evolution equations

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Initial-Boundary Value Problems for Model Equations for the Propagation of Long Waves

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1. INTRODUCTION

This paper is concerned with initial- and boundary-value problems for evolution equations of the form

$$u_t + u_x + P(u)_x - \nu u_{xx} - \alpha^2 u_{xxt} = 0, \tag{1.1}$$

where $u = u(x,t)$ is a real-valued function of the two real variables x and t, and subscripts adorning u connote partial differentiation. Here P is a smooth, real-valued function of one real variable which will be suitably restricted later, and ν and α are non-negative real numbers. Such equations are often called pseudo-parabolic and the very particular form appearing in (1.1) arises in the modeling of unidirectional long waves in nonlinear dispersive systems. In case P is quadratic, (1.1) was studied by Peregrine [28] and Benjamin *et al.* [4] as an alternative to the well-known Korteweg-de Vries equation [18]

$$u_t + u_x + uu_x + u_{xxx} = 0. \tag{1.2}$$

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Equations of type (1.1) have significant advantages over those of type (1.2) when it comes to the imposition of the non-homogeneous boundary conditions arising in practice, especially when numerical techniques for the approximation of solutions are implemented. This point is discussed in some detail in Bona *et al.* [13] and Bona and Winther [17, 18] in the context of modeling surface waves in a flume generated by a wavemaker, and it arises again in the analysis presented by Albert and Bona [1] of the relation between a general class of models of type (1.1) and their Korteweg-de Vries-type analogues.

Analysis of initial- and boundary-value problems for equations like (1.1) began with the paper of Bona and Bryant [5] on the regularized long wave equation

$$u_t + u_x + uu_x - u_{xxt} = 0. \tag{1.3}$$

When (1.3) is used as a model for waves in a channel, the variable x is proportional to distance in the direction of propagation, t is proportional to time and u represents the deviation of the surface of the fluid from its rest position as it sustains the two-dimensional propagation of small-amplitude long waves. In [5], equation (1.3) was posed in a quarter plane $\{(x,t) : x \geq 0, t \geq 0\}$ with a single boundary condition at $x = 0$, and a theory of existence, uniqueness and continuous dependence established. Posing (1.1) in this form is of somewhat more practical interest than the more commonly considered pure initial-value problem in which u is specified for all x at some fixed time, say $t = 0$.

The theory in [5] was extended in [6] to initial- and two-point boundary-value problem for the equation

$$u_t + u_x + uu_x - \nu u_{xx} - \alpha^2 u_{xxt} = 0 \tag{1.4}$$

posed on a bounded interval with the solution specified at the right- and left-hand endpoints of the interval. These results are of especial interest in regard to the construction and analysis of numerical schemes for (1.4) since numerically feasible approximation schemes are necessarily applied on bounded intervals. The results in [5] and [6] are further developed in various ways in [19, 26, 27]. For example, in [19] Dang and Tran study the more general initial- and two-point boundary-value problem

$$\begin{aligned}
u_t + P(u)_x + G(u) - F(u,u_x,x,t)_x - (b(x,t)u_{xt})_x &= H(x,t), && x,t \in [0,1]\times[0,T],\\
u(x,0) &= f(x), && x \in [0,1],\\
u(0,t) = g(t), \qquad u(1,t) &= h(t), && t \in [0,T].
\end{aligned} \tag{1.5}$$

In all of the works just cited, either the nonlinear term P is allowed to grow at most quadratically or the boundary conditions are taken to be homogeneous.

Considerable interest has been shown recently in dispersive, dissipative evolution equations with nonlinearities that grow a rates higher than quadratic (cf. [7, 8, 21, 22, 23]). Perhaps the foremost reason for this attention stems from efforts to understand the interaction between these three, competing effects. In addition, mathematical issues arise for nonlinearities of order higher than quadratic that are not easily understood. Of course one can also broaden the perspective and consider at the same time more general dispersive and dissipative processes.

In this paper, we study well-posedness of equation (1.1) for the initial- and two-point boundary-value problem

$$\begin{cases} \qquad u(x,0) = f(x), & x \in [a,b], \\ u(a,t) = g(t), \quad u(b,t) = h(t), & t \in [0,T], \end{cases} \tag{1.6a}$$

and for the quarter-plane problem

$$\begin{cases} u(x,0) = f(x), & x \in \mathbb{R}^+, \\ u(0,t) = g(t), & t \in [0,T]. \end{cases} \tag{1.6b}$$

It will be shown that both of these problems for the equation (1.1) possesses at least locally in time a unique classical solution which depends continuously on ν in $\mathbb{R}^+$ and on variations of the data f, g, and h within their respective function classes. The local existence theory for the initial-boundary-value problems is relatively straightforward, and does not depend on the detailed structive of P. A theory that is global in time is more difficult, and depends upon the derivation of *a priori* bounds on local solutions. The provision of such bounds appears to need a growth condition on P, namely that its grow at infinity at a rate which is not more than quartic.

In addition to establishing the well-posedness of (1.1)-(1.6), we consider the degradation of the wave in case the parameter ν is actually positive. This aspect has already received some attention in the case of the pure initial-value problem on the whole line $\mathbb{R}$ (cf. [2, 9, 20]) and the periodic initial-value problem (see [7]). Especially the decay problem on the entire line is decidedly non-trivial, but all these results rely upon the homogeneity of the boundary conditions. Here we study decay in the more practically interesting setting of the two-point boundary-value problem (1.6a). Our results in this arena are interesting in their own right, but in addition they justify certain modelling considerations that arose in [13] in connection with water waves in channels. There is also interest in decay theory for the quarter-plane problem (1.6b), but the results in hand for this context appear not to be sharp, and consequently they will not be reported here.

This paper is organized as follows. In Section 2 basic notation is reviewed and the main theorem of well posedness is stated. In Section 3, the quarter-plane problem (1.1)-(1.6b) is studied. Local existence is proved by converting the differential equation (1.1) into an equivalent integral equation. *A priori* bounds that apply uniformly on compact subsets of the temporal variable are then derived in the presence of a growth condition on P, and these are used to extend the local solutions indefinitely. The continuous dependence of the solution on variations in the initial- and boundary-data follows readily from the proof of local existence. The two-point boundary-value problem (1.1)-(1.6a) is considered in Section 4. The theory for this problem is not dissimilar to that for the quarter-plane problem, and hence the presentation is abbreviated. In case $\nu > 0$, the decay theory for the two-point boundary-value problem (1.1)-(1.6a) is then discussed in the final part of Section 4.

2. NOTATION AND STATEMENT OF THE MAIN RESULTS

Throughout the paper, all functions will be real-valued. For any Banach space X, the associated norm will be denoted $||\cdot||_X$ except for a few abbreviations noted below. Spaces that arise in our analysis include the standard spaces $C^k(\bar{\Omega})$ for Ω a bounded open set in $\mathbb{R}^+$, $k = 0, 1, 2, \cdots$, $L_p(\Omega)$ for $1 \leq p \leq \infty$, and the L_2-based Sobolev spaces $H^m(\Omega)$ for $m = 0, 1, 2, \cdots$ (cf. Lions [25], Treves [30]). If Ω is an unbounded open set in $\mathbb{R}^+$, $C_b^k(\bar{\Omega})$ is defined exactly as $C^k(\bar{\Omega})$ except that the function and its first k derivatives are required to be bounded.

In the analysis of the initial- and boundary-value problem (1.1)-(1.6), the spaces $H^m(\Omega)$ will occur often with m a positive integer and $\Omega = \mathbb{R}^+ = (0, +\infty)$, $\Omega = (0, 1)$ or $\Omega = (0, T)$. Because of their frequent occurrence, it is convenient to abbreviate their norms thusly:

$$||\cdot||_m = ||\cdot||_{H^m(\mathbb{R}^+)}, \qquad \text{or} \qquad ||\cdot||_m = ||\cdot||_{H^m(0,1)} \qquad \text{and} \qquad |\cdot|_{m,T} = ||\cdot||_{H^m(0,T)}. \tag{2.1}$$

If $m = 0$, the subscript m will be omitted altogether, so that

$$||\cdot|| = ||\cdot||_{L_2(\mathbb{R}^+)}, \qquad \text{or} \qquad ||\cdot|| = ||\cdot||_{L_2(0,1)} \qquad \text{and} \qquad |\cdot|_T = ||\cdot||_{0,T}. \tag{2.2}$$

Let X be a Banach space, T be a positive real number and $1 \leq p \leq +\infty$. Then $L_p(0, T; X)$ denotes the Banach space of all measurable functions $u : (0, T) \to X$, such that $t \to ||u(t)||_X$ is in $L_p(0, T)$. Similarly, by $C(0, T; X)$, we denote the subspace of $L_\infty(0, T; X)$ of all continuous functions $u : [0, T] \to X$.

For $\Omega = \mathbb{R}^+$ or $\Omega = [0,1]$, the abbreviation $\mathcal{B}_T^{k,l}$ will be employed for the functions $u:\Omega\times[0,T]\to\mathbb{R}$ such that $\partial_x^i\partial_t^j u\in C(0,T;C_b)$ for $0\le i\le k$, and $0\le j\le l$. This Banach space will carry the norm

$$\|u\|_{\mathcal{B}_T^{k,l}} = \sum_{\substack{0\le i\le k\\ 0\le j\le l}} \|\partial_x^i\partial_t^j u\|_{C(0,T;C_b)}.$$

The space $\mathcal{B}_T^{0,0}$ will be abbreviated simply $\mathcal{B}_T$ and its norm is just that of $L_\infty(\Omega\times[0,T])$.

In Sections 3 and 4, the following results are proved. For simplicity, and because all the interesting examples are thereby covered, it is assumed that P is a C^∞-function, though finite regularity assumptions suffice for most of our theory. The result is stated informally here, with precise versions provided later in the technical sections of the paper.

MAIN RESULTS. *Let $T>0$, $f\in C_b^2(\mathbb{R}^+)\cap H^2(\mathbb{R}^+)$ and $g\in C^1(0,T)$ be given and suppose $f(0)=g(0)$. If the growth of P is no more than quartic at infinity, then there exists a unique solution u of (1.1) for the quarter-plane problem (1.6a) in the space $\mathcal{B}_T^{2,1}\cap C(0,T;H^2(\mathbb{R}^+))$. The solution depends continuously on the initial and boundary data, and on $\nu\ge 0$. Similar results hold for the initial- and two-point boundary-value problem (1.1)-(1.6b). If $\nu>0$, and with appropriate decay assumptions on the boundary data g and h, the solution of (1.1)-(1.6b) tends to zero as t tends to infinity.*

3. WELL-POSEDNESS IN THE QUARTER-PLANE

In this section, interest will be focused on the initial- and boundary-value problem

$$u_t + u_x + P(u)_x - \nu u_{xx} - \alpha^2 u_{xxt} = 0, \qquad \text{for } x,\, t\ge 0, \tag{3.1a}$$

$$u(x,0) = f(x), \qquad \text{for } x\ge 0, \tag{3.1b}$$

$$u(0,t) = g(t), \qquad \text{for } t\ge 0. \tag{3.1c}$$

For consistency, the restriction

$$u(0,0) = f(0) = g(0), \tag{3.2}$$

will be imposed throughout the discussion and we take it that $\nu\ge 0$ and $\alpha\ne 0$.

By converting the differential equation (3.1a) with initial condition (3.1b) and boundary condition (3.1c) into an integral equation and applying the contraction mapping theorem to the integral equation, the existence of a local solution may be established. This local solution is extended to a global solution by appeal to an *a priori* estimation of smooth solutions of (3.1). The results of continuous dependence follow from the local theory.

3.1. Local Solution

To obtain a local existence theorem, we first convert the problem (3.1) into an integral equation. The argument closely parallels that given in detail in [5], and consequently many of the calculations are abbreviated.

Equation (3.1a) may be regarded as an ordinary differential equation for u_t by considering $\nu u_{xx} - u_x - P(u)_x$ as an external force. Solving this equation for u_t, formally integrating the solution by parts and then integrating from 0 to t, there appears the relation

$$u(x,t) = \exp(-t\nu/\alpha^2)f(x) + \tilde{g}(t)e^{-x/\alpha} + \mathbb{B}(u)(x,t), \tag{3.3}$$

where

$$\begin{aligned}\mathbb{B}(u)(x,t) = & \int_0^t \int_0^{+\infty} \exp(-\nu(t-\tau)/\alpha^2)K(x,\xi)\big[P(u(\xi,\tau)) + u(\xi,\tau)\big]d\xi d\tau \\ & - \frac{\nu}{\alpha}\int_0^t \int_0^{+\infty} \exp(-\nu(t-\tau)/\alpha^2)L(x,\xi)u(\xi,\tau)d\xi d\tau,\end{aligned} \tag{3.4}$$

$$K(x,\xi) = \frac{1}{2\alpha^2}\big[\exp(-(x+\xi)/\alpha) + sgn(x-\xi)\exp(-|x-\xi|/\alpha)\big], \tag{3.5}$$

$$L(x,\xi) = \frac{1}{2\alpha^2}\big[\exp(-(x+\xi)/\alpha) - \exp(-|x-\xi|/\alpha)\big], \tag{3.6}$$

and

$$\tilde{g}(t) = g(t) - \exp(-t\nu/\alpha^2)g(0). \tag{3.7}$$

Define the operator $\mathbb{A}$ by

$$(\mathbb{A}u)(x,t) = \exp(-t\nu/\alpha^2)f(x) + \tilde{g}(t)e^{-x/\alpha} + \mathbb{B}(u)(x,t). \tag{3.8}$$

Then assuming that $f \in C_b(\mathbb{R}^+)$ and $g \in C(0,T)$, the operator $\mathbb{A}$ maps a function $u \in \mathcal{B}_T$ into itself since K is integrable. If T is chosen small enough, $\mathbb{A}$ is a contraction mapping of a ball centered at the origin in $\mathcal{B}_T$ into itself. This observation leads immediately to the following proposition.

PROPOSITION 3.1. *Let $T > 0$, $f \in C_b(\mathbb{R}^+)$ and $g \in C(0,T)$. Then there exists a positive constant*

$$T' = T'(||f||_{C_b(\mathbb{R}^+)}, ||g||_{C(0,T)})$$

such that for any T_0 with $T_0 \leq \min\{T', T\}$, there is a unique solution of (3.3) in $\mathcal{B}_{T_0}$. If $f \in H^1(\mathbb{R}^+)$, then there is a positive constant $T'(||f||_1, ||g||_{C(0,T)})$ such that for any $T_0 \leq \min\{T', T\}$ there is a unique solution of (3.3) in $C(0, T_0; H^1(\mathbb{R}^+))$. In either case, for T sufficiently small, the mapping that associates to initial and boundary data (f, g) the solution u of (3.3) is continuous from $C_b(\mathbb{R}^+) \times C(0,T)$ into $\mathcal{B}_T$ or from $H^1(\mathbb{R}^+) \times C(0,T)$ into $C(0,T; H^1(\mathbb{R}^+))$.

As mentioned, this proposition may be established by choosing positive values T' and R so that $\mathbb{A}$ is a contraction mapping of the ball of radius R centered at the origin in $\mathcal{B}_{T'}$. The crucial estimate is that for $v, w \in \mathcal{B}_{T'}$ with $||v||_{\mathcal{B}_{T'}}, ||w||_{\mathcal{B}_{T'}} \leq R$, then

$$||\mathbb{A}v - \mathbb{A}w||_{\mathcal{B}_{T'}} \leq C(R)T'||v - w||_{\mathcal{B}_{T'}}$$

where the constant $C(R)$ is an absolute constant connected with norms of K and L times $\max_{|z| \leq R} |P'(z)|$. Once this inequality is in hand, the proof follows exactly the lines worked out in [1] or [5].

REMARK 3.2: The time interval T' for which $\mathbb{A}$ is inferred to be contractive depends inversely on $||f||_1$ and $||g||_{C(0,T)}$. If the boundary data g is given in $C(0,T)$ and it is known somehow that $||u(\cdot, t)||_1$ is bounded on bounded time intervals, then the contraction-mapping argument used to obtain Proposition 3.1 may be iterated to produce a solution of (3.3) defined for all t in $[0, T]$. This remark applies even if $T = +\infty$.

If $u \in \mathcal{B}_T$ is a solution of (3.3) and the boundary and initial data has regularity beyond just being bounded, continuous and consistent as in (3.2), it follows readily from the representation $u = \mathbb{A}u$ that u possesses additional regularity. The arguments leading to this conclusion parallel those spelled out in [4] and [5], and consequently we content ourselves with a statement of these useful results.

LEMMA 3.3. *Suppose $f \in C_b^k(\mathbb{R}^+)$ and $g \in C^l(0,T)$ with $f(0) = g(0)$ where $k \geq 1$, $l \geq 0$ and $k > l$. Let $u \in \mathcal{B}_T$ be a solution of (3.3). Then $u \in \mathcal{B}_T^{k,l}$, and, moreover, if $k \geq 2$, $l \geq 1$, then u is a classical solution of (3.1) on $\mathbb{R}^+ \times [0,T]$. Similarly, if $f \in H^k(\mathbb{R}^+)$ for some $k > 1$ and $u \in C(0,T; H^1(\mathbb{R}^+))$ is the solution of (3.3) guaranteed by Proposition 3.1, then $u \in C(0,T; H^k(\mathbb{R}^+))$. If $f \in C_b^k(\mathbb{R}^+)$, $g \in C^l(0,T)$ where $k \geq 1$, $l \geq 0$, $k > l$ and $f^{(j)}(x) \to 0$ as $x \to +\infty$ for $0 \leq j \leq k$, then the solution u of (3.1) has the property that*

$$\partial_x^j \partial_t^i u(x,t) \to 0 \qquad \text{as} \qquad x \to +\infty,$$

uniformly for $0 \leq t \leq T$, for $0 \leq j \leq k$, $0 \leq i \leq l$. In all cases, the mapping that associates the solution u to the initial and boundary data (f, g) is a continuous mapping between the function classes from which (f, g) and u are drawn.

With these preliminary results in hand, we are ready to undertake the derivation of *a priori* bounds that allow the local solution of (3.1) obtained via the contraction-mapping argument in Proposition 3.1 to be extended to arbitrary time intervals $[0, T]$.

3.2. Global Solutions

Suppose there is to hand a classical solution of (3.1) at least on a time interval $[0, T]$ for some $T > 0$. The following lemmas are aimed at extending this local solution to an arbitrary time interval.

The first foray will be into the situation with relatively weak assumptions on the initial data and with no dissipative effects. It will be handy in this result and later to define $\Lambda(s)$ by the specification

$$\frac{d\Lambda}{ds} = P(s), \qquad \Lambda(0) = 0. \tag{3.9}$$

LEMMA 3.4. *Let $T > 0$, $f \in C_b^2(\mathbb{R}^+) \cap H^1(\mathbb{R}^+)$ and $g \in C^1(0, T)$ be given with $f(0) = g(0)$. Suppose that $\Lambda(s)$ satisfies the one-sided growth condition*

$$\limsup_{|s| \to \infty} |s|^{-4} \Lambda(s) \leq 0. \tag{$*$}$$

Let $u(x, t)$ be the classical solution of (3.1) with $\nu = 0$ on $\mathbb{R}^+ \times [0, T']$ whose existence and regularity is guaranteed by Proposition 3.1 and Lemma 3.3. Then there exists a constant a_1 only depending on $\|f\|_1$ and $|g|_{1,T}$ such that $\|u(\cdot, t)\|_1 \leq a_1$.

PROOF: Multiply (3.1a) with $\nu = 0$ by $2u(x, t)$ and integrate the result over $\mathbb{R}^+ \times [0, t)$. After integrations by parts and using Lemma 3.3 to dismiss the boundary contributions at infinity, it appears that

$$\|u(\cdot, t)\|_1^2 = \|f\|_1^2 + \int_0^t \left[2Q(g(\tau)) + g(\tau)^2 - 2\alpha^2 g(\tau) u_{xt}(0, \tau)\right] d\tau, \tag{3.10}$$

where $Q(u) = \int_0^u \lambda P'(\lambda) d\lambda$. The Cauchy-Schwarz inequality then implies that

$$\|u(\cdot, t)\|_1^2 \leq C(\|f\|_1, |g|_{1,T}) + 2\alpha^2 |g|_T \left(\int_0^t u_{xt}^2(0, \tau) d\tau \right)^{\frac{1}{2}} \tag{3.11}$$

for $0 \le t \le T'$, where here and subsequently C will denote various constants that depend only on norms of the auxiliary data. Continue by multiplying (3.1a) with $\nu = 0$ by $2\alpha^2 u_{xt}(x,t) - 2P(u)$ and integrating the result over $\mathbb{R}^+ \times [0,t)$. After integrations by parts and using again Lemma 3.3, there obtains the relation

$$\begin{aligned} &\alpha^2 \|u_x(\cdot,t)\|^2 + \alpha^4 \int_0^t u_{xt}^2(0,\tau)d\tau - 2\int_0^{+\infty} \Lambda(u(x,t))dx \\ &= \alpha^2\|f'\|^2 - 2\int_0^{+\infty} \Lambda(f(x))dx \\ &+ \int_0^t \Big[\alpha^2(g'(\tau))^2 + 2\alpha^2 P(g(\tau))u_{xt}(0,\tau) - P^2(g(\tau)) - 2\Lambda(g(\tau))\Big]d\tau. \end{aligned} \tag{3.12}$$

It is deduced that

$$\begin{aligned} \alpha^4 \int_0^t u_{xt}^2(0,\tau)d\tau &\le C(\|f\|_1, |g|_{1,T}) + 2\|\Lambda(u)\|_{L_1}^2 \\ &\le C(\|f\|_1, |g|_{1,T}) + 2\|u\|^2 \tilde{E}(\|u\|^{\frac{1}{2}}\|u_x\|^{\frac{1}{2}}), \end{aligned} \tag{3.13}$$

where

$$\Lambda(\lambda) = \lambda^2 E(\lambda), \qquad \tilde{E}(r) = \sup_{|\lambda|\le r} E(\lambda),$$

by use of the elementary inequality

$$\|u(\cdot,t)\|_{L_\infty} \le \sqrt{2}\|u(\cdot,t)\|^{\frac{1}{2}}\|u_x(\cdot,t)\|^{\frac{1}{2}}. \tag{3.14}$$

By using the assumption $(*)$, (3.13) may be further simplified to

$$\int_0^t u_{xt}^2(0,\tau)d\tau \le C(\|f\|_1, |g|_{1,T}, \delta) + \delta\|u\|^2\|u\|_1^2, \tag{3.15}$$

for any $\delta > 0$. Substituting (3.15) into (3.11), it is concluded that there is a constant a_1 such that

$$\|u(\cdot,t)\|_1^2 \le C(\|f\|_1, |g|_{1,T}) = a_1, \tag{3.16}$$

where a_1 only depends on $\|f\|_1$ and $|g|_{1,T}$. The lemma is proved. □

REMARK 3.5: It is easy to see that if $\Lambda \le 0$, for example when $P(u) = -u^{2m-1}$ where m is a positive integer, one immediately obtains H^1-bounds without resort to growth conditions. Even in case Λ is unrestricted in sign, it still follows from (3.11) and (3.13) that $\|u(\cdot,t)\|_1$ is bounded, independently of t, provided the initial data f and the boundary data g are small enough in $H^1(\mathbb{R}^+)$ and $H^1(0,T)$, respectively. This follows since (3.10) and (3.13) together imply that

$$\|u(\cdot,t)\|_1^2\Big(1 - \delta\tilde{E}(\|u\|_1)\Big) \le C(\|f\|_1, |g|_{1,T}),$$

where $\delta = |g|_T$. If δ is small enough relative to $||f||_1$ and if $||f||_1$ and $|g|_{1,T}$ are likewise not too large, then this last inequality provides a t-independent bound on $||u(\cdot,t)||_1$. Also note that estimates (3.11) and (3.15) yield a bound on $||u(\cdot,t)||_1$ that only depends on the norms of the auxiliary data and not explicitly on T. When boundary data g is specified in $H^1(\mathbb{R}^+)$, the solution $u(\cdot,t)$ of (3.1) with $\nu = 0$ is therefore bounded in $H^1(\mathbb{R}^+)$ independently of t. Finally, it is worth note that estimate (3.16), when unraveled implies that $||u(\cdot,t)||_1$ grows more or less linearly with the energy supplied by the wavemaker (cf. [5]). This satisfying state of affairs points to a certain consistency of the equation that lends credibility to its status as a model of real physical phenomena. □

If stronger conditions on the initial value are given, the order of growth required of P can be as high as quartic while still maintaining a satisfactory theory of well-posedness. The following lemma applies even if $\nu > 0$.

LEMMA 3.6. *Suppose $f \in C_b^2(\mathbb{R}^+) \cap H^2(\mathbb{R}^+)$, $g \in C^1(0,T)$ for some $T > 0$ and $f(0) = g(0)$. It is presumed that P satisfies the growth condition*

$$\limsup_{|s|\to\infty} |s|^{-2}|P''(s)| \le \varepsilon \tag{$**$}$$

for some non-negative constant ε. Let $u(x,t)$ be a classical solution of (3.1) up to the boundary on $\mathbb{R}^+ \times [0,T']$ for some $T' \le T$, as guaranteed in Lemma 3.3. Then for all $t \in [0,T']$, $u(\cdot,t) \in H^2(\mathbb{R}^+)$ and there exists a constant a_2 depending only on T, $||f||_2$ and $|g|_{C^1(0,T)}$ such that for $0 \le t \le T'$, $||u(\cdot,t)||_2 \le a_2$.

PROOF: Multiply (3.1a) by $2u(x,t)$ and integrate the result over $\mathbb{R}^+ \times [0,t)$. After integration by parts and using Lemma 3.3, there obtains

$$\begin{aligned} &||u(\cdot,t)||^2 + \alpha^2||u_x(\cdot,t)||^2 + \nu\int_0^t ||u_x(\cdot,\tau)||^2 d\tau \\ &= ||f||^2 + \alpha^2||f'||^2 - 2\alpha^2 g(t)u_x(0,t) + 2\alpha^2 g(0)f'(0) \\ &+ \int_0^t \left[2Q(g(\tau)) + g(\tau)^2 - 2\nu g(\tau)u_x(0,\tau) + 2\alpha^2 g'(\tau)u_x(0,\tau)\right] d\tau. \end{aligned} \tag{3.17}$$

By using some elementary inequalities, including (3.14) applied to u_x, we infer that

$$\begin{aligned} &\left|\int_0^t -2\nu g(\tau)u_x(0,\tau)d\tau\right| \\ &\le \int_0^t 2\sqrt{2}\nu|g(\tau)|(||u_x(\cdot,\tau)||\,||u_{xx}(\cdot,\tau)||)^{\frac{1}{2}} d\tau \\ &\le 2^{\frac{3}{2}}\nu\left(\int_0^t |g(\tau)|^{\frac{4}{3}}||u_{xx}(\cdot,\tau)||^{\frac{2}{3}} d\tau\right)^{\frac{3}{4}}\left(\int_0^t ||u_x(\cdot,\tau)||^2 d\tau\right)^{\frac{1}{4}} \\ &\le \frac{\nu}{2}\int_0^t ||u_x(\cdot,\tau)||^2 d\tau + C\nu|g|_T^{\frac{4}{3}}\left(\int_0^t ||u_{xx}(\cdot,\tau)||^2 d\tau\right)^{\frac{1}{3}}. \end{aligned} \tag{3.18}$$

Similarly one has

$$
\begin{aligned}
&|\int_0^t \alpha^2 g'(\tau)u_x(0,\tau)d\tau| \\
&\le \int_0^t \alpha^2|g'(\tau)|\sqrt{2}||u_x(\cdot,\tau)||^{\frac{1}{2}}||u_{xx}(\cdot,\tau)||^{\frac{1}{2}}d\tau \\
&\le \sqrt{2}\alpha^2\left(\int_0^t |g'(\tau)|^{\frac{3}{2}}d\tau\right)^{\frac{2}{3}}\left(\int_0^t ||u_x(\cdot,\tau)||^6 d\tau\right)^{\frac{1}{12}}\left(\int_0^t ||u_{xx}(\cdot,\tau)||^2 d\tau\right)^{\frac{1}{4}} \\
&\le C_T|g|_{1,T}\left[\left(\int_0^t ||u_x(\cdot,\tau)||^6 d\tau\right)^{\frac{1}{3}} + \left(\int_0^t ||u_{xx}(\cdot,\tau)||^2 d\tau\right)^{\frac{1}{3}}\right],
\end{aligned}
\tag{3.19}
$$

and

$$
|2\alpha^2 g(t)u_x(0,t)| \le \frac{\alpha^2}{2}||u_x(\cdot,t)||^2 + \frac{3\alpha^2}{2}|g(t)|^{\frac{4}{3}}||u_{xx}(\cdot,t)||^{\frac{2}{3}}. \tag{3.20}
$$

Using (3.18), (3.19) and (3.20) in (3.17) yields

$$
\begin{aligned}
&||u(\cdot,t)||_1^2 + \nu\int_0^t ||u_x(\cdot,\tau)||^2 d\tau \\
&\le C(||f||_1, |g|_{C^1(0,T)}) + C|g(t)|^{\frac{4}{3}}||u_{xx}(\cdot,t)||^{\frac{2}{3}} \\
&+ C_T|g|_{1,T}\left[\left(\int_0^t ||u_x(\cdot,\tau)||^6 d\tau\right)^{\frac{1}{3}} + \left(\int_0^t ||u_{xx}(\cdot,\tau)||^2 d\tau\right)^{\frac{1}{3}}\right],
\end{aligned}
\tag{3.21}
$$

or, what is the same,

$$
\begin{aligned}
&||u(\cdot,t)||_1^6 + \nu^3\Big(\int_0^t ||u_x(\cdot,\tau)||^2 d\tau\Big)^3 \\
&\le C(||f||_1, |g|_{C^1(0,T)}) + C|g(t)|^4||u_{xx}(\cdot,t)||^2 \\
&+ C_T|g|_{1,T}^3\int_0^t \big[||u_x(\cdot,\tau)||^6 + ||u_{xx}(\cdot,\tau)||^2\big]d\tau.
\end{aligned}
\tag{3.22}
$$

Now multiply (3.1a) by $2u_{xx}(x,t)$ and integrate the result over $\mathbb{R}^+\times[0,t)$. After integration by parts and using Lemma 3.3, we reach the equation

$$
\begin{aligned}
&||u_x(\cdot,t)||^2 + \alpha^2||u_{xx}(\cdot,t)||^2 + \nu\int_0^t ||u_{xx}(\cdot,\tau)||^2 d\tau \\
&= ||f'||^2 + \alpha^2||f''||^2 - \int_0^t\int_0^{+\infty} P''(u)u_x^3 dx d\tau \\
&- \int_0^t \big[2g'(\tau)u_x(0,\tau) + u_x^2(0,\tau) + \frac{1}{2}P'(g(\tau))u_x^2(0,\tau)\big]d\tau.
\end{aligned}
\tag{3.23}
$$

By using (**), (3.14) and some other elementary inequalities, we deduce from (3.23) that

$$\begin{aligned}
&\left|\int_0^t \int_0^{+\infty} P''(u)u_x^3\,dx\,d\tau\right| \\
&\le \int_0^t \int_0^{+\infty} \varepsilon|u^2u_x^3|\,dx\,d\tau \\
&\le \int_0^t \varepsilon||u(\cdot,\tau)||_{L_\infty}^2 ||u_x(\cdot,\tau)||_{L_\infty} ||u_x(\cdot,\tau)||^2 d\tau \\
&\le \int_0^t C(\varepsilon)||u(\cdot,\tau)||\,||u_x(\cdot,\tau)||^{7/2}||u_{xx}(\cdot,\tau)||^{1/2} d\tau \\
&\le \int_0^t C(\varepsilon)\big[||u(\cdot,\tau)||_1^6 + ||u_{xx}(\cdot,\tau)||^2\big] d\tau.
\end{aligned} \tag{3.24}$$

Using (3.24) in (3.23), one obtains

$$\begin{aligned}
&||u_{xx}(\cdot,t)||^2 + \nu\int_0^t ||u_{xx}(\cdot,\tau)||^2 d\tau \\
&\le C(||f||_2, |g|_{C^1(0,T)}) + C(\alpha)||u_x(\cdot,t)||^2 \\
&\quad + \int_0^t C(\varepsilon)\big[||u(\cdot,\tau)||_1^6 + ||u_{xx}(\cdot,\tau)||^2\big] d\tau.
\end{aligned} \tag{3.25}$$

If (3.25) is multiplied by a suitable constant and the result added to (3.22), there appears

$$\begin{aligned}
&||u(\cdot,t)||_1^6 + C(|g|_{C^1(0,T)})||u_{xx}(\cdot,t)||^2 \\
&\quad + C(\nu, |g|_{C^1(0,T)})\left[\left(\int_0^t ||u_x(\cdot,\tau)||^2 d\tau\right)^3 + \int_0^t ||u_{xx}(\cdot,\tau)||^2 d\tau\right] \\
&\le C(||f||_2, |g|_{C^1(0,T)}) \\
&\quad + \int_0^t C(\nu,\varepsilon,|g|_{C^1(0,T)},T)\big[||u(\cdot,\tau)||_1^6 + ||u_{xx}(\cdot,\tau)||^2\big] d\tau.
\end{aligned} \tag{3.26}$$

An appeal to Gronwall's lemma now concludes the proof. □

REMARK 3.7: From (3.10) and (3.17), one sees immediately that an H^1-bound can be obtained without a growth restriction on P if a homogeneous boundary condition $g \equiv 0$ is posed.

REMARK 3.8: The condition in Lemma 3.6 on the growth of P is not sharp. One may use a more general version of Gronwall's inequality applied to (3.24) to handle cases where the growth of P is a little bit stronger than assumed in (**). While this strengthening of Lemma 3.6 is probably not of practical importance, it is perhaps worth recording.

LEMMA 3.9. *(A particular case of Theorem 3 in* [3], *Chapter 4, §5) Let $T > 0$ and let G be an increasing positive function defined on $[0, T]$ which is bounded away from 0. Let u be a continuous function on $[0,T]$ such that*

$$u(t) \leq k + \int_0^t G(u(s))ds \qquad \text{for } 0 \leq t \leq T,$$

where k is a positive constant. Then

$$u(t) \leq Q^{-1}(t) \qquad \text{for } 0 \leq t \leq T_1,$$

where $Q(t) = \int_k^t \frac{ds}{G(s)}$, range $G = [0, T^*]$, $T^* \in (0, +\infty]$ *and* $T_1 = \min\{T, T^*\}$.

COROLLARY 3.10. *Let f and g satisfy the conditions in Lemma 3.6. Suppose the nonlinearity P satisfies the growth condition*

$$\limsup_{|s|\to\infty} \frac{|P''(s)|}{|s|^2 \log(2 + |s^6|)} \leq \varepsilon_1, \tag{***}$$

for some constant ε_1. Let $u(x,t)$ be a classical solution of (3.1) up to the boundary on $\mathbb{R}^+ \times [0, T']$ where $T' \leq T$. Then for all $t \in [0, T']$ $u(\cdot, t) \in H^2(\mathbb{R}^+)$ and there exists a constant a_2' only depending on $T, \|f\|_2$ and $|g|_{C^1(0,T)}$ such that

$$\|u(\cdot, t)\|_2 \leq a_2'. \tag{3.27}$$

PROOF: The hypothesis $(***)$ implies that

$$\begin{aligned} &|\int_0^t \int_0^{+\infty} P''(u) u_x^3 dx d\tau| \\ &\leq C(\varepsilon_1) \int_0^t (\|u(\cdot,\tau)\|_1^6 + \|u_{xx}(\cdot,\tau)\|^2) \log(2 + \|u(\cdot,\tau)\|_1^6) d\tau. \end{aligned} \tag{3.28}$$

Using (3.28) in (3.23) yields a new version of (3.24), namely

$$\begin{aligned} \|u(\cdot,t)\|_1^6 + C_2\|u_{xx}(\cdot,t)\|^2 + C_3\Big(\int_0^t \|u_x(\cdot,\tau)\|^2 d\tau\Big)^3 + C_4 \int_0^t \|u_{xx}(\cdot,\tau)\|^2 d\tau \\ \leq C_5(\|f\|_2, |g|_{C^1(0,T)}) + \int_0^t C_6(\varepsilon_1, \nu, |g|_{C^1(0,T)}) \big[\|u(\cdot,\tau)\|_1^6 \\ + \|u_{xx}(\cdot,\tau)\|^2\big] \log(2 + \|u(\cdot,\tau)\|_1^6) d\tau. \end{aligned} \tag{3.29}$$

Applying Lemma 3.9 to the above inequality gives (3.27). □

An immediate consequence of the just derived *a priori* bounds is our main existence theorem.

Theorem 3.11. *Let f and g be given with $f(0) = g(0)$, and suppose that $T > 0$. Let the nonlinearity P in equation (3.1) be specified and assume that Λ is defined as before by $\Lambda'(z) = P(z)$ for $z \in \mathbb{R}$ and $\Lambda(0) = 0$.*

(1) If $f \in C_b^2(\mathbb{R}^+) \cap H^1(\mathbb{R}^+)$, $g \in C^1(0,T)$, Λ satisfies the one-sided growth condition

$$\limsup_{|s|\to\infty} |s|^{-4}\Lambda(s) \leq 0, \tag{$*$}$$

and $\Lambda \in C^2(\mathbb{R}^+)$, then the system (3.1) with $\nu = 0$ has a unique solution $u \in \mathcal{B}_T^{2,1} \cap C(0,T;H^1(\mathbb{R}^+))$ corresponding to the auxiliary specification of f as initial data and g as boundary data. Moreover, if $g \in C^1(\mathbb{R}^+) \cap H^1(\mathbb{R}^+)$, then $u \in C_b(\mathbb{R}^+;H^1(\mathbb{R}^+))$.

(2) If $f \in C_b^2(\mathbb{R}^+) \cap H^2(\mathbb{R}^+)$, $g \in C^1(0,T)$, P satisfies the growth condition

$$\limsup_{|s|\to\infty} |s|^{-2}|P''(s)| \leq \varepsilon, \tag{$**$}$$

for some constant ε and $P \in C^2(\mathbb{R}^+)$, then the equation (3.1) has a unique solution $u \in \mathcal{B}_T^{2,1} \cap C(0,T;H^2(\mathbb{R}^+))$ corresponding to the initial and boundary conditions (f,g).

(3) If $f \in C_b^r(\mathbb{R}^+) \cap H^k(\mathbb{R}^+)$ and $g \in C^s(0,T)$ where $k \geq 2$, $r \geq 2$, $s \geq 1$, $r > s$, then the solution u of (3.1) corresponding to initial data f and boundary data g lies in $\mathcal{B}_T^{r,s} \cap C(0,T;H^k(\mathbb{R}^+))$.

(4) In all the above cases, the solution u depends continuously on variations of the auxiliary data. That is, the mapping that assigns to (f,g) the associated solution of (3.1) is continuous from the function class of the initial data to the function class of the solution. In case (2) and (3), the solution also depends continuously on ν.

(5) If the boundary condition g is the zero function, then the above conclusion hold without the growth conditions $(*)$ or $(**)$ on the nonlinearity P.

By Lemma 3.3 and Proposition 3.2, the initial-boundary-value problem (3.1) has a solution u in $\mathcal{B}_T^{2,1} \cap C(0,T_0;H^k(\mathbb{R}^+))$ for $k = 1,2$, for small T_0. Then Lemma 3.5 and Lemma 3.6 show that on any finite time interval $[0,T]$, $||u(\cdot,t)||_1$ and hence $||u(\cdot,t)||_{C_b}$, is uniformly bounded. Thus the quantity $||u(\cdot,t)||_{C_b} + 2||g||_{C(0,T)}$ is uniformly bounded for $0 \leq t \leq T$. This in turn determines a lower bound on how far a solution, defined already on $[0,T_0]$, can be extended by an application of the local existence result in Proposition 3.1 and Proposition 3.2. As this term is bounded above, then the extension length is bounded below by a positive constant. By iteration of the existence proof of Proposition 3.1 and Proposition 3.2 one can extend the solution u from $[0,T_0]$ to $[0,T]$ in a finite number of temporal steps. □

If $\nu \neq 0$, it would be expected that solutions of (3.1) decay in time. The following result will be useful for estimating temporal decay.

COROLLARY 3.12. *Let $f \in C_b^2(\mathbb{R}^+) \cap H^2(\mathbb{R}^+)$, $g \in C^1(\mathbb{R}^+) \cap H^1(\mathbb{R}^+)$, $\nu > 0$ and suppose P to satisfy the condition (**) in such a way that if equality occurs in (**), then either ε is sufficiently small, ν is sufficiently big, or the boundary data g is small in the sense that $|g|_{1,+\infty}$ is sufficiently small. The corresponding solution u of (3.1) then has the properties that $u_x, u_{xx} \in L_2(\mathbb{R}^+ \times \mathbb{R}^+)$ and $u \in C_b(\mathbb{R}^+; H^2)$.*

PROOF: Since $\nu > 0$, the left-hand side of (3.19) may be estimated as

$$\begin{aligned} &|\int_0^t \alpha^2 g'(\tau) u_x(0,\tau) d\tau| \\ &\leq \int_0^t \sqrt{2}\alpha^2 |g'(\tau)| \cdot ||u_x(\cdot,\tau)||^{\frac{1}{2}} ||u_{xx}(\cdot,\tau)||^{\frac{1}{2}} d\tau \\ &\leq \frac{\nu}{4}\int_0^t ||u_x(\cdot,\tau)||^2 d\tau + \frac{C|g|_{1,+\infty}^{\frac{4}{3}}}{\nu^{\frac{1}{3}}} \left(\int_0^t ||u_{xx}(\cdot,\tau)||^2 d\tau\right)^{\frac{1}{3}}. \end{aligned} \tag{3.30}$$

Because of this new estimate in regard to (3.19), (3.22) can be rewritten as

$$\begin{aligned} &||u(\cdot,t)||_1^6 + \nu^3 \Big(\int_0^t ||u_x(\cdot,\tau)||^2 d\tau\Big)^3 \\ &\leq C_1(||f||_1, |g|_{1,+\infty}) + C_2 |g|_{1,+\infty}^4 ||u_{xx}(\cdot,t)||^2 \\ &\quad + \frac{C_3 |g|_{1,+\infty}^4}{\nu} \int_0^t ||u_{xx}(\cdot,\tau)||^2 d\tau. \end{aligned} \tag{3.31}$$

The elementary inequality (3.14) and the hypothesis (**) show that

$$|\int_0^t \int_0^{+\infty} P''(u) u_x^3 dx d\tau| \leq \int_0^t \Big[\frac{C_4 \varepsilon^{\frac{4}{3}}}{\nu^{\frac{1}{3}}} ||u(\cdot,\tau)||^{\frac{4}{3}} ||u_x(\cdot,\tau)||^{\frac{14}{3}} + \frac{\nu}{4} ||u_{xx}(\cdot,\tau)||^2\Big] d\tau. \tag{3.32}$$

Using (3.32) and Young's inequality, the inequality in (3.23) may be revised to read

$$\begin{aligned} &||u_{xx}(\cdot,t)||^2 + \nu \int_0^t ||u_{xx}(\cdot,\tau)||^2 d\tau \\ &\leq C_5(||f||_2, |g|_{1,+\infty}) + \delta ||u_x(\cdot,t)||^6 + \int_0^t \frac{C_4 \varepsilon^{\frac{4}{3}}}{\nu^{\frac{1}{3}}} ||u(\cdot,\tau)||^{\frac{4}{3}} ||u_x(\cdot,\tau)||^{\frac{14}{3}} d\tau, \end{aligned} \tag{3.33}$$

for any $\delta > 0$. Multiplying (3.33) by the constant C^* determined by the relation

$$C^* \frac{|g|_{1,+\infty}^4}{\nu^2} = \max\{1 + C_3 \frac{|g|_{1,+\infty}^4}{\nu^2}, 1 + C_2 |g|_{1,+\infty}^4\},$$

where C_2 and C_3 are the constants appearing earlier in the proof, and then adding the result to (3.31), it is deduced that

$$\begin{aligned} &||u(\cdot,t)||_1^6 + C_7(\nu, |g|_{1,+\infty}) ||u_{xx}(\cdot,t)||^2 + \nu^2 C_8 |g|_{1,+\infty}^4 \Big(\int_0^t ||u_x(\cdot,\tau)||^2 d\tau\Big)^3 \\ &+ C_9(\nu, |g|_{1,+\infty}) \int_0^t ||u_{xx}(\cdot,\tau)||^2 d\tau \\ &\leq C_{10}(||f||_2, |g|_{1,+\infty}, \nu) + \varepsilon^{\frac{4}{3}} C_{11} \frac{|g|_{1,+\infty}^4}{\nu^{\frac{4}{3}}} \int_0^t ||u(\cdot,\tau)||^{\frac{4}{3}} ||u_x(\cdot,\tau)||^{\frac{14}{3}} d\tau. \end{aligned} \tag{3.34}$$

It is then straightforward to show that

$$\begin{aligned}
&\int_0^t C_{11}\varepsilon^{\frac{4}{3}}\frac{|g|_{1,+\infty}^4}{\nu^{\frac{4}{3}}}\|u(\cdot,\tau)\|^{\frac{4}{3}}\|u_x(\cdot,\tau)\|^{\frac{14}{3}}d\tau\\
&\le C_{11}\varepsilon^{\frac{4}{3}}\frac{|g|_{1,+\infty}^4}{\nu^{\frac{4}{3}}}\|u\|_{C(\mathbb{R}^+;L_2)}^{\frac{4}{3}}\|u_x\|_{C(\mathbb{R}^+;L_2)}^{\frac{8}{3}}\int_0^t\|u_x(\cdot,\tau)\|^2d\tau\\
&\le\frac{2C_{11}^{\frac{3}{2}}\varepsilon^2|g|_{1,+\infty}^4}{3C_8^2\nu^3}\|u\|_{C(\mathbb{R}^+;L_2)}^2\|u_x\|_{C(\mathbb{R}^+;L_2)}^4\\
&\quad+\frac{C_8\nu^2|g|_{1,+\infty}^4}{3}\Big(\int_0^t\|u_x(\cdot,\tau)\|^2d\tau\Big)^3.
\end{aligned}\tag{3.35}$$

Putting (3.35) in (3.34) and assuming that $\frac{2C_{11}^{\frac{3}{2}}\varepsilon^2|g|_{1,+\infty}^4}{3C_8^2\nu^3}\le\frac{1}{2}$, it follows that

$$\begin{aligned}
&\|u\|_{C(\mathbb{R}^+;H^1)}^6+C_6\|u_{xx}\|_{C(\mathbb{R}^+;L_2)}^2+C_7\Big(\int_0^{+\infty}\|u_x(\cdot,\tau)\|^2d\tau\Big)^3\\
&\quad+C_8\int_0^{+\infty}\|u_{xx}(\cdot,\tau)\|^2d\tau\le C_9(\|f\|_2,|g|_{1,+\infty},\nu).
\end{aligned}\tag{3.36}$$

Hence the corollary is proved. □

LEMMA 3.13. *Let u be the solution of (3.1a) corresponding to initial data $f\in H^2(\mathbb{R}^+)$ and boundary data $g\in H^1(\mathbb{R}^+)$. If P satisfies the conditions delineated in Corollary 3.12 and $\nu>0$, then*

$$\|u_x(\cdot,t)\|,\ \|u_{xx}(\cdot,t)\|\to 0,\qquad \text{as}\qquad t\to+\infty,$$

and

$$\|u(\cdot,t)\|_{L_\infty}\to 0\qquad \text{as}\qquad t\to+\infty.$$

PROOF: By Corollary 3.12, it is known that $\|u_x(\cdot,t)\|$, $\|u_{xx}(\cdot,t)\|$ lie in $L_2(\mathbb{R}^+)$. From (3.23), it then follows that both $\|u_x(\cdot,t)\|$ and $\|u_{xx}(\cdot,t)\|$ have limits as t tends to infinity. In consequence, these two limits at infinity must be zero. Noting that

$$\|u(\cdot,t)\|_{L_\infty}^2\le 2\|u(\cdot,t)\|\|u_x(\cdot,t)\|\le 2C(\|f\|_2,|g|_{1,+\infty},\nu)\|u_x(\cdot,t)\|$$

and remarking that the right-hand side of this inequality tends to zero as $t\to+\infty$, the result follows. □

LEMMA 3.14. *Let u be the solution of (3.1a) with $\nu>0$ corresponding to initial data $f\in H^2(\mathbb{R}^+)$ and boundary data $g\in H^2(\mathbb{R}^+)$ and suppose P satisfies the condition in Corollary 3.12. It then follows that u_t and u_{xt} lie in $L_2(\mathbb{R}^+\times\mathbb{R}^+)$.*

PROOF: Multiply (3.1a) by u_t and integrate the result over $\mathbb{R}^+\times[0,t)$. After integration by parts and using Lemma 3.3, we are reduced to

$$\begin{aligned}
&\frac{\nu}{2}||u_x(\cdot,t)||^2 + \int_0^t \left[||u_t(\cdot,\tau)||^2 + \alpha^2||u_{xt}(\cdot,\tau)||^2\right]d\tau \\
&= \frac{\nu}{2}||f'||^2 + \int_0^t \left[\alpha^2 g''(\tau) - \nu g'(\tau)\right]u_x(0,\tau)d\tau - \alpha^2 g'(t)u_x(0,t) \\
&\quad + \alpha^2 g'(0)f(0) - \int_0^t\int_0^{+\infty} u_t\big(u_x + P(u)_x\big)dxd\tau \\
&\le C(||f||_2, |g|_{2,+\infty}) + \frac{1}{2}\int_0^t ||u_t(\cdot,\tau)||^2 d\tau \\
&\quad + C|g|_{2,+\infty}\left(\int_0^t ||u_x(\cdot,\tau)||^2 d\tau\right)^{\frac{1}{4}}\left(\int_0^t ||u_{xx}(\cdot,\tau)||^2 d\tau\right)^{\frac{1}{4}} \\
&\quad + \frac{1}{2}(1+||P'(u)||^2_{L_\infty})\int_0^t ||u_x(\cdot,\tau)||^2 d\tau,
\end{aligned} \tag{3.37}$$

where (3.14) has been used. The result follows from (3.37) since u_x, $u_{xx}\in L_2(\mathbb{R}^+\times\mathbb{R}^+)$ and $||P'(u)||_{L_\infty} \le C(||f||_1, |g|_{1,\infty})$. □

3.3. Continuous Dependence

Attention is now given to showing that solutions of (3.1) depend continuously on the specified data. That is, small perturbations of the initial and boundary data f and g lead to small perturbations of the corresponding solution. This is a very important aspect of model equations for waves that apply in regimes where breaking and other singularity formation are not countenanced. Indeed, this property is crucial if laboratory measurements are to be successfully compared with numerical approximations of the solutions of the model.

Let (f_1, g_1) and (f_2, g_2) be two sets of data for problem (3.1). Theorem 3.11 shows that if $f_i\in C_b^2(\mathbb{R}^+)\cap H^1(\mathbb{R}^+)$ (respectively, $f_i\in C_b^2(\mathbb{R}^+)\cap H^2(\mathbb{R}^+)$) and $g_i\in C^1(0,T)$, $i=1,2$, then with suitable growth restrictions on P, the corresponding solutions u_1 and u_2 of (3.1) lie in $\mathcal{B}_T^{2,1}\cap C(0,T;H^1)$ (respectively, $\mathcal{B}_T^{2,1}\cap C(0,T;H^2)$).

For $k=1,2$, let $\mathbb{U}_k$ denote the mapping that takes the auxiliary specifications ν, f and g into the corresponding solutions of (3.1). Thus $\mathbb{U}_k$ maps X_k into Y_k, $k=1,2$, where

$$X_1 = \{(f,g):\ (f,g)\in H^1(\mathbb{R}^+)\cap C_b^2(\bar{\mathbb{R}}^+)\times C^1(0,T)\},$$
$$X_2 = \{(\nu,f,g):\ (\nu,f,g)\in\mathbb{R}^+\times H^2(\mathbb{R}^+)\cap C_b^2(\bar{\mathbb{R}}^+)\times C^1(0,T)\}$$

and

$$Y_k = \mathcal{B}_T^{2,1}\cap C(0,T;H^k).$$

THEOREM 3.15. *The mapping $\mathbb{U}_k$ defined above is continuous, $k = 1, 2$.*

PROOF: We deal only with showing that $\mathbb{U}_2$ is continuous. Similar arguments suffice to show that $\mathbb{U}_1$ is continuous.

It is first shown that $\mathbb{U}_2$ is continuous for a fixed positive ν. Let $(\nu, f_i, g_i) \in X_2$, and $u_i = \mathbb{U}_2(\nu, f_i, g_i)$ for $i = 1, 2$ and define w to be $w = u_1 - u_2$. Then w satisfies the initial- and boundary-value problem

$$\begin{aligned} w_t + w_x + \{P(u_1) - P(u_2)\}_x - \nu w_{xx} - \alpha^2 w_{xxt} &= 0, & &\text{for } x,\ t \geq 0, & (3.38a)\\ w(x,0) &= f(x), & &\text{for } x \geq 0, & (3.38b)\\ w(0,t) &= g(t), & &\text{for } t \geq 0. & (3.38c) \end{aligned}$$

where

$$f(x) = f_1(x) - f_2(x), \ \ g(t) = g_1(t) - g_2(t).$$

Multiply equation (3.38a) by $2w - 2w_{xx}$ and integrate the result over $\mathbb{R}^+ \times [0, t)$. After integration by parts, one has

$$\begin{aligned} &||w(\cdot,t)||^2 + (1+\alpha^2)||w_x(\cdot,t)||^2 + \alpha^2||w_{xx}(\cdot,t)||^2 \\ &\quad + \int_0^t \Big(w_x^2(0,\tau) + 2\nu\big[||u_x(\cdot,\tau)||^2 + ||u_{xx}(\cdot,\tau)||^2\big]\Big)d\tau \\ &= ||f||^2 + (1+\alpha^2)||f'||^2 + \alpha^2||f''||^2 - 2\alpha^2 g(t)w_x(0,t) + 2\alpha^2 g(0)f'(0) \\ &\quad + \int_0^t \big[g^2(\tau) + 2(\alpha^2-1)g'(\tau)w_x(0,\tau) - 2\nu g(\tau)w_x(0,\tau)\big]\,d\tau \\ &\quad - \int_0^t\int_0^{+\infty} 2\big[w(x,\tau) - w_{xx}(x,\tau)\big]\big(P(u_1(x,\tau)) - P(u_2(x,\tau))\big)_x\,dx\,d\tau. \end{aligned} \tag{3.39}$$

By Lemma 3.6, there exists a constant $\bar{a}_2$ which only depends upon $T, ||f_i||_2$ and $|g_i|_{1,T}$ such that

$$||u_i(\cdot,t)||_2 \leq \bar{a}_2, \qquad (i = 1,2). \tag{3.40}$$

From (3.39), (3.40) and some elementary inequalities, including (3.14) applied to w_x, it follows that

$$||w(\cdot,t)||_2^2 \leq ||f||_2^2 + C_1||g||^2_{C^1(0,T)} + C_2(\gamma(\mathbf{B}_T))\int_0^t ||w(\cdot,\tau)||_2^2 d\tau, \tag{3.41}$$

where $\gamma(\mathbf{B}_T)$ is a constant such that $|P(z_1) - P(z_2)| \leq \gamma(\mathbf{B}_T)|z_1 - z_2|$ and $\mathbf{B}_T = \{w \in \mathcal{B}_T : ||w||_{\mathcal{B}_T} \leq 2\bar{a}_2\}$. By Gronwall's lemma, it follows from (3.41) that

$$||w(\cdot,t)||_2^2 \leq C(||f||_2^2 + ||g||^2_{C^1(0,T)}),$$

or what is the same,

$$||u_1 - u_2||^2_{C(0,T;H^2)} \leq C(||f_1 - f_2||_2^2 + ||g_1 - g_2||^2_{C^1(0,T)}). \tag{3.42}$$

By a Sobolev embedding theorem, (3.42) implies that

$$||u_1 - u_2||_{\mathcal{B}_T} \leq C(||f_1 - f_2||_2 + ||g_1 - g_2||_{C^1(0,T)}). \tag{3.43}$$

In consequence, the mapping $\mathbb{U}_2 \colon X_2 \to \mathcal{B}_T$ is continuous, at least for fixed ν.

To prove that $\mathbb{U}_2 \colon X_2 \to \mathcal{B}_T^{2,1}$ is continuous for fixed ν, write (3.38a) as an integral equation, analogous to (3.8), namely

$$\begin{aligned} w(x,t) &= \exp(-t\nu/\alpha^2)f(x) + \tilde{g}(t)e^{-x/\alpha} \\ &+ \int_0^t \int_0^{+\infty} \exp(-\nu(t-\tau)/\alpha^2)K(x,\xi)\big[P(u_1(\xi,\tau)) - P(u_2(\xi,\tau)) + w(\xi,\tau)\big]\,d\xi d\tau \\ &- \frac{\nu}{\alpha}\int_0^t \int_0^{+\infty} \exp(-\nu(t-\tau)/\alpha^2)L(x,\xi)w(\xi,\tau)\,d\xi d\tau, \end{aligned} \tag{3.44}$$

where $\tilde{g}(t) = g(t) - \exp(-t\nu/\alpha^2)g(0)$ and K and L are as in (3.5) and (3.6), respectively. By using (3.43) and (3.44), along with the assumption $\tilde{g}(t) \in C^1(0,T)$ and $f(x) \in H^2(\mathbb{R}^+) \cap C_b^2(\mathbb{R}^+)$, one can easily show that

$$||w||_{\mathcal{B}_T^{2,1}} \leq C(||f_1 - f_2||_2 + ||g_1 - g_2||_{C^1(0,T)}). \tag{3.45}$$

As an example, by differentiating (3.44) with respect to t, it follows that

$$\begin{aligned} w_t(x,t) &= -\frac{\nu}{\alpha^2}\exp(-t\nu/\alpha^2)f(x) + \tilde{g}'(t)e^{-x/\alpha} \\ &- \frac{\nu}{\alpha^2}\int_0^t \int_0^{+\infty} \exp(-\nu(t-\tau)/\alpha^2)K(x,\xi)\Big[P(u_1(\xi,\tau)) \\ &\qquad\qquad - P(u_2(\xi,\tau)) + w(\xi,\tau)\Big]\,d\xi d\tau \\ &+ \int_0^{+\infty} K(x,\xi)\big[P(u_1(\xi,\tau)) - P(u_2(\xi,\tau)) + w(\xi,\tau)\big]\,d\xi d\tau \\ &+ \frac{\nu^2}{\alpha^3}\int_0^t \int_0^{+\infty} \exp(-\nu(t-\tau)/\alpha^2)L(x,\xi)w(\xi,\tau)\,d\xi d\tau \\ &- \frac{\nu}{\alpha}\int_0^{+\infty} L(x,\xi)w(\xi,\tau)\,d\xi d\tau. \end{aligned} \tag{3.46}$$

Notice that

$$\begin{aligned} \int_0^{+\infty} |K(x,\xi)|\,d\xi &= \frac{1}{2\alpha^2}\int_0^{+\infty} |\exp(-(x+\xi/\alpha) + sgn(x-\xi)\exp(-|x-\xi|/\alpha)|\,d\xi \\ &= \frac{1}{2\alpha^2}\int_x^{+\infty} |\exp(-(x+\xi/\alpha) - \exp((x-\xi)/\alpha)|\,d\xi \\ &\qquad + \frac{1}{2\alpha^2}\int_0^x (\exp(-(x+\xi/\alpha) + \exp((\xi-x)/\alpha))\,d\xi \\ &\leq \frac{1}{\alpha}, \end{aligned} \tag{3.47}$$

and similarly,

$$\int_0^{+\infty} |L(x,\xi)|d\xi \le \frac{2}{\alpha}. \tag{3.48}$$

Using some elementary inequalities, (3.46), together with (3.47) and (3.48) give

$$||w_t||_{\mathcal{B}_T} \le \frac{\nu}{\alpha^2}||f||_2 + ||\tilde{g}||_{C^1(0,T)} + C(T\nu + 1 + \gamma(\mathbf{B}_T))||w||_{\mathcal{B}_T}. \tag{3.49}$$

By using (3.43), one then obtains

$$||w_t||_{\mathcal{B}_T} \le C(||f_1 - f_2||_2 + ||g_1 - g_2||_{C^1(0,T)}). \tag{3.50}$$

Differentiating (3.44) with respect to x and then using (3.43) and the same sort of considerations that led to (3.50), one sees that

$$||w_x||_{\mathcal{B}_T} \le C(||f_1 - f_2||_2 + ||g_1 - g_2||_{C^1(0,T)}). \tag{3.51}$$

Differentiating (3.44) with respect to x again and arguing as above yields

$$||w_{xx}||_{\mathcal{B}_T} \le C(||f_1 - f_2||_2 + ||g_1 - g_2||_{C^1(0,T)}). \tag{3.52}$$

Similar results may be obtained with regard to u_{xt} and u_{xxt}, and in this manner (3.45) is verified.

Next, we show that $\mathbb{U}_2$ is continuous in the variable ν when the initial and boundary data are fixed. Let $v_i = \mathbb{U}_2(\nu_i, f_2, g_2)$ for $i = 1, 2$, and let $y = v_1 - v_2$. Then y satisfies the initial- and boundary-value problem

$$y_t + y_x + \{P(v_1) - P(v_2)\}_x - (\nu_1 - \nu_2)(v_2)_{xx} - \nu_1 y_{xx} - \alpha^2 y_{xxt} = 0, \quad \text{for } x,\, t \ge 0, \tag{3.53a}$$

$$y(x,0) = 0, \qquad \text{for } x \ge 0, \tag{3.53b}$$

$$y(0,t) = 0, \qquad \text{for } t \ge 0. \tag{3.53c}$$

Multiply (3.53a) by $2y - 2y_{xx}$ and integrate the result over $\mathbb{R}^+$. After integration by parts and using Lemma 3.6 applied to the solutions v_1 and v_2, one has

$$\begin{aligned}\frac{d}{dt}\Big(||y||^2 + (1+\alpha^2)||y_x||^2 + \alpha^2||y_{xx}||^2\Big) + y_x^2(0,t) + 2\nu_1\big(||y_x||^2 + ||y_{xx}||^2\big) \\ = \int_0^{+\infty} [2y - 2y_{xx}]\big[(\nu_1 - \nu_2)(v_2)_{xx} - \{P(v_1) - P(v_2)\}_x\big]dx \\ \le \Big(|\nu_1 - \nu_2|||v_2||_{\mathcal{B}_T^{2,1}}||y||_2 + \gamma(\mathbf{B}_T)||y||_2^2\Big) \\ \le C^*(|\nu_1 - \nu_2| \cdot ||y||_2 + ||y||_2^2),\end{aligned} \tag{3.54}$$

where $C^* = max\{\gamma(\mathbf{B}_T), ||v_2||_{\mathcal{B}_T^{2,1}}\}$. Since $y(x,0) = 0$, it follows from Gronwall's lemma that

$$||y||_2 \leq |\nu_1 - \nu_2|(\exp(C^* c(\alpha)t) - 1)$$

for $0 \leq t \leq T$, where $c(\alpha) = \frac{1}{\min\{1,\alpha^2\}}$. By a Sobolev embedding theorem, this in turn shows that

$$||v_1 - v_2||_{\mathcal{B}_T} \leq C|\nu_1 - \nu_2|. \tag{3.55}$$

Using of an integral equation version of (3.53), the last inequality may be extended to a bound of the form

$$||v_1 - v_2||_{\mathcal{B}_T^{2,1}} \leq C|\nu_1 - \nu_2|, \tag{3.56}$$

thus showing that $\mathbb{U}_2$ is a Lipschitz continuous function of the parameter ν.

The triangle inequality applied thusly,

$$\begin{aligned} ||\mathbb{U}_2(\nu_1, f_1, g_1) &- \mathbb{U}_2(\nu_2, f_2, g_2)||_{\mathcal{B}_T^{2,1}\cap C(0,T;H^2(\mathbb{R}^+))} \\ &\leq ||\mathbb{U}_2(\nu_1, f_1, g_1) - \mathbb{U}_2(\nu_1, f_2, g_2)||_{\mathcal{B}_T^{2,1}\cap C(0,T;H^2(\mathbb{R}^+))} \\ &\qquad + ||\mathbb{U}_2(\nu_1, f_2, g_2) - \mathbb{U}_2(\nu_2, f_2, g_2)||_{\mathcal{B}_T^{2,1}\cap C(0,T;H^2(\mathbb{R}^+))}, \end{aligned}$$

allows one to infer from (3.45) and (3.57) that

$$\mathbb{U}_2 \colon X_2 = \{[H^2(\mathbb{R}^+) \cap C_b^2(\mathbb{R}^+)] \times C^1(0,T)\} \longmapsto Y = \mathcal{B}_T^{2,1} \cap C(0,T; H^2(\mathbb{R}^+))$$

is continuous. This concludes the proof of the theorem. □

COROLLARY 3.16. *Let $\mathbb{U}$ denote the mapping that associates with the triple (ν, f, g) the corresponding solution of (3.1). For any $k \geq 2$ and $l \geq 1$,*

$$\begin{aligned} \mathbb{U} : \ X = \{(\nu, f, g) : \ (\nu, f, g) \in \mathbb{R}^+ \times H^k(\mathbb{R}^+) \cap C_b^k(\bar{\mathbb{R}}^+) \times C^l(0,T)\} \\ \longmapsto Y = \mathcal{B}_T^{k,l} \cap C(0,T; H^k(\mathbb{R}^+)), \end{aligned}$$

and this correspondence is continuous. □

4. WELL-POSEDNESS OF TWO-POINT BOUNDARY-VALUE PROBLEM

In this section, interest will be focused on the initial- and two-point boundary-value problem

$$u_t + u_x + P(u)_x - \nu u_{xx} - \alpha^2 u_{xxt} = 0, \qquad \text{for } x \in (0,1),\ t \geq 0, \tag{4.1a}$$

$$u(x,0) = f(x), \qquad \text{for } x \in [0,1], \tag{4.1b}$$

$$u(0,t) = g(t), \qquad \text{for } t \geq 0. \tag{4.1c}$$

$$u(1,t) = h(t), \qquad \text{for } t \geq 0. \tag{4.1d}$$

For consistency, the restrictions

$$u(0,0) = f(0) = g(0), \quad \text{and} \quad u(1,0) = f(1) = h(0) \tag{4.2}$$

will be imposed and it will be supposed that $\nu \geq 0$ and $\alpha \neq 0$. Throughout this section, Ω will stand for the interval $[0,1]$.

4.1. Local Solutions

By converting the differential equation with initial condition (4.1b) and boundary conditions (4.1c) and (4.1d) into an integral equation as in (3.3) and applying the contraction-mapping theorem to this formulation of the problem, a local solution can be established. The argument closely parallels that worked out above for (3.1) (see also [6]), and consequently we content ourselves with a statement of the conclusions that may be derived by this approach.

PROPOSITION 4.1. *Let $T > 0$, $f \in C(\Omega)$, g and $h \in C(0,T)$, and P locally Lipschitz continuous. Then there exists a positive constant $T' = T'(||f||_{C_b(\Omega)}, ||g||_{C(0,T)})$ such that for any t with $0 < t < T_0 = min(T',T)$, there is a unique solution of (4.1) in $\mathcal{B}_{T_0}$. If $f \in C^k(\Omega)$, g and $h \in C^l(0,T)$ where $k \geq 2$, $l \geq 1$ and $k > l$, then the corresponding solution u of (4.1) is an element of $\mathcal{B}_{T_0}^{k,l}$ and is a classical solution of the initial- and boundary-value problem (4.1).*

4.2. Global Solutions

The local solution of (4.1) whose existence was just confirmed can be extended to a global solution by appeal to appropriate *a priori* estimates. The *a priori* bounds which allow one to extend the local solution of (4.1) to arbitrary time intervals is derived by energy estimates just as was done for (3.1). We have been unable to derive an analog of Lemma 3.4 for the two-point boundary-value problem (4.1). Similar problems were noted for nonhomogeneous boundary-value problem for KdV (see [6, 17]). Thus we are not able to obtain the helpful $H^1(\mathbb{R}^+)$-bound that is the outcome of Lemma 3.4 in the present circumstances. However, $H^2(\mathbb{R}^+)$-bounds can be derived as is now demonstrated.

LEMMA 4.2. *Suppose that $f \in C^2(\Omega) \cap H^2(\Omega)$, g, $h \in C^1(0,T)$ with $f(0) = g(0)$ and $f(1) = h(0)$, and that $P \in C^2(\mathbb{R})$ satisfies the growth condition*

$$\limsup_{|s|\to\infty} |s|^{-2}|P''(s)| \leq \varepsilon \tag{**}$$

for some finite constant ε. Let $u(x,t)$ be a classical solution of (4.1) up to the boundary on $\Omega\times[0,T]$. Then for all $t\in[0,T)$ $(T\le T_0), u(\cdot,t)\in H^2(\Omega)$ and there exists a constant a_2 only depending on T, $\|f\|_2$, $|g|_{C^1(0,T)}$ and $|h|_{C^1(0,T)}$ such that $\|u(\cdot,t)\|_2\le a_2$.

Lemma 4.2 for the problem (4.1) may be proved in exactly the same way as we proved Lemma 3.6 for the problem (3.1). The principal difference is that the term $u_x(1,t)$ arises on account of integrations by parts. This quantity can be controlled by use of the analogue

$$||v||_{C_b(0,1)}\le[\|v\|(\|v\|+2\|v'\|)]^{\frac{1}{2}} \tag{4.3}$$

of (3.14). Applying this relation and an estimate like that appearing in (3.18), a suitable differential inequality may be derived to which Gronwall's lemma applies and yields the desired results.

Using the bound obtained from Lemma 4.2, the local solution obtained in Proposition 4.1 can be extended to arbitrary time intervals. Results of continuous dependence on the data f, g and h and on the parameter ν can be obtained just as for (3.1). Thus the following theorem for (4.1) emerges.

THEOREM 4.3. *Let there be given $T>0$, $f\in C^k(\Omega)\cap H^k(\Omega)$, g, $h\in C^l(0,T)$, where $k\ge 2$, $l\ge 1$. Suppose that $f(0)=g(0)$, $f(1)=h(0)$, and that nonlinearity P is smooth and satisfies the conditions (**) specified in Lemma 4.2. Then the initial-boundary-value problem (4.1) has a unique solution $u\in\mathcal{B}_T^{k,l}$. Let $\mathbb{U}$ denote the mapping that associates to ν and the triple (f,g,h) of data in (4.1b, c, d) the corresponding solutions of (4.1). Then $\mathbb{U}$ is a continuous mapping of X into Y where*

$$X=\{(\nu,f,g,h):\ (\nu,f,g,h)\in\mathbb{R}^+\times C^k(\Omega)\times C^l(0,T)\times C^l(0,T)\}\quad\text{and}$$
$$Y=\mathcal{B}_T^{k,l}.$$

The following result is similar to Corollary 3.12. It is useful for estimating decay in the temporal variable in case $\nu>0$.

COROLLARY 4.4. *Let initial data f defined on $\Omega=(0,1)$ be given and boundary data g, h defined on all of $\mathbb{R}^+$ specified. Suppose that (f,g,h) satisfies the hypotheses in Lemma 4.2 for any $T>0$, and that in addition g, $h\in H^1(\mathbb{R}^+)$. Suppose $\nu>0$ and that either ε is small enough, ν is big enough, or that the boundary data g and h are small enough in $H^1(\mathbb{R}^+)$. If u is the solution of (4.1) corresponding to (f,g,h), then u_x, $u_{xx}\in L_2(\Omega\times\mathbb{R}^+)$ and $||u(\cdot,t)||_{L_\infty}\to 0$ as $t\to+\infty$.*

4.3. Decay Rates

The asymptotic behaviour as $t \to +\infty$ of solutions of (4.1) will be considered in this subsection. To simplify the presentation, it is assumed that $(P(x))_x = u^p u_x$. It is worth note that the proof of decay obtained for the pure initial-value problem posed on all of $\mathbb{R}$ in [2, 9] can be taken over intact in case g, $h \equiv 0$. However, since Ω is bounded, stronger temporal decay is expected than the algebraic rates obtained in [2, 9] in case $\nu > 0$. Moreover, the imposition of homogeneous boundary conditions at both sides of the spatial domain is artificial. In consequence, we undertake now a direct derivation of temporal decay rates for the initial-boundary-value problem (4.1).

LEMMA 4.5. *Suppose $\nu > 0$ and that the auxiliary data (f, g, h) and P satisfy the conditions specified in Corollary 4.4. Let u be the associated solution of (4.1). Then, $u \in C_b(\mathbb{R}^+; H^2(\Omega)) \cap L_2(\mathbb{R}^+; H^2(\Omega))$ and u_{xt}, $u_{xxt} \in L_2(\Omega \times \mathbb{R}^+)$.*

PROOF: First multiply (4.1a) by the combination $2(b_1 - x)u(x,t)$ for some positive constant b_1 to be specified presently, and then integrate the result over Ω. After simplification, we reach the relation

$$\begin{aligned}
&\int_0^1 \left[u(x,t)^2 + 2\nu(b_1 - x)u_x^2(x,t)\right]dx + \frac{d}{dt}\int_0^1 \left[(b_1 - x)[u(x,t)^2 + \alpha^2 u_x(x,t)^2]\right]dx \\
&= b_1\Big[2\alpha^2 g(t)u_{xt}(0,t) + 2\nu g(t)u_x(0,t) - g^2(t) - \frac{2}{p+2}g^{p+2}(t)\Big] \\
&\quad - (b_1 - 1)\Big[2\alpha^2 h(t)u_{xt}(1,t) + 2\nu h(t)u_x(1,t) - h^2(t) - \frac{2}{p+2}h^{p+2}(t)\Big] \\
&\quad + \nu h(t)^2 - \nu g(t)^2 + 2\alpha^2 h(t)h'(t) - 2\alpha^2 g(t)g'(t) \\
&\quad - \int_0^1 \left[\frac{2}{p+2}u^{p+2}(x,t) + 2\alpha^2 u_x(x,t)u_t(x,t)\right]dx.
\end{aligned} \tag{4.4}$$

Equation (4.1a) gives the crude inequality

$$\begin{aligned}
||u_t(\cdot,t)||^2 \le\, & C(|g'(t)| + |h'(t)|) \\
& + C\Big[||u_x(\cdot,t)||^2 + ||u_{xx}(\cdot,t)||^2 + ||u_{xt}(\cdot,t)||^2 + ||u_{xxt}(\cdot,t)||^2\Big].
\end{aligned} \tag{4.5}$$

Applying (4.3) to u_x and u_{xt}, using (4.5) in (4.4) and making other elementary estimates leads to

$$\begin{aligned}
&\int_0^1 \left[u(x,t)^2 + 2\nu(b_1 - x)u_x^2(x,t)\right]dx + \frac{d}{dt}\int_0^1 \left[(b_1 - x)[u(x,t)^2 + \alpha^2 u_x(x,t)^2]\right]dx \\
&\le C(\delta)(|g(t)|^2 + |h(t)|^2 + |g'(t)|^2 + |h'(t)|^2) + \frac{2}{p+2}||u(\cdot,t)||_{L_\infty}^p ||u(\cdot,t)||^2 \\
&\quad + C(\delta)||u_x(\cdot,t)||^2 + \delta\big[||u_{xx}(\cdot,t)||^2 + ||u_{xt}(\cdot,t)||^2 + ||u_{xxt}(\cdot,t)||^2\big],
\end{aligned} \tag{4.6}$$

which holds for any $\delta > 0$.

Multiply (4.1a) by u_{xx} and integrate the result over Ω. After integration by parts and using Lemma 4.2, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||u_x(\cdot,t)||^2 + \frac{\alpha^2}{2}\frac{d}{dt}||u_{xx}(\cdot,t)||^2 + \nu||u_{xx}(\cdot,t)||^2 \\
&= [\frac{1}{2}u_x^2(1,t) + h'(t)u_x(1,t)] - [\frac{1}{2}u_x^2(0,t) + g'(t)u_x(0,t)] - \int_0^1 u^p u_x u_{xx} dx \\
&\leq C(|g'(t)|^2 + |h'(t)|^2) + C(\nu)(1 + ||u(\cdot,t)||_{L_\infty}^{2p})||u_x(\cdot,t)||^2 + \frac{\nu}{2}||u_{xx}(\cdot,t)||^2,
\end{aligned} \tag{4.7}$$

where inequalities (4.3) has been used to control the terms $u_x(0,t)$ and $u_x(1,t)$.

Multiply (4.1a) by u_{xxt} and integrate the result over Ω. After integration by parts and use of (4.3), it is adduced that

$$\begin{aligned}
&\frac{\nu}{2}\frac{d}{dt}||u_{xx}(\cdot,t)||^2 + \alpha^2||u_{xxt}(\cdot,t)||^2 + ||u_{xt}(\cdot,t)||^2 \\
&= h'(t)u_{xt}(1,t) - g'(t)u_{xt}(0,t) + \int_0^1 [u^p u_x u_{xxt} + u_x u_{xxt}]dx \\
&\leq C(|g'(t)|^2 + |h'(t)|^2) + C(\alpha)(1 + ||u(\cdot,t)||_{L_\infty}^{2p})||u_x(\cdot,t)||^2 \\
&\quad + \frac{1}{2}||u_{xt}(\cdot,t)||^2 + \frac{\alpha^2}{2}||u_{xxt}(\cdot,t)||^2.
\end{aligned} \tag{4.8}$$

Adding (4.6), (4.7) and (4.8) together yields the differential inequality

$$\begin{aligned}
&\int_0^1 [u(x,t)^2 + 2\nu(b_1 - x)u_x^2(x,t)]\,dx + \frac{\nu}{2}||u_{xx}(\cdot,t)||^2 + \frac{\alpha^2}{2}||u_{xxt}(\cdot,t)||^2 + \frac{1}{2}||u_{xt}(\cdot,t)||^2 \\
&+ \frac{d}{dt}\Big[\int_0^1 [(b_1 - x)[u(x,t)^2 + \alpha^2 u_x(x,t)^2]]\,dx + \frac{1}{2}||u_x(\cdot,t)||^2 + \Big(\frac{\alpha^2}{2} + \frac{\nu}{2}\Big)||u_{xx}(\cdot,t)||^2\Big] \\
&\leq C(|g(t)|^2 + |h(t)|^2 + |g'(t)|^2 + |h'(t)|^2) + \frac{2}{p+2}||u(\cdot,t)||_{L_\infty}^p ||u(\cdot,t)||^2 \\
&\quad + \big[C(\delta) + C(\nu)(1 + ||u(\cdot,t)||_{L_\infty}^{2p}) + C(\alpha)(1 + ||u(\cdot,t)||_{L_\infty}^{2p})\big]||u_x(\cdot,t)||^2 \\
&\quad + \delta\big[||u_{xx}(\cdot,t)||^2 + ||u_{xt}(\cdot,t)||^2 + ||u_{xxt}(\cdot,t)||^2\big].
\end{aligned} \tag{4.9}$$

First, choose T large enough so that, simultaneously,

$$||u(\cdot,t)||_{L_\infty}^{2p} \leq 1 \quad \text{and} \quad \frac{2}{p+2}||u(\cdot,t)||_{L_\infty}^p \leq \frac{1}{2},$$

for $t > T$. This is possible since $||u(\cdot,t)||_{L_\infty} \to 0$ for $t \to +\infty$. Next, choose the constant b_1 in (4.9) so that

$$b_1 \geq \max\Big\{2, \frac{C(\delta) + 2C(\nu) + 2C(\alpha)}{\nu}\Big\}.$$

Finally, choose δ small, say

$$\delta = \frac{1}{4}\min\{\alpha^2, \nu, 1\}.$$

With these choice, (4.9) implies that

$$\begin{aligned} &\beta||u(\cdot,t)||_2^2 + \frac{d}{dt}||u(\cdot,t)||_2^2 \\ &+ C_1||u_{xt}(\cdot,t)||^2 + C_2||u_{xxt}(\cdot,t)||^2 \\ &\leq C(|g(t)|^2 + |h(t)|^2 + |g'(t)|^2 + |h'(t)|^2), \end{aligned} \tag{4.10}$$

for all $t > T$, where β, C_1 and C_2 are some positive constants depending on α, δ and ν. Integrating (4.10) over $[T, t)$, the result stated in the lemma comes into view upon applying Theorem 4.3 to $||u(\cdot, T)||_2$. □

If stronger hypotheses are posited on the boundary conditions, stronger results can be derived. In fact, if one multiplies (4.10) by βt and adds the result to (4.10), there appears

$$\begin{aligned} &t\beta^2||u(\cdot,t)||_2^2 + \frac{d}{dt}\Big((1+t\beta)||u(\cdot,t)||_2^2\Big) \\ &+ t\beta(C_1||u_{xt}(\cdot,t)||^2 + C_2||u_{xxt}(\cdot,t)||^2) \\ &\leq C(t\beta+1)(|g(t)|^2 + |h(t)|^2 + |g'(t)|^2 + |h'(t)|^2), \end{aligned} \tag{4.11}$$

for $t > T$. If $t[|g(t)|^2 + |h(t)|^2 + |g'(t)|^2 + |h'(t)|^2] \in L_1(\mathbb{R}^+)$, then integrating (4.11) over $[T, t)$ leads to

$$\begin{aligned} &(1+t\beta)||u(\cdot,t)||_2^2 + \int_T^t \tau\beta^2||u(\cdot,\tau)||_2^2 d\tau \\ &\leq (1+T\beta)||u(\cdot,T)||_2^2 + C\int_T^t (\tau\beta+1)(|g(\tau)|^2 + |h(\tau)|^2 + |g'(\tau)|^2 + |h'(\tau)|^2)d\tau, \end{aligned} \tag{4.12}$$

for $t > T$. From (4.12), one sees that

$$||u(\cdot,t)||_2^2 = 0(t^{-1})$$

as $t \to +\infty$, by using Theorem 4.3 applied to $||u(\cdot, T)||_2$.

Repeating this procedure, one may deduce that if $t^n(|g(t)|^2 + |h(t)|^2 + |g'(t)|^2 + |h'(t)|^2) \in L_1(T, +\infty)$ for some $T > 0$, then $||u(\cdot,t)||_2^2 = 0(t^{-n})$. In consequence, it is seen that

$$||u(\cdot,t)||_{L_\infty} = 0(t^{-\frac{n}{2}}),$$

as $t \to +\infty$, since $||u(\cdot,t)||_{L_\infty}^2 \leq ||u(\cdot,t)||(||u(\cdot,t)|| + 2||u_x(\cdot,t)||)$.

If equation (4.10) is multiplied by $e^{\beta t}$ and the result integrated over $[T,t)$, there is derived

$$e^{\beta t}||u(\cdot,t)||_2^2 \le e^{\beta T}||u(\cdot,T)||_2^2 + \int_T^{+\infty} Ce^{\beta\tau}\big[|g(\tau)|^2 + |h(\tau)|^2 + |g'(\tau)|^2 + |h'(\tau)|^2\big]d\tau. \tag{4.13}$$

If $e^{\beta t}(|g(t)|^2 + |h(t)|^2 + |g'(t)|^2 + |h'(t)|^2) \in L_1(T,+\infty)$, then (4.13) yields

$$||u(\cdot,t)||_2^2 = 0(e^{-\beta t}), \tag{4.14}$$

for $t > T$, again by using Theorem 4.3 applied to $||u(\cdot,T)||_2$. Moreover combining (4.3) and (4.14) shows that

$$||u(\cdot,t)||_{L_\infty} = 0(e^{-\frac{\beta t}{2}}),$$

as $t \to +\infty$.

In particular, if (4.1) is endowed with homogeneous boundary condition, the decay of solutions of (4.1) is exponential. The results just explained are summarized in the following corollary.

COROLLARY 4.6. *Let the initial condition f lie in $C_b^2(\Omega)$ and suppose the nonlinearity satisfies condition (**) and the other stipulations in Corollary 4.4.*

(1) If the boundary data g and h are in $H^1(\mathbb{R}^+)$ and $t^{\frac{n}{2}}g$ and $t^{\frac{n}{2}}h$ are also in $H^1(\mathbb{R}^+)$, then $||u(\cdot,t)||_2^2 = 0(t^{-n})$ and

$$||u(\cdot,t)||_{L_\infty} = 0(t^{-\frac{n}{2}}),$$

as $t \to +\infty$, $n = 0,1,2,\cdots$.

(2) If $e^{\frac{\beta t}{2}}g$ and $e^{\frac{\beta t}{2}}h$ are in $H^1(\mathbb{R}^+)$, then $||u(\cdot,t)||_2^2 = 0(e^{-\beta t})$ and

$$||u(\cdot,t)||_{L_\infty} = 0(e^{-\frac{\beta t}{2}}),$$

as $t \to +\infty$. □

5. CONCLUSION

In this paper, initial- and boundary-value problems for the generalized regularized long-wave equation have been considered. Both the quarter-plane problem in which an initial condition together with boundary data at the end of a semi-infinite stretch of the medium and the two-point boundary-value problem in which initial conditions are coupled with boundary conditions at both ends of a finite extent of the medium of propagation are shown to be well posed subject only to suitable smoothness together with obvious compatibility conditions. Extending earlier work on these problems, results of global existence, uniqueness and continuous dependence of the solutions on the initial and boundary data is established for nonlinearities that grow at infinity no faster than quartically in the case of nonhomogeneous boundary conditions, and for smooth nonlinearities of any growth rate in the case of homogeneous boundary conditions.

When dissipative effects modeled by the term $-\nu u_{xx}$ are present, solutions of the two-point boundary-value problems are found to decay at rates up to exponential. Indeed, the rates of decay are determined by the evanescence of the boundary data for large values of t. This is in contrast to the sharp decay rates that obtain for the pure initial-value problem (see [2] and [9]), which are only algebraic. Preliminary decay results are also obtained for the quarter-plane problem, and these appear similar to those that obtain for the pure initial-value problem, reflecting the difference between bounded and unbounded domains.

This work points to a number of open questions. As far as well posedness is concerned, we have not yet determined if the quartic restriction on the growth rate of the nonlinearity P imposed in our theory is merely an artifact of our proof, or whether singularities in solutions may form in finite time for nonlinearities not respecting this restriction. The decay results for the quarter-plane problem could use some refinement and it could certainly be interesting to treat more general forms of both dispersion and dissipation.

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