University Park, PA 16802

Laihan Luo
Department of Mathematics, The Pennsylvania State University, University Park, PA 16802

Abstract. The asymptotic behavior of solutions to the initial-value problem for the generalized Korteweg-de Vries-Burgers equation

$$u_t + u_x + u^p u_x - \nu u_{xx} + u_{xxx} = 0$$

and the generalized regularized long-wave-Burgers equation

$$u_t + u_x + u^p u_x - \nu u_{xx} - u_{xxt} = 0$$

is studied for $\nu > 0$ and $p \geq 2$. The decay rate of solutions of these equations is that exhibited by solutions of the linearized equation in which the nonlinear term is simply dropped.

1. Introduction. This paper is concerned with the decay of solutions of the damped wave equations

$$u_t + u_x + u^p u_x - \nu u_{xx} + u_{xxx} = 0, \quad (x \in \mathbb{R},\ t > 0) \tag{1.1}$$

and

$$u_t + u_x + u^p u_x - \nu u_{xx} - u_{xxt} = 0, \quad (x \in \mathbb{R},\ t > 0) \tag{1.2}$$

with initial-value conditions

$$u(x,0) = f(x), \quad (x \in \mathbb{R}). \tag{1.3}$$

In the above equations, $u = u(x,t)$ is a real-valued function of the two real variables x and t, subscripts adorning u connote partial differentiation, ν is a positive number and p is a positive integer.

Such equations arise as mathematical models for the unidirectional propagation of nonlinear, dispersive, long waves. In this sort of application, u is typically an amplitude or a velocity, x is proportional to distance in the direction of propagation and t is proportional to elapsed time. Important special cases of (1.1) and (1.2) are the well known Korteweg-de Vries equation (KdV equation)

$$u_t + u_x + uu_x + u_{xxx} = 0, \tag{1.4}$$

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derived originally as a model for waves propagating on the surface of a canal (Korteweg & de Vries [17]) and the alternative regularized long-wave equation (RLW equation)

$$u_t + u_x + uu_x - u_{xxt} = 0, \tag{1.5}$$

put forward by Peregrine [22] and Benjamin *et al.* [2]. These equations feature a balance between nonlinear and dispersive effects, but take no account of dissipation.

The addition of a dissipative term to the wave equations (1.4) and (1.5) becomes necessary when damping effects are comparable with the effects of nonlinearity and dispersion. This certainly occurs when modelling water waves on laboratory scales and in the near-shore zones of large bodies of water ([6, 19, 20, 21]). It also occurs when consideration is given to problems concerning bore propagation (see [7, 8, 13]). The modelling of dissipative effects at the same level as nonlinear and dispersive effects in equations of KdV- or RLW-type is a somewhat delicate matter (see [24]), and consequently the KdV-Burgers equation (KdV-B equation which is (1.1) with $p = 1$) or the RLW-Burgers equation (RLW-B equation which is (1.2) with $p = 1$) has gained some acceptance when there is need of combining nonlinear, dispersive and dissipative effects into a single evolution equation (see e.g., [6, 12]).

When an *ad hoc* dissipative term is appended to a KdV- or RLW-type equation, it is important in using the resulting model in practical situations to understand the decay rates thereby implied when initial data of finite total energy is posed. This issue was the genesis of the study by Amick *et al.* [1] of (1.1) and (1.2) in the special case $p = 1$. It was shown in the last-mentioned reference that solutions of the initial-value problem for either (1.1) or (1.2) with $p = 1$ with initial data in $L_1(\mathbb{R})$ only decay algebraically in time. Indeed, it turns out that if u is a solution of (1.1) or (1.2) corresponding to suitably restricted data, then the $L_2(\mathbb{R})$-norm of u decays at the rate $t^{-1/4}$ as $t \to +\infty$, and this rate is sharp for a generic class of initial data. This is the same rate that obtains via Fourier analysis for the linearized equations, but there is a subtle difference between the linear and nonlinear problems explained in [1, §5].

The results in [1] have been usefully generalized by making allowance for more general nonlinearities, dispersion relations, and linear dissipative mechanisms (c.f. [3, 4, 10, 11, 27]). Recently, considerable attention has been given to the generalized KdV equation (GKdV equation, which is (1.1) with $\nu = 0$) and the generalized RLW equation (GRLW equation, which is (1.2) with $\nu = 0$) (e.g., [5, 15, 16]). These studies have focused on understanding the interaction between nonlinearity and dispersion by keeping the relatively simple dispersive terms u_{xxx} or $-u_{xxt}$ and varying the strength of the nonlinearity by changing the value of p. Some very interesting aspects of wave propagation have been turned up in these studies. Also attention has been given to whether or not properties observed for solutions of the GKdV or GRLW equation persist in the presence of dissipative effects.

It is our purpose here to add to the discussion outlined above. An analysis is carried out to determine the temporal decay rates of solutions of (1.1) and (1.2) in case $\nu > 0$ and the initial data has finite energy, by which is meant that it belongs to a suitable Sobolev space. Attention will be given entirely to the cases where $p \geq 2$ not treated by Amick *et al.* [1] (the case of asymptotically weak nonlinearity in the parlance introduced by Dix [11]). The techniques used here to obtain sharp results are different from those in [1] where a central role was played by the Cole-Hopf transformation. This change of variables is no longer effective if $p > 1$ and so more general methods are required. In consequence, the proof of our results is considerably less intricate than that presented in [1], though some of the very detailed information derived in [1, §3] seems not to be available using the robust arguments favored here.

The outcome of our analysis is that solutions decay at temporal rates identical to those of the linear equations, just as for the case $p = 1$. For the GKdV-Burgers equation, our conclusions are the same as those one may derive by specializing the general theory developed by Dix [11]. Both our results and those of Dix considerably improve the earlier theory of Biler [3]. The proof offered here of decay of solutions of (1.1) is technically simpler than that presented in [11], but doesn't have the range of applicability. It is remarkably similar to one that is effective for solutions of (1.2). Consequently, the presentation here focuses on the new results for the GRLW-Burgers equation (1.2) and provides only a brief outline of the theory for decay of solutions of the GKdV-Burgers equation (1.1). Decay results for (1.2) for relatively high order nonlinearities ($p \geq 4$) have also been given by Zhang [27] using different estimates than those appearing in Section 3 below.

The paper is organized as follows. In Section 2 the notation is set and the theoretical results stated. Section 3 contains some preliminary technicalities. In Section 4, some non-optimal results are derived for solutions of (1.2). With the help of the preliminary results and the non-optimal decay results, the sharp theory is derived in Section 5. Section 6 recounts briefly the analogous results for equation (1.1).

2. Notation and statement of the main results. In this paper, all functions will be real-valued. For an arbitrary Banach space X, the associated norm will be denoted $\|\cdot\|_X$ except for a few convenient abbreviations to be introduced now. The L_p-norm of a function f which is pth-power absolutely integrable on $\mathbb{R}$ is denoted by $|f|_p$ for $1 \leq p < \infty$, and similarly $|f|_\infty = \|f\|_{L_\infty}$. If $m \geq 0$ is an integer, $W_p^m(\mathbb{R})$ will be the Sobolev space consisting of those $L_p(\mathbb{R})$-functions whose first m generalized derivatives lie in $L_p(\mathbb{R})$, equipped with the usual norm,

$$\|f\|_{W_p^m(\mathbb{R})} = \sum_{k=0}^{m} |f^{(k)}|_p.$$

The case $p = 2$ deserves the special notation $H^m(\mathbb{R})$. The norm of f in $H^m(\mathbb{R})$ will be noted simply $\|f\|_m$. The space $C_b^k(\mathbb{R})$ connotes the functions defined on $\mathbb{R}$ whose first k derivatives are bounded, continuous functions. The class $C_b^\infty = C_b^\infty(\mathbb{R}) = \cap_{k\geq 0} C_b^k(\mathbb{R})$ will not be given a topological structure. Let X be a Banach space, T be a positive real number and $1 \leq p \leq +\infty$. The symbol $L_p(0, T; X)$ connotes the Banach space of all measurable functions u: $(0, T) \to X$, such that $t \to \|u(t)\|_X$ is in $L_p(0, T)$, with the norm

$$\|u\|_{L_p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{\frac{1}{p}}, \quad \text{if} \quad 1 \leq p < +\infty$$

and

$$\|u\|_{L_\infty(0,T;X)} = \underset{0<t<T}{\text{essential supremum}} \, (\|u(t)\|_X).$$

Similarly, $C(0,T;X)$ denotes the subspace of $L_\infty(0,T;X)$ of all continuous functions $u:[0,T]\to X$ with the norm

$$\|u\|_{C(0,T;X)} = \sup_{0\le t\le T} \|u(t)\|_X.$$

If $T=\infty$, then $C_b(\mathbb{R}^+;X)$ denotes the bounded continuous mappings $u:\mathbb{R}^+\to X$. This is a Banach space with the norm

$$\|u\|_{C_b(\mathbb{R}^+;X)} = \sup_{\mathbb{R}^+} \|u(t)\|_X.$$

Finally, the Fourier transform $\hat{f}$ of a function f is defined by

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)dx.$$

As already mentioned, the principal thrust of the present work is to obtain sharp decay rates for solution of (1.2) in case $\nu>0$. Precise statements of our results appear in Sections 5 and 6. However, the following informal statement serves to provide a goal the reader may usefully bear in mind as the theory is developed.

Main result. Let $p\ge 2$, $\nu>0$, and consider initial data f that is suitably restricted in smoothness and evanescence as $x\to\pm\infty$. Then there is a unique solution u of (1.2) corresponding to the initial value f and u decays to zero as $t\to+\infty$ in various norms. In particular, there are constants C_j, $1\le j\le 3$, such that

$$\begin{aligned} |u(\cdot,t)|_2 &\le C_1(1+t)^{-1/4},\\ |u(\cdot,t)|_\infty &\le C_2(1+t)^{-1/2},\\ |u_x(\cdot,t)|_2 &\le C_3(1+t)^{-3/4}, \end{aligned} \tag{2.1}$$

for all $t\ge 0$. If $p<4$, the same result holds regarding equation (1.1). If $p\ge 4$ and the datum f is not too large, then global existence and the decay rates (2.1) still hold.

3. Some preliminary results. The well-posedness theory and a couple of technical results connected with the linear semigroup corresponding to (1.2) without its nonlinear term are presented here. All of these will find use later in our development.

The first result is essentially in Kato [15] and extends earlier work in [9], [14] and [25].

Proposition 3.1. *Suppose $F:\mathbb{R}\to\mathbb{R}$ is C^∞, and consider the initial-value problem*

$$u_t + u_x + (F(u))_x - \nu u_{xx} + u_{xxx} = 0, \quad (x\in\mathbb{R},\ t\ge 0), \tag{3.1a}$$

$$u(x,0) = f(x), \qquad (x\in\mathbb{R}). \tag{3.1b}$$

(a) *Problem (3.1) is locally well-posed in $H^r(\mathbb{R})$ for any $r>\frac{3}{2}$.*
(b) *If $\limsup_{|s|\to\infty}|s|^{-6}\Lambda(s)\le 0$, where $\Lambda'(s)=F(s)$ and $\Lambda(0)=0$, then problem (3.1) is globally well-posed in $H^r(\mathbb{R})$ for all $r\ge 2$. Moreover, the solution u lies in $C_b(\mathbb{R}^+;H^1(\mathbb{R}))$.*

(c) *For any nonlinearity F, there exists a number $\gamma_F > 0$ such that* (3.1) *is globally well posed in $H^r(\mathbb{R}) \cap \{f : \|f\|_1 < \gamma_F\}$. For data in the last-mentioned class, the corresponding solution u lies in $C_b(\mathbb{R}^+; H^1(\mathbb{R}))$.*

(d) *If $f \in W_1^k(\mathbb{R}) \cap H^r(\mathbb{R})$ and $r \geq 2$, $\nu > 0$, then $u \in C(0,T; W_1^k(\mathbb{R}))$ for any $T > 0$ for which the solution exists in $H^r(\mathbb{R})$.*

It deserves remark that more subtle results are available for certain classes of GKdV equations with $\nu = 0$. A good summary of this recent theory may be found in Kenig, Ponce & Vega [16]. The relatively straightforward results quoted above suffice for the present purposes.

Similar theory is available for equations of type (1.2). The following is a straightforward generalization of the results in [1].

Proposition 3.2. *Suppose $F : \mathbb{R} \to \mathbb{R}$ is C^∞ and consider the initial-value problem*

$$u_t + u_x + (F(u))_x - \nu u_{xx} - u_{xxt} = 0, \quad (x \in \mathbb{R},\ t \geq 0), \tag{3.2a}$$

$$u(x,0) = f(x), \qquad (x \in \mathbb{R}). \tag{3.2b}$$

Problem (3.2) *is globally well-posed in $H^s(\mathbb{R})$ for $s \geq 1$. The solution lies in $C_b(\mathbb{R}^+; H^1(\mathbb{R}))$ and in $C^k(0,T; H^s(\mathbb{R}))$ for all $k \geq 0$. If $f \in W_1^k(\mathbb{R})$ then u and all of its temporal derivatives lie in $C(0,T; W_1^k(\mathbb{R}))$, for any $T > 0$.*

The linearized RLW-Burgers initial-value problem

$$w_t + w_x - \nu w_{xx} - w_{xxt} = 0, \tag{3.3a}$$

$$w(x,0) = f(x), \tag{3.3b}$$

was also discussed in [1]. This problem can easily be solved by formally taking the Fourier transform of equation (3.3a) with respect to the spatial variable x. One deduces that for $f \in L_2$,

$$\hat{w}(y,t) = \exp\big(\frac{-\nu y^2 t - iyt}{1+y^2}\big)\hat{w}(y,0), \tag{3.4}$$

and therefore that

$$w(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\big(\frac{-\nu y^2 t - iyt}{1+y^2} + iyx\big)\hat{f}(y)\,dy. \tag{3.5}$$

The integral on the right-hand side of (3.5) will be denoted by $S(t)f(x)$. Here are some straightforward results about the decay of solutions of (3.3) (see [1, §4]).

Lemma 3.3. *If $f \in H^1(\mathbb{R}) \cap L_1(\mathbb{R})$, then*

(a) $$\lim_{t\to\infty} t^{\frac{1}{2}} \int_{-\infty}^{\infty} w^2(x,t)\,dx = \lim_{t\to\infty} t^{\frac{1}{2}} |S(t)f(x)|_2^2 = (8\nu\pi)^{-\frac{1}{2}} \Big(\int_{-\infty}^{\infty} f(x)dx\Big)^2,$$

and

(b) $$\lim_{t\to\infty} t^{\frac{3}{2}} \int_{-\infty}^{\infty} w_x^2(x,t)\,dx = (128\nu^3\pi)^{-\frac{1}{2}} \Big(\int_{-\infty}^{\infty} f(x)dx\Big)^2.$$

The following lemma will be useful in establishing the main theorem about equations of type (1.2).

Lemma 3.4. *Let ϕ be defined by its Fourier transform $\hat{\phi}$ as*

$$\hat{\phi}(y,r) = \frac{1}{1+y^2}\exp\big(\big(\frac{\nu y^2+iy}{1+y^2}\big)r\big).$$

Then it follows that

$$\sup_{t\geq 0}\int_{-\infty}^{\infty}|\phi(x,-t)|dx < \infty. \tag{3.6}$$

If instead, $\hat{\phi}$ is given by

$$\hat{\phi}(y,r) = \frac{y}{1+y^2}\exp\big(\big(\frac{\nu y^2+iy}{1+y^2}\big)r\big),$$

then it follows that

$$\sup_{t\geq 0} t^{1/2}\int_{-\infty}^{\infty}|\phi(x,-t)|dx < \infty. \tag{3.7}$$

Proof. The proofs of (3.6) and (3.7) are very similar, and so we content ourselves with a demonstration of the former. The estimation of $|\phi(x,-t)|_1$ is made by breaking the range of integration into pieces as follows:

$$\int_{-\infty}^{\infty}|\phi(x,-t)|\,dx = \int_{|x|\leq 1}|\phi(x,-t)|\,dx + \int_{|x|\geq 1}|\phi(x,-t)|\,dx. \tag{3.8}$$

Since the range of integration in the first term on the right-hand side of (3.8) is bounded, it suffices to show that $|\phi(x,-t)|$ is bounded on this range, independently of $t\geq 0$. But by the definition of ϕ, we see that

$$\begin{aligned}|\phi(x,-t)| &= \Big|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{ixy}}{1+y^2}\exp\big(\frac{-\nu y^2 t - iyt}{1+y^2}\big)\,dy\Big| \\ &\leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{1+y^2}\exp\big(\frac{-\nu y^2 t}{1+y^2}\big)\,dy \leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{1+y^2}\,dy \leq C.\end{aligned}$$

(N.B. Sharper estimation shows that $|\phi(x,-t)|\leq Ct^{-1/2}$ as $t\to\infty$.)

To control the second term on the right-hand side of (3.8), one may follow a line of argument suggested in [1]. For convenience, write $h(y,t)$ for $\hat{\phi}(y,-t)$. Integration by parts shows that

$$\sqrt{2\pi}\phi(x,-t) = -\int_{-\infty}^{\infty}\frac{e^{ixy}}{ix}\partial_y h(y,t)\,dy = -\int_{-\infty}^{\infty}\frac{e^{ixy}}{x^2}\partial_y^2 h(y,t)\,dy.$$

If $|y|\geq 1$, it is easy to check that

$$|\partial_y^2 h(y,t)| \leq \frac{Ct^2 e^{-\frac{\nu t}{2}}}{(1+|y|)^3},$$

and hence that

$$\sqrt{2\pi}|\phi(x,-t)| \le \Big|\int_{|y|\ge 1} \frac{e^{ixy}}{x^2}\partial_y^2 h(y,t)\,dy\Big| + \Big|\int_{|y|\le 1} \frac{e^{ixy}}{x^2}\partial_y^2 h(y,t)\,dy\Big|$$
$$\le C\frac{t^2}{x^2}e^{-\frac{\nu t}{2}} + \Big|\int_{-1}^{1} \frac{e^{ixy}}{x^2}\partial_y^2 h(y,t)\,dy\Big|.$$

Note that since

$$\partial_y h(y,t) = \big[-\frac{2y}{(1+y^2)^2} - t\frac{2\nu y + i - y^2 i}{(1+y^2)^3}\big]\exp\big(-\frac{\nu y^2 t + iyt}{1+y^2}\big), \tag{3.9}$$

it follows that

$$|\partial_y h(\pm 1, t)| \le C\frac{t}{x^2}e^{-\frac{\nu t}{2}},$$

and therefore that

$$\sqrt{2\pi}|\phi(x,-t)| \le C\frac{t^2}{x^2}e^{-\frac{\nu t}{2}} + \Big|\int_{-1}^{1} \frac{ie^{ixy}}{x}\partial_y h(y,t)\,dy\Big|. \tag{3.10}$$

Let us define

$$H(x,t) = \int_{-1}^{1} \frac{ie^{ixy}}{x}\partial_y h(y,t)\,dy,$$

and then rewrite it as

$$H(x,t) = \frac{i}{x}\int_{-1}^{1} e^{ixy-ity}\big[-\frac{2y}{(1+y^2)^2} - t\frac{2\nu y + i - y^2 i}{(1+y^2)^3}\big]\exp\big(\frac{-\nu y^2 t + iy^3 t}{1+y^2}\big)dy. \tag{3.11}$$

Integrating by parts twice leads to the estimate

$$|H(x,t)| \le \frac{Ct^{1/2}}{|x(x-t)|} + \frac{Ct^{3/2}}{|x(x-t)^2|}.$$

Hence to prove the lemma, it suffices to show that

$$\int_{|x|\ge 1} |H(x,t)|dx \le C.$$

Divide the range of integration into four pieces, namely $(-\infty,-1)$, $(1, t-\sqrt{t})$, $(t+\sqrt{t},\infty)$ and $(t-\sqrt{t}, t+\sqrt{t})$. The arguments for the integral over the first three intervals are similar, and therefore only one is worked out in detail.

$$\int_{-\infty}^{-1} |H(x,t)|dx \le C\int_{-\infty}^{-1}\Big(\frac{t^{1/2}}{|x(x-t)|} + \frac{t^{3/2}}{|x(x-t)^2|}\Big)dx$$
$$= Ct^{-1/2}\int_{1/t}^{\infty}\Big(\frac{1}{y(y+1)} + \frac{1}{y(y+1)^2}\Big)dy \le C\frac{\ln t}{t^{1/2}} \le C,$$

for values of t away from 0. To estimate the integral over $(t-\sqrt{t}, t+\sqrt{t})$, use (3.11) to ascertain that

$$|H(x,t)| \leq \frac{C}{|x|}\int_0^1 (1+t)\exp\big(\frac{-\nu y^2 t}{2}\big)\,dy \leq Ct^{1/2}|x|^{-1},$$

whence

$$\int_{t-\sqrt{t}}^{t+\sqrt{t}} |H(x,t)|\,dx \leq Ct^{1/2}\Big(\ln(t+\sqrt{t}) - \ln(t-\sqrt{t})\Big) \leq C.$$

The proof of the lemma is complete. □

4. Some non-optimal results. In this section, interest will be focused on the decay rate experienced by solutions of the initial-value problem (1.2)–(1.3). Although some of the following results are true for $p=1$, it is assumed that $p \geq 2$ henceforth. We will prove here that $|u(\cdot,t)|_4 = 0(t^{-\frac{1}{8}})$ and $|u_x(\cdot,t)|_2 = 0(t^{-\frac{1}{4}})$ as $t \to +\infty$. In Section 5, these preliminary rates will aid in obtaining optimal rates.

Lemma 4.1. *Let $f \in H^2(\mathbb{R})$. Then $u_x, u_{xx} \in L_2(\mathbb{R}\times\mathbb{R}^+)$ and $u \in C_b(\mathbb{R}^+; H^2)$.*

Proof. Multiply equation (1.2) by u and integrate over $\mathbb{R}\times[0,t]$. After integrations by parts, there appears the equation

$$\|u(\cdot,t)\|_1^2 + 2\nu\int_0^t |u_x(\cdot,\tau)|_2^2 d\tau = \|f\|_1^2. \tag{4.1}$$

Hence, $\|u(\cdot,t)\|_1 \leq \|f\|_1$ for all $t \geq 0$ and $u_x \in L_2(\mathbb{R}\times\mathbb{R}^+)$. In particular, it follows that

$$|u(\cdot,t)|_\infty^2 \leq |u(\cdot,t)|_2|u_x(\cdot,t)|_2 \leq \|f\|_1^2. \tag{4.2}$$

Multiply equation (1.2) by u_{xx} and integrate over $\mathbb{R}\times[0,t]$. Following integration by parts, one obtains

$$\begin{aligned} |u_x(\cdot,t)|_2^2 + |u_{xx}(\cdot,t)|_2^2 + 2\nu\int_0^t |u_{xx}(\cdot,\tau)|_2^2\,d\tau \\ = |f'|_2^2 + |f''|_2^2 + \int_0^t\int_{-\infty}^\infty 2u^p u_x u_{xx}\,dx\,d\tau. \end{aligned} \tag{4.3}$$

Because of (4.2), it follows that

$$\begin{aligned} \int_{-\infty}^\infty 2u^p u_x u_{xx}\,dx &\leq \nu|u_{xx}(\cdot,t)|_2^2 + \frac{1}{\nu}\int_{-\infty}^\infty u^{2p}u_x^2\,dx \\ &\leq \nu|u_{xx}(\cdot,t)|_2^2 + \frac{1}{\nu}|u(\cdot,t)|_\infty^{2p}|u_x(\cdot,t)|_2^2 \\ &\leq \nu|u_{xx}(\cdot,t)|_2^2 + \frac{1}{\nu}\|f\|_1^{2p}|u_x(\cdot,t)|_2^2. \end{aligned} \tag{4.4}$$

Hence, (4.3) takes the form

$$|u_x(\cdot,t)|_2^2 + |u_{xx}(\cdot,t)|_2^2 + \nu\int_0^t |u_{xx}(\cdot,\tau)|_2^2\,d\tau \leq \|f\|_2^2 + \frac{1}{\nu}\|f\|_1^{2p}\int_0^t |u_x(\cdot,\tau)|_2^2\,d\tau. \tag{4.5}$$

Since $u_x \in L_2(\mathbb{R}\times\mathbb{R}^+)$, (4.5) implies that $u_{xx} \in L_2(\mathbb{R}\times\mathbb{R}^+)$. □

Corollary 4.2. *Let u be the solution of (1.2) corresponding to initial data $f \in H^2(\mathbb{R})$. Then it follows that*

$$|u_x(\cdot,t)|_2, \quad |u_{xx}(\cdot,t)|_2 \to 0, \qquad \text{as} \qquad t \to +\infty,$$

and

$$|u(\cdot,t)|_\infty \to 0 \qquad \text{as} \qquad t \to +\infty.$$

Proof. First note that $U(t) = |u_x(\cdot,t)|_2^2 + |u_{xx}(\cdot,t)|_2^2$ has a limit U_∞ as $t \to +\infty$ since both the temporal integrals in (4.3) are convergent as $t \to \infty$ because of (4.4) and (4.5). Since both summands comprising $U(t)$ lie in $L_1(\mathbb{R}^+)$ on account of Lemma 4.1, it follows that the limit U_∞ must be zero, and thus the first statement is verified. It then follows that

$$|u(\cdot,t)|_\infty^2 \le |u(\cdot,t)|_2 |u_x(\cdot,t)|_2 \le \|f\|_1 |u_x(\cdot,t)|_2,$$

and since the right-hand side of this inequality tends to zero as $t \to +\infty$, the result follows. □

Lemma 4.3. *Let u be the solution of (1.2) corresponding to initial data $f \in H^2(\mathbb{R})$. Then*

$$u_t, \; u_{xt} \in L_2(\mathbb{R} \times \mathbb{R}^+).$$

Proof. Multiply (1.2) by u_t and integrate the result over $\mathbb{R} \times [0,t)$. After integration by parts and using Lemma 4.1, one adduces that

$$\begin{aligned}
&\frac{\nu}{2}|u_x(\cdot,t)|_2^2 + \int_0^t \left[|u_t(\cdot,\tau)|_2^2 + |u_{xt}(\cdot,\tau)|_2^2\right] d\tau \\
=&\frac{\nu}{2}|f'|_2^2 - \int_0^t \int_{-\infty}^{\infty} u_t(u_x + u^p u_x)\, dx\, d\tau \\
\le&\frac{\nu}{2}|f'|_2^2 + \frac{1}{2}\int_0^t |u_t(\cdot,\tau)|_2^2 d\tau + \frac{1}{2}\int_0^t \int_{-\infty}^{\infty} (1+u^p)^2 u_x^2\, dx\, d\tau \\
\le&\frac{\nu}{2}|f'|_2^2 + \frac{1}{2}\int_0^t |u_t(\cdot,\tau)|_2^2 d\tau + \frac{1}{2}(1+\|f\|_1^p)^2 \int_0^t |u_x(\cdot,\tau)|_2^2 d\tau.
\end{aligned} \tag{4.6}$$

Since u_x, $u_{xx} \in L_2(\mathbb{R} \times \mathbb{R}^+)$ by Lemma 4.1, the result follows. □

The next results will yield decay rates, albeit non-optimal ones. These will be parlayed into optimal rates in the next section.

Lemma 4.4. *If u is the solution of (1.2) corresponding to initial data $f \in H^2(\mathbb{R})$, then*

$$\lim_{t \to +\infty} t^{\frac{1}{2}} \left[|u_x(\cdot,t)|_2^2 + |u(\cdot,t)|_4^4\right] = 0,$$

and

$$\lim_{t \to +\infty} t^{\frac{1}{2}} \int_t^{+\infty} |u_{xt}(\cdot,t)|_2^2 = 0.$$

Proof. Multiply (1.2) by the combination $u_t + u_x + \frac{1}{\nu}u_{xt}$ and integrate the result over $\mathbb{R}$. After integration by parts, one has

$$(\frac{\nu}{2} + \frac{1}{2\nu})\frac{d}{dt}\{|u_x(\cdot,t)|_2^2\} + |u_{xt}(\cdot,t)|_2^2 + |u_t(\cdot,t) + u_x(\cdot,t)|_2^2 = -\int_{-\infty}^{\infty}\{(u_t + u_x)u^p u_x + \frac{1}{\nu}u_{xt}u^p u_x\}dx. \tag{4.7}$$

Then (1.2) is multiplied by u^3 and the result integrated over $\mathbb{R}$ to obtain

$$\frac{1}{4}\frac{d}{dt}|u(\cdot,t)|_4^4 + 3\nu|u(\cdot,t)u_x(\cdot,t)|_2^2 = -3\int_{-\infty}^{\infty}\{u^2 u_x u_{xt}\}dx. \tag{4.8}$$

Add (4.7) and (4.8) together and then use Young's inequality to derive the inequality

$$\begin{aligned}(\frac{\nu}{2} + \frac{1}{2\nu})&\frac{d}{dt}|u_x(\cdot,t)|_2^2 + \frac{1}{4}\frac{d}{dt}|u(\cdot,t)|_4^4 + |u_{xt}(\cdot,t)|_2^2 \\ &+ 3\nu|u(\cdot,t)u_x(\cdot,t)|_2^2 + |u_t(\cdot,t) + u_x(\cdot,t)|_2^2 \\ &= -\int_{\infty}^{\infty}\{(u_t + u_x + \frac{1}{\nu}u_{xt})u^p u_x + 3u^2 u_x u_{xt}\}dx \\ &\leq \frac{1}{2}|u_t(\cdot,t) + u_x(\cdot,t)|_2^2 + \frac{1}{2}|u_{xt}(\cdot,t)|_2^2 \\ &\quad + [(\frac{1}{\nu^2} + \frac{1}{2})|u(\cdot,t)|_\infty^{2p-2} + 9|u(\cdot,t)|_\infty^2]|u(\cdot,t)u_x(\cdot,t)|_2^2.\end{aligned} \tag{4.9}$$

Note that since $2p - 2 > 0$, one may use Corollary 4.2 to infer the existence of a positive value T such that for $t \geq T$,

$$(\frac{1}{\nu^2} + \frac{1}{2})|u(\cdot,t)|_\infty^{2p-2} \leq \nu \quad \text{and} \quad 9|u(\cdot,t)|_\infty^2 \leq \nu. \tag{4.10}$$

In consequence of (4.9) and (4.10), it is assured that

$$\frac{d(\Gamma(t))}{dt} + \nu|u(\cdot,t)u_x(\cdot,t)|_2^2 + \frac{1}{2}|u_{xt}(\cdot,t)|_2^2 \leq 0, \tag{4.11}$$

for $t \geq T$, where

$$\Gamma(t) = (\frac{\nu}{2} + \frac{1}{2\nu})|u_x(\cdot,t)|_2^2 + \frac{1}{4}|u(\cdot,t)|_4^4.$$

Since $|u(\cdot,t)|_2$ is bounded, u_x and $u_{xx} \in L_2(\mathbb{R}\times\mathbb{R}^+)$, and

$$|u(\cdot,t)|_4^4 \leq |u(\cdot,t)|_\infty^2|u(\cdot,t)|_2^2 \leq |u_x(\cdot,t)|_2|u(\cdot,t)|_2^3,$$

it is inferred that $\Gamma(t) \in L_2(\mathbb{R}^+)$. At this point, the line of argument in [1] may be followed to finish the proof of the lemma. In fact, because of (4.11), one has

$$\int_\tau^{+\infty}\Gamma^2(s)ds \geq \int_\tau^t \Gamma^2(s)ds \geq (t-\tau)\Gamma^2(t), \qquad \text{for } t > \tau \text{ and } \tau \geq T.$$

It follows that

$$\int_{\tau}^{+\infty} \Gamma^2(s)ds \geq \limsup_{t\to+\infty} \, t\Gamma^2(t)$$

for any τ large enough. Since $\Gamma(t) \in L_2(\mathbb{R}^+)$, the left-hand side of the above inequality can be made as small as desired by choosing τ large enough, and thus one has the desired result

$$\lim_{t\to+\infty} t^{\frac{1}{2}}\left(|u_x(\cdot,t)|_2^2 + |u(\cdot,t)|_4^4\right) = 0. \tag{4.12}$$

To obtain the second result, choose T so large that for $t \geq T$, the inequality (4.9) yields

$$\frac{d\Gamma(t)}{dt} + \frac{1}{2}|u_{xt}(\cdot,t)|_2^2 \leq 0. \tag{4.13}$$

Integrating this relation over the temporal interval $[t,+\infty)$ for $t \geq T$, it appears that

$$-\Gamma(t) + \frac{1}{2}\int_t^{+\infty} |u_{xt}(\cdot,\tau)|_2^2\, d\tau \leq 0,$$

and hence by the result (4.12),

$$\int_t^{+\infty} |u_{xt}(\cdot,\tau)|_2^2 d\tau = o(t^{-\frac{1}{2}}) \qquad \text{as } t\to+\infty.$$

The proof of the lemma is thus complete. □

From Lemma 4.4, a decay rate for the L_∞-norm of the solution u can be derived. Indeed, for any $x \in \mathbb{R}$,

$$|u(x,t)|^3 = 3\left|\int_x^{+\infty} u^2 u_x dx\right| \leq \frac{3}{2}\int_{-\infty}^{\infty}(u^4+u_x^2)dx = o(t^{-\frac{1}{2}})$$

as $t\to+\infty$, and it follows that

$$|u(\cdot,t)|_\infty = o(t^{-\frac{1}{6}}) \quad \text{as } t\to+\infty.$$

Corollary 4.5. *Let u be the solution of* (1.2) *satisfying the conditions in Lemma* 4.1 *for the initial data f. Then*

$$|u(\cdot,t)|_\infty = o(t^{-\frac{1}{6}}) \quad \textit{as} \quad t\to+\infty.$$

The following lemma plays a key role in the proof of our main theorem.

Lemma 4.6. *Let u be the solution of* (1.2) *corresponding to the initial data $f \in H^2(\mathbb{R}) \cap W_1^2(\mathbb{R})$. Then for any fixed, positive constant α,*

$$\sup_{0\leq t<\infty} t\int_{|y|\leq\sqrt{\frac{\alpha}{t}}} y^2|\hat{u}(y,t)|^2 dy < \infty.$$

Proof. Differentiate equation (1.2) with respect to x, take the Fourier transform of the resulting relation with respect to the spatial variable x, and solve the resulting ordinary differential equation by the variation of constants formula to reach the expression

$$
\begin{aligned}
y\hat{u}(y,t) &= y\exp\big(\frac{-\nu y^2 t - iyt}{1+y^2}\big)\hat{f}(y) \\
&\quad - \frac{i}{p+1}\int_0^t \frac{y^2}{1+y^2}\exp\big(\frac{-\nu y^2 - iy}{1+y^2}(t-\tau)\big)\widehat{u^{p+1}}(y,\tau)d\tau.
\end{aligned} \tag{4.14}
$$

It follows that

$$
\begin{aligned}
|y\hat{u}(y,t)|^2 &\le 2y^2\exp\big(\frac{-2\nu y^2 t}{1+y^2}\big)|\hat{f}(y)|^2 \\
&\quad + \frac{2}{(p+1)^2}\Big(\int_0^t \frac{y^2}{1+y^2}\exp\big(\frac{-\nu y^2}{1+y^2}(t-\tau)\big)\big|\widehat{u^{p+1}}(y,\tau)\big|d\tau\Big)^2 \\
&\le 2y^2\exp\big(\frac{-2\nu y^2 t}{1+y^2}\big)|\hat{f}(y)|^2 \\
&\quad + \frac{2t}{(p+1)^2}\int_0^t \frac{y^4}{(1+y^2)^2}\exp\big(\frac{-2\nu y^2}{1+y^2}(t-\tau)\big)\big|\widehat{u^{p+1}}(y,\tau)\big|^2 d\tau.
\end{aligned} \tag{4.15}
$$

Note that

$$
\int_{|y|\le\sqrt{\frac{\alpha}{t}}} y^2\exp\big(\frac{-2\nu y^2 t}{1+y^2}\big)|\hat{f}(y)|^2 dy \le C(|f|_2)t^{-1}. \tag{4.16}
$$

Because of Lemma 4.4 and the assumption that $p \ge 2$, it transpires that

$$
|\widehat{u^{p+1}}(\cdot,\tau)|_\infty \le |u^{p+1}(\cdot,\tau)|_1 \le |u(\cdot,\tau)|_\infty^{p-2}|u(\cdot,\tau)|_4^2|u(\cdot,\tau)|_2 \le C\tau^{-1/4}, \tag{4.17}
$$

as $\tau \to +\infty$, where C is a constant. It follows from this that for $t \ge \alpha$,

$$
\begin{aligned}
&\int_{|y|\le\sqrt{\frac{\alpha}{t}}}\int_0^t \frac{y^4}{(1+y^2)^2}\exp\big(\frac{-2\nu y^2}{1+y^2}(t-\tau)\big)\big|\widehat{u^{p+1}}(y,\tau)\big|^2 d\tau dy \\
&\quad \le C\big(\frac{\alpha}{\sqrt{t}}\big)^4\int_0^t \frac{1}{\sqrt{\tau}}\int_{|y|\le\sqrt{\frac{\alpha}{t}}}\exp\big(-\nu y^2(t-\tau)\big)dyd\tau \\
&\quad \le Ct^{-2}\int_0^t \frac{1}{\sqrt{\tau}}\int_{-\infty}^{\infty} e^{-\nu y^2(t-\tau)}dyd\tau \\
&\quad \le Ct^{-2}\int_0^t \frac{1}{\sqrt{t-\tau}}\frac{1}{\sqrt{\tau}}d\tau \\
&\quad \le Ct^{-2}\int_0^1 \frac{dw}{\sqrt{w(1-w)}} \le Ct^{-2}.
\end{aligned} \tag{4.18}
$$

Combining (4.15), (4.16) and (4.18) gives

$$
\int_{|y|\le\sqrt{\frac{\alpha}{t}}} |y\hat{u}(y,t)|^2 dy \le Ct^{-1}, \tag{4.19}
$$

where C is independent of t. The lemma is proved. □

Lemma 4.7. *If $f \in H^2(\mathbb{R}) \cap W_1^2(\mathbb{R})$ and $p \geq 2$, then the solution of (1.2) corresponding to the initial data f satisfies*

$$t^{\frac{1}{2}}|u_x(\cdot,t)|_2 \leq C \quad \text{and} \quad t^{\frac{1}{2}}|u_{xx}(\cdot,t)|_2 \leq C,$$

for all $t \geq 0$, where C is independent of t.

Proof. If (1.2) is multiplied by u^5 and the result integrated over $\mathbb{R}$, there appears

$$\frac{1}{6}\frac{d}{dt}|u(\cdot,t)|_6^6 + 5\nu|u^2(\cdot,t)u_x(\cdot,t)|_2^2 = -\int_{-\infty}^{\infty} 5u^4 u_x u_{xt}\,dx. \tag{4.20}$$

Multiply (4.20) by a constant b and add the result and (4.7) together. Using Young's inequality then implies that

$$\begin{aligned}
&(\frac{\nu}{2}+\frac{1}{2\nu})\frac{d}{dt}|u_x(\cdot,t)|_2^2 + \frac{b}{6}\frac{d}{dt}|u(\cdot,t)|_6^6 + |u_{xt}(\cdot,t)|_2^2 \\
&\quad + 5b\nu|u^2(\cdot,t)u_x(\cdot,t)|_2^2 + |u_t(\cdot,t)+u_x(\cdot,t)|_2^2 \\
&= -\int_{\infty}^{\infty}\big[(u_t+u_x+\frac{1}{\nu}u_{xt})u^p u_x + 5bu^4u_xu_{xt}\big]dx \\
&\leq \frac{1}{2}|u_t(\cdot,t)+u_x(\cdot,t)|_2^2 + \frac{1}{2}|u_{xt}(\cdot,t)|_2^2 \\
&\quad + \big[(\frac{1}{\nu^2}+\frac{1}{2})|u(\cdot,t)|_\infty^{2p-4} + 25b^2|u(\cdot,t)|_\infty^4\big]|u^2(\cdot,t)u_x(\cdot,t)|_2^2.
\end{aligned} \tag{4.21}$$

Since $p \geq 2$, we may choose b large enough that

$$4b\nu \geq (\frac{1}{\nu^2}+\frac{1}{2})\sup_{0\leq t<\infty}|u(\cdot,t)|_\infty^{2p-4} + 1. \tag{4.22}$$

Then choose T so that for $t \geq T$,

$$25b^2|u(\cdot,t)|_\infty^4 \leq 1. \tag{4.23}$$

With these restrictions on b and T, it is assured that

$$\frac{d}{dt}\big((\frac{\nu}{2}+\frac{1}{2\nu})|u_x(\cdot,t)|_2^2 + \frac{b}{6}|u(\cdot,t)|_6^6\big) + b\nu|u^2(\cdot,t)u_x(\cdot,t)|_2^2 \leq 0, \tag{4.24}$$

for $t \geq T$. Moreover, if (1.2) is multiplied by u_{xx} and then integrated over $\mathbb{R}$, one sees readily that

$$\begin{aligned}
&\frac{d}{dt}\big(|u_x(\cdot,t)|_2^2 + |u_{xx}(\cdot,t)|_2^2\big) + 2\nu|u_{xx}(\cdot,t)|_2^2 \\
&= \int_{-\infty}^{\infty} 2u^p u_x u_{xx}\,dx \leq \nu|u_{xx}(\cdot,t)|_2^2 + \frac{1}{\nu}|u(\cdot,t)|_\infty^{2p-4}|u^2(\cdot,t)u_x(\cdot,t)|_2^2.
\end{aligned} \tag{4.25}$$

Now multiply (4.24) by a constant and add the result and (4.25) together to reach the inequality

$$\frac{d}{dt}\big(|u_x(\cdot,t)|_2^2 + |u_{xx}(\cdot,t)|_2^2 + |u(\cdot,t)|_6^6\big) \leq -A|u_{xx}(\cdot,t)|_2^2 - B|u^2(\cdot,t)u_x(\cdot,t)|_2^2. \tag{4.26}$$

Formula (4.26) holds for $t \geq T$, where A, B and T are suitably chosen positive constants. The inequality (4.26) leads to the related differential inequality

$$\begin{aligned} &\frac{d}{dt}\Big(t^2\big(|u_x(\cdot,t)|_2^2 + |u_{xx}(\cdot,t)|_2^2 + |u(\cdot,t)|_6^6\big)\Big) \\ &\leq 2t\big(|u_x(\cdot,t)|_2^2 - \frac{tA}{4}|u_{xx}(\cdot,t)|_2^2\big) + 2t\big(|u(\cdot,t)|_6^6 - \frac{tB}{2}|u^2(\cdot,t)u_x(\cdot,t)|_2^2\big), \end{aligned} \tag{4.27}$$

if $t \geq \bar{T} = max\{T, \frac{4}{A}\}$. By using Parseval's theorem and Lemma 4.6, the first term on the right hand-side of (4.27) can be bounded above, independently of $t \geq \bar{T}$; viz.

$$t\big(|u_x(\cdot,t)|_2^2 - \frac{tA}{4}|u_{xx}(\cdot,t)|_2^2\big) \leq t\int_{|y|\leq\sqrt{\frac{4}{At}}} y^2|\hat{u}(y,t)|^2dy \leq C. \tag{4.28}$$

If we let $v(x,t) = u^3(x,t)$, then $v_x = 3u^2u_x$. Using Parseval's theorem again, the second term on the right hand-side of (4.27) can be bounded above as follows:

$$t\big(|u(\cdot,t)|_6^6 - \frac{tB}{2}|u^2(\cdot,t)u_x(\cdot,t)|_2^2\big) \leq t\int_{|y|\leq\sqrt{\frac{18}{Bt}}} |\hat{v}(y,t)|^2dy \leq C, \tag{4.29}$$

since

$$|\hat{v}(\cdot,t)|_\infty \leq |u(x,t)|_3^3 \leq |u(\cdot,t)|_4^2|u(\cdot,t)|_2 \leq Ct^{-1/4}.$$

Because of (4.28) and (4.29), (4.27) reduces to

$$\frac{d}{dt}\Big(t^2\big(|u_x(\cdot,t)|_2^2 + |u_{xx}(\cdot,t)|_2^2 + |u(\cdot,t)|_6^6\big)\Big) \leq C,$$

whence

$$|u_x(\cdot,t)|_2^2 + |u_{xx}(\cdot,t)|_2^2 + |u(\cdot,t)|_6^6 \leq Ct^{-1}.$$

Corollary 4.8. *If u is the solution of* (1.2) *corresponding to initial data f in $H^2(\mathbb{R}) \cap W_1^2(\mathbb{R})$, then $|u(\cdot,t)|_\infty = o(t^{-\frac{1}{4}})$ as $t \to +\infty$.*

Proof. This follows from the last result since

$$|u(\cdot,t)|_\infty^2 \leq |u_x(\cdot,t)|_2|u(\cdot,t)|_2 \leq Ct^{-1/2}|u(\cdot,t)|_2. \tag{4.30}$$

5. Decay rates for the GRLW-Burgers equation. With the help of the non-optimal results derived in Section 4 and the preliminary results in Section 3, we are now ready to prove the main result concerning the RLW-Burgers equation (1.2).

Theorem 5.1. *If $f \in H^2(\mathbb{R}) \cap W_1^2(\mathbb{R})$ and $p \geq 2$, then the solution of* (1.2) *corresponding to initial data f satisfies*

$$|u(\cdot,t)|_2 \leq C(1+t)^{-\frac{1}{4}}, \tag{5.1}$$

for all $t \geq 0$, where C is independent of t.

Proof. According to [1, Theorem 5.1], the desired result (5.1) is equivalent to showing that

$$\sup_{0\leq t<\infty} |u(\cdot,t))|_1 < \infty. \tag{5.2}$$

Proceeding as in the derivation of (4.14), except without first differentiating, leads to the following formula for the solution u :

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\big(\frac{-\nu y^2 t - iyt}{1+y^2} + iyx\big)\hat{f}(y)\,dy$$
$$- \frac{i}{\sqrt{2\pi}(p+1)}\int_0^t\int_{-\infty}^{\infty} \frac{y}{1+y^2}\exp\big(\frac{-\nu y^2 - iy}{1+y^2}(t-\tau) + iyx\big)\widehat{u^{p+1}}(y,\tau)\,dy\,d\tau. \tag{5.3}$$

The first term on the right-hand side of (5.3) is in $L_1(\mathbb{R})$ because of Lemma 3.4. In fact, if one lets

$$\begin{aligned} h(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\big(\frac{-\nu y^2 t - iyt}{1+y^2} + iyx\big)\hat{f}(y)\,dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{1+y^2}\exp\big(\frac{-\nu y^2 t - iyt}{1+y^2} + iyx\big)(1+y^2)\hat{f}(y)\,dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi(x-y,-t)\big(f(y) - f''(y)\big)dy, \end{aligned} \tag{5.4}$$

where $\hat{\phi}$ is defined in Lemma 3.4, then

$$\int_{-\infty}^{\infty} |h(x,t)|dx \leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} |\phi(x,-t)|\,dx \int_{-\infty}^{\infty} \big(|f(y)| + |f''(y)|\big)\,dy. \tag{5.5}$$

Lemma 3.4 asserts that

$$\int_{-\infty}^{\infty} |\phi(x,-t)|dx \leq C,$$

and so (5.5) implies that

$$\int_{-\infty}^{\infty} |h(x,t)| \leq C(\|f\|_{W_1^2(\mathbb{R})}). \tag{5.6}$$

In a similar manner, the second term on the right-hand side of (5.3) can be represented as

$$g(x,t) = \frac{i}{(p+1)\sqrt{2\pi}}\int_0^t\int_{-\infty}^{\infty} \psi(x-y,\tau-t)u^{p+1}(y,\tau)\,dy\,d\tau, \tag{5.7}$$

where

$$\hat{\psi}(y,-r) = \frac{y}{1+y^2}\exp\big(\frac{-\nu y^2 r - iyr}{1+y^2}\big).$$

In consequence, one sees that

$$\int_{-\infty}^{\infty} |g(x,t)|\,dx \le C\int_0^t |\psi(\cdot,\tau-t)|_1 \int_{-\infty}^{\infty} |u^{p+1}(y,\tau)|\,dy\,d\tau. \tag{5.8}$$

Note first that by Lemma 3.4,

$$|\psi(\cdot,-r)|_1 \le Cr^{-1/2}$$

for $r > 0$. Note also that

$$\int_{-\infty}^{\infty} |u^{p+1}(y,\tau)|\,dy \le C|u(\cdot,\tau)|_\infty^p |u(\cdot,\tau)|_1 \le C\tau^{-1/2}|u(\cdot,\tau)|_1, \tag{5.9}$$

where use has been made of Corollary 4.8 in the last step. Hence the left-hand side of (5.8) can be estimated above by

$$\int_{-\infty}^{\infty} |g(x,t)|\,dx \le C\int_0^t \frac{1}{\sqrt{t-\tau}}\frac{1}{\sqrt{\tau}}|u(\cdot,\tau)|_1\,d\tau. \tag{5.10}$$

Use of (5.5) and (5.10) leads to

$$|u(\cdot,t)|_1 \le C_1 + C_2\int_0^t \frac{1}{\sqrt{t-\tau}}\frac{1}{\sqrt{\tau}}|u(\cdot,\tau)|_1\,d\tau. \tag{5.11}$$

An application of Gronwall's lemma then gives

$$|u(\cdot,t)|_1 \le C_1\exp\Big(C_2\int_0^t \frac{1}{\sqrt{t-\tau}}\frac{1}{\sqrt{\tau}}\,d\tau\Big) \le C_1\exp\Big(C_2\int_0^1 \frac{dw}{\sqrt{(1-w)w}}\Big),$$

and the theorem is proved. □

Corollary 5.2. *If $f \in H^2(\mathbb{R})\cap W_1^2(\mathbb{R})$ and $p \ge 2$, then the solution of the initial-value problem for* (1.2) *with initial data f satisfies*

$$(a) \qquad t^{\frac{3}{4}}|u_x(\cdot,t)|_2 \le C, \quad t^{\frac{3}{4}}|u(\cdot,t)|_4^2 \le C,$$

and

$$(b) \qquad t^{\frac{1}{2}}|u(\cdot,t)|_\infty \le C,$$

for all $t \ge 0$, where the constants are independent of t.

Proof. A suitable linear combination of (4.11) and (4.25) leads to the differential inequality

$$\frac{d}{dt}\big(|u_x(\cdot,t)|_2^2 + |u_{xx}(\cdot,t)|_2^2 + |u(\cdot,t)|_4^4\big) \le -A|u_{xx}(\cdot,t)|_2^2 - B|u(\cdot,t)u_x(\cdot,t)|_2^2,$$

for $t \ge T$, where A, B and T are positive constants. It follows that

$$\begin{aligned}&\frac{d}{dt}\Big(t^2\big(|u_x(\cdot,t)|_2^2 + |u_{xx}(\cdot,t)|_2^2 + |u(\cdot,t)|_4^4\big)\Big)\\ &\le 2t\big(|u_x(\cdot,t)|_2^2 - \frac{tA}{4}|u_{xx}(\cdot,t)|_2^2 + |u(\cdot,t)|_4^4 - \frac{tB}{2}|u(\cdot,t)u_x(\cdot,t)|_2^2\big),\end{aligned} \tag{5.12}$$

for $t \geq \bar{T} = \max\{T, \frac{4}{A}\}$. By using Parseval's relation, it transpires that

$$\begin{aligned} t\big(|u_x(\cdot,t)|_2^2 - \frac{tA}{4}|u_{xx}(\cdot,t)|_2^2 + |u(\cdot,t)|_4^4 - \frac{tB}{2}|u(\cdot,t)u_x(\cdot,t)|_2^2\big) \\ \leq t\int_{|y|\leq\sqrt{\frac{4}{tA}}} y^2|\hat{u}(y,t)|^2dy + t\int_{|y|\leq\sqrt{\frac{8}{tB}}} |\widehat{u^2}(y,t)|^2dy \\ \leq Ct\Big(\sqrt{\frac{4}{tA}}\Big)^3 + Ct\sqrt{\frac{8}{tB}}\Big(\frac{1}{\sqrt{t}}\Big)^2 \leq \frac{C}{t^{1/2}}, \end{aligned} \tag{5.13}$$

where recourse has been taken to the inequality

$$|\widehat{u^2}(\cdot,t)|_\infty \leq |u^2(\cdot,t)|_1 = |u(\cdot,t)|_2^2 \leq \frac{C}{t^{1/2}}, \tag{5.14}$$

which is a consequence of Theorem 5.1. Using (5.13) in (5.12) shows that

$$\frac{d}{dt}\Big(t^2\big(|u_x(\cdot,t)|_2^2 + |u_{xx}(\cdot,t)|_2^2 + |u(\cdot,t)|_4^4\big)\Big) \leq \frac{C}{t^{1/2}},$$

from which one ascertains that

$$|u_x(\cdot,t)|_2^2 + |u_{xx}(\cdot,t)|_2^2 + |u(\cdot,t)|_4^4 \leq \frac{C}{t^{3/2}}. \tag{5.15}$$

Part (a) is thereby established.

To prove part (b), note that

$$|u(\cdot,t)|_\infty^2 \leq |u(\cdot,t)|_2|u_x(\cdot,t)|_2 \leq Ct^{-\frac{1}{4}}t^{-\frac{3}{4}} = Ct^{-1},$$

by using Theorem 5.1 and (5.15). Hence

$$|u(\cdot,t)|_\infty \leq Ct^{-\frac{1}{2}},$$

and the corollary is proved.

6. Decay rates for the GKdV-Burgers equation. Subject to the provisos mentioned in the informal statement of the principal results at the end of §2, the decay rates of solutions to the generalized Korteweg-de Vries-Burgers equation (1.1) are the same as those exhibited by solutions to (1.2). The overall structure of the proof of this assertion follows the general lines presented above in regard to the generalized regularized long-wave-Burgers equation (1.2). In consequence, we content ourselves with an outline of the steps in the proof with especial emphasis on aspects that differ from those appearing above.

The first and most crucial difference between (1.1) and (1.2) is visible already in Proposition 3.1 and Proposition 3.2. Unlike the situation that obtains for equation (1.2), equation (1.1) appears to admit a t-independent bound on the $H^1(\mathbb{R})$-norm of a solution only if $p < 4$ or if $\|f\|_1$ is not too large in case $p \geq 4$ (see [5]).

Lemma 3.3 has an exactly analogous statement and proof for equation (1.1) (see again [1, §4]). The analogous version of Lemma 3.4 also holds with

$$\hat{\phi}(y,r) = \exp\Big((\nu y^2 + iy - iy^3)r\Big) \quad \text{and} \quad \hat{\phi}(y,r) = y\exp\Big((\nu y^2 + iy - iy^3)r\Big)$$

replacing those that appear above (3.6) and (3.7), respectively. The proof is a little more complicated, but still straightforward.

The important Lemma 4.1 holds for the global solutions u of (1.1) whose existence is charted in Proposition 3.1. The proof relies on the same energy-type estimates, except that the t-independent H^1-bound promised in Proposition 3.1 is needed since the analog of (4.1) is

$$|u(\cdot,t)|_2^2 + 2\nu\int_0^t |u_x(\cdot,\tau)|_2^2\,d\tau = |f|_2^2, \tag{6.1}$$

and this relation on its own does not provide the essential L_∞-bound in (4.2). Moreover, the analog of formula (4.3) only contains $|u_x(\cdot,t)|_2^2 + 2\nu\int_0^t |u_{xx}(\cdot,\tau)|_2^2 d\tau$ on the left-hand side. Hence it is convenient to multiply (1.1) also by u_{xxxx} and integrate over $\mathbb{R}\times[0,t]$ to reach the relationship

$$\begin{aligned} &|u_{xx}(\cdot,t)|_2^2 + 2\nu\int_0^t |u_{xxx}(\cdot,\tau)|_2^2\,d\tau \\ &= 2\int_0^t\int_{-\infty}^{+\infty} u^p u_{xx}u_{xxx}\,dx\,d\tau + 2p\int_0^t\int_{-\infty}^{\infty} u^{p-1}u_x^2 u_{xxx}\,dx\,d\tau \\ &\le \nu\int_0^t |u_{xxx}(\cdot,\tau)|_2^2\,d\tau + \frac{C}{\nu}\int_0^t |u_{xx}(\cdot,\tau)|_2^2\,d\tau, \end{aligned} \tag{6.2}$$

where the constant C depends on the t-independent H^1-bound on u guaranteed by Proposition 3.1. One readily deduces from these relations that $u_x, u_{xx}, u_{xxx} \in L_2(\mathbb{R}\times\mathbb{R}^+)$, $u \in C_b(\mathbb{R}^+; H^2(\mathbb{R}))$, and the conclusions that $|u_x(\cdot,t)|_2$, $|u_{xx}(\cdot,t)|_2$, $|u(\cdot,t)|_\infty \to 0$ as $t\to+\infty$. With this information in hand, it is easy to demonstrate that $u_t \in L_2(\mathbb{R}\times\mathbb{R}^+)$, while the relation $u_{xxx}\in L_2(\mathbb{R}\times\mathbb{R}^+)$ takes the place of the condition on u_{xt} in Lemma 4.3. A new version of Lemma 4.4 valid for solutions of (1.1) now follows by essentially the same of energy inequalities (multiply (1.1) by $u_t + u_x - \frac{1}{\nu}u_{xx}$ and by u^3 and proceed as before). Naturally, the second conclusion is replaced by

$$\lim_{t\to+\infty} t^{\frac{1}{2}}\int_t^\infty |u_{xx}(\cdot,\tau)|_2^2 d\tau = 0.$$

Corollary 4.5 is then inferred to hold as stated for global solutions of (1.1). The same is true of Lemma 4.6, and again, the proof differs only slightly from that presented in Section 4 for solutions of (1.2). With these results in hand, Lemma 4.7 and Corollary 4.8 follow by the same arguments for solutions of (1.1) as for solutions of (1.2).

The main result as it applies to solutions of (1.1) is then deduced as it was for solutions of (1.2) by use of the preliminary results analogous to those in Section 4 together with the representation

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\big(-\nu y^2 t - iyt + iy^3 t + ixy\big) \hat{f}(y) dy$$
$$- \frac{i}{\sqrt{2\pi}(p+1)} \int_0^t \int_{-\infty}^{\infty} y \exp\Big((-\nu y^2 - iy + iy^3)(t-\tau) + ixy\Big) \widehat{u^{p+1}}(y,\tau) dy d\tau,$$

obtained by taking the Fourier transform of (1.1) with respect to the spatial variable x.

It is worth summarizing the foregoing commentary in a formal statement.

Theorem 6.1. *Suppose $p \geq 2$ in* (1.1) *and that the initial data f lies in $H^2(\mathbb{R}) \cap W_1^2(\mathbb{R})$. Suppose also that either $p < 4$ or that $\|f\|_1 < \gamma_F$ where $F(z) = z^{p+1}/(p+1)$ and γ_F is the ceiling featured in Proposition* 3.1. *Then the solution u of* (1.1) *corresponding to the initial data f satisfies*

$$|u(\cdot,t)|_2 \leq C(1+t)^{\frac{1}{4}}, \qquad |u_x(\cdot,t)|_2 \leq C(1+t)^{\frac{3}{4}}, \tag{6.3}$$

for all $t \geq 0$, where the constants C are independent of t.

7. Conclusion. The rate of decay to the quiescent state $u \equiv 0$ of solutions of the generalized Korteweg-de Vries-Burgers equation and solutions of the generalized regularized long-wave equation has been studied. Attention has been concentrated on the case of nonlinearities of cubic order and higher, so complementing the earlier studies of Biler [3] and Amick *et al.* [1], and paralleling the current work of Dix [11] and Zhang [27].

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