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MORE RESULTS ON THE DECAY OF SOLUTIONS TO NONLINEAR, DISPERSIVE WAVE EQUATIONS

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Abstract. The asymptotic behaviour of solutions to the generalized regularized long-wave-Burgers equation

$$u_t + u_x + u^p u_x - \nu u_{xx} - u_{xxt} = 0 \tag{*}$$

is considered for $\nu > 0$ and $p \geq 1$. Complementing recent studies which determined sharp decay rates for these kind of nonlinear, dispersive, dissipative wave equations, the present study concentrates on the more detailed aspects of the long-term structure of solutions. Scattering results are obtained which show enhanced decay of the difference between a solution of (∗) and an associated linear problem. This in turn leads to explicit expressions for the large-time asymptotics of various norms of solutions of these equations for general initial data for $p > 1$ as well as for suitably restricted data for $p \geq 1$. Higher-order temporal asymptotics of solutions are also obtained. Our techniques may also be applied to the generalized Korteweg-de Vries-Burgers equation

$$u_t + u_x + u^p u_x - \nu u_{xx} + u_{xxx} = 0, \tag{**}$$

and in this case our results overlap with those of Dix. The decay of solutions in the spatial variable x for both (∗) and (∗∗) is also considered.

1. Introduction. This paper has as its genesis our earlier study [6] which was motivated by the work of Amick *et al.* [2] and Dix [13], and which determined the temporal decay of various norms of solutions of the pure initial-value problem for the generalized Korteweg-de Vries-Burgers (GKdV-B) equation

$$u_t + u_x + u^p u_x - \nu u_{xx} + u_{xxx} = 0, \quad (x \in \mathbb{R},\ t > 0) \tag{1.1}$$

and the generalized regularized long-wave-Burgers (GRLW-B) equation

$$u_t + u_x + u^p u_x - \nu u_{xx} - u_{xxt} = 0, \quad (x \in \mathbb{R},\ t > 0). \tag{1.2}$$

Here $u = u(x,t)$ is a real-valued function of the two real variables x and t, subscripts adorning u connote partial differentiation, ν is a positive number, p is a positive integer, and the initial condition

$$u(x,0) = f(x), \quad (x \in \mathbb{R}), \tag{1.3}$$

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is assumed to be suitably smooth and evanescent as $x \to \pm\infty$.

Special cases of (1.1) and (1.2) are the well known Korteweg-de Vries equation (KdV equation)

$$u_t + u_x + uu_x + u_{xxx} = 0, \tag{1.4}$$

derived originally by Korteweg & de Vries [19] as a model for waves propagating on the surface of a canal and the alternative regularized long-wave equation (RLW equation)

$$u_t + u_x + uu_x - u_{xxt} = 0, \tag{1.5}$$

put forward by Peregrine [24] and Benjamin *et al.* [3]. The equations (1.4) and (1.5) feature a balance between nonlinear and dispersive effects, but take no account of dissipation.

In many practical situations, damping effects are comparable in strength to nonlinear and dispersive effects [7, 20, 21, 22], and in such cases the models (1.4) and (1.5) require a dissipative term if good predictive power is desired. This is especially true in problems involving bore propagation (cf. [8, 15]), but it is also the case for water waves in all but the largest spatial scales. A popular model dissipative term is $-\nu u_{xx}$, $\nu > 0$, which has often been appended to (1.4) and (1.5) when the need to account for damping arises (cf. [7, 14, 15]), though it must be acknowledged that when a range of wavenumbers are present in the disturbance under consideration, and depending on the physical situation being considered, more accurate appendages may be appropriate (cf. [16, 26]). The operator $-\nu\partial_x^2$ has the advantage over some of its more accurate counterparts of being local, a property that greatly facilitates its analysis.

Whatever dissipative correction is added, it is important to understand the corresponding decay rate experienced by solutions (see [7]). Accordingly, and because it is an interesting question in its own right, the asymptotic behavior of solutions of equations like (1.1) and (1.2) has been the object of a number of recent works (cf. [2, 4, 6, 12, 13, 23, 28]). An example of the outcome of these studies is that if u is a solution of (1.1) or (1.2) corresponding to a generic class of reasonably smooth L_2-data, then the L_2-norm of u tends to zero as t tends to infinity at the rate $t^{-\frac{1}{4}}$ (see [2] for $p = 1$, and [6, 13, 28] for larger values of p). The value $-\frac{1}{4}$ is exactly the exponent of decay experienced by the L_2-norm of solutions of the linear problem obtained by dropping the nonlinear term in (1.1) or (1.2).

When $p = 1$, there is a subtle difference between the linear and the nonlinear equation for (1.1) and (1.2) (see [2]). Because of the nonlinearity, the value of the solution of (1.1) is smaller (as measured by $t^{1/2}\int_{-\infty}^{\infty} u^2(x,t)dx$) than that of the corresponding solution to the linear equation, at least for most initial data. However, when the power of the nonlinearity is larger ($p \geq 2$), this difference disappears. More precisely, we will show that the asymptotic behavior of the solutions is, to lowest order, exactly the same as that of the corresponding linearized equation in case $p \geq 2$.

In [2], it was also pointed out for the case $p = 1$ that if initial data has zero mass, then solutions of (1.1) and (1.2) decay faster than solutions of these equations with generic initial data, by which we mean data restricted only by membership in a Sobolev function class, but without auxiliary specifications like zero mass. Dix also showed in [13] that a suitable restriction of the Fourier transform of the initial data near the origin leads to an enhanced decay rate for the corresponding solution of

(1.1). Indeed, his investigation encompassed a broad class of nonlinear, dispersive, dissipative evolution equations of the type exemplified by (1.1), and his work very clearly points the way to the theory contained herein. In fact, it appears that if the initial data has the property that its Fourier transform vanishes at the origin like $|y|^\alpha$, $0 \le \alpha \le 1$, say, as $y \to 0$, then for $p \ge 1$, the decay rate of the corresponding solutions of (1.1) and (1.2) will increase by $\frac{\alpha}{2}$ over what can be expected of solutions corresponding to generic initial data. This result is particularly interesting when $0 < \alpha < 1$ and $p = 1$. Like the decay of solutions of (1.1) and (1.2) with higher-order nonlinearity ($p > 1$), the asymptotic form of the solution when $0 < \alpha < 1$ and $p = 1$ consists exactly of the solution of the linear equation plus a term that vanishes at higher order as t becomes unboundedly large. As a corollary, if $\alpha > \frac{1}{2}$, then solutions of (1.1) and (1.2) lie in the space $L_2(\mathbb{R} \times \mathbb{R}^+)$. While the decay results for norms of solutions obtained for (1.1) in case $p \ge 2$ are already contained in the previously mentioned work of Dix, the method of proof implied in our sketch is different and perhaps more transparent, though it doesn't have the scope evinced in [13]. Going beyond the lowest order of the decay, some details of the higher-order, large-time asymptotics are also provided by our theory.

An interesting consequence of the higher-order expansion obtains for any $p \ge 1$ when $\alpha = 1$ so the Fourier transform of the initial data vanishes at the origin like $|y|$. It transpires in this case that the L_2-norms of solutions of (1.1) and (1.2) have long-time asymptotic form $Ct^{-\frac{3}{4}}$ where $C^2 = \lim_{t \to +\infty} t^{\frac{3}{2}} \int_{-\infty}^{\infty} u^2(x,t)dx$. The constant C may be computed explicitly (see (1.14)-(1.15) below) to depend on the first moment of the initial data f and on the the double integral

$$\int_0^\infty \int_{-\infty}^\infty u^{p+1}(x,t)dxdt. \tag{1.6}$$

It does not coincide with the constant associated in the same way to the linearized initial-value problem. Furthermore, in this special case, the L_2-norm of the difference between solutions of (1.1) or (1.2) and the corresponding linear equation also has large-time asymptotic form $C't^{-\frac{3}{4}}$, where C' depends only on the double integral in (1.6) (see (1.13) below) and is not equal to zero in general. This suggests that if initial data has zero mass and its first few moments about the origin are also zero, or more generally speaking if $\alpha > 1$, then solutions of (1.1) and (1.2) only decay like $t^{-\frac{3}{4}}$ in L_2-norm. However, with the same initial data, the L_2-norm of solutions of the linear equations corresponding to (1.1) and (1.2) decays like $t^{-\frac{1+2\alpha}{4}}$, which is a faster decay than $t^{-\frac{3}{4}}$ when $\alpha > 1$.

In their wide-ranging discussion [23], Naumkin & Shishmarev considered the large-time asymptotics of solutions of the equation

$$w_t + w_x^2 + w_{xxx} - w_{xx} = 0, \tag{1.7}$$

and other more general dissipative and dispersive equations with initial data in $H^\infty(\mathbb{R})$. The decay of solutions of (1.7) in $L_\infty(\mathbb{R})$-norm is shown in [23] to have the form $C_\infty t^{-\frac{1}{2}}$ as t tends to $+\infty$. An explicit formula for the constant C_∞ is given by a series whose summands depend only on the initial data. To obtain (1.7) from (1.1) with $p = 1$, one simply integrates the evolution equation with respect to x. Of course, entailed in this procedure and the fact that the initial data for (1.7) lies in $L_2(\mathbb{R})$ is the implicit, zero-mass presumption that $\hat{f}(y) = iyg(y)$ for some $g \in L_1(\mathbb{R})$, say. Indeed, in order to apply their perturbation arguments, additional restrictions on g are needed (see [23, p.180 condition (1.6)]). Once this point is noted, the decay rates obtained in [23] square with those reported elsewhere. The constant C_∞ has in [23] a very complicated expression. However, without the additional conditions imposed on g in [23], the constant C_∞ can also be obtained following the lines developed here, although the dependence upon the initial data is implicit in that the quantity in (1.6) (with p =1) arises.

In addition to studying the temporal decay of solutions of (1.1) and (1.2), attention is also given to the spatial decay of solutions as x tends to $\pm\infty$. Consideration of spatial decay arises when one tries to devise error bounds for numerical approximations of solutions of the initial-value problem posed on all of $\mathbb{R}$ and, recently, in the study of the dispersive blow-up phenomenon (see [9]). The results obtained here show an interesting interplay between spatial and temporal decay. Note that when the interplay between spatial and temporal asymptotics is considered, it is important to keep the linear convective term u_x in (1.1) or (1.2) if the results are to be of interest in laboratory settings. (This term can always be eliminated by passing to a moving frame of reference, but in such a frame the spatial asymptotics depend on time.) This point is telling in the analysis in [23] where the fact that (1.7) has no linear term seems to be an important ingredient.

The script concentrates on the theory for equation (1.2). After some basic lemmas which are established in Section 2, the main results of temporal decay for solutions of equation (1.2) are stated and proved in Section 3. The theory about spatial decay is worked out in Section 4. In the last section, the results for equation (1.1) are stated and commentary is provided about their proof, and in particular, points where the proofs differ from those provided for equation (1.2).

The notation to be used henceforth is mostly standard. The $L_p(\mathbb{R})$-norm of a Lebesgue-measurable function f on $\mathbb{R}$ is denoted by $|f|_p$ for $1 \le p \le \infty$. If $m \ge 0$ is an integer, $W_p^m(\mathbb{R})$ will be the Sobolev space consisting of those $L_p(\mathbb{R})$-functions whose first m generalized derivatives lie in $L_p(\mathbb{R})$, with the usual norm denoted $|f|_{W_p^m(\mathbb{R})}$. The case $p = 2$ has the special notation $H^m(\mathbb{R})$. The norm of f in $H^m(\mathbb{R})$ will be connoted $||f||_m$, and of course $\|\cdot\|_0 = |\cdot|_2$. When we discuss decay to zero at infinity in the spatial variable, it will be convenient to use weighted L_p-norms. In particular, the spaces $\mathcal{L}_p^n(\Omega)$ and $\mathcal{L}_p^n = \mathcal{L}_p^n(\mathbb{R})$ will arise in our analysis. These are the subspaces of $L_p(\Omega)$ and $L_p(\mathbb{R})$ consisting of those functions f such that

$$|f|_{\mathcal{L}_p^n(\Omega)} = \left(\int_\Omega (1+x^2)^{\frac{n}{2}} |f(x)|^p dx\right)^{\frac{1}{p}}, \qquad \text{or}$$

$$|f|_{p,n} = \left(\int_{\mathbb{R}} |x|^n |f(x)|^p dx\right)^{\frac{1}{p}},$$

is finite. The weighted Sobolev space $\mathcal{H}_n^m(\Omega)$ consisting of those $\mathcal{L}_2^n(\Omega)$-functions whose first m generalized derivatives lie in $\mathcal{L}_2^n(\Omega)$ is equipped with the norm

$$\|f\|_{\mathcal{H}_n^m(\Omega)} = \left(\sum_{k=0}^m |f^{(k)}(\cdot)|^2_{\mathcal{L}_2^n(\Omega)}\right)^{\frac{1}{2}}.$$

For smooth functions f with compact support in $\mathbb{R}$, define the Fourier transform of f to be $\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx$; this definition is extended in the usual way to the set of all tempered distributions on $\mathbb{R}$.

In Sections 3 and 4, the primary goal is to prove the following results about solutions of (1.2).

Theorem. *Let $f \in H^2(\mathbb{R}) \cap W_1^2(\mathbb{R})$ and suppose $p \geq 1$ and $\nu > 0$. If w is the solution of the initial-value problem obtained from (1.2) by dropping the nonlinear term but maintaining the same initial data and u is the solution of (1.2)-(1.3), then $|u(\cdot,t)|_2$ and $|w(\cdot,t)|_2$ both decay like $t^{-\frac{1}{4}}$. If $p \geq 2$, then for $l = 0, 1$,*

$$\begin{aligned}\lim_{t\to+\infty} t^{l+\frac{1}{2}}|\partial_x^l u(\cdot,t)|_2^2 &= \lim_{t\to+\infty} t^{l+\frac{1}{2}}|\partial_x^l w(\cdot,t)|_2^2 \\ &= \frac{1}{(8\nu\pi)^{\frac{1}{2}}(4\nu)^l}\Big(\int_{-\infty}^{\infty} f(x)dx\Big)^2.\end{aligned} \tag{1.8}$$

Moreover, if $\int_{-\infty}^{\infty} f(x) = 0$ and the Fourier transform $\hat{f}$ of f satisfies the inequality

$$|\hat{f}(y)| \leq C|y|^\alpha, \tag{1.9}$$

for small values of y, where $0 \leq \alpha \leq 1$ and C is a positive constant, then there are constants C' and C'' such that the solution u of (1.2) with $p \geq 1$ has the properties

$$|u(\cdot,t)|_2 \leq C'(1+t)^{-\frac{1+2\alpha}{4}} \qquad \text{and} \qquad |u_x(\cdot,t)|_2 \leq C''(1+t)^{-\frac{3+2\alpha}{4}}, \tag{1.10}$$

for $t \geq 0$. If $0 < \alpha < 1$, then there exist constants C_f^l, $l = 0, 1$, which depend only on the solution w such that

$$\lim_{t\to+\infty} t^{\frac{1+2(l+\alpha)}{2}}|\partial_x^l u(\cdot,t)|_2^2 = \lim_{t\to+\infty} t^{\frac{1+2(l+\alpha)}{2}}|\partial_x^l w(\cdot,t)|_2^2 = C_f^l. \tag{1.11}$$

In addition, if $\hat{f}(y) = y^\alpha \hat{g}(y)$ for some $g \in L_1(\mathbb{R})$, then

$$\begin{aligned}\lim_{t\to+\infty} t^{\frac{1+2(l+\alpha)}{2}}|\partial_x^l u(\cdot,t)|_2^2 &= \lim_{t\to+\infty} t^{\frac{1+2(l+\alpha)}{2}}|\partial_x^l w(\cdot,t)|_2^2 \\ &= \frac{\Gamma(\alpha+l+\frac{1}{2})}{2\pi(2\nu)^{\alpha+l+\frac{1}{2}}}\Big(\int_{-\infty}^{\infty} g(x)dx\Big)^2,\end{aligned} \tag{1.12}$$

where Γ connotes the Gamma function. If $\alpha = 1$, then one has

$$\lim_{t\to+\infty} t^{\frac{3}{2}}|u(\cdot,t) - w(\cdot,t)|_2^2 = \frac{1}{4\nu(8\nu\pi)^{\frac{1}{2}}(p+1)^2}\Big(\int_0^\infty\int_{-\infty}^{\infty} u^{p+1}dxdt\Big)^2. \tag{1.13}$$

If $\hat{f}(y) = iy\hat{g}(y)$ for some $g \in L_1(\mathbb{R})$, then

$$\lim_{t\to+\infty} t^{\frac{3}{2}}|u(\cdot,t)|_2^2 = \frac{1}{4\nu(8\nu\pi)^{\frac{1}{2}}}\Big(\int_{-\infty}^{\infty} g(x)dx - \int_0^\infty\int_{-\infty}^{\infty}\frac{u^{p+1}(x,t)}{1+p}dxdt\Big)^2. \tag{1.14}$$

In particular, if $xf(x) \in L_1(\mathbb{R})$ and $\frac{d}{dx}g(x) = f(x)$ with $xg(x) \to 0$ as $x \to \pm\infty$, it follows that

$$\lim_{t\to+\infty} t^{\frac{3}{2}}|u(\cdot,t)|_2^2 = \frac{1}{4\nu(8\nu\pi)^{\frac{1}{2}}}\Big(\int_{-\infty}^{\infty} xf(x)dx + \int_0^\infty\int_{-\infty}^{\infty}\frac{u^{p+1}(x,t)}{p+1}dxdt\Big)^2. \tag{1.15}$$

If $f \in W_1^2(\mathbb{R}) \cap \mathcal{H}_n^2(\mathbb{R})$, $p \geq 1$ and $\nu > 0$, then for any $T > 0$, equation (1.2) has a unique solution $u \in L_\infty(0,T;\mathcal{H}_n^2(\mathbb{R}))$ corresponding to the initial data f. Furthermore, the solution u has the properties that $\|u_{xt}(\cdot,t)\|_{\mathcal{L}_2^1(\mathbb{R})}$ and $\|u_{xx}(\cdot,t)\|_{\mathcal{L}_2^1(\mathbb{R})}$ lie in $L_2(\mathbb{R}^+)$. Specifically, if initial data f satisfies (1.9) and $\alpha > \frac{1}{2}$, then $u \in L_2(\mathbb{R}\times\mathbb{R}^+) \cap C_b(\mathbb{R}^+;\mathcal{H}_1^2(\mathbb{R}))$, and $u_x \in L_2(\mathbb{R}^+;\mathcal{H}_1^1(\mathbb{R}))$.

In Section 5, similar results are obtained for solutions of (1.1).

2. Some Preliminary Results. Henceforth, it will be taken for granted that the generalized KdV-B equation (1.1) and the generalized RLW-B equation (1.2) with the initial condition (1.3) are well-posed in certain function spaces. Indeed, Proposition 3.1 and 3.2 in [6] suffice for the discussion here. For the nonce, attention will be focussed upon the decay properties of solutions of (1.2).

The linearized RLW-Burgers equation

$$w_t + w_x - \nu w_{xx} - w_{xxt} = 0, \tag{2.1a}$$

$$w(x,0) = f(x) \tag{2.1b}$$

has been discussed in [2]. The initial-value problem (2.1) can be solved by formally taking the Fourier transform of equation (2.1a) with respect to the spatial variable x. It is thereby seen that for any $f \in L_2$,

$$\hat{w}(y,t) = \exp\Big(\frac{-\nu y^2 t - iyt}{1+y^2}\Big)\hat{w}(y,0), \tag{2.2}$$

and consequently

$$w(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\Big(\frac{-\nu y^2 t - iyt}{1+y^2} + iyx\Big)\hat{f}(y)dy. \tag{2.3}$$

Here are cited the principal results about the decay of solutions of (2.1).

Lemma 2.1. *If $f \in H^r \cap L_1$, where $r \geq 1$ and*

$$|\hat{f}(y)| \leq C|y|^\alpha \tag{2.4}$$

for small values of y, where $\alpha \geq 0$ and C is a positive constant, then

$$\sup_{0\leq t\leq\infty} t^{\alpha+l+\frac{1}{2}} \int_{-\infty}^{\infty} [\partial_x^l w(x,t)]^2 dx < \infty, \tag{2.5}$$

for $0 \leq l \leq r$. In particular, if

$$|\hat{f}(y)| = |y|^\alpha |\hat{g}(y)| \tag{2.6}$$

for some $g \in L_1(\mathbb{R})$, then

$$\lim_{t\to+\infty} t^{\alpha+l+\frac{1}{2}} \int_{-\infty}^{\infty} [\partial_x^l w(x,t)]^2 dx = \frac{\Gamma(\alpha+l+\frac{1}{2})}{2\pi(2\nu)^{\alpha+l+\frac{1}{2}}}\Big(\int_{-\infty}^{\infty} g(x)dx\Big)^2, \tag{2.7}$$

where Γ *connotes, as before, the Gamma function. Thus if we have*

$$\int_{-\infty}^{\infty} |x|^j |f(x)| dx < \infty, \quad \text{for} \quad 0 \leq j \leq k \quad \text{and} \\ x^{k+1} f(x) \to 0 \quad \text{as } x \to \pm\infty, \quad \text{with} \quad \int_{-\infty}^{\infty} x^j f(x) dx = 0, \tag{2.8}$$

for $0 \leq j \leq k-1$, *then for* $0 \leq l \leq r$,

$$\lim_{t \to +\infty} t^{k+l+\frac{1}{2}} \int_{-\infty}^{\infty} [\partial_x^l w(x,t)]^2 dx = \frac{1 \cdot 3 \cdot 5 \cdots (2(k+l)-1)}{(8\nu\pi)^{\frac{1}{2}} (4\nu)^{k+l}} \Big(\int_{-\infty}^{\infty} x^k f(x) dx \Big)^2 \tag{2.9}$$

Proof. A special case of (2.9) is proved in [2]. Here, more general results are established that will find use presently.

We start proving (2.5) by multiplying (2.1a) by $2w$ and then integrating the result over $\mathbb{R}$ and integrating by parts to obtain the exact relation

$$\frac{d}{dt}(|w(\cdot,t)|_2^2 + |w_x(\cdot,t)|_2^2) + 2\nu |w_x(\cdot,t)|_2^2 = 0. \tag{2.10}$$

The use of (2.10) shows that for $t \geq \frac{1+\alpha}{\nu}$,

$$\begin{aligned} &\frac{d}{dt}\big(t^{1+\alpha}[|w(\cdot,t)|_2^2 + |w_x(\cdot,t)|_2^2]\big) \\ &\quad = t^\alpha \Big((1+\alpha)|w(\cdot,t)|_2^2 + [(1+\alpha) - 2\nu t]|w_x(\cdot,t)|_2^2 \Big) \\ &\quad \leq t^\alpha \int_{\infty}^{+\infty} \Big((1+\alpha) + [(1+\alpha) - 2\nu t] y^2 \Big) |\hat{w}(y,t)|^2 dy \\ &\quad \leq t^\alpha \int_{|y| \leq \sqrt{\frac{(1+\alpha)}{2\nu t - (1+\alpha)}}} \exp\big(-\frac{2\nu y^2 t}{1+y^2}\big) |\hat{f}(y)|^2 dy \\ &\quad \leq C t^\alpha \int_{|y| \leq \sqrt{\frac{(1+\alpha)}{2\nu t - (1+\alpha)}}} y^{2\alpha} \exp\big(-\frac{2\nu y^2 t}{1+y^2}\big) dy \\ &\quad \leq C t^\alpha \Big(\sqrt{\frac{(1+\alpha)}{2\nu t - (1+\alpha)}} \Big)^{2\alpha+1} \leq C t^{-\frac{1}{2}}, \end{aligned} \tag{2.11}$$

where Parseval's theorem, the representation (2.3) for w, and assumption (2.4) on the initial data f have been used in the first, the second and the third step, respectively. Integrate (2.11) once to obtain

$$t^{1+\alpha}[|w(\cdot,t)|_2^2 + |w_x(\cdot,t)|_2^2] \leq C(1 + t^{\frac{1}{2}}),$$

whence

$$t^{\alpha+\frac{1}{2}} |w(\cdot,t)|_2^2 \leq C.$$

Similarly, differentiate (2.1a) once and then multiply the result by $2w_x$. Integrate the latter result over $\mathbb{R}$ to reach the exact relation

$$\frac{d}{dt}(|\partial_x w(\cdot,t)|_2^2 + |\partial_x^2 w(\cdot,t)|_2^2) + 2\nu |\partial_x^2 w(\cdot,t)|_2^2 = 0. \tag{2.12}$$

By using (2.12), Parseval's theorem, the representation (2.3) and assumption (2.4), it then follows that for $t \geq \frac{2+\alpha}{\nu}$,

$$\begin{aligned}
&\frac{d}{dt}(t^{2+\alpha}[|\partial_x w(\cdot,t)|_2^2 + |\partial_x^2 w(\cdot,t)|_2^2]) \\
&\quad = t^{1+\alpha}\Big((2+\alpha)|\partial_x w(\cdot,t)|_2^2 + [(2+\alpha) - 2\nu t]|\partial_x^2 w(\cdot,t)|_2^2\Big) \\
&\quad \leq t^{1+\alpha}\int_{\infty}^{+\infty}\Big((2+\alpha) + [(2+\alpha) - 2\nu t]y^2\Big)y^2|\hat{w}(y,t)|^2 dy \\
&\quad \leq t^{1+\alpha}\int_{|y|\leq\sqrt{\frac{(2+\alpha)}{2\nu t-(2+\alpha)}}} \exp\Big(-\frac{2\nu y^2 t}{1+y^2}\Big)y^2|\hat{f}(y)|^2 dy \\
&\quad \leq Ct^{1+\alpha}\int_{|y|\leq\sqrt{\frac{(2+\alpha)}{2\nu t-(2+\alpha)}}} y^{2\alpha+2}dy \\
&\quad \leq Ct^{1+\alpha}\Big(\sqrt{\frac{(2+\alpha)}{2\nu t-(2+\alpha)}}\Big)^{2\alpha+3} \leq Ct^{-\frac{1}{2}}.
\end{aligned} \tag{2.13}$$

Integrate (2.13) once to obtain

$$t^{2+\alpha}[|\partial_x w(\cdot,t)|_2^2 + |\partial_x^2 w(\cdot,t)|_2^2] \leq C(1+t^{\frac{1}{2}}),$$

from whence it follows that

$$t^{\frac{3}{2}+\alpha}|\partial_x w(\cdot,t)|_2^2 \leq C.$$

Continuing in this vein, it is deduced that if the initial data f satisfies condition (2.4), the L_2-norm of $\partial_x^l w$ is bounded in the form (2.5) for any integer $l \geq 0$.

In particular, the use of Parseval's theorem, the representation (2.3), assumption (2.4) and the dominated convergence theorem shows that for $0 \leq l \leq r$,

$$\begin{aligned}
&\lim_{t\to+\infty} t^{\alpha+l+\frac{1}{2}}|\partial_x^l w(\cdot,t)|_2^2 = \lim_{t\to+\infty} t^{\alpha+l+\frac{1}{2}}\int_{-\infty}^{\infty}|y|^{2l}|\hat{w}(y,t)|^2 dy \\
&\quad = \lim_{t\to+\infty} t^{\alpha+l+\frac{1}{2}}\int_{-\infty}^{\infty}|y|^{2l}e^{-\frac{2\nu y^2 t}{1+y^2}}|\hat{f}(y)|^2 dy \\
&\quad = \lim_{t\to+\infty} t^{\alpha+l+\frac{1}{2}}\int_{-\infty}^{\infty}|y|^{2(\alpha+l)}e^{-\frac{2\nu y^2 t}{1+y^2}}|\hat{g}(y)|^2 dy \\
&\quad = \frac{1}{(2\nu)^{\alpha+l+\frac{1}{2}}}\lim_{t\to+\infty}\int_{-\infty}^{\infty} s^{2(\alpha+l)}e^{-\frac{s^2}{1+s^2/2\nu t}}|\hat{g}(\frac{s}{\sqrt{2\nu t}})|^2 ds \\
&\quad = \frac{|\hat{g}(0)|^2}{(2\nu)^{\alpha+l+\frac{1}{2}}}\int_{-\infty}^{\infty} s^{2(\alpha+l)}e^{-s^2}ds \\
&\quad = \frac{|\hat{g}(0)|^2}{(2\nu)^{\alpha+l+\frac{1}{2}}}\int_{0}^{\infty} z^{\alpha+l-\frac{1}{2}}e^{-z}dz \\
&\quad = \frac{\Gamma(\alpha+l+\frac{1}{2})}{(2\nu)^{\alpha+l+\frac{1}{2}}}|\hat{g}(0)|^2 = \frac{\Gamma(\alpha+l+\frac{1}{2})}{2\pi(2\nu)^{\alpha+l+\frac{1}{2}}}\Big(\int_{-\infty}^{\infty} g(x)dx\Big)^2.
\end{aligned} \tag{2.14}$$

In particular, if $f = \frac{d^k}{dx^k}g$ where $g \in L_1$, say, then $\hat{f} = (-iy)^k\hat{g}$, so that α can be taken at least as large as k in (2.4).

If f satisfies properties (2.8), then $\hat{f} \in C^k$ and $\hat{f}$, $\hat{f}'$, $\cdots$, $\hat{f}^{(k)}$ are bounded and continuous. Moreover, $\hat{f}(0) = \hat{f}'(0) = \cdots = \hat{f}^{(k-1)}(0) = 0$ because of the second set of relations in (2.8). Hence Taylor's theorem with remainder implies (2.6) to hold with $\alpha = k$ and $\hat{g}$ such that

$$|\hat{g}(0)| = \left|\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^k f(x)dx\right|.$$

The result (2.9) follows from the use of this information in (2.14) and the lemma is proved. □

When the case $p = 1$ is studied in Sections 3 and 5, use will be made of the well known Cole-Hopf transformation

$$W = -\nu \log V. \tag{2.15}$$

While it does not lead to the same level of simplification that obtains when it is applied to Burgers equation, this transformation nevertheless proved to be effective when it was used in [2] on nonlinear, dispersive, dissipative evolution equations. It turns out here also to be useful as a preliminary step to apply the Cole-Hopf transformation to the linear equation (2.1).

If w solves (2.1), let $\bar{w}(x,t) = \frac{1}{2}w(x+t,t)$, so that

$$\bar{w}_t - \nu\bar{w}_{xx} + \bar{w}_{xxx} - \bar{w}_{xxt} = 0. \tag{2.16}$$

From Lemma 2.1, one easily sees that $\bar{w} \in C_b(\mathbb{R}^+; L_1(\mathbb{R}))$. Then the function $W(x,t)$ defined by

$$W(x,t) = \int_{-\infty}^{x} \bar{w}(y,t)dy, \tag{2.17}$$

is uniformly bounded on $\mathbb{R} \times \mathbb{R}^+$ and satisfies

$$W_t(x,t) - \nu W_{xx}(x,t) - \frac{1}{2}w_{xt}(x+t,t) = 0. \tag{2.18}$$

Defining V via the transformation (2.15), a short computation shows that

$$V_t - \nu V_{xx} = RV \equiv G(x,t), \tag{2.19}$$

where $R(x,t) = -\frac{1}{\nu}W_x^2 - \frac{1}{2\nu}w_{xt}(x+t,t)$. Because $V(x,t) = \exp(-\nu^{-1}W(x,t))$, W is bounded and $\nu > 0$, one has

$$0 < \inf_{t\geq 0}\inf_{x\in\mathbb{R}} V(x,t) \leq \sup_{t\geq 0}\sup_{x\in\mathbb{R}} V(x,t) < \infty. \tag{2.20}$$

Let $Y \equiv V_x$ and remark that Y satisfies the initial-value problem

$$Y_t = \nu Y_{xx} + G_x, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{2.21a}$$

$$Y(x,0) = -\frac{f(x)}{\nu}\exp\left(-\frac{1}{\nu}\int_{\infty}^{x} f(x)dx\right) = F(x), \qquad x \in \mathbb{R}. \tag{2.21b}$$

Note that from (2.15) and (2.17), one has the relation

$$\frac{1}{2}w = \bar{w} = W_x = -\nu\frac{V_x}{V} = -\nu\frac{Y}{V}. \tag{2.22}$$

Corollary 2.2. *Let $f \in H^2 \cap W_1^2$, and suppose $|\hat{f}(y)| \le C|y|^\alpha$ for small values of y, where $\alpha > 0$ and C is a positive constant. There exists $\beta_0 > 0$ such that if γ is a fixed positive constant, then for $t \ge \gamma$,*

$$\int_{|y|\le\sqrt{\frac{\gamma}{t}}} \exp(-2\nu y^2 t)|\hat{F}(y)|^2 dy \le C_1(1+t)^{-(\frac{1}{2}+\beta_0)} + o(t^{-(\beta_0+\frac{1}{2})}), \tag{2.23}$$

as $t \to +\infty$, where F is defined in (2.21b) and C_1 is independent of t.

Proof. . Take the Fourier transform of (2.21a) with respect to the spatial variable x and solve the resulting ordinary differential equation to reach the integral equation

$$\begin{aligned}\hat{Y}(y,t) &= \exp(-\nu y^2 t)\hat{F}(y) + iy\int_0^t \exp(-\nu y^2(t-\tau))\hat{G}(y,\tau)d\tau \\ &= \exp(-\nu y^2 t)\hat{F}(y) \\ &\quad -\frac{1}{\nu}\int_0^t e^{-\nu y^2(t-\tau)}\Big[iy[\widehat{W_x^2 V}(y,\tau) - \frac{\widehat{w_t V_x}(y,\tau)}{2}] - \frac{y^2}{2}\widehat{w_t V}(y,\tau)\Big]d\tau.\end{aligned} \tag{2.24}$$

Elementary inequalities applied to (2.24) lead to estimate

$$\begin{aligned}\exp(-2\nu y^2 t)|\hat{F}(y)|^2 &\le 4|\hat{Y}(y,t)|^2 + \frac{y^2}{\nu^2}\Big(\int_0^t \exp(-\nu y^2(t-\tau))|\widehat{w_t V_x}(y,\tau)|d\tau\Big)^2 \\ &+ \frac{y^4 t}{\nu^2}\int_0^t \exp(-2\nu y^2(t-\tau))|\widehat{w_t V}(y,\tau)|^2 d\tau \\ &+ \frac{4y^2}{\nu^2}\Big(\int_0^t \exp(-\nu y^2(t-\tau))|\widehat{W_x^2 V}(y,\tau)|d\tau\Big)^2.\end{aligned} \tag{2.25}$$

First note that by using the relations in (2.22) and Lemma 2.1, one obtains

$$\int_{|y|\le\sqrt{\frac{\gamma}{t}}} |\hat{Y}(y,t)|^2 dy \le |Y(\cdot,t)|_2^2 \le C|w(\cdot,t)|_2^2 \le Ct^{-(\frac{1}{2}+\alpha)}, \tag{2.26}$$

for large t. The use of the relations (2.22), the property (2.20) of V and Lemma 2.1 shows that

$$|V_x(\cdot,t)|_2 \le \frac{|V(\cdot,t)|_\infty}{\nu}|\bar{w}(\cdot,t)|_2 \le C|w(\cdot,t)|_2 \le C(1+t)^{-\frac{1+2\alpha}{4}}. \tag{2.27}$$

By using equation (2.1a) and Lemma 2.1, one sees immediately that

$$|w_t(\cdot,t)|_2 \le C|w_x(\cdot,t)|_2 \le C(1+t)^{-\frac{3+2\alpha}{4}}. \tag{2.28}$$

Applying (2.27) and (2.28), it is then shown that

$$|\widehat{w_t V_x}(y,\tau)| \le \frac{1}{\sqrt{2\pi}}|w_t V_x(\cdot,\tau)|_1 \le \frac{1}{\sqrt{2\pi}}|w_t(\cdot,\tau)|_2|V_x(\cdot,\tau)|_2 \le C(1+t)^{-(1+\alpha)}.$$

Hence for $t \geq \gamma$, the use of the above inequality gives

$$\begin{aligned}\int_{|y|\leq\sqrt{\frac{\gamma}{t}}} y^2\Big(\int_0^t \exp(-\nu y^2(t-\tau))\big|\widehat{w_t V_x}(y,\tau)\big|d\tau\Big)^2 dy \\ \leq C\Big(\int_0^t \frac{d\tau}{(1+\tau)^{1+\alpha}}\Big)^2 \int_{|y|\leq\sqrt{\frac{\gamma}{t}}} y^2 dy \\ \leq C\Big(\sqrt{\frac{\gamma}{t}}\Big)^3 = Ct^{-\frac{3}{2}}.\end{aligned} \tag{2.29}$$

Similarly, by using (2.28) and the property (2.20) of V, there obtains the estimate

$$\begin{aligned}\int_{|y|\leq\sqrt{\frac{\gamma}{t}}} ty^4 \int_0^t \exp(-2\nu y^2(t-\tau))\big|\widehat{w_t V}(y,\tau)\big|^2 d\tau dy \\ \leq Ct\Big(\sqrt{\frac{\gamma}{t}}\Big)^4 \int_0^t \int_{|y|\leq\sqrt{\frac{\gamma}{t}}} \big|\widehat{w_t V}(y,\tau)\big|^2 dy d\tau \\ \leq Ct^{-1}\int_0^t |w_t V(\cdot,\tau)|_2^2 d\tau \leq Ct^{-1}\int_0^t |w_t(\cdot,\tau)|_2^2 d\tau \\ \leq Ct^{-1}\int_0^t (1+\tau)^{-(\frac{3}{2}+\alpha)} d\tau = Ct^{-1}.\end{aligned} \tag{2.30}$$

Finally, since $W_x = \bar{w}$, (2.20) may be used again to show

$$\begin{aligned}\int_{|y|\leq\sqrt{\frac{\gamma}{t}}} y^2\Big(\int_0^t \exp(-\nu y^2(t-\tau))\big|\widehat{W_x^2 V}(y,\tau)\big|d\tau\Big)^2 dy \leq C\Big(\sqrt{\frac{\gamma}{t}}\Big)^3\Big(\int_0^t |w(\cdot,\tau)|_2^2 d\tau\Big)^2 \\ \leq Ct^{-\frac{3}{2}}\Big(\int_0^t (1+\tau)^{-(\frac{1}{2}+\alpha)} d\tau\Big)^2 \\ \leq \begin{cases} Ct^{-\frac{1}{2}-2\alpha}, & \text{if } \alpha < \frac{1}{2}, \\ Ct^{-\frac{3}{2}}(\log(t+1))^2, & \text{if } \alpha = \frac{1}{2}, \\ Ct^{-\frac{3}{2}}, & \text{if } \alpha > \frac{1}{2}. \end{cases}\end{aligned} \tag{2.31}$$

Choosing $\beta_0 = \min\{\alpha, \frac{1}{2}\}$, it follows from (2.26), (2.29), (2.30) and (2.31) that

$$\int_{|y|\leq\sqrt{\frac{\gamma}{t}}} \exp(-2\nu y^2 t)|\hat{F}(y)|^2 dy \leq Ct^{-(\beta_0+\frac{1}{2})} + o(t^{-(\beta_0+\frac{1}{2})}),$$

as $t \to +\infty$, for all $t \geq \gamma$. The lemma is thereby proved. □

Finally, results are recalled which were proved in [2, 6] for the nonlinear equation (1.2).

Theorem 2.3. *Let $f \in H^2 \cap W_1^2$, $p \geq 1$ and $\nu > 0$ be given. Then the solution u of (1.2) corresponding to the initial data f satisfies*

$$|u(\cdot,t)|_2 \leq C(1+t)^{-\frac{1}{4}}, \quad |u(\cdot,t)|_\infty \leq C(1+t)^{-\frac{1}{2}}\ |u_x(\cdot,t)|_2 \leq C(1+t)^{-\frac{3}{4}} \tag{2.32}$$

for all $t \geq 0$, where the constants C depend only on norms of f, and are therefore independent of t. Moreover, we have that

$$u_t,\ u_{xx},\ u_{xt}\ \in L_2(\mathbb{R} \times \mathbb{R}^+) \ \text{ and } \ |u(\cdot,t)|_1\ \in L_\infty(\mathbb{R}^+). \tag{2.33}$$

3. The Temporal Decay of Solutions of the GRLW-B Equation. In this section further results on the temporal decay of solutions of (1.2) are obtained. For a generic class of initial data, various optimal decay results have been obtained in [2, 4, 6, 12, 13, 28]. Guided by the observations in [2, 13], we here concentrate on results that obtain for more restricted initial data. In particular, the assumption that the initial data has zero added mass is essential in all that is accomplished in this section. It is worth comment that many disturbances on the surface of water in both laboratory and field settings are initiated by processes that exactly or very closely conform to this assumption. As in previous studies, the so-called 'balanced' case $p = 1$ is handled differently from the case $p > 1$ of asymptotically weak nonlinearity. The results in the first part of this section rely on the presumption that $p > 1$ whilst the somewhat more challenging case $p = 1$ is reserved for the remainder of the section.

The first result is a helpful technical lemma.

Lemma 3.1. *Let $p \geq 2$, $f \in H^2 \cap W_1^2$ and suppose that*

$$|\hat{f}(y)| \leq C|y|^\alpha \tag{3.1}$$

for small values of y, where $0 \leq \alpha \leq 1$ and C is a positive constant. Then for any fixed $\gamma > 0$, the solution u of (1.2) corresponding to the initial data f satisfies

$$\int_{|y|\leq\sqrt{\frac{\gamma}{t}}} |\hat{u}(y,t)|^2 dy \leq \begin{cases} C_f t^{-(\alpha+\frac{1}{2})}, & \text{if } 0 \leq \alpha < 1, \\ \Big[C_f + C_N(\log(1+t))^2\Big] t^{-\frac{3}{2}}, & \text{if } \alpha = 1, \end{cases} \tag{3.2}$$

for all $t \geq \gamma$, where both C_f and C_N are independent of t. The constant C_f depends only on the initial data f while C_N depends on p.

Remark 3.2. Since $f \in L_1$, $\hat{f} \in C_b$ and so if (3.1) holds for $|y| \leq \delta$, say, by suitably enlarging the constant C it may be supposed that (3.1) holds for all values of y.

Proof.. Take the Fourier transform of (1.2) with respect to the spatial variable x and solve the resulting ordinary differential equation to reach the integral equation

$$\begin{aligned} \hat{u}(y,t) = &\exp\Big(\frac{-\nu y^2 t - iyt}{1+y^2}\Big)\hat{f}(y) \\ &- \frac{i}{p+1}\int_0^t \frac{y}{1+y^2}\exp\Big(\frac{-\nu y^2 - iy}{1+y^2}(t-\tau)\Big)\widehat{u^{p+1}}(y,\tau)d\tau. \end{aligned} \tag{3.3}$$

It follows readily from (3.3) that

$$\begin{aligned} |\hat{u}(y,t)|^2 \leq &\ 2\exp\Big(\frac{-2\nu y^2 t}{1+y^2}\Big)|\hat{f}(y)|^2 \\ &+ \frac{2y^2}{(p+1)^2(1+y^2)^2}\Big(\int_0^t \exp\Big(\frac{-\nu y^2}{1+y^2}(t-\tau)\Big)\big|\widehat{u^{p+1}}(y,\tau)\big|d\tau\Big)^2. \end{aligned} \tag{3.4}$$

Note that for large t, it is apparent that

$$\begin{aligned}\int_{|y|\le\sqrt{\frac{\gamma}{t}}}\exp\big(\frac{-2\nu y^2t}{1+y^2}\big)|\hat{f}(y)|^2dy &\le C^2\int_{|y|\le\sqrt{\frac{\gamma}{t}}}|y|^{2\alpha}\exp\big(\frac{-2\nu y^2t}{1+y^2}\big)dy\\ &\le C_f t^{-\frac{1+2\alpha}{2}},\end{aligned}\tag{3.5}$$

because of hypothesis (3.1). Note also that for $t\ge\gamma$, one obtains

$$\begin{aligned}&\int_{|y|\le\sqrt{\frac{\gamma}{t}}}\frac{2y^2}{(p+1)^2(1+y^2)^2}\Big(\int_0^t\exp\big(\frac{-\nu y^2}{1+y^2}(t-\tau)\big)\big|\widehat{u^{p+1}}(y,\tau)\big|d\tau\Big)^2dy\\ &\le C\int_{|y|\le\sqrt{\frac{\gamma}{t}}}\frac{y^2dy}{(1+y^2)^2}\Big(\int_0^t\frac{d\tau}{(1+\tau)^{\frac{p}{2}}}\Big)^2\\ &\le C\big(\sqrt{\frac{\gamma}{t}}\big)^3\Big(\int_0^t\frac{d\tau}{(1+\tau)^{\frac{p}{2}}}d\tau\Big)^2\le\begin{cases}C_Nt^{-\frac{3}{2}}, & \text{if } p\ge 3,\\ C_Nt^{-\frac{3}{2}}(\log(t+1))^2, & \text{if } p=2,\end{cases}\end{aligned}\tag{3.6}$$

where we have used that

$$|\widehat{u^{p+1}}(y,\tau)|\le\frac{1}{\sqrt{2\pi}}|u^{p+1}(\cdot,\tau)|_1\le\frac{1}{\sqrt{2\pi}}|u(\cdot,\tau)|_\infty^p|u(\cdot,\tau)|_1\tag{3.7}$$

and the fact, garnered from the previous theory (e.g. [6, Corollary 5.2]) that

$$|u(\cdot,t)|_\infty\le C(1+t)^{-\frac{1}{2}}.\tag{3.8}$$

Combining (3.4), (3.5) and (3.6) shows that

$$\int_{|y|\le\sqrt{\frac{\gamma}{t}}}|\hat{u}(y,t)|^2dy\le\begin{cases}C_ft^{-(\alpha+\frac{1}{2})}+C_Nt^{-\frac{3}{2}}, & \text{if } 0\le\alpha<1,\\ C_ft^{-\frac{3}{2}}+C_N(\log(1+t))^2t^{-\frac{3}{2}}, & \text{if } \alpha=1,\end{cases}\tag{3.9}$$

where C_f and C_N are independent of t. The constant C_f depends only on the initial data f, or more precisely on the term $\hat{w}(y,t)=\exp\big(\frac{-\nu y^2t-iyt}{1+y^2}\big)\hat{f}(y)$. This proves the lemma. □

With Lemma 3.1 in hand, it will be shown that the decay rate of solutions of (1.2) in L_2-norm increases by order $\frac{\alpha}{2}$ if the initial data f satisfies condition (3.1) with $0\le\alpha\le1$. In preparation, the following lemma is established for α in the range $0\le\alpha<1$.

Lemma 3.3. *Let $p\ge2$ and $f\in H^2\cap W_1^2$ be such that*

$$|\hat{f}(y)|\le C|y|^\alpha$$

for small values of y, where $0\le\alpha<1$ and C is a positive constant. Then the solution u of (1.2) satisfies

$$\begin{aligned}&|u(\cdot,t)|_2\le C_ft^{-\frac{1+2\alpha}{4}}+o(t^{-\frac{1+2\alpha}{4}})\quad\text{and}\\ &|u_x(\cdot,t)|_2\le C_ft^{-\frac{3+2\alpha}{4}}+o(t^{-\frac{3+2\alpha}{4}}),\end{aligned}\tag{3.10}$$

as $t \to +\infty$, where C_f is a constant which depends only on the solution of (2.1) with initial data f.

Proof. If (1.2) is multiplied by the combination $u_t + u_x + \frac{1}{\nu}u_{xt} + bu^{2p+1}$ for a constant b to be specified presently, and the result integrated over $\mathbb{R}$, there appears after suitable integrations by parts the relation

$$\begin{aligned}
&(\frac{\nu}{2}+\frac{1}{2\nu})\frac{d}{dt}|u_x(\cdot,t)|_2^2 + \frac{b}{2(p+1)}\frac{d}{dt}|u(\cdot,t)|_{2p+2}^{2p+2} + |u_{xt}(\cdot,t)|_2^2 \\
&\quad + (2p+1)b\nu|u^p(\cdot,t)u_x(\cdot,t)|_2^2 + |u_t(\cdot,t)+u_x(\cdot,t)|_2^2 \\
&\qquad = -\int_{\infty}^{\infty}\Big[(u_t+u_x+\frac{1}{\nu}u_{xt})u^pu_x + (2p+1)bu^{2p}u_xu_{xt}\Big]dx \\
&\qquad \le \frac{1}{2}|u_t(\cdot,t)+u_x(\cdot,t)|_2^2 + \frac{1}{2}|u_{xt}(\cdot,t)|_2^2 \\
&\qquad\quad + \{(\frac{1}{4\nu^2}+\frac{1}{2}) + (2p+1)^2b^2|u(\cdot,t)|_\infty^p\}|u^p(\cdot,t)u_x(\cdot,t)|_2^2,
\end{aligned} \tag{3.11}$$

where Young's inequality has been brought to bear. First, choose b large enough that

$$2pb\nu \ge (\frac{1}{4\nu^2}+\frac{1}{2}) + 1. \tag{3.12}$$

Then choose T such that for $t \ge T$,

$$(2p+1)^2b^2|u(\cdot,t)|_\infty^p \le 1. \tag{3.13}$$

For $t \ge T$, it is then assured that

$$\frac{d}{dt}\Big[(\frac{\nu}{2}+\frac{1}{2\nu})|u_x(\cdot,t)|_2^2 + \frac{b}{2(p+1)}|u(\cdot,t)|_{2p+2}^{2p+2}\Big] + b\nu|u^p(\cdot,t)u_x(\cdot,t)|_2^2 \le 0. \tag{3.14}$$

If (1.2) is multiplied by u_{xx} and the result integrated over $\mathbb{R}$, it follows that

$$\begin{aligned}
\int_{-\infty}^{\infty} 2u^pu_xu_{xx}dx &= \frac{d}{dt}\big(|u_x(\cdot,t)|_2^2 + |u_{xx}(\cdot,t)|_2^2\big) + 2\nu|u_{xx}(\cdot,t)|_2^2 \\
&\le \nu|u_{xx}(\cdot,t)|_2^2 + \frac{1}{\nu}|u^p(\cdot,t)u_x(\cdot,t)|_2^2.
\end{aligned} \tag{3.15}$$

Finally, multiplying (3.14) by an appropriate constant and then adding the result and (3.15) yields the differential inequality

$$\frac{d}{dt}\big(|u_x(\cdot,t)|_2^2 + |u_{xx}(\cdot,t)|_2^2 + |u(\cdot,t)|_{2p+2}^{2p+2}\big) \le -A|u_{xx}(\cdot,t)|_2^2 - B|u^p(\cdot,t)u_x(\cdot,t)|_2^2, \tag{3.16}$$

which is valid for $t \ge T$, where A and B are suitably chosen, positive constants and T is as above. The differential inequality (3.16) implies that

$$\begin{aligned}
\frac{d}{dt}\Big(t^{2+\alpha}\big[|u_x(\cdot,t)|_2^2 + |u_{xx}(\cdot,t)|_2^2 + |u(\cdot,t)|_{2p+2}^{2p+2}\big]\Big) &\le t^{1+\alpha}\Big[(2+\alpha)|u_x(\cdot,t)|_2^2 \\
&- \frac{tA}{2}|u_{xx}(\cdot,t)|_2^2 + (2+\alpha)|u(\cdot,t)|_{2p+2}^{2p+2} - tB|u^p(\cdot,t)u_x(\cdot,t)|_2^2\Big],
\end{aligned} \tag{3.17}$$

if $t \geq \tilde{T} = \max(T, \frac{2(2+\alpha)}{A})$. By using Parseval's theorem, the first term on the right-hand side of (3.17) can be estimated as

$$\begin{aligned}
t^{1+\alpha}\big((2+\alpha)|u_x(\cdot,t)|_2^2 - \frac{tA}{2}|u_{xx}(\cdot,t)|_2^2\big) &\leq t^{1+\alpha}\int_{|y|\leq\sqrt{\frac{2(2+\alpha)}{At}}} y^2|\hat{u}(y,t)|^2 dy \\
&\leq t^{1+\alpha}\Big(\sqrt{\frac{2(2+\alpha)}{At}}\Big)^2 \int_{|y|\leq\sqrt{\frac{2(2+\alpha)}{At}}} |\hat{u}(y,t)|^2 dy \\
&\leq C_f t^{-\frac{1}{2}} + C_N t^{-(\frac{3}{2}-\alpha)},
\end{aligned} \tag{3.18}$$

where Lemma 3.1 has been applied in the last inequality in (3.18) under the restriction $0 \leq \alpha < 1$ and C_N is independent of t.

If one defines $v(x,t) = u^{p+1}(x,t)$, then $v_x = (p+1)u^p u_x$. Using Parseval's theorem again, this time to estimate the second term on the right-hand side of (3.17), it is seen that

$$\begin{aligned}
&t^{1+\alpha}\big[(2+\alpha)|u(\cdot,t)|_{2p+2}^{2p+2} - tB|u^p(\cdot,t)u_x(\cdot,t)|_2^2\big] \\
&\leq t^{1+\alpha}\int_{|y|\leq\frac{(p+1)\sqrt{2+\alpha}}{\sqrt{Bt}}} |\hat{v}(y,t)|^2 dy \leq C_N t^{-(p-1-\alpha+\frac{1}{2})},
\end{aligned} \tag{3.19}$$

since

$$|\hat{v}(\cdot,t)|_\infty \leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} |u(x,t)|^{p+1} dx \leq \frac{1}{\sqrt{2\pi}}|u(\cdot.t)|_\infty^p |u(\cdot,t)|_1 \leq Ct^{-\frac{p}{2}}.$$

The use of (3.18) and (3.19) reduces (3.17) to

$$\frac{d}{dt}\Big(t^{2+\alpha}\big[|u_x(\cdot,t)|_2^2 + |u_{xx}(\cdot,t)|_2^2 + |u(\cdot,t)|_{2p+2}^{2p+2}\big]\Big) \leq C_f t^{-\frac{1}{2}} + C_N t^{-(\frac{3}{2}-\alpha)},$$

from which it follows immediately that for $0 \leq \alpha < 1$,

$$|u_x(\cdot,t)|_2^2 + |u_{xx}(\cdot,t)|_2^2 + |u(\cdot,t)|_{2p+2}^{2p+2} \leq C_f t^{-(\alpha+\frac{3}{2})} + o(t^{-(\alpha+\frac{3}{2})}). \tag{3.20}$$

Multiply equation (1.2) by $2u$ and integrate the result over $\mathbb{R}$. After integrations by parts, there appears the exact relation

$$\frac{d}{dt}\Big(|u(\cdot,t)|_2^2 + |u_x(\cdot,t)|_2^2\Big) + 2\nu|u_x(\cdot,t)|_2^2 = 0. \tag{3.21}$$

The use of equation (3.21) and Parseval's formula yields

$$\begin{aligned}
&\frac{d}{dt}\Big(t^{1+\alpha}\big[|u(\cdot,t)|_2^2 + |u_x(\cdot,t)|_2^2\big]\Big) = (1+\alpha)t^\alpha\big(|u(\cdot,t)|_2^2 + |u_x(\cdot,t)|_2^2\big) - 2\nu t^{1+\alpha}|u_x(\cdot,t)|_2^2 \\
&= t^\alpha\Big((1+\alpha)|\hat{u}(\cdot,t)|_2^2 - 2\nu t\int_{-\infty}^{\infty} y^2|\hat{u}(y,t)|^2 dy\Big) + (1+\alpha)t^\alpha|u_x(\cdot,t)|_2^2 \\
&\leq t^\alpha\int_{|y|<\sqrt{\frac{1+\alpha}{2\nu t}}} |\hat{u}(y,t)|^2 dy + (1+\alpha)t^\alpha|u_x(\cdot,t)|_2^2 \\
&\leq C_f t^{-\frac{1}{2}} + C_N t^{-(\frac{3}{2}-\alpha)} + (1+\alpha)t^\alpha|u_x(\cdot,t)|_2^2.
\end{aligned} \tag{3.22}$$

Again Lemma 3.1 has been applied in the last inequality in (3.22).

By integrating (3.22) with respect to t over the interval (T, t), there appears

$$t^{1+\alpha}|u(\cdot,t)|_2^2 \le C + C_f t^{1/2} + \int_T^t \tau^\alpha |u_x(\cdot,\tau)|_2^2 d\tau. \tag{3.23}$$

Because of (3.20) one has $t^\alpha |u_x(\cdot,t)|_2^2 \in L_1(\mathbb{R}^+)$. Hence (3.23) yields

$$|u(\cdot,t)|_2 \le C_f t^{-\frac{1+2\alpha}{4}} + o(t^{-\frac{1+2\alpha}{4}}),$$

as $t \to +\infty$. □

It will be shown in the next lemma that for $\alpha = 1$, the decay of solutions of (1.2) in the L_2-norm goes as $t^{-\frac{3}{4}}$ rather than $\log(1+t)t^{-\frac{3}{4}}$. However, unlike the results for the case $0 \le \alpha < 1$, the asymptotic nature of solutions of (1.2) for $\alpha = 1$ is not the same as that of solutions of the corresponding linear equation. Indeed, suppose u and w are solutions of (1.2) and (2.1), respectively, both corresponding to initial data f satisfying (3.1) with $\alpha = 1$. Then $|u(\cdot,t)|_2$ and $|w(\cdot,t)|_2$ both decay at the rate $t^{-\frac{3}{4}}$, but

$$\lim_{t\to+\infty} \frac{|u(\cdot,t)|_2}{|w(\cdot,t)|_2} \ne 1$$

in general.

Lemma 3.4. *Let $p \ge 2$, $f \in H^2 \cap W_1^2$ and suppose that (3.1) holds. Then the solution of (1.2) with initial data f satisfies*

$$\begin{aligned} |u(\cdot,t)|_2 &\le C_\alpha^1 t^{-\frac{1+2\alpha}{4}} + o(t^{-\frac{1+2\alpha}{4}}) \quad \text{and} \\ |u_x(\cdot,t)|_2 &\le C_\alpha^2 t^{-\frac{3+2\alpha}{4}} + o(t^{-\frac{3+2\alpha}{4}}) \end{aligned} \tag{3.24}$$

as $t \to +\infty$, where C_α^1 and C_α^2 have the form

$$\begin{cases} C_f, & \text{if } 0 \le \alpha < 1, \\ C_f + C_N, & \text{if } \alpha = 1. \end{cases} \tag{3.25}$$

Corollary 3.5. *If $p \ge 2$ and f satisfies the conditions in Lemma 3.4, then the corresponding solution of (1.2) satisfies*

$$|u(\cdot,t)|_\infty \le C_\alpha t^{-\frac{1+\alpha}{2}} + o(t^{-\frac{1+\alpha}{2}}), \tag{3.26}$$

as $t \to +\infty$, for a suitable constant C_α of the form described in (3.25).

Proof. The inequality (3.26) follows immediately from those in (3.24) because

$$|u(\cdot,t)|_\infty^2 \le |u(\cdot,t)|_2 |u_x(\cdot,t)|_2 \le C_\alpha (1+t)^{-(1+\alpha)}. \qquad \square$$

Proof of Lemma 3.4. If $0 \le \alpha < 1$, the advertised result is already in Lemma 3.3. Consider now the case $\alpha = 1$. Then (3.24) holds for any α_0 within the interval

$(0,1)$, say $\alpha_0 = \frac{1}{2}$, and therefore (3.26) holds for $\alpha = \alpha_0 = \frac{1}{2}$. Consequently, there is a constant $C > 0$ such that the solution u of (1.2) with initial data f satisfies

$$|u(\cdot,t)|_\infty \le C(1+t)^{-\frac{3}{4}}, \tag{3.27}$$

for all $t \ge 0$. The decay estimate (3.27) shows that

$$|\widehat{u^{p+1}}(y,t)| \le \frac{1}{\sqrt{2\pi}}|u(\cdot,t)|_\infty^p |u(\cdot,t)|_1 \le C(1+t)^{-\frac{3p}{4}} \tag{3.28}$$

Hence for $t \ge \gamma$ where $\gamma > 0$ is fixed, one has

$$\begin{aligned} \int_{|y|\le\sqrt{\frac{\gamma}{t}}} & \frac{2y^2}{(p+1)^2(1+y^2)^2}\Big(\int_0^t \exp\big(\frac{-\nu y^2}{1+y^2}(t-\tau)\big)|\widehat{u^{p+1}}(y,\tau)|d\tau\Big)^2 dy \\ & \le C\int_{|y|\le\sqrt{\frac{\gamma}{t}}} \frac{y^2 dy}{(1+y^2)^2}\Big(\int_0^t \frac{d\tau}{(1+\tau)^{\frac{3p}{4}}}\Big)^2 \\ & \le C\Big(\sqrt{\frac{\gamma}{t}}\Big)^3\Big(\int_0^t \frac{d\tau}{(1+\tau)^{\frac{3p}{4}}}\Big)^2 \le C_N t^{-\frac{3}{2}}. \end{aligned} \tag{3.29}$$

For $\alpha = 1$ and the same value of γ, it follows at once that

$$\begin{aligned} \int_{|y|\le\sqrt{\frac{\gamma}{t}}} & \exp\big(\frac{-2\nu y^2 t}{1+y^2}\big)|\hat{f}(y)|^2 dy \\ & \le C\int_{|y|\le\sqrt{\frac{\gamma}{t}}} |y|^2 \exp\big(\frac{-2\nu y^2 t}{1+y^2}\big)dy \le C_f t^{-\frac{3}{2}}. \end{aligned} \tag{3.30}$$

Combining (3.3), (3.29) and (3.30) leads to the conclusion

$$\int_{|y|\le\sqrt{\frac{\gamma}{t}}} |\hat{u}(y,t)|^2 dy \le C_f t^{-\frac{3}{2}} + C_N t^{-\frac{3}{2}} \le C_\alpha t^{-\frac{3}{2}}. \tag{3.31}$$

If one chooses $\gamma = \frac{6}{A}$, then (3.31) implies that

$$\begin{aligned} t^2\big(3|u_x(\cdot,t)|_2^2 - \frac{tA}{2}|u_{xx}(\cdot,t)|_2^2\big) & \le t^2\int_{|y|\le\sqrt{\frac{6}{At}}} y^2|\hat{u}(y,t)|^2 dy \\ \le t^2\Big(\sqrt{\frac{6}{At}}\Big)^2 & \int_{|y|\le\sqrt{\frac{6}{At}}} |\hat{u}(y,t)|^2 dy \le C_\alpha t^{-\frac{1}{2}}. \end{aligned} \tag{3.32}$$

Making use of (3.27) again, it is seen that

$$|\hat{v}(\cdot,t)|_\infty \le \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} |u(x,t)|^{p+1}dx \le \frac{1}{\sqrt{2\pi}}|u(\cdot,t)|_\infty^p |u(\cdot,t)|_1 \le Ct^{-\frac{3p}{4}}, \tag{3.33}$$

where $v(x,t) = u^{p+1}(x,t)$. It transpires because of Parseval's theorem and (3.33) that

$$t^2\big(3|u(\cdot,t)|_{2p+2}^{2p+2} - tB|u^p(\cdot,t)u_x(\cdot,t)|_2^2\big) \le t^2\int_{|y|\le\frac{(p+1)\sqrt{3}}{\sqrt{Bt}}} |\hat{v}(y,t)|^2 dy \le Ct^{-\frac{3(p-1)}{2}}. \tag{3.34}$$

Using (3.32) and (3.34), the differential inequality (3.17) in Lemma 3.3 may be seen to imply that

$$\frac{d}{dt}\Big(t^3\left[|u_x(\cdot,t)|_2^2+|u_{xx}(\cdot,t)|_2^2+|u(\cdot,t)|_{2p+2}^{2p+2}\right]\Big)\le C_\alpha t^{-\frac{1}{2}}+Ct^{-\frac{3(p-1)}{2}}, \tag{3.35}$$

whence

$$|u_x(\cdot,t)|_2^2+|u_{xx}(\cdot,t)|_2^2+|u(\cdot,t)|_{2p+2}^{2p+2}\le C_\alpha^2 t^{-\frac{5}{2}}+Ct^{-3}, \tag{3.36}$$

where C_α^2 takes the form C_f+C_N.

Finally, following the line of argument leading to (3.21), (3.22) and (3.23), but using (3.31) and (3.36), it is concluded at once that if $\alpha=1$, then

$$t^2|u(\cdot,t)|_2^2\le C+C_\alpha^1 t^{1/2},$$

for all t and suitable constants C and C_α^1, where C_α^1 takes the form in (3.25). The last inequality and (3.36) imply the estimates in (3.24), and the lemma is proved. □

One can see from Lemma 3.4 that for $p\ge 2$ and $0\le\alpha<1$, the nonlinear term decays faster than the terms related to initial data f as $t\to+\infty$. Hence the decay behavior of solutions of (1.2) as $t\to+\infty$ is exactly the same as that of the corresponding linear equation which was determined already in Section 2. The resulting state of affairs is summarized in the following theorem.

Theorem 3.6. *Let $p\ge 2$, $f\in H^2\cap W_1^2$ and suppose*

$$|\hat f(y)|\le C|y|^\alpha,$$

for small values of y, where $0\le\alpha\le 1$ and C is a fixed constant. Then the solution u of (1.2) corresponding to the initial data f has the properties

$$\sup_{0\le t<\infty} t^{\alpha+l+\frac{1}{2}}\int_{-\infty}^{\infty}[\partial_x^l u(x,t)]^2dx<\infty, \tag{3.37}$$

for $l=0,1$. In particular, if $0\le\alpha<1$, then there exists a constant C_f^l which depends only on the solution of (2.1) with the same initial data, such that for $l=0,1$,

$$\lim_{t\to+\infty} t^{\alpha+l+\frac{1}{2}}\int_{-\infty}^{\infty}[\partial_x^l u(x,t)]^2dx=C_f^l. \tag{3.38}$$

If $|\hat f(y)|=|y|^\alpha|\hat g(y)|$ for some $g\in L_1(\mathbb{R})$, then

$$C_f^l=\frac{\Gamma(\alpha+l+\frac{1}{2})}{2\pi(2\nu)^{\alpha+l+\frac{1}{2}}}\Big(\int_{-\infty}^{\infty}g(x)dx\Big)^2. \tag{3.39}$$

In case $\alpha=0$, for $l=0,1$. we have

$$\lim_{t\to+\infty} t^{l+\frac{1}{2}}\int_{-\infty}^{\infty}[\partial_x^l u(x,t)]^2dx=\frac{1}{(8\nu\pi)^{\frac{1}{2}}(4\nu)^l}\Big(\int_{-\infty}^{\infty}f(x)dx\Big)^2, \tag{3.40}$$

Remark 3.7. Equations (3.37) and (3.38) follow directly from the estimates in Lemma 3.3 and Lemma 3.4. The precise values shown in (3.39) and (3.40) can be obtained by further calculations. These will appear at the end of this section together with the results for equation (1.2) in case $p = 1$.

If $p = 1$, the decay rate of the difference $u - w$ in L_2-norm is the same as that of the individual components u and w (see [2]) and the limit of $u - w$ in the sense of $\lim_{t\to+\infty} t^{\frac{1}{4}}|(u-w)(\cdot,t)|_2$ is not equal to zero in general. But if initial data f satisfies condition (3.1) for some positive α, then the decay rate of the difference $u-w$ in L_2-norm is higher than that of the individual components u and w. In other words, it might be expected in this case that the decay behavior of solutions of (1.2) is the same as that of solutions of the associated linear equation (2.1). Proving this to be the case is our next task, after which we show that the conclusions of Theorem 3.6 also hold when $p = 1$. In case $\alpha = 1$, the decay result appears already in [2], but with a somewhat less transparent proof.

Let $p = 1$ in the equation (1.2) and set $\bar{u}(x,t) = \frac{1}{2}u(x+t,t)$. Then $\bar{u}$ satisfies the equation

$$\bar{u}_t + 2\bar{u}\bar{u}_x - \nu\bar{u}_{xx} + \bar{u}_{xxx} - \bar{u}_{xxt} = 0. \tag{3.41}$$

It follows from (2.33) in Theorem 2.3 that $\bar{u} \in C_b(\mathbb{R}^+; L_1(\mathbb{R}))$, so the function $U(x,t)$ defined by

$$U(x,t) = \int_{-\infty}^{x} \bar{u}(y,t)dy \tag{3.42}$$

is uniformly bounded on $\mathbb{R} \times \mathbb{R}^+$ and satisfies

$$U_t(x,t) + (U_x(x,t))^2 - \nu U_{xx}(x,t) - \frac{1}{2}u_{xt}(x+t,t) = 0. \tag{3.43}$$

Let $U = -\nu \log v$ so that

$$v_t - \nu v_{xx} = Rv \equiv g(x,t), \tag{3.44}$$

where $R(x,t) = -\frac{1}{2\nu}u_{xt}(x+t,t)$. Because U is bounded and $v(x,t) = \exp(-\frac{U(x,t)}{\nu})$, it transpires that

$$0 < \inf_{t\geq 0}\inf_{x\in\mathbb{R}} v(x,t) \leq \sup_{t\geq 0}\sup_{x\in\mathbb{R}} v(x,t) < \infty. \tag{3.45}$$

If $Y \equiv v_x$, a computation shows that

$$Y_t = \nu Y_{xx} + g_x, \qquad (x,t) \in \mathbb{R}\times\mathbb{R}^+, \tag{3.46a}$$

$$Y(x,0) = -\frac{f(x)}{\nu}\exp\left(-\frac{1}{\nu}\int_{\infty}^{x} f(x)dx\right) = F(x), \qquad x \in \mathbb{R}. \tag{3.46b}$$

The next step is to show that if $\alpha > 0$, there is a $\beta > 0$ such that

$$t^{\frac{1+2\beta}{4}}|\bar{u}(\cdot,t)|_2 = \frac{1}{2}t^{\frac{1+2\beta}{4}}|u(\cdot,t)|_2 \leq C,$$

for all $t \geq 0$, where C is independent of t. Because of the relation

$$\bar{u}(x,t) = U_x(x,t) = -\nu\frac{v_x(x,t)}{v(x,t)} = -\nu\frac{Y(x,t)}{v(x,t)} \tag{3.47}$$

and the property (3.45) of v, it suffices to establish the result

$$t^{\frac{1+2\beta}{4}}|Y(\cdot,t)|_2 \leq C. \tag{3.48}$$

The next lemma corresponds to Lemma 3.1, but for the transformed equation (3.46a).

Lemma 3.8. *Let $f \in H^2 \cap W_1^2$ and suppose that $|\hat{f}(y)|$ satisfies (3.1). Then for any fixed $\gamma > 0$, the solution u of (1.2) with $p = 1$ corresponding to the initial data f satisfies*

$$\int_{|y| \le \sqrt{\frac{\gamma}{t}}} |\hat{Y}(y,t)|^2 dy \le C_\beta t^{-(\beta + \frac{1}{2})} \tag{3.49}$$

for all $t \ge \gamma$, where C_β is independent of t and $\beta = \min\{\alpha, \frac{1}{2}\}$.

Proof. Take the Fourier transform of (3.46a) with respect to the spatial variable x and solve the resulting ordinary differential equation to obtain

$$\hat{Y}(y,t) = \exp(-\nu y^2 t)\hat{F}(y) + iy \int_0^t \hat{g}(y,\tau)d\tau = \exp(-\nu y^2 t)\hat{F}(y)$$
$$+ \frac{1}{2\nu} \int_0^t \exp(-\nu y^2(t-\tau)) \Big[iy\widehat{u_t v_x}(y,\tau) + y^2 \widehat{u_t v}(y,\tau) \Big] d\tau. \tag{3.50}$$

Elementary estimates applied to (3.50) give

$$|\hat{Y}(y,t)|^2 \le 2 \exp(-2\nu y^2 t)|\hat{F}(y)|^2 + \frac{y^2}{\nu^2}\Big(\int_0^t \exp(-\nu y^2(t-\tau))|\widehat{u_t v_x}(y,\tau)| d\tau \Big)^2$$
$$+ \frac{y^4 t}{\nu^2} \int_0^t \exp(-2\nu y^2(t-\tau))|\widehat{u_t v}(y,\tau)|^2 d\tau. \tag{3.51}$$

Note that by Corollary 2.2, if $\beta_0 = \min\{\alpha, \frac{1}{2}\} > 0$, then for large t,

$$\int_{|y| \le \sqrt{\frac{\gamma}{t}}} \exp(-2\nu y^2 t)|\hat{F}(y)|^2 dy \le C t^{-(\frac{1}{2} + \beta_0)}. \tag{3.52}$$

By using property (3.45) of v, the relation (3.47) and Theorem 2.3, one has

$$|v_x(\cdot,t)|_2 \le \frac{|v(\cdot,t)|_\infty}{\nu} |\bar{u}(\cdot,t)|_2 \le C|u(\cdot,t)|_2 \le C(1+t)^{-\frac{1}{4}}.$$

By using equation (1.2) with $p = 1$, one sees that

$$|u_t(\cdot,t)|_2 \le C|u_x(\cdot,t)|_2 \le C(1+t)^{-\frac{3}{4}}, \tag{3.53}$$

whence

$$|\widehat{v_x u_t}(y,\tau)| \le \frac{1}{\sqrt{2\pi}} |v_x u_t(\cdot,\tau)|_1 \le \frac{1}{\sqrt{2\pi}} |v_x(\cdot,\tau)|_2 |u_t(\cdot,\tau)|_2 \le C(1+t)^{-1}. \tag{3.54}$$

The decay rate (3.54) leads one to conclude that for $t \ge \gamma$,

$$\int_{|y| \le \sqrt{\frac{\gamma}{t}}} y^2 \Big(\int_0^t \exp(-\nu y^2(t-\tau))|\widehat{v_x u_t}(y,\tau)| d\tau \Big)^2 dy \le C\Big(\int_0^t \frac{d\tau}{(1+\tau)} \Big)^2 \int_{|y| \le \sqrt{\frac{\gamma}{t}}} y^2 dy$$
$$\le C\Big(\sqrt{\frac{\gamma}{t}}\Big)^3 (\log(t+1))^2 = C t^{-\frac{3}{2}} (\log(1+t))^2. \tag{3.55}$$

Similarly, by using property (3.45) of v and (3.53), one can show

$$\begin{aligned}
\int_{|y|\le\sqrt{\frac{\gamma}{t}}} ty^4 \int_0^t \exp(-2\nu y^2(t-\tau))\left|\widehat{u_t v}(y,\tau)\right|^2 d\tau dy \\
&\le Ct\Big(\sqrt{\frac{\gamma}{t}}\Big)^4 \int_0^t \int_{|y|\le\sqrt{\frac{\gamma}{t}}} \left|\widehat{u_t v}(y,\tau)\right|^2 dy d\tau \\
&\le Ct^{-1}\int_0^t |u_t v(\cdot,\tau)|_2^2 d\tau \le Ct^{-1}\int_0^t |u_t(\cdot,\tau)|_2^2 d\tau \\
&\le Ct^{-1}\int_0^t (1+\tau)^{-\frac{3}{2}} d\tau = Ct^{-1}.
\end{aligned} \tag{3.56}$$

Choosing $\beta=\beta_0=\min\{\alpha,\frac{1}{2}\}$, it follows from (3.52), (3.55) and (3.56) that

$$\int_{|y|\le\sqrt{\frac{\gamma}{t}}} |\hat{Y}(y,t)|^2 dy \le C_\beta t^{-(\beta+\frac{1}{2})},$$

for all $t\ge\gamma$, and the lemma is proved. □

Lemma 3.9. *Let f satisfy the conditions in Lemma 3.8. Then, there are positive numbers β and C_β, independent of t such that the solution u of (1.2) corresponding to initial data f satisfies*

$$|u(\cdot,t)|_2^2 \le C_\beta(1+t)^{-(\frac{1}{2}+\beta)} \tag{3.57}$$

for all $t\ge 0$.

Proof.. Because of (3.45) and the relation (3.47), It is only necessary to show that $|Y(\cdot,t)|_2^2 \le C_\beta(1+t)^{-(\frac{1}{2}+\beta)}$ for some positive number β. ¿From (3.46a), one can derive

$$\frac{d}{dt}|Y(\cdot,t)|_2^2 + 2\nu|Y_x(\cdot,t)|_2^2 = -2\int_{-\infty}^{\infty} Y_x(x,t)g(x,t)dx,$$

from which it follows that

$$\frac{d}{dt}|Y(\cdot,t)|_2^2 + \nu|Y_x(\cdot,t)|_2^2 \le \frac{1}{\nu}|g(\cdot,t)|_2^2. \tag{3.58}$$

Note that by Corollary 5.3 in [2], one has

$$|u_{xx}(\cdot,t)|_2^2 \le C(1+t)^{-\frac{5}{2}}. \tag{3.59}$$

Let $\beta=\min\{\alpha,\frac{1}{2}\}$ so that $\beta\le\frac{1}{2}$. The use of equation (1.2) and (3.59) yields

$$t^{1+\beta}|u_{xt}(\cdot,t)|_2^2 \le Ct^{1+\beta}|u_{xx}(\cdot,t)|_2^2 \le Ct^{-1}, \tag{3.60}$$

while the relation (3.58) implies

$$\begin{aligned}
\frac{d}{dt}(t^{1+\beta}|Y(\cdot,t)|_2^2) &\le t^\beta\left[(1+\beta)|Y(\cdot,t)|_2^2 - \nu t|Y_x(\cdot,t)|_2^2\right] + Ct^{1+\beta}|g(\cdot,t)|_2^2 \\
&\le t^\beta\int_{-\infty}^{\infty}[(1+\beta)-\nu t y^2]|\hat{Y}(y,t)|^2 dy + Ct^{1+\beta}|v(\cdot,t)u_{xt}(\cdot,t)|_2^2 \\
&\le t^\beta\int_{|y|\le\sqrt{\frac{1+\beta}{\nu t}}} |\hat{Y}(y,t)|^2 dy + Ct^{1+\beta}|u_{xt}(\cdot,t)|_2^2 \\
&\le Ct^{-\frac{1}{2}} + Ct^{-1},
\end{aligned} \tag{3.61}$$

where (3.49) and (3.60) are used in the last inequality. It follows immediately from (3.61) that

$$|Y(\cdot,t)|_2^2 \le Ct^{-(\frac{1}{2}+\beta)}.$$

with β as above one has

$$|u(\cdot,t)|_2^2 = 4|\bar{u}(\cdot,t)|_2^2 = 4\nu^2\Big|\frac{Y(\cdot,t)}{v(\cdot,t)}\Big|_2^2 \le C|Y(\cdot,t)|_2^2 \le C_\beta t^{-(\frac{1}{2}+\beta)},$$

by again using (3.45) and (3.47). □

With Lemmas 3.8 and 3.9 in hand, we are ready to show that the decay rate of solutions of (1.2) in case $p=1$ and $\alpha>0$ is the same as that of solutions of the corresponding linear equation (2.1).

Theorem 3.10. *Let $p=1$, $f\in H^2\cap W_1^2$ and suppose*

$$|\hat{f}(y)| \le C|y|^\alpha,$$

for small values of y, where $0\le\alpha\le 1$ and C is a positive constant. Then the solution u of (1.2) corresponding to the initial data f has the properties

$$\sup_{0\le t<\infty} t^{\alpha+l+\frac{1}{2}}\int_{-\infty}^{\infty}[\partial_x^l u(x,t)]^2dx < \infty, \tag{3.62}$$

for $l=0,1$. In particular, if $0<\alpha<1$, there exist constants C_f^l which depend only on the solution of the linear equation (2.1) associated with (1.2) with the initial data f, such that for $l=0,1$,

$$\lim_{t\to+\infty} t^{\alpha+l+\frac{1}{2}}\int_{-\infty}^{\infty}[\partial_x^l u(x,t)]^2dx = C_f^l. \tag{3.63}$$

In particular, if $|\hat{f}(y)| = |y|^\alpha|\hat{g}(y)|$ for some $g\in L_1(\mathbb{R})$, then

$$C_f^l = \frac{\Gamma(\alpha+l+\frac{1}{2})}{2\pi(2\nu)^{\alpha+l+\frac{1}{2}}}\Big(\int_{-\infty}^{\infty} g(x)dx\Big)^2. \tag{3.64}$$

Proof. In case $\alpha=0$, the estimate (3.62) is in [2]. Suppose $0<\alpha\le 1$. From Lemma 3.9, there is a number $\beta>0$ and a constant C_β such that, for $t\ge 0$,

$$|u(\cdot,t)|_2^2 \le C_\beta t^{-(\frac{1}{2}+\beta)}.$$

We show that β can be chosen to be α and that C_β depends only on the solution of the linear equation (2.1) if $0<\alpha<1$.

Let $p=1$ in (1.2) and take the Fourier transform of the equation with respect to the spatial variable x, then solve the resulting ordinary differential equation to come to

$$\begin{aligned}\hat{u}(y,t) &= \exp\Big(\frac{-\nu y^2t - iyt}{1+y^2}\Big)\hat{f}(y)\\ &-\frac{i}{2}\int_0^t \frac{y}{1+y^2}\exp\Big(\frac{-\nu y^2 - iy}{1+y^2}(t-\tau)\Big)\widehat{u^2}(y,\tau)d\tau.\end{aligned} \tag{3.65}$$

This equation leads at once to the inequality

$$|\hat{u}(y,t)|^2 \le 2\exp\Big(\frac{-2\nu y^2 t}{1+y^2}\Big)|\hat{f}(y)|^2 + \frac{y^2}{2(1+y^2)^2}\Big(\int_0^t \exp\Big(\frac{-\nu y^2}{1+y^2}(t-\tau)\Big)\big|\widehat{u^2}(y,\tau)\big|d\tau\Big)^2. \tag{3.66}$$

Note that for $t \ge \frac{2\gamma}{\nu}$, we have

$$\int_{|y|\le\sqrt{\frac{\gamma}{\nu t-\gamma}}} \exp\Big(\frac{-2\nu y^2 t}{1+y^2}\Big)|\hat{f}(y)|^2 dy \le C^2 \int_{|y|\le\sqrt{\frac{\gamma}{\nu t-\gamma}}} |y|^{2\alpha}\exp\Big(\frac{-2\nu y^2 t}{1+y^2}\Big)dy \le C_f t^{-\frac{1+2\alpha}{2}} \tag{3.67}$$

because of hypothesis (3.1). Note also that by using Lemma 3.9, one obtains

$$|\widehat{u^2}(y,\tau)| \le \frac{1}{\sqrt{2\pi}}|u^2(\cdot,\tau)|_1 = \frac{1}{\sqrt{2\pi}}|u(\cdot,\tau)|_2^2 \le C(1+t)^{-(\frac{1}{2}+\beta)},$$

for $\beta = \min\{\alpha, \frac{1}{2}\} > 0$. The use of the last inequality shows that for $t \ge \frac{2\gamma}{\nu}$,

$$\begin{aligned}
&\int_{|y|\le\sqrt{\frac{\gamma}{\nu t-\gamma}}} \frac{y^2}{2(1+y^2)^2}\Big(\int_0^t \exp\Big(\frac{-\nu y^2}{1+y^2}(t-\tau)\Big)\big|\widehat{u^2}(y,\tau)\big|d\tau\Big)^2 dy \\
&\le C\int_{|y|\le\sqrt{\frac{\gamma}{\nu t-\gamma}}} \frac{y^2 dy}{(1+y^2)^2}\Big(\int_0^t \frac{d\tau}{(1+\tau)^{\frac{1}{2}+\beta}}\Big)^2 \\
&\le \begin{cases} C\Big(\sqrt{\frac{\gamma}{\nu t-\gamma}}\Big)^3 t^{1-2\alpha} \le Ct^{-(\frac{1}{2}+2\alpha)}, & \text{if } \alpha < \frac{1}{2}, \\ C\Big(\sqrt{\frac{\gamma}{\nu t-\gamma}}\Big)^3 (\log(1+t))^2 \le Ct^{-\frac{3}{2}}(\log(1+t))^2, & \text{if } \alpha \ge \frac{1}{2}, \end{cases}
\end{aligned} \tag{3.68}$$

because $\beta \le \frac{1}{2}$. Using the estimates (3.67) and (3.68), one can conclude from (3.66) that

$$\int_{|y|\le\sqrt{\frac{\gamma}{\nu t-\gamma}}} |\hat{u}(y,t)|^2 dy \le \begin{cases} C_f t^{-(\frac{1}{2}+\alpha)} + C_N t^{-(\frac{1}{2}+2\alpha)}, & \text{if } \alpha < \frac{1}{2}, \\ C_f t^{-(\frac{1}{2}+\alpha)} + C_N t^{-\frac{3}{2}}(\log(1+t))^2, & \text{if } \alpha \ge \frac{1}{2}. \end{cases} \tag{3.69}$$

It is easily seen from equation (3.21) that for t large enough,

$$\begin{aligned}
\frac{d}{dt}t^{1+\alpha}(|u(\cdot,t)|_2^2 + |u_x(\cdot,t)|_2^2) &= t^\alpha\Big((1+\alpha)[|u(\cdot,t)|_2^2 + |u_x(\cdot,t)|_2^2] - \nu t|u_x(\cdot,t)|_2^2\Big) \\
&= t^\alpha \int_{-\infty}^{\infty} [(1+\alpha)(1+y^2) - \nu t y^2]|\hat{u}(y,t)|^2 dy \\
&\le t^\alpha \int_{|y|\le\sqrt{\frac{1+\alpha}{\nu t-(1+\alpha)}}} |\hat{u}(y,t)|^2 dy \\
&\le \begin{cases} C_f t^{-\frac{1}{2}} + C_N t^{-(\frac{1}{2}+\alpha)}, & \text{if } \alpha < \frac{1}{2}, \\ C_f t^{-\frac{1}{2}} + C_N t^{-(\frac{3}{2}-\alpha)}(\log(1+t))^2, & \text{if } \alpha \ge \frac{1}{2}, \end{cases}
\end{aligned} \tag{3.70}$$

where Parseval's formula has been used in the second equality and (3.69) has been used in the last inequality with $\gamma = 1+\alpha$. It follows from (3.70) that the asymptotic behavior of the solution of (1.2) in L_2-norm is given in the form

$$|u(\cdot,t)|_2^2 \le C_f t^{-(\frac{1}{2}+\alpha)} + o(t^{-(\frac{1}{2}+\alpha)}) \qquad \text{as } t \to +\infty, \quad \text{if } 0<\alpha\le 1-\epsilon, \tag{3.71}$$

where $0<\epsilon\le\frac{1}{4}$, say, and C_f depends only on the solution of (2.1) with initial data f. Further if $\alpha \ge 1-\epsilon$ with, say, $\epsilon = \frac{1}{4}$, then (3.71) yields

$$|u(\cdot,t)|_2^2 \le Ct^{-\frac{5}{4}}. \tag{3.72}$$

With this new estimate in hand, it follows that for $t \ge \frac{2\gamma}{\nu}$,

$$\begin{aligned}
&\int_{|y|\le\sqrt{\frac{\gamma}{\nu t-\gamma}}} \frac{y^2}{2(1+y^2)^2}\Big(\int_0^t \exp\big(\frac{-\nu y^2}{1+y^2}(t-\tau)\big)\big|\widehat{u^2}(y,\tau)\big|d\tau\Big)^2 dy\\
&\le C\int_{|y|\le\sqrt{\frac{\gamma}{\nu t-\gamma}}} \frac{y^2dy}{(1+y^2)^2}\Big(\int_0^t \frac{d\tau}{(1+\tau)^{\frac{5}{4}}}\Big)^2\\
&\le C\Big(\sqrt{\frac{\gamma}{\nu t-\gamma}}\Big)^3 \le Ct^{-\frac{3}{2}},
\end{aligned} \tag{3.73}$$

where we have used

$$|\widehat{u^2}(y,\tau)| \le \frac{1}{\sqrt{2\pi}}|u^2(\cdot,\tau)|_1 = \frac{1}{\sqrt{2\pi}}|u(\cdot,\tau)|_2^2 \le C(1+t)^{-\frac{5}{4}}.$$

Taking account of (3.67) and (3.73), one concludes from (3.66) that

$$\int_{|y|\le\sqrt{\frac{\gamma}{\nu t-\gamma}}} |\hat{u}(y,t)|^2 dy \le C_f t^{-(\frac{1}{2}+\alpha)} + C_N t^{-\frac{3}{2}}. \tag{3.74}$$

Then (3.70) can be estimated as

$$\frac{d}{dt}\Big(t^{1+\alpha}[|u(\cdot,t)|_2^2 + |u_x(\cdot,t)|_2^2]\Big) \le C_f t^{-\frac{1}{2}} + C_N t^{-(\frac{3}{2}-\alpha)}, \tag{3.75}$$

by using the new inequality (3.74). It follows immediately that when $1-\epsilon\le\alpha\le 1$, then as $t\to+\infty$,

$$|u(\cdot,t)|_2^2 \le \begin{cases} C_f t^{-(\frac{1}{2}+\alpha)} + o(t^{-(\frac{1}{2}+\alpha)}), & \text{if } 1-\epsilon\le\alpha<1,\\ (C_f+C_N)t^{-\frac{3}{2}}, & \text{if } \alpha = 1. \end{cases} \tag{3.76}$$

Combining (3.71) and (3.76), the asymptotic behavior of the solution of (1.2) in the L_2-norm is given in the form

$$|u(\cdot,t)|_2^2 \le \begin{cases} C_f t^{-(\frac{1}{2}+\alpha)} + o(t^{-(\frac{1}{2}+\alpha)}), & \text{if } \alpha<1,\\ C_\alpha t^{-\frac{3}{2}}, & \text{if } \alpha = 1, \end{cases} \tag{3.77}$$

as $t \to +\infty$, where C_f and C_α are constant, C_f depends only on the solution of (2.1) with initial data f and C_α depends on the norm of nonlinear term and the initial data f.

Next, remark that the assumption $p \geq 2$ in Lemma 3.3 was not used in deriving (3.16). It follows, therefore, that

$$\frac{d}{dt}(|u_x(\cdot,t)|_2^2 + |u_{xx}(\cdot,t)|_2^2 + |u(\cdot,t)|_4^4) \leq -A|u_{xx}(\cdot,t)|_2^2 - B|u(\cdot,t)u_x(\cdot,t)|_2^2, \quad (3.78)$$

for $t \geq T$, where A, B and T are suitably chosen, positive constants. The relation (3.78) implies that

$$\begin{aligned} \frac{d}{dt}\Big(t^{2+\alpha}\big[|u_x(\cdot,t)|_2^2 + |u_{xx}(\cdot,t)|_2^2 + |u(\cdot,t)|_4^4\big]\Big) \\ \leq t^{1+\alpha}\Big[(2+\alpha)|u_x(\cdot,t)|_2^2 - \frac{tA}{2}|u_{xx}(\cdot,t)|_2^2 \\ + (2+\alpha)|u(\cdot,t)|_4^4 - tB|u(\cdot,t)u_x(\cdot,t)|_2^2\Big], \end{aligned} \quad (3.79)$$

if $t \geq \bar{T} = \max\{T, \frac{2(2+\alpha)}{A}\}$. By using Parseval's theorem, the first term on the right-hand side of (3.79) can be estimated as follows:

$$\begin{aligned} t^{1+\alpha}\Big((2+\alpha)|u_x(\cdot,t)|_2^2 - \frac{tA}{2}|u_{xx}(\cdot,t)|_2^2\Big) &\leq t^{1+\alpha}\int_{|y|\leq\sqrt{\frac{2(2+\alpha)}{At}}} y^2|\hat{u}(y,t)|^2 dy \\ &\leq t^{1+\alpha}\frac{2(2+\alpha)}{At}\int_{|y|\leq\sqrt{\frac{2(2+\alpha)}{At}}} |\hat{u}(y,t)|^2 dy \\ &\leq C_f t^{-\frac{1}{2}} + C_N t^{-(\frac{3}{2}-\alpha)}, \end{aligned} \quad (3.80)$$

where (3.74) has been applied in the last inequality, and C_N is a constant which is independent of t. To estimate the second term on the right-hand side of (3.79), let $v(x,t) = u^2(x,t)$. Using Parseval's theorem again, one has

$$t^{1+\alpha}\big((2+\alpha)|u(\cdot,t)|_4^4 - tB|u(\cdot,t)u_x(\cdot,t)|_2^2\big) \leq t^{1+\alpha}\int_{|y|\leq\frac{2\sqrt{2+\alpha}}{\sqrt{Bt}}} |\hat{v}(y,t)|^2 dy \leq C_N t^{-(\frac{1}{2}+\alpha)} \quad (3.81)$$

because

$$|\hat{v}(\cdot,t)|_\infty \leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} |u(x,t)|^2 dx \leq C t^{-(\frac{1}{2}+\alpha)}.$$

Inequalities (3.80) and (3.81) permit the reduction of (3.79) to

$$\begin{aligned} \frac{d}{dt}\Big(t^{2+\alpha}\big[|u_x(\cdot,t)|_2^2 + |u_{xx}(\cdot,t)|_2^2 + |u(\cdot,t)|_4^4\big]\Big) \\ \leq C_f t^{-\frac{1}{2}} + C_N(t^{-(\frac{1}{2}+\alpha)} + t^{-(\frac{3}{2}-\alpha)}), \end{aligned}$$

from which it follows immediately that

$$|u_x(\cdot,t)|_2^2 + |u_{xx}(\cdot,t)|_2^2 + |u(\cdot,t)|_4^4 \leq \begin{cases} C_f t^{-(\alpha+\frac{3}{2})} + C_N(t^{-(\frac{3}{2}+2\alpha)} + t^{-(2+\alpha)}), & \text{if } \alpha < \frac{1}{2}, \\ C_f t^{-2} + C_N t^{-\frac{5}{2}}\log t, & \text{if } \alpha = \frac{1}{2}, \\ C_f t^{-(\alpha+\frac{3}{2})} + C_N(t^{-(2+\alpha)} + t^{-\frac{5}{2}}), & \text{if } \frac{1}{2} < \alpha \leq 1. \end{cases} \quad (3.82)$$

Hence the asymptotic behavior of u_x in the L_2-norm is given by

$$|u_x(\cdot,t)|_2^2 \le \begin{cases} C_f t^{-(\frac{3}{2}+\alpha)} + o(t^{-(\frac{3}{2}+\alpha)}), & \text{if } \alpha < 1, \\ C_\alpha t^{-\frac{5}{2}} + o(t^{-\frac{5}{2}}), & \text{if } \alpha = 1, \end{cases} \tag{3.83}$$

where C_f depends only on the solution of (2.1) with initial data f and C_α is a constant depending on the norm of the nonlinear term and on the initial data f. □

Next, we compute the precise values of the constants C_f^l appearing in Theorem 3.6 and Theorem 3.10. The results hold for all $p \ge 1$. The following Corollary will be useful in this endeavor.

Corollary 3.11. *Let $p \ge 1$, $f \in H^2 \cap W_1^2$ and suppose*

$$|\hat{f}(y)| \le C|y|^\alpha$$

for small values of y, where $0 \le \alpha \le 1$ and C is a positive constant. Then the solution u of (1.2) and the solution w of (2.1) corresponding to the initial data f have the properties that for $l = 0, 1$,

$$|\partial_x^l(u-w)(\cdot,t)|_2^2 \le \begin{cases} Ct^{-(\frac{3}{2}+l)}, & \text{if } p \ge 3;\ \text{or } p=2,\ \alpha \ne 0;\ \text{or } p=1,\ \alpha > \frac{1}{2}, \\ Ct^{-(\frac{3}{2}+l)}(\log(1+t))^2, & \text{if } p=2,\ \alpha=0;\ \text{or } p=1,\ \alpha=\frac{1}{2}, \\ Ct^{-(\frac{1}{2}+2\alpha+l)}, & \text{if } p=1 \text{ and } \alpha<\frac{1}{2}. \end{cases} \tag{3.84}$$

Proof. Note first that from (2.2) and (3.3), $\hat{w} - \hat{u}$ has the representation

$$\hat{w}(y,t) - \hat{u}(y,t) = \frac{i}{p+1}\int_0^t \frac{y}{1+y^2}\exp\left(\frac{-\nu y^2 - iy}{1+y^2}(t-\tau)\right)\widehat{u^{p+1}}(y,\tau)d\tau. \tag{3.85}$$

Note also that, by Theorem 3.6 if $p > 1$ and Theorem 3.10 if $p = 1$, one has

$$|u(\cdot,t)|_2^2 \le Ct^{-(\frac{1}{2}+\alpha)} \quad \text{and} \quad |u_x(\cdot,t)|_2^2 \le Ct^{-(\frac{3}{2}+\alpha)}. \tag{3.86}$$

Straightforward interpolation then implies

$$|u(\cdot,t)|_\infty \le Ct^{-\frac{1+\alpha}{2}}. \tag{3.87}$$

Hence, by using (3.85), (3.86) and (3.87), one shows that

$$
\begin{aligned}
\int_{|y|\le\sqrt{\frac{7}{t}}} |\hat{w}(y,t) - \hat{u}(y,t)|^2 dy &\le C \int_{|y|\le\sqrt{\frac{7}{t}}} y^2 \Big(\int_0^t \exp\big(\frac{-\nu y^2}{1+y^2}(t-\tau)\big) \widehat{u^{p+1}}(y,\tau) d\tau \Big)^2 dy \\
&\le C \int_{|y|\le\sqrt{\frac{7}{t}}} y^2 dy \Big(\int_0^t |u(\cdot,\tau)|_\infty^{p-1} |u(\cdot,\tau)|_2^2 d\tau \Big)^2 \\
&\le C t^{-\frac{3}{2}} \Big(\int_0^t (1+\tau)^{-(\frac{1}{2}+\alpha+\frac{(1+\alpha)(p-1)}{2})} d\tau \Big)^2 \\
&\le \begin{cases} C t^{-\frac{3}{2}}, & \text{if } \frac{1}{2}+\alpha+\frac{(1+\alpha)(p-1)}{2} > 1, \\ C t^{-\frac{3}{2}} (\log(1+t))^2, & \text{if } \frac{1}{2}+\alpha+\frac{(1+\alpha)(p-1)}{2} = 1, \\ C t^{-(\frac{1}{2}+2\alpha+(1+\alpha)(p-1))}, & \text{if } \frac{1}{2}+\alpha+\frac{(1+\alpha)(p-1)}{2} < 1, \end{cases} \qquad (3.88) \\
&= \begin{cases} C t^{-\frac{3}{2}}, & \text{if } p \ge 3; \text{ or } p=2,\ \alpha \ne 0; \text{ or } p=1,\ \alpha > \frac{1}{2}, \\ C t^{-\frac{3}{2}} (\log(1+t))^2, & \text{if } p=2,\ \alpha=0; \text{ or } p=1,\ \alpha=\frac{1}{2}, \\ C t^{-\frac{1}{2}+2\alpha}, & \text{if } p=1 \text{ and } \alpha<\frac{1}{2}, \end{cases} \\
&= C t^{-(\frac{1}{2}+\alpha+\delta)},
\end{aligned}
$$

where δ is defined as

$$
\delta = \begin{cases} 1-\alpha, & \text{if } p \ge 3; \text{ or } p=2,\ \alpha \ne 0; \text{ or } p=1,\ \alpha > \frac{1}{2}, \\ \alpha, & \text{if } p=1,\ \alpha < \frac{1}{2}, \end{cases}
$$

while in the other cases $\delta = \delta(t)$ is almost equal to 1 when $p = 2,\ \alpha = 0$, and is almost equal to $\frac{1}{2}$ when $p = 1,\ \alpha = \frac{1}{2}$. That is to say, δ is defined in such a way that $t^{-(\frac{1}{2}+\alpha+\delta)} = t^{-\frac{3}{2}}(\log(1+t))^2$ in the latter cases.

The differential inequality

$$
\frac{d}{dt}[|(u-w)(\cdot,t)|_2^2 + |\partial_x(u-w)(\cdot,t)|_2^2] + \nu|\partial_x(u-w)(\cdot,t)|_2^2 \le \frac{1}{\nu}|u^{p+1}(\cdot,t)|_2^2 \qquad (3.89)
$$

is easily derived from (1.2) and (2.1). Inequality (3.89) is equivalent to

$$
\begin{aligned}
\frac{d}{dt}\Big(t^{1+\alpha+\delta}[|(u-w)(\cdot,t)|_2^2 + |\partial_x(u-w)(\cdot,t)|_2^2]\Big) &\le t^{\alpha+\delta}(1+\alpha+\delta)[|(u-w)(\cdot,t)|_2^2 \\
&+ |\partial_x(u-w)(\cdot,t)|_2^2] - t^{1+\alpha+\delta}\big(\nu|\partial_x(u-w)(\cdot,t)|_2^2 - \frac{1}{\nu}|u^{p+1}(\cdot,t)|_2^2\big) \qquad (3.90)
\end{aligned}
$$

and the right-hand side of (3.90) is bounded above by

$$
\begin{aligned}
&C_1 t^{\alpha+\delta} \int_{|y|\le\sqrt{\frac{1+\alpha+\delta}{t\nu-(1+\alpha+\delta)}}} |(\hat{u}-\hat{w})(y,t)|^2 dy + C_2 t^{1+\alpha+\delta} |u(\cdot,t)|_2^2 |u(\cdot,t)|_\infty^{2p} \\
&\le C_1 t^{-\frac{1}{2}} + C_2 t^{-(\frac{1}{2}+p-1+p\alpha-\delta)}.
\end{aligned}
$$

The estimate (3.88) and (3.86) have been used in the last inequality of (3.90). Note that $p - 1 + p\alpha - \delta \ge 0$ by the definition of δ. It follows immediately from (3.90) that

$$
t^{\frac{1}{2}+\alpha+\delta}|(u-w)(\cdot,t)|_2^2 \le C.
$$

Again, direct appeal to (1.2) and (2.1) leads to

$$\frac{d}{dt}[|\partial_x(u-w)(\cdot,t)|_2^2+|\partial_x^2(u-w)(\cdot,t)|_2^2]+\nu|\partial_x^2(u-w)(\cdot,t)|_2^2 \le \frac{1}{\nu}|u^p u_x(\cdot,t)|_2^2, \quad (3.91)$$

which is equivalent to

$$\begin{aligned}\frac{d}{dt}\Big(t^{2+\alpha+\delta}\left[|\partial_x(u-w)(\cdot,t)|_2^2+|\partial_x^2(u-w)(\cdot,t)|_2^2\right]\Big) &\le t^{1+\alpha+\delta}(2+\alpha+\delta)[|\partial_x(u-w)(\cdot,t)|_2^2+|\partial_x^2(u-w)(\cdot,t)|_2^2] \\ &\quad - t^{2+\alpha+\delta}\nu|\partial_x^2(u-w)(\cdot,t)|_2^2+\frac{t^{2+\alpha+\delta}}{\nu}|u^p u_x(\cdot,t)|_2^2.\end{aligned} \quad (3.92)$$

By using (3.88), the first two terms on the right-hand side of (3.92) can be bounded above by

$$\begin{aligned}&t^{1+\alpha+\delta}(2+\alpha+\delta)[|\partial_x(u-w)(\cdot,t)|_2^2+|\partial_x^2(u-w)(\cdot,t)|_2^2]-t^{2+\alpha+\delta}\nu|\partial_x^2(u-w)(\cdot,t)|_2^2 \\ &\quad \le C_1 t^{1+\alpha+\delta}\int_{|y|\le\sqrt{\frac{2+\alpha+\delta}{t\nu-(2+\alpha+\delta)}}} y^2|(\hat{u}-\hat{w})(y,t)|^2dy \le C_1 t^{-\frac{1}{2}}.\end{aligned} \quad (3.93)$$

Then, using (3.86) and (3.87), the last term on the right-hand side of (3.92) can be bounded as follows:

$$\frac{t^{2+\alpha+\delta}}{\nu}|u^p u_x(\cdot,t)|_2^2 \le C_2 t^{2+\alpha+\delta}|u_x(\cdot,t)|_2^2|u(\cdot,t)|_\infty^{2p} \le C_2 t^{-(\frac{1}{2}+p-1+p\alpha-\delta)} \quad (3.94)$$

Applying (3.93) and (3.94) reduces (3.92) to the simple inequality

$$\frac{d}{dt}\Big(t^{2+\alpha+\delta}\left[|\partial_x(u-w)(\cdot,t)|_2^2+|\partial_x^2(u-w)(\cdot,t)|_2^2\right]\Big) \le C_1 t^{-\frac{1}{2}}+C_2 t^{-(\frac{1}{2}+p-1+p\alpha-\delta)} \quad (3.95)$$

Since $p-1+p\alpha-\delta \ge 0$, (3.95) yields

$$t^{\frac{3}{2}+\alpha+\delta}|\partial_x(u-w)(\cdot,t)|_2^2 \le C, \quad (3.96)$$

and the corollary is established. □

Corollary 3.12. *Let $p \ge 1$ and suppose f to satisfy the conditions specified in Corollary 3.11. Then the solution u of (1.2) and the solution w of (2.1) corresponding to the initial data f have the properties*

$$\begin{aligned}&\lim_{t\to+\infty} t^{\frac{1}{2}+l+\alpha}\Big||\partial_x^l u(\cdot,t)|_2^2-|\partial_x^l w(\cdot,t)|_2^2\Big| \\ &\quad =\begin{cases}0, & \text{if } 0<\alpha<1 \text{ and } p=1;\text{ or } p>1 \text{ and } 0\le\alpha<1 \\ C_N^l, & \text{if } \alpha=0 \text{ and } p=1;\text{ or } \alpha=1 \text{ and } p\ge 1,\end{cases}\end{aligned} \quad (3.97)$$

where C_N^l is a constant and $l=0,1$.

Proof. From Corollary 3.11, it follows that

$$\sup_{0\le t<\infty} t^{\frac{1}{2}+l+\alpha+\delta}|\partial_x^l(u-w)(\cdot,t)|_2^2<\infty, \quad (3.98)$$

where δ is not equal to zero when $p=1,\ 0<\alpha<1$, and when $p>1,\ 0\le\alpha<1$. Because of the triangle inequality,

$$\begin{aligned}\left||\partial_x^l u(\cdot,t)|_2^2-|\partial_x^l w(\cdot,t)|_2^2\right| &\le |\partial_x^l(u-w)(\cdot,t)|_2|\partial_x^l(u+w)(\cdot,t)|_2\\ &\le |\partial_x^l(u-w)(\cdot,t)|_2[|\partial_x^l u(\cdot,t)|_2+|\partial_x^l w(\cdot,t)|_2].\end{aligned}\tag{3.99}$$

Note also that from Theorem 3.6 if $p\ge 2$ and Theorem 3.10 if $p=1$, respectively, one has

$$\lim_{t\to+\infty} t^{\frac{1+2(l+\alpha)}{4}}|\partial_x^l u(\cdot,t)|_2=C\tag{3.100}$$

for some non-negative constant C. It follows from (3.99) that for $l=0,1$,

$$\begin{aligned}&\lim_{t\to+\infty} t^{\frac12+l+\alpha}\left||\partial_x^l u(\cdot,t)|_2^2-|\partial_x^l w(\cdot,t)|_2^2\right|\\ &\le \lim_{t\to+\infty} t^{\frac{1+2(l+\alpha)}{4}}|\partial_x^l(u-w)(\cdot,t)|_2(t^{\frac{1+2(l+\alpha)}{4}}[|\partial_x^l u(\cdot,t)|_2+|\partial_x^l w(\cdot,t)|_2])\\ &=\begin{cases}0, & \text{if } 0<\alpha<1 \text{ and } p=1;\text{ or } p>1 \text{ and } 0\le\alpha<1,\\ C_N^l, & \text{if } \alpha=0 \text{ and } p=1;\text{ or } \alpha=1 \text{ and } p\ge 1,\end{cases}\end{aligned}$$

by using (3.98) and (3.100), and the Corollary is proved. □

Remark 3.13. Corollary 3.12 shows that the asymptotic behavior of the solution of (1.2) is exactly the same as that of the solution of (2.1) in case either (1.2) has higher-order nonlinearity ($p>1$) or the initial data has zero mass.

When $\alpha=1$, the L_2-norm of the solution u of (1.2) and the solution w of (2.1) both decay like $t^{-\frac34}$, but it turns out their asymptotic states $\lim_{t\to+\infty} t^{\frac34}|\cdot|_2$ are different. In the penultimate result of Section 3, we compute the weighted limits of $u-w$ and u in L_2-norm when $\alpha=1$. The results imply that further decay of solutions of equation (1.2) depends on the nonlinear term.

Corollary 3.14. *Let f satisfy the conditions in Corollary 3.11 and suppose $\alpha=1$. Then the difference between the solution u of (1.2) for $p\ge1$ and the solution w of (2.1), both with initial value f, has the property*

$$\lim_{t\to+\infty} t^{\frac32}|u(\cdot,t)-w(\cdot,t)|_2^2=\frac{1}{4\nu(8\nu\pi)^{\frac12}(p+1)^2}\Big(\int_0^\infty\int_{-\infty}^\infty u^{p+1}dxdt\Big)^2.\tag{3.101}$$

If $\hat f(y)=iy\hat g(y)$ for some $g\in L_1(\mathbb{R})$, then

$$\lim_{t\to+\infty} t^{\frac32}|u(\cdot,t)|_2^2=\frac{1}{4\nu(8\nu\pi)^{\frac12}}\Big(\int_{-\infty}^\infty g(x)dx-\int_0^\infty\int_{-\infty}^\infty \frac{u^{p+1}(x,t)}{p+1}dxdt\Big)^2.\tag{3.102}$$

In particular, if $xf(x)\in L_1(\mathbb{R})$ and $\frac{d}{dx}g(x)=f(x)$ with $xg(x)\to0$ as $x\to\pm\infty$, one has

$$\lim_{t\to+\infty} t^{\frac32}|u(\cdot,t)|_2^2=\frac{1}{4\nu(8\nu\pi)^{\frac12}}\Big(\int_{-\infty}^\infty xf(x)dx+\int_0^\infty\int_{-\infty}^\infty \frac{u^{p+1}(x,t)}{p+1}dxdt\Big)^2.\tag{3.103}$$

Proof. Since $\alpha = 1$, Theorem 3.6 for $p \geq 2$ and Theorem 3.10 for $p = 1$ show that

$$|u(\cdot,t)|_2 \leq C(1+t)^{-\frac{3}{4}}, \quad \text{and} \quad |u_x(\cdot,t)|_2 \leq C(1+t)^{-\frac{5}{4}},$$

for $t \geq 0$. Hence for $p \geq 1$, one has

$$|u^{p+1}(\cdot,t)|_1 \leq |u(\cdot,t)|_\infty^{p-1} |u(\cdot,t)|_2^2 \leq C(1+t)^{-\frac{3}{2}}.$$

In consequence, the right-hand side of (3.101) is a finite number. Note that

$$\begin{aligned} \Big|\int_{t^{\frac{1}{2}-\epsilon}}^{t} e^{-\frac{\nu y^2+iy}{1+y^2}(t-\tau)} \widehat{u^p u_x}(y,\tau)d\tau\Big| &\leq \int_{t^{\frac{1}{2}-\epsilon}}^{t} |u^p u_x(\cdot,\tau)|_1 d\tau \\ &\leq C\int_{t^{\frac{1}{2}-\epsilon}}^{t} (1+\tau)^{-2}d\tau \leq C(1+t^{\frac{1}{2}-\epsilon})^{-2} \leq Ct^{2\epsilon-1}, \end{aligned} \tag{3.104}$$

where ϵ is a small positive number that will be determined momentarily, and the inequality

$$|u^p u_x(\cdot,\tau)|_1 \leq |u(\cdot,\tau)|_\infty^{p-1} |u(\cdot,\tau)|_2 |u_x(\cdot,\tau)|_2 \leq C(1+\tau)^{-2} \tag{3.105}$$

has been used at the second step in (3.104).

With this information in hand, one determines that

$$\begin{aligned} &\lim_{t\to+\infty} t^{\frac{3}{2}} \Big|\int_{t^{\frac{1}{2}-\epsilon}}^{t} \frac{1}{1+y^2} \exp\big(\frac{-\nu y^2 - iy}{1+y^2}(t-\tau)\big)\widehat{u^p u_x}(y,\tau)d\tau\Big|_2^2 \\ &\leq \lim_{t\to+\infty} Ct^{\frac{3}{2}}t^{2\epsilon-1}\Big|\int_{t^{\frac{1}{2}-\epsilon}}^{t} \frac{e^{-\frac{\nu y^2+iy}{1+y^2}(t-\tau)}}{(1+y^2)^2}\widehat{u^p u_x}(y,\tau)d\tau\Big|_1 \\ &\leq \lim_{t\to+\infty} Ct^{\frac{1}{2}+2\epsilon}\int_{t^{\frac{1}{2}-\epsilon}}^{t} \Big|\frac{\exp\big(-\frac{\nu y^2+iy}{1+y^2}(t-\tau)\big)}{(1+y^2)^2}\Big|_1 |u^p u_x(\cdot,\tau)|_1 d\tau \\ &\leq \lim_{t\to+\infty} Ct^{\frac{1}{2}+2\epsilon}\int_{t^{\frac{1}{2}-\epsilon}}^{t} |u^p u_x(\cdot,\tau)|_1 d\tau \leq \lim_{t\to+\infty} Ct^{\frac{1}{2}+2\epsilon}t^{2\epsilon-1} = 0, \end{aligned} \tag{3.106}$$

if $\epsilon > 0$ is chosen small enough, say $0 < \epsilon < \frac{1}{8}$, where the estimate (3.104) has been used at the first step, while

$$\Big|\frac{\exp\big(-\frac{\nu y^2+iy}{1+y^2}(t-\tau)\big)}{(1+y^2)^2}\Big|_1 \leq C \tag{3.107}$$

and (3.105) have been used at the third and the last step, respectively.

One can also compute that

$$\begin{aligned} &\lim_{t\to+\infty} t^{\frac{3}{2}}\Big|\int_0^{t^{\frac{1}{2}-\epsilon}} \frac{1}{1+y^2}\exp\big(\frac{-\nu y^2-iy}{1+y^2}(t-\tau)\big)\widehat{u^p u_x}(y,\tau)d\tau\Big|_2^2 \\ &= \lim_{t\to+\infty} t^{\frac{3}{2}}\Big|\frac{i}{p+1}\int_0^{t^{\frac{1}{2}-\epsilon}} \frac{y}{1+y^2}\exp\big(\frac{-\nu y^2-iy}{1+y^2}(t-\tau)\big)\widehat{u^{p+1}}(y,\tau)d\tau\Big|_2^2 \\ &= \lim_{t\to+\infty} t^{\frac{3}{2}}\int_{-\infty}^{\infty} \frac{y^2}{(p+1)^2(1+y^2)^2}\Big|\int_0^{t^{\frac{1}{2}-\epsilon}} e^{-\frac{\nu y^2+iy}{1+y^2}(t-\tau)}\widehat{u^{p+1}}(y,\tau)d\tau\Big|^2 dy \\ &= \lim_{t\to+\infty}\int_{-\infty}^{\infty} \frac{s^2 e^{-\frac{2\nu s^2}{1+s^2/t}}}{(p+1)^2(1+\frac{s^2}{t})^2}\Big|\int_0^{t^{\frac{1}{2}-\epsilon}} e^{\frac{\nu s^2+is\sqrt{t}}{t+s^2}\tau}\widehat{u^{p+1}}(\frac{s}{\sqrt{t}},\tau)d\tau\Big|^2 ds \end{aligned}$$

hence we have

$$\begin{aligned}
&\lim_{t\to+\infty} t^{\frac{3}{2}}\Big|\int_0^{t^{\frac{1}{2}-\epsilon}} \frac{1}{1+y^2}\exp\big(\frac{-\nu y^2-iy}{1+y^2}(t-\tau)\big)\widehat{u^p u_x}(y,\tau)d\tau\Big|_2^2 \\
&= \lim_{t\to+\infty}\int_{-\infty}^{\infty}\frac{s^2 e^{-\frac{2\nu s^2}{1+s^2/t}}}{(p+1)^2(1+\frac{s^2}{t})^2}\Big|\int_0^{t^{\frac{1}{2}-\epsilon}} e^{\frac{\nu s^2+is\sqrt{t}}{t+s^2}\tau}\widehat{u^{p+1}}(\frac{s}{\sqrt{t}},\tau)d\tau\Big|^2 ds \\
&= \frac{1}{(p+1)^2}\int_{-\infty}^{\infty} s^2 e^{-2\nu s^2}\Big|\int_0^{\infty}\widehat{u^{p+1}}(0,\tau)d\tau\Big|^2 ds \\
&= \frac{1}{(p+1)^2}\int_{-\infty}^{\infty} s^2 e^{-2\nu s^2}ds\Big(\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\int_{-\infty}^{\infty} u^{p+1}(x,\tau)dxd\tau\Big)^2 \qquad (3.108)\\
&= \frac{1}{2\pi(p+1)^2(2\nu)^{\frac{3}{2}}}\int_0^{\infty} s^{\frac{1}{2}}e^{-s}ds\Big(\int_0^{\infty}\int_{-\infty}^{\infty} u^{p+1}(x,\tau)dxd\tau\Big)^2 \\
&= \frac{1}{4\nu(8\nu\pi)^{\frac{1}{2}}(p+1)^2}\Big(\int_0^{\infty}\int_{-\infty}^{\infty} u^{p+1}(x,\tau)dxd\tau\Big)^2,
\end{aligned}$$

because

$$\exp\big(\frac{\nu s^2+is\sqrt{t}}{t+s^2}\tau\big)\to 1, \qquad (3.109)$$

as $t\to+\infty$ for any fixed s and $\tau\in[0,t^{\frac{1}{2}-\epsilon}]$. The substitution $s=y\sqrt{t}$ has been used at the third step in (3.108).

The use of (3.106) and (3.108) shows that if $\theta(y,t,\tau)=\frac{1}{1+y^2}e^{-\frac{\nu y^2+iy}{1+y^2}(t-\tau)}\widehat{u^p u_x}(y,\tau)$, then

$$\begin{aligned}
&\lim_{t\to+\infty} t^{\frac{3}{2}}\Big|\int_{-\infty}^{\infty} Re\Big(\Big(\int_0^{t^{\frac{1}{2}-\epsilon}}\theta(y,t,\tau)d\tau\Big)\Big(\int_{t^{\frac{1}{2}-\epsilon}}^{t}\theta(y,t,\tau)d\tau\Big)\Big)dy\Big| \\
&\leq \lim_{t\to+\infty} t^{\frac{3}{2}}\Big|\int_0^{t^{\frac{1}{2}-\epsilon}}\theta(y,t,\tau)d\tau\Big|_2\Big|\int_{t^{\frac{1}{2}-\epsilon}}^{t}\theta(y,t,\tau)d\tau\Big|_2=0, \qquad (3.110)
\end{aligned}$$

where Re stands for the real part of its argument.

Apply Parseval's Theorem to (3.85), and then use (3.106), (3.108) and (3.110) to obtain

$$\begin{aligned}
&\lim_{t\to+\infty} t^{\frac{3}{2}}|u(\cdot,t)-w(\cdot,t)|_2^2=\lim_{t\to+\infty} t^{\frac{3}{2}}|\hat{u}(y,t)-\hat{w}(y,t)|_2^2 \\
&=\lim_{t\to+\infty} t^{\frac{3}{2}}\Big|\int_0^t\frac{1}{1+y^2}\exp\big(\frac{-\nu y^2-iy}{1+y^2}(t-\tau)\big)\widehat{u^p u_x}(y,\tau)d\tau\Big|_2^2 \\
&=\lim_{t\to+\infty} t^{\frac{3}{2}}\int_{-\infty}^{\infty}\frac{1}{(1+y^2)^2}\Big|\Big(\int_0^{t^{\frac{1}{2}-\epsilon}}+\int_{t^{\frac{1}{2}-\epsilon}}^{t}\Big)e^{-\frac{\nu y^2+iy}{1+y^2}(t-\tau)}\widehat{u^p u_x}(y,\tau)d\tau\Big|^2 dy
\end{aligned}$$

hence

$$\begin{aligned}
&\lim_{t\to+\infty} t^{\frac{3}{2}}|u(\cdot,t)-w(\cdot,t)|_2^2\\
&= \lim_{t\to+\infty} t^{\frac{3}{2}}\int_{-\infty}^{\infty}\frac{1}{(1+y^2)^2}\Big|\int_0^{t^{\frac{1}{2}-\epsilon}} e^{-\frac{\nu y^2+iy}{1+y^2}(t-\tau)}\widehat{u^pu_x}(y,\tau)d\tau\Big|^2dy\\
&+\lim_{t\to+\infty} t^{\frac{3}{2}}\int_{-\infty}^{\infty}\frac{1}{(1+y^2)^2}\Big|\int_{t^{\frac{1}{2}-\epsilon}}^{t} e^{-\frac{\nu y^2+iy}{1+y^2}(t-\tau)}\widehat{u^pu_x}(y,\tau)d\tau\Big|^2dy\\
&+\lim_{t\to+\infty} t^{\frac{3}{2}}\int_{-\infty}^{\infty}2Re\Big(\Big(\int_0^{t^{\frac{1}{2}-\epsilon}}\theta(y,t,\tau)d\tau\Big)\Big(\int_{t^{\frac{1}{2}-\epsilon}}^{t}\theta(y,t,\tau)d\tau\Big)\Big)dy \qquad (3.111)\\
&= \lim_{t\to+\infty}\int_{-\infty}^{\infty}\frac{s^2e^{-\frac{2\nu s^2}{1+s^2/t}}}{(p+1)^2(1+\frac{s^2}{t})^2}\Big|\int_0^{t^{\frac{1}{2}-\epsilon}}\exp\big(\frac{\nu s^2+is\sqrt{t}}{t+s^2}\tau\big)\widehat{u^{p+1}}(\frac{s}{\sqrt{t}},\tau)d\tau\Big|^2ds\\
&= \frac{1}{(p+1)^2}\int_{-\infty}^{\infty}s^2e^{-2\nu s^2}\Big|\int_0^{\infty}\widehat{u^{p+1}}(0,\tau)d\tau\Big|^2ds\\
&= \frac{1}{4\nu(8\nu\pi)^{\frac{1}{2}}(p+1)^2}\Big(\int_0^{\infty}\int_{-\infty}^{\infty}u^{p+1}(x,\tau)dxd\tau\Big)^2.
\end{aligned}$$

If $\hat{f}(y)=iy\hat{g}(y)$ for some $g\in L_1(\mathbb{R})$, then by using the representation (3.3) for a solution u of (1.2) and following the line of argument laid out from (3.104) to (3.111), one obtains

$$\begin{aligned}
&\lim_{t\to+\infty} t^{\frac{3}{2}}|u(\cdot,t)|_2^2 = \lim_{t\to+\infty} t^{\frac{3}{2}}|\hat{u}(\cdot,t)|_2^2\\
&= \lim_{t\to+\infty} t^{\frac{3}{2}}\Big|\hat{w}(y,t)-\int_0^t\frac{e^{-\frac{\nu y^2+iy}{1+y^2}(t-\tau)}}{1+y^2}\widehat{u^pu_x}(y,\tau)d\tau\Big|_2^2\\
&= \lim_{t\to+\infty} t^{\frac{3}{2}}\Big|\hat{f}(y)e^{-\frac{\nu y^2+iy}{1+y^2}t}-\Big(\int_0^{t^{\frac{1}{2}-\epsilon}}+\int_{t^{\frac{1}{2}-\epsilon}}^{t}\frac{e^{-\frac{\nu y^2+iy}{1+y^2}(t-\tau)}}{1+y^2}\widehat{u^pu_x}(y,\tau)d\tau\Big)\Big|_2^2\\
&= \lim_{t\to+\infty} t^{\frac{3}{2}}\Big|iy\hat{g}(y)e^{-\frac{\nu y^2+iy}{1+y^2}t}-\int_0^{t^{\frac{1}{2}-\epsilon}}\frac{iye^{\frac{\nu y^2+iy}{1+y^2}(t-\tau)}}{(1+p)(1+y^2)}\widehat{u^{p+1}}(y,\tau)d\tau\Big|_2^2 \qquad (3.112)\\
&= \lim_{t\to+\infty}\int_{-\infty}^{\infty}s^2e^{-\frac{2\nu s^2}{1+s^2/t}}\Big|\hat{g}(\frac{s}{\sqrt{t}})-\int_0^{t^{\frac{1}{2}-\epsilon}}\frac{e^{\frac{\nu s^2+is\sqrt{t}}{t+s^2}\tau}\widehat{u^{p+1}}(\frac{s}{\sqrt{t}},\tau)}{(1+p)(1+s^2/t)}d\tau\Big|^2ds\\
&= \int_{-\infty}^{\infty}s^2e^{-2\nu s^2}ds\Big|\hat{g}(0)-\int_0^{\infty}\frac{\widehat{u^{p+1}}(0,\tau)}{(p+1)}d\tau\Big|^2\\
&= \frac{1}{4\nu(8\nu\pi)^{\frac{1}{2}}}\Big(\int_{-\infty}^{\infty}g(x)dx-\int_0^{\infty}\int_{-\infty}^{\infty}\frac{u^{p+1}(x,t)}{p+1}dxdt\Big)^2,
\end{aligned}$$

where again we have used that $e^{\frac{\nu s^2+is\sqrt{t}}{t+s^2}\tau}\to 1$ as $t\to+\infty$, for any fixed s and $\tau\in[0,t^{\frac{1}{2}-\epsilon}]$.

Furthermore, if $xf(x)\in L_1(\mathbb{R})$ and $\frac{d}{dx}g(x)=f(x)$ with $xg(x)\to 0$ as $x\to\pm\infty$, then

$$\int_{-\infty}^{\infty}g(x)dx=-\int_{-\infty}^{\infty}xf(x)dx,$$

and so (3.112) becomes

$$\lim_{t\to+\infty} t^{\frac{3}{2}}|u(\cdot,t)|_2^2 = \frac{1}{4\nu(8\nu\pi)^{\frac{1}{2}}}\Big(\int_{-\infty}^{\infty} xf(x)dx + \int_0^{\infty}\int_{-\infty}^{\infty}\frac{u^{p+1}(x,t)}{p+1}dxdt\Big)^2.$$

The Corollary is proved. □

Remark 3.15. If the initial data f satisfies $|\hat{f}(y)| \leq C|y|^\alpha$ for small values of y, where $\alpha > 1$ and C is a positive constant, then the decay of solution of (1.2) depends only on u^{p+1} because, by Lemma 2.1, the solution w of (2.1) decays in L_2-norm faster than $t^{-\frac{3}{4}}$.

This section is concluded with another corollary which will find use in Section 4.

Corollary 3.16. *Let f satisfy the conditions in Lemma 3.1. If $\alpha > \frac{1}{2}$, then the solution $u(x,t)$ of (1.2) for $p \geq 1$ is in $L_2(\mathbb{R}\times\mathbb{R}^+)$.*

4. Decay in the Spatial Variable. In this section, consideration is given to the decay of solutions of (1.1) or (1.2) in the spatial variable x. Such decay results are interesting in their own right, and they often prove useful in the analysis of detailed aspects of solutions (cf. [9]). Spatial decay is conveniently enunciated in terms of weighted-norm spaces, and this point of view leads to the development of a theory for the initial-value problem for (1.1) or (1.2) concluded in such spaces. Some theory for equation (1.1) has already appeared, for example in [1] and in the just-cited paper of [9]. We intend to add to these results in Section 5. In the present section, attention is focussed on the initial-value problem for (1.2) set in the weighted Sobolev class $\mathcal{H}_n^m = \mathcal{H}_n^m(\mathbb{R})$ introduced in Section 1 of functions $f:\mathbb{R}\to\mathbb{R}$ such that its i^{th} derivative $f^{(i)}$ is square-integrable with respect to the measure $(1+x^2)^{\frac{n}{2}}dx$ for $0 \leq i \leq m$.

Lemma 4.1. *Let f be given in $\mathcal{H}_1^2(\mathbb{R})$ and $p \geq 1$. Then corresponding to f, equation (1.2) has a unique solution u which lies in $L_\infty(0,T;\mathcal{H}_1^2(\mathbb{R}))$ for arbitrary $T > 0$.*

The local existence and uniqueness for such solution can be obtained by following the line of argument in [3]. To guarantee that the local solution can be continued forward indefinitely in time while maintaining membership in $\mathcal{H}_1^2(\mathbb{R})$, *a priori* bounds are needed. The following lemma is useful in this regard. Its proof is the same as that of [3] for the case $p = 1$ and $\nu = 0$, and so is omitted.

Lemma 4.2. *Let $f \in C_b^k(\mathbb{R})$ where $k \geq 2$ and let a positive integer n be given. Suppose $x^n f^{(m)}(x) \to 0$ as $x \to \pm\infty$ for $0 \leq m \leq k$. Then if the solution u of (1.2) for $p \geq 1$ exists on the temporal interval $[0,T]$, it has the property*

$$x^n\partial_x^i\partial_t^j u(x,t) \to 0 \qquad as \qquad x\to\pm\infty, \tag{4.1}$$

for $0 \leq i \leq m$, $0 \leq j \leq k$.

Proof of Lemma 4.1. Multiply (1.2) by $2|x|u(x,t)$ and then integrate the result over $\mathbb{R}\times[0,t]$. Integration by parts, and using Lemma 4.2 to conclude the boundary terms vanish, leads to

$$\begin{aligned}
&|u(\cdot,t)|_{2,1}^2 + |u_x(\cdot,t)|_{2,1}^2 + 2\nu\int_0^t |u_x(\cdot,\tau)|_{2,1}^2 d\tau \\
&= |f|_{2,1}^2 + |f'|_{2,1}^2 + \int_0^t\int_{-\infty}^{\infty} \operatorname{sgn}(x)\{u^2 + \frac{2}{p+2}u^{p+2} - 2uu_{xt} - 2\nu uu_x\}dxd\tau \\
&\le C(\|f\|_{\mathcal{H}_1^1(\mathbb{R})}) + C\int_0^t [|u(\cdot,\tau)|_2^2 + |u_x(\cdot,\tau)|_2^2 + |u_{xt}(\cdot,\tau)|_2^2]d\tau \\
&\le C(\|f\|_{\mathcal{H}_1^1(\mathbb{R})}, T)
\end{aligned} \tag{4.2}$$

(see (1.6) for the definition of the norm). Next, multiply (1.2) by $2|x|u_{xx}(x,t)$ and integrate the result over $\mathbb{R}\times[0,t]$. Integration by parts as above and use of the fact that u_x and u_t lie in $L_2(\mathbb{R}\times\mathbb{R}^+)$ (see Thm. 2.3) gives

$$\begin{aligned}
&|u_x(\cdot,t)|_{2,1}^2 + |u_{xx}(\cdot,t)|_{2,1}^2 + 2\nu\int_0^t |u_{xx}(\cdot,\tau)|_{2,1}^2 d\tau \\
&= |f'|_{2,1}^2 + |f''|_{2,1}^2 - \int_0^t\int_{-\infty}^{\infty} \{\operatorname{sgn}(x)(2u_xu_t + u_x^2) - 2|x|u_{xx}u^pu_x\}dxd\tau \\
&\le C(\|f\|_{\mathcal{H}_1^2(\mathbb{R})}) + C\int_0^t [|u_{xx}(\cdot,\tau)|_{2,1}|u_x(\cdot,\tau)|_{2,1}|u(\cdot,\tau)|_\infty^p]d\tau \\
&\le C(\|f\|_{\mathcal{H}_1^2(\mathbb{R})}) + \nu\int_0^t |u_{xx}(\cdot,\tau)|_{2,1}^2 d\tau + C\int_0^t |u_x(\cdot,\tau)|_{2,1}^2|u(\cdot,\tau)|_\infty^{2p}d\tau.
\end{aligned} \tag{4.3}$$

By using the bound (4.2) in (4.3), there obtains

$$|u_{xx}(\cdot,t)|_{2,1}^2 + \int_0^t |u_{xx}(\cdot,\tau)|_{2,1}^2 d\tau \le C(\|f\|_{\mathcal{H}_1^2(\mathbb{R})}, T),$$

for any $T > 0$. With *a priori* bounds in hand, the local $\mathcal{H}_1^2$-solution can be extended to any time interval $[0,T]$. This proves the lemma. □

Continuing to argue in the same vein proves the following extension of Lemma 4.1.

Corollary 4.3. *Let initial data $f \in \mathcal{H}_n^k(\mathbb{R})$ be given and suppose $p \ge 1$. Then corresponding to f, equation (1.2) has a unique solution $u \in L_\infty(0,T;\mathcal{H}_n^k)$ for arbitrary $T > 0$.*

The next step in our development is to obtain combined spatial and temporal decay results, and thereby bounds on weighted norms of solutions that are independent of T.

Lemma 4.4. *Let there be given initial data $f \in \mathcal{H}_1^2(\mathbb{R})$ and suppose $p \ge 1$. Let u be the solution of (1.2) associated to f. Then $|u_{xt}(\cdot,t)|_{2,1}$, $|u_{xx}(\cdot,t)|_{2,1} \in L_2(\mathbb{R}^+)$, and $|u_x(\cdot,t)|_{2,1}$, $|u(\cdot,t)|_{4,1} \in L_\infty(\mathbb{R}^+)$.*

Proof. In light of Lemma 4.1, It is only necessary to obtain bounds on spatial norms of $u(\cdot,t)$ for large values of t. First note that

$$\Big|\int_{-\infty}^{\infty} 2|x|u_{xx}u^pu_xdx\Big| \le \nu|u_{xx}(\cdot,t)|_{2,1}^2 + C|u(\cdot,\tau)u_x(\cdot,\tau)|_{2,1}^2|u(\cdot,\tau)|_\infty^{2(p-1)}.$$

Preceding as in the derivation of (4.3) except integrating over $\mathbb{R} \times [T,t]$, and using the last estimate gives the inequalities

$$\begin{aligned} &|u_x(\cdot,t)|_{2,1}^2 + |u_{xx}(\cdot,t)|_{2,1}^2 + \nu \int_T^t |u_{xx}(\cdot,\tau)|_{2,1}^2 d\tau \\ &\leq C||u(\cdot,T)||_{\mathcal{H}_1^2(\mathbb{R})}^2 + C\int_T^t |u(\cdot,\tau)u_x(\cdot,\tau)|_{2,1}^2 |u(\cdot,\tau)|_\infty^{2(p-1)} d\tau \\ &\leq C(||f||_{\mathcal{H}_1^2(\mathbb{R})}, T) + C\int_T^t |u(\cdot,\tau)u_x(\cdot,\tau)|_{2,1}^2 |u(\cdot,\tau)|_\infty^{2(p-1)} d\tau. \end{aligned} \tag{4.4}$$

Next, multiply (1.2) by $|x|u^3(x,t)$ and integrate the result over $\mathbb{R} \times [T,t]$. After integration by parts and using Lemma 4.1, one has

$$\begin{aligned} &\frac{1}{4}|u(\cdot,t)|_{4,1}^4 + 3\nu \int_T^t |u(\cdot,\tau)u_x(\cdot,\tau)|_{2,1}^2 d\tau = \frac{1}{4}|u(\cdot,T)|_{4,1}^4 + \frac{\nu}{2}\int_T^t u^4(0,\tau)d\tau \\ &+ \int_T^t \int_{-\infty}^\infty \{\operatorname{sgn}(x)(\frac{1}{4}u^4 - u^3 u_{xt} + \frac{1}{p+4}u^{p+4}) - 3|x|u^2 u_x u_{xt}\} dx d\tau. \end{aligned} \tag{4.5}$$

Because of the temporal decay results in Theorem 2.3, it is adduced that

$$\int_T^t |u(\cdot,\tau)|_\infty^2 |u(\cdot,\tau)|_2^2 d\tau \leq \int_T^t C\tau^{-3/2} d\tau \leq C(T)$$

and

$$\int_T^t |u(0,\tau)|^4 d\tau \leq \int_T^t |u(\cdot,\tau)|_\infty^4 d\tau \leq \int_T^t C\tau^{-3/2} d\tau \leq C(T).$$

Lemma 4.2, Theorem 2.3 and the above estimates combine to show that all terms on the right-hand side of (4.5) except the last one can be bounded above in a simple way:

$$\begin{aligned} &\frac{1}{4}|u(\cdot,T)|_{4,1}^4 + \frac{\nu}{2}\int_T^t u^4(0,\tau)d\tau + \Big|\int_T^t \int_{-\infty}^\infty \operatorname{sgn}(x)(\frac{1}{4}u^4 - u^3 u_{xt} + \frac{1}{p+4}u^{p+4}) dx d\tau\Big| \\ &\leq C(|f|_{2,1}^2, T) + \int_T^t C\Big[|u(\cdot,\tau)|_\infty^2 |u(\cdot,\tau)|_2^2 + |u(\cdot,\tau)|_\infty^{p+2} |u(\cdot,\tau)|_2^2 \\ &\quad + |u_{xt}(\cdot,\tau)|_2 |u(\cdot,\tau)|_\infty^2 |u(\cdot,\tau)|_2\Big] d\tau \\ &\leq C(||f||_{\mathcal{H}_1^2(\mathbb{R})}, T). \end{aligned} \tag{4.6}$$

By Young's inequality, the last term in (4.5) can be bounded as follows:

$$\Big|\int_T^t \int_{-\infty}^\infty 3|x|u^2 u_x u_{xt} dx d\tau\Big| \leq \int_T^t \Big[\frac{3\nu}{2}|u(\cdot,\tau)u_x(\cdot,\tau)|_{2,1}^2 + C^*(\nu)|u(\cdot,\tau)|_\infty^2 |u_{xt}(\cdot,\tau)|_{2,1}^2\Big] d\tau \tag{4.7}$$

The inequalities (4.5), (4.6) and (4.7) lead to

$$\begin{aligned} &\frac{1}{4}|u(\cdot,t)|_{4,1}^4 + \frac{3\nu}{2}\int_T^t |u(\cdot,\tau)u_x(\cdot,\tau)|_{2,1}^2 d\tau \\ &\qquad \leq C(||f||_{\mathcal{H}_1^2(\mathbb{R})}, T) + \int_T^t C^*(\nu)|u(\cdot,\tau)|_\infty^2 |u_{xt}(\cdot,\tau)|_{2,1}^2 d\tau. \end{aligned} \tag{4.8}$$

Finally, multiply (1.2) by the combination $|x|(u_t + u_x + \frac{1}{\nu}u_{xt})$ and integrate the result over $\mathbb{R} \times [T,t]$. After simplification, there appears the inequality

$$\begin{aligned}
&(\frac{\nu}{2} + \frac{1}{2\nu})|u_x(\cdot,t)|_{2,1}^2 + \int_T^t \Big[|u_{xt}(\cdot,\tau)|_{2,1}^2 + |u_t(\cdot,\tau) + u_x(\cdot,\tau)|_{2,1}^2\Big] d\tau \\
&= (\frac{\nu}{2} + \frac{1}{2\nu})|u_x(\cdot,T)|_{2,1}^2 + \int_T^t \frac{1}{2}u_t^2(0,\tau)d\tau + \frac{1}{2}\int_{-\infty}^{\infty} \operatorname{sgn}(x)\big[u_x^2(x,T) - u_x^2(x,t)\big]dx \\
&+ \int_T^t \Big[\int_{-\infty}^{\infty} \Big(\operatorname{sgn}(x)\big[\frac{1}{2\nu}u_t^2 - \frac{1}{2\nu}u_{xt}^2 - \frac{\nu}{2}u_x^2 - \nu u_t u_x\big] \\
&- |x|\big[(u_t + u_x + \frac{1}{\nu}u_{xt})u^p u_x\big]\Big)dx\Big] d\tau \\
&\le C(\|f\|_{\mathcal{H}_1^1(\mathbb{R})}, T) + \int_T^t \Big[\frac{1}{2}|u_t(\cdot,\tau) + u_x(\cdot,\tau)|_{2,1}^2 + \frac{1}{2}|u_{xt}(\cdot,\tau)|_{2,1}^2 \\
&+ (\frac{1}{2} + \frac{1}{2\nu^2})|u(\cdot,\tau)|_\infty^{2(p-1)}|u(\cdot,\tau)u_x(\cdot,\tau)|_{2,1}^2 + C\big(|u_t(\cdot,\tau)|_2^2 \\
&+ |u_x(\cdot,\tau)|_2^2 + |u_{xt}(\cdot,\tau)|_2^2\big)\Big] d\tau,
\end{aligned} \tag{4.9}$$

where the elementary relation

$$|u_t(\cdot,t)|_\infty^2 \le |u_t(\cdot.t)|_2 |u_{xt}(\cdot,t)|_2,$$

has been used to estimate the boundary term.

Multiply (4.8) by a suitable constant b then add the result, (4.4) and (4.9) together to come to the inequality

$$\begin{aligned}
&(1 + \frac{\nu}{2} + \frac{1}{2\nu})|u_x(\cdot,t)|_{2,1}^2 + |u_{xx}(\cdot,t)|_{2,1}^2 + \frac{b}{4}|u(\cdot,t)|_{4,1}^4 + \int_T^t \Big[\frac{3b\nu}{2}|u(\cdot,\tau)u_x(\cdot,\tau)|_{2,1}^2 \\
&+ \nu|u_{xx}(\cdot,\tau)|_{2,1}^2 + |u_{xt}(\cdot,\tau)|_{2,1}^2 + |u_t(\cdot,\tau) + u_x(\cdot,\tau)|_{2,1}^2\Big] d\tau \\
&\le C(\|f\|_{\mathcal{H}_1^2(\mathbb{R})}, T) + \int_T^t \Big[C(\nu)|u(\cdot,\tau)|_\infty^{2(p-1)}|u(\cdot,\tau)u_x(\cdot,\tau)|_{2,1}^2 \\
&+ bC^*(\nu)|u(\cdot,\tau)|_\infty^2 |u_{xt}(\cdot,\tau)|_{2,1}^2\Big] d\tau.
\end{aligned} \tag{4.10}$$

Since $|u(\cdot,t)|_\infty$ is bounded and $|u(\cdot,t)|_\infty \to 0$ as $t \to +\infty$, it is possible to choose b large enough so that $\frac{b\nu}{2} \ge C(\nu)|u(\cdot,t)|_\infty^{2(p-1)}$. Next, choose T large enough that $2C^*(\nu)b|u(\cdot,t)|_\infty^2 \le 1$. With these choices of b and T, (4.10) gives the bound

$$\begin{aligned}
&(1 + \frac{\nu}{2} + \frac{1}{2\nu})|u_x(\cdot,t)|_{2,1}^2 + |u_{xx}(\cdot,t)|_{2,1}^2 + \frac{b}{4}|u(\cdot,t)|_{4,1}^4 + \int_T^t \Big[b\nu|u(\cdot,\tau)u_x(\cdot,\tau)|_{2,1}^2 \\
&\qquad + \nu|u_{xx}(\cdot,\tau)|_{2,1}^2 + \frac{1}{2}|u_{xt}(\cdot,\tau)|_{2,1}^2 + |u_t(\cdot,\tau) + u_x(\cdot,\tau)|_{2,1}^2\Big] d\tau \\
&\qquad \le C(\|f\|_{\mathcal{H}_1^2(\mathbb{R})}, T),
\end{aligned} \tag{4.11}$$

valid for any $t > T$. The lemma is thereby proved. □

The last result in this section connects spatial decay as $x \to \pm\infty$ to temporal decay as $t \to +\infty$. Spatial decay can be linked to whether or not a solution u lies in $L_2(\mathbb{R} \times \mathbb{R}^+)$. This in turn may be related to temporal decay via Corollary 3.16.

Lemma 4.5. *Let the initial data $f \in \mathcal{H}_1^2(\mathbb{R})$ be such that $|\hat{f}(y)| \leq C|y|^\alpha$, for small values of y, where $\alpha > \frac{1}{2}$ and C is a positive constant. Then the solution $u(x,t)$ of (1.2) with $p \geq 1$ corresponding to f lies in $C_b(\mathbb{R}^+; \mathcal{H}_1^2(\mathbb{R}))$ and $u_x \in L_2(\mathbb{R}^+; \mathcal{H}_1^1(\mathbb{R}))$.*

Proof. First note that because of Theorem 2.3 and Corollary 3.16, u_x, u_{xt} and u are in the space $L_2(\mathbb{R} \times \mathbb{R}^+)$. In consequence, (4.2) can be rewritten as

$$\begin{aligned}
&|u(\cdot,t)|_{2,1}^2 + |u_x(\cdot,t)|_{2,1}^2 + 2\nu \int_0^t |u_x(\cdot,\tau)|_{2,1}^2 d\tau = |f|_{2,1}^2 + |f'|_{2,1}^2 \\
&+ \int_0^t \int_{-\infty}^{\infty} \operatorname{sgn}(x)\big[u^2 + \frac{2}{p+2}u^{p+2} - 2uu_{xt} - 2\nu u u_x\big] dx d\tau \\
&\leq ||f||_{\mathcal{H}_1^1(\mathbb{R})}^2 + C\int_0^t \big[|u(\cdot,\tau)|_2^2 + |u_x(\cdot,\tau)|_2^2 + |u_{xt}(\cdot,\tau)|_2^2\big] d\tau \leq C(||f||_{\mathcal{H}_1^1(\mathbb{R})}^2),
\end{aligned} \tag{4.12}$$

and (4.3) can be estimated as

$$\begin{aligned}
&|u_x(\cdot,t)|_{2,1}^2 + |u_{xx}(\cdot,t)|_{2,1}^2 + 2\nu \int_0^t |u_{xx}(\cdot,\tau)|_{2,1}^2 d\tau = |f'|_{2,1}^2 + |f''|_{2,1}^2 \\
&- \int_0^t \int_{-\infty}^{\infty} \big[\operatorname{sgn}(x)(2u_x u_t + u_x^2) - 2|x| u_{xx} u^p u_x\big] dx d\tau \\
&\leq C(||f||_{\mathcal{H}_1^2(\mathbb{R})}) + C\int_0^t \big[|u_{xx}(\cdot,\tau)|_{2,1} |u_x(\cdot,\tau)|_{2,1} |u(\cdot,\tau)|_\infty^p\big] d\tau \\
&\leq C(||f||_{\mathcal{H}_1^2(\mathbb{R})}) + \int_0^t \big[\nu |u_{xx}(\cdot,\tau)|_{2,1}^2 + C|u_x(\cdot,\tau)|_{2,1}^2 |u(\cdot,\tau)|_\infty^{2p}\big] d\tau.
\end{aligned} \tag{4.13}$$

Using (4.12) in (4.13) shows that

$$|u_x(\cdot,t)|_{2,1}^2 + |u_{xx}(\cdot,t)|_{2,1}^2 + \nu \int_0^t |u_{xx}(\cdot,\tau)|_{2,1}^2 d\tau \leq C(||f||_{\mathcal{H}_1^2(\mathbb{R})}^2), \tag{4.14}$$

for any $t \geq 0$. Inequalities (4.12) and (4.14) give the required results, and the lemma is proved. □

5. Decay Results for GKdV-B Equation. In [2] and [6] it was shown that there are many similarities with regard to decay between solutions of the GKdV-Burgers equation and solutions of the GRLW-Burgers equation. The present section is devoted to deriving results analogous to those appearing in Section 3 and 4 for the GKdV-Burgers equation (1.1). Because the theory closely resembles that for (1.2), we content ourselves with a sketch that emphasizes the points of departure from the development for (1.2).

The solution u for (1.1) has the form

$$\begin{aligned}
u(x,t) =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\phi}_0(y,-t) e^{ixy} \hat{f}(y) dy \\
&- \frac{i}{\sqrt{2\pi}(p+1)} \int_0^t \int_{-\infty}^{\infty} e^{ixy} \hat{\phi}_1(y,\tau - t) \widehat{u^{p+1}}(y,\tau) dy d\tau,
\end{aligned} \tag{5.1}$$

obtained by taking the Fourier transform of (1.1) with respect to the spatial variable x, where

$$\hat{\phi}_0(y,r) = \exp\Big((\nu y^2 + iy - iy^3)r\Big), \hat{\phi}_1(y,r) = y\exp\Big((\nu y^2 + iy - iy^3)r\Big). \quad (5.2)$$

The linearized GKdV-B equation has the same property, stated in Lemma 2.1, as the linearized GRLW-B equation. The function $R(x,t)$ appearing in (2.19) is replaced by $R(x,t) = W_x^2 - \frac{1}{2\nu}w_{xx}(x+t,t)$. One may then follow the earlier line of argument to obtain results for (1.1) corresponding to Corollary 2.2. The results proved in [2, 6] for (1.1) can be stated as Theorem 2.3 in Section 2, but there are two main differences. One is that there are certain restrictions on the nonlinear term and initial data for a t-independent bound on the $H^1(\mathbb{R})$-norm of a solution of (1.1). One needs either that $p < 4$ or that $||f||_1$ is not too large in case $p \geq 4$ (see [5, 17]). The other is that the solution of (1.1) has the property $u_{xxx} \in L_2(\mathbb{R}^+)$ instead of $u_{xt} \in L_2(\mathbb{R}^+)$. Similarly, after applying the Cole-Hopf transformation to the linearized GKdV-B equation, the transformed initial data has the same decay as (2.32).

Following the development in Section 3, it is straightforward to derive Lemma 3.1 for (1.1) by using the representation (5.1). To obtain Lemma 3.3 for (1.1), note that the main inequality which the solution of (1.1) satisfies is obtained by multiplying (1.1) by $u_t + u_x - \frac{1}{\nu}u_{xx} + bu^{2p+1}$ and by u_{xxxx}, and then integrating over $\mathbb{R} \times [0,t]$. With these lemmas in hand, one is ready to establish analogues of Lemma 3.4 and Theorem 3.6 for (1.1). By using the Cole-Hopf transformation on (1.1) with $p = 1$, one can obtain the results for (1.1) corresponding to those for (1.2) in Lemma 3.8. Note that the function g in Lemma 3.8 is in this case equal to $-\frac{1}{2\nu}u_{xx}(x+t,t)v(x,t)$. Following the argument put forth in Lemma 3.9 and Theorem 3.10 for (1.2), one shows that the decay of solutions of (1.1) in L_2-norm has the form $t^{-\frac{1+2\alpha}{4}}$ for any $0 \leq \alpha \leq 1$. The decay of the difference between solutions of (1.1) and the corresponding linear equation, in L_2-norm, is the same in this case as in Corollary 3.12. There is a little difference in the proof of Corollary 3.14 for (1.1). Because of the difference in the kernels in the integral representation of solutions, it is useful to estimate the last two inequalities in (3.106) in the following way:

$$\begin{aligned} \int_{t^{\frac{1}{2}-\epsilon}}^t \big|\exp\big((-\nu y^2 - iy + iy^3)(t-\tau)\big)\big|_1 |u^p u_x(\cdot,\tau)|_1 d\tau &\leq \int_{t^{\frac{1}{2}-\epsilon}}^t \frac{C}{\sqrt{t-\tau}(1+\tau)^2} d\tau \\ &\leq \frac{C}{(1+t^{\frac{1}{2}-\epsilon})^{3/2}} \int_{t^{\frac{1}{2}-\epsilon}}^t \frac{1}{\sqrt{(t-\tau)(1+\tau)}} d\tau \\ &\leq Ct^{-\frac{3}{4}+\frac{3\epsilon}{2}} \int_0^1 \frac{1}{\sqrt{(1-\tau)\tau}} d\tau \leq Ct^{-\frac{3}{4}+\frac{3\epsilon}{2}}, \end{aligned} \quad (5.3)$$

because the analogue of (3.107) can be estimated as

$$\big|\exp\big((-\nu y^2 - iy + iy^3)(t-\tau)\big)\big|_1 \leq \frac{C}{\sqrt{t-\tau}}. \quad (5.4)$$

Hence if $\epsilon > 0$ is chosen smaller than that in (3.106) for equation (1.2) (for instance,

$0 < \epsilon < \frac{1}{14}$), one still obtains

$$
\begin{aligned}
&\lim_{t\to+\infty} t^{\frac{3}{2}}\Big|\int_{t^{\frac{1}{2}-\epsilon}}^{t} \exp\big((-\nu y^2 - iy + iy^3)(t-\tau)\big)\widehat{u^p u_x}(y,\tau)d\tau\Big|_2^2 \\
&\le \lim_{t\to+\infty} Ct^{\frac{3}{2}}t^{2\epsilon-1}\Big|\int_{t^{\frac{1}{2}-\epsilon}}^{t} \exp\big(-\nu y^2 - iy + iy^3)(t-\tau)\big)\widehat{u^p u_x}(y,\tau)d\tau\Big|_1 \\
&\le \lim_{t\to+\infty} Ct^{\frac{1}{2}+2\epsilon}\int_{t^{\frac{1}{2}-\epsilon}}^{t} \big|\exp\big((-\nu y^2 - iy + iy^3)(t-\tau)\big)\big|_1 \big|u^p u_x(\cdot,\tau)\big|_1 d\tau \\
&\le \lim_{t\to+\infty} Ct^{\frac{1}{2}+2\epsilon}t^{\frac{3\epsilon}{2}-\frac{3}{4}} = 0,
\end{aligned}
\tag{5.5}
$$

so that (3.106) is valid for solutions of equation (1.1). The other estimates follow the line set forth in Corollary 3.14, but replacing $\exp\big(-\frac{\nu y^2+iy}{1+y^2}(t-\tau)\big)/1+y^2$ with $\exp\big(-(\nu y^2 + iy - iy^3)(t-\tau)\big)$.

Lemma 4.1, Lemma 4.2 and Corollary 4.3 are the same for (1.1) as for (1.2). As mentioned above, either p is restricted to be less than 4 for equation (1.1), or else the initial data must be suitably small in H^1-norm. A new version of Lemma 4.4 is that $|u_{xx}(\cdot,t)|_{2,1}$, $|u_{xxx}(\cdot,t)|_{2,1} \in L_2(\mathbb{R}^+)$ and $|u_x(\cdot,t)|_{2,1}$, $|u(\cdot,t)|_{4,1} \in L_\infty(\mathbb{R}^+)$. The important inequality (4.9) is obtained by multiplying (1.1) by $|x|(u_t + u_x - \frac{1}{\nu}u_{xx})$. After integrating over $\mathbb{R}$ and simplifying the result, one has

$$
\begin{aligned}
&(\frac{\nu}{2} + \frac{1}{2\nu})\frac{d}{dt}|u_x(\cdot,t)|_{2,1}^2 + \frac{1}{2}|u_t(\cdot,t) + u_x(\cdot,t)|_{2,1}^2 + \frac{1}{2}|u_{xx}(\cdot,t)|_{2,1}^2 \\
&\qquad \le C_1(\nu)\big(|u_{xxx}(\cdot,t)|_{2,1}^2 + |u(\cdot,t)|_\infty^{2(p-1)}|u(\cdot,\tau)u_x(\cdot,\tau)|_{2,1}^2\big) \\
&\qquad\quad + C(|u_t(\cdot,t)|_2^2 + |u_x(\cdot,t)|_2^2 + |u_{xx}(\cdot,t)|_2^2).
\end{aligned}
\tag{5.6}
$$

Multiply (1.1) by $|x|u^3$ and then integrate the result over $\mathbb{R}$. After integration by parts and simplifying, there obtains

$$
\begin{aligned}
&\frac{1}{4}\frac{d}{dt}|u(\cdot,t)|_{4,1}^4 + \frac{3\nu}{2}|u(\cdot,t)u_x(\cdot,t)|_{2,1}^2 \\
&\le C_2(\nu)|u(\cdot,t)|_\infty^2|u_{xx}(\cdot,t)|_{2,1}^2 \quad + C(|u(\cdot,t)|_4^4 + |u_x(\cdot,t)|_2^2 + |u_{xx}(\cdot,t)|_2^2).
\end{aligned}
\tag{5.7}
$$

To make progress, another inequality is needed to control $|u_{xxx}(\cdot,t)|_{2,1}$. Differentiate (1.1) once and multiply the result by $2|x|u_{xxx}$. Then integrate the result over $\mathbb{R}$. After integration by parts, one comes to

$$
\begin{aligned}
&\frac{d}{dt}|u_{xx}(\cdot,t)|_{2,1}^2 + 2\nu|u_{xxx}(\cdot,t)|_{2,1}^2 \\
&\quad = \int_{-\infty}^{\infty}\Big[2|x|u_{xxx}(u^p u_{xx} + pu^{p-1}u_x^2) - \mathrm{sgn}(x)\big(2u_{xx}u_{xt} + u_{xx}^2 + u_{xxx}^2\big)\Big]dx \\
&\quad \le \nu|u_{xxx}(\cdot,t)|_{2,1}^2 + C_3(\nu)|u(\cdot,t)|_\infty^{2p}|u_{xx}(\cdot,t)|_{2,1}^2 \\
&\quad\quad + C_3(\nu)|u(\cdot,t)|_\infty^{2(p-1)}|u_x(\cdot,t)|_\infty^2|u_x(\cdot,t)|_{2,1}^2 \\
&\quad\quad + C(|u_{xx}(\cdot,t)|_2^2 + |u_{xxx}(\cdot,t)|_2^2 + |u_{xt}(\cdot,t)|_2^2).
\end{aligned}
\tag{5.8}
$$

Multiply (5.7) and (5.8) by a suitable constant b, then add the results and (5.6) together to yield

$$\begin{aligned}
&\frac{d}{dt}\Big[(\frac{\nu}{2}+\frac{1}{2\nu})|u_x(\cdot,t)|_{2,1}^2+\frac{b}{4}|u(\cdot,t)|_{4,1}^4+b|u_{xx}(\cdot,t)|_{2,1}^2\Big]\\
&+\frac{1}{2}|u_{xx}(\cdot,t)|_{2,1}^2+\frac{3b\nu}{2}|u(\cdot,t)u_x(\cdot,t)|_{2,1}^2+b\nu|u_{xxx}(\cdot,t)|_{2,1}^2\\
&\qquad\le C_1(\nu)|u_{xxx}(\cdot,t)|_{2,1}^2+C_1(\nu)|u(\cdot,t)|_\infty^{2(p-1)}|u(\cdot,\tau)u_x(\cdot,\tau)|_{2,1}^2\\
&\qquad\quad+b\Big[C_2(\nu)+C_3(\nu)|u(\cdot,t)|_\infty^{2(p-1)}\Big]|u(\cdot,t)|_\infty^2|u_{xx}(\cdot,t)|_{2,1}^2\\
&\qquad\quad+C_3(\nu)b|u(\cdot,t)|_\infty^{2(p-1)}|u_x(\cdot,t)|_\infty^2|u_x(\cdot,t)|_{2,1}^2\\
&\qquad\quad+C\Big(|u_t(\cdot,t)|_2^2+|u_x(\cdot,t)|_2^2+|u_{xx}(\cdot,t)|_2^2+|u_{xxx}(\cdot,t)|_2^2+|u_{xt}(\cdot,t)|_2^2\Big).
\end{aligned}\tag{5.9}$$

Choose b large enough that

$$b\nu\ge 1+C_1(\nu)\qquad\text{and}\qquad\frac{b\nu}{2}\ge C_1(\nu)|u(\cdot,t)|_\infty^{2(p-1)}.$$

Then for this fixed b, choose T large enough that for $t\ge T$,

$$b\Big[C_2(\nu)+C_3(\nu)|u(\cdot,t)|_\infty^{2(p-1)}\Big]|u(\cdot,t)|_\infty^2\le\frac{1}{4}.$$

With such values of b and T, (5.9) implies

$$\begin{aligned}
&\frac{d}{dt}\Big[(\frac{\nu}{2}+\frac{1}{2\nu})|u_x(\cdot,t)|_{2,1}^2+\frac{b}{4}|u(\cdot,t)|_{4,1}^4+b|u_{xx}(\cdot,t)|_{2,1}^2\Big]\\
&\qquad+\frac{1}{4}|u_{xx}(\cdot,t)|_{2,1}^2+b\nu|u(\cdot,t)u_x(\cdot,t)|_{2,1}^2+|u_{xxx}(\cdot,t)|_{2,1}^2\\
&\qquad\le C_3(\nu)b|u(\cdot,t)|_\infty^{2(p-1)}|u_x(\cdot,t)|_\infty^2|u_x(\cdot,t)|_{2,1}^2\\
&\qquad\quad+C\Big(|u_t(\cdot,t)|_2^2+|u_x(\cdot,t)|_2^2+|u_{xx}(\cdot,t)|_2^2+|u_{xxx}(\cdot,t)|_2^2+|u_{xt}(\cdot,t)|_2^2\Big),
\end{aligned}\tag{5.10}$$

for $t>T$. Note that

$$|u_x(\cdot,t)|_\infty^2\le|u_x(\cdot,t)|_2|u_{xx}(\cdot,t)|_2\le\frac{1}{2}\big[|u_x(\cdot,t)|_2^2+|u_{xx}(\cdot,t)|_2^2\big],\tag{5.11}$$

and since $|u_x(\cdot,t)|_2^2$ and $|u_{xx}(\cdot,t)|_2^2$ lie in $L_1(\mathbb{R}^+)$, it follows that $|u_x(\cdot,t)|_\infty^2$ is in $L_1(\mathbb{R}^+)$. Further, note that $|u_t(\cdot,t)|_2^2$, $|u_{xxx}(\cdot,t)|_2^2$ and $|u_{xt}(\cdot,t)|_2^2$ lie in $L_1(\mathbb{R}^+)$. With these facts in hand, an application of Gronwall's lemma to (5.10) gives

$$\begin{aligned}
&\Big[(\frac{\nu}{2}+\frac{1}{2\nu})|u_x(\cdot,t)|_{2,1}^2+\frac{b}{4}|u(\cdot,t)|_{4,1}^4+b|u_{xx}(\cdot,t)|_{2,1}^2\Big]\\
&\qquad+\int_T^t\Big[\frac{1}{4}|u_{xx}(\cdot,\tau)|_{2,1}^2+b\nu|u(\cdot,t)u_x(\cdot,\tau)|_{2,1}^2+|u_{xxx}(\cdot,\tau)|_{2,1}^2\Big]d\tau\\
&\le C(\|f\|_{\mathcal{H}_1^2(\mathbb{R})},T)\exp\Big(C\int_T^t|u_x(\cdot,\tau)|_\infty^2d\tau\Big)\le C(\|f\|_{\mathcal{H}_1^2(\mathbb{R})},T),
\end{aligned}\tag{5.12}$$

for $t>T$.

If the initial data f satisfies the condition in Lemma 4.5, then the solution u of (1.1) is in $C_b(\mathbb{R}^+;\mathcal{H}_1^2(\mathbb{R}))\cap L_2(\mathbb{R}\times\mathbb{R}^+)$ and $u_x\in L_2(\mathbb{R}^+;\mathcal{H}_1^2(\mathbb{R}))$. The preceding discussion of (1.1) is summarized in the following theorem.

Theorem 5.1. *Let $f \in H^2(\mathbb{R}) \cap W_1^2(\mathbb{R})$ and suppose $p \geq 1$ and $\nu > 0$. Further suppose that if $p \geq 4$, the initial data f is suitably small in H^1-norm. If w is the solution of the evolution equation obtained from (1.1) by dropping the nonlinear term but maintaining the same initial data and u is the solution of (1.1)-(1.3), then $|u(\cdot,t)|_2$ and $|w(\cdot,t)|_2$ both decay like $t^{-\frac{1}{4}}$. If $p \geq 2$, then*

$$\lim_{t\to+\infty} t^{l+\frac{1}{2}}|\partial_x^l u(\cdot,t)|_2^2 = \lim_{t\to+\infty} t^{l+\frac{1}{2}}|\partial_x^l w(\cdot,t)|_2^2 = \frac{1}{(8\nu\pi)^{\frac{1}{2}}(4\nu)^l}\Big(\int_{-\infty}^{\infty} f(x)dx\Big)^2 \tag{5.13}$$

for $l = 0, 1$.

Moreover, if $\int_{-\infty}^{\infty} f(x) = 0$, and the Fourier transform $\hat{f}$ of f satisfies the inequality

$$\big|\hat{f}(y)\big| \leq C|y|^\alpha, \tag{5.14}$$

for small y, where $0 \leq \alpha \leq 1$ and C is a positive constant, then the solution u of (1.1) corresponding to f has the properties

$$|u(\cdot,t)|_2 \leq C(1+t)^{-\frac{1+2\alpha}{4}} \qquad \text{and} \qquad |u_x(\cdot,t)|_2 \leq C(1+t)^{-\frac{3+2\alpha}{4}}, \tag{5.15}$$

for $t \geq 0$. If $0 < \alpha < 1$, there exist some non-negative constants C_f^l which depend only on the initial data f such that

$$\lim_{t\to+\infty} t^{\frac{1+2(l+\alpha)}{2}}|\partial_x^l u(\cdot,t)|_2^2 = \lim_{t\to+\infty} t^{\frac{1+2(l+\alpha)}{2}}|\partial_x^l w(\cdot,t)|_2^2 = C_f^l, \tag{5.16}$$

for $l = 0, 1$. In addition, if $\hat{f}(y) = y^\alpha \hat{g}(y)$ for some $g \in L_1(\mathbb{R})$, then

$$\lim_{t\to+\infty} t^{\frac{1+2(l+\alpha)}{2}}|\partial_x^l u(\cdot,t)|_2^2 = \lim_{t\to+\infty} t^{\frac{1+2(l+\alpha)}{2}}|\partial_x^l w(\cdot,t)|_2^2 = \frac{\Gamma(\alpha+l+\frac{1}{2})}{2\pi(2\nu)^{\alpha+l+\frac{1}{2}}}\Big(\int_{-\infty}^{\infty} g(x)dx\Big)^2. \tag{5.17}$$

If $\alpha = 1$, then

$$\lim_{t\to+\infty} t^{\frac{3}{2}}|u(\cdot,t) - w(\cdot,t)|_2^2 = \frac{1}{4\nu(8\nu\pi)^{\frac{1}{2}}(p+1)^2}\Big(\int_0^\infty\int_{-\infty}^{\infty} u^{p+1}dxdt\Big)^2. \tag{5.18}$$

If $\hat{f}(y) = iy\hat{g}(y)$ for some $g \in L_1(\mathbb{R})$, then

$$\lim_{t\to+\infty} t^{\frac{3}{2}}|u(\cdot,t)|_2^2 = \frac{1}{4\nu(8\nu\pi)^{\frac{1}{2}}}\Big(\int_{-\infty}^{\infty} g(x)dx - \int_0^\infty\int_{-\infty}^{\infty}\frac{u^{p+1}(x,t)}{1+p}dxdt\Big)^2. \tag{5.19}$$

In particular, if $xf(x) \in L_1(\mathbb{R})$ and $\frac{d}{dx}g(x) = f(x)$ with $xg(x) \to 0$ as $x \to \pm\infty$, one has

$$\lim_{t\to+\infty} t^{\frac{3}{2}}|u(\cdot,t)|_2^2 = \frac{1}{4\nu(8\nu\pi)^{\frac{1}{2}}}\Big(\int_{-\infty}^{\infty} xf(x)dx + \int_0^\infty\int_{-\infty}^{\infty}\frac{u^{p+1}(x,t)}{p+1}dxdt\Big)^2. \tag{5.20}$$

If $f \in W_1^2(\mathbb{R}) \cap \mathcal{H}_n^2(\mathbb{R})$, $p \geq 1$ and $\nu > 0$, then for any $T > 0$, equation (1.1) has a unique solution $u \in L_\infty(0,T;\mathcal{H}_n^2(\mathbb{R}))$ corresponding to the initial data f. Furthermore, the solution u has the properties that $\|u_{xx}(\cdot,t)\|_{\mathcal{L}_2^1(\mathbb{R})}$ and $\|u_{xxx}(\cdot,t)\|_{\mathcal{L}_2^1(\mathbb{R})}$ lie in $L_2(\mathbb{R}^+)$. Specifically, if initial data f satisfies (5.14) and $\alpha > \frac{1}{2}$, then $u \in L_2(\mathbb{R}\times\mathbb{R}^+) \cap C_b(\mathbb{R}^+;\mathcal{H}_1^2(\mathbb{R}))$, and $u_x \in L_2(\mathbb{R}^+;\mathcal{H}_1^2(\mathbb{R}))$.

6. Conclusion. We have studied the long-time temporal and the spatial decay of solutions of the initial-value problem for the KdV-B equation and the RLW-B equation (which are (1.1) and (1.2) with $p = 1$, respectively) as well as the same properties of their generalized versions with $p > 1$. For the two ranges of nonlinearity ($p = 1$ and $p > 1$) the study is carried out in different ways. The value of p together with the value of α specifying at what rate, if any, the Fourier transform of the initial data vanishes at the origin are the two aspects that determine the long-time asymptotics of solutions. Our results, combined with the earlier theory in [2] ($p = 1$ and $\alpha = 0$), provide explicit decay structures of solutions of these equations.

It has been shown that for $0 \leq \alpha \leq 1$, the L_2-norm of solutions of these equations decays like $t^{-\frac{1}{4}-\frac{\alpha}{2}}$ as $t \to +\infty$ if the Fourier transform of the initial data vanishes at the origin at order α. The decay of the difference between solutions of the nonlinear equation and the corresponding linear equation in L_2-norm is $t^{-(\frac{1+2\alpha+2\delta}{4})}$ for an appropriately defined δ. This latter result implies that when either $p > 1$ or $\alpha > 0$, the decay behavior of solutions of (1.1) and (1.2) is the same as that of the corresponding linear equations in that $\delta > 0$. Moreover, if the Fourier transform of the initial data vanishes at the origin at order $\alpha > \frac{1}{2}$ and $p \geq 1$, the corresponding solution lies in the space $L_2(\mathbb{R}\times\mathbb{R}^+)$. In consequence of this fact, weighted H^2-norms (with the weight $(1+x^2)^{\frac{1}{4}}$) of solutions of these equations are bounded uniformly in t. This interaction between spatial and temporal decay seems interesting.

When $\alpha = 1$ so the Fourier transform of initial data f has the form $|\hat{f}(y)| = C|y|$ at the origin, the L_2-norm of solutions of the nonlinear equations has precisely the asymptotic form $C_N t^{-\frac{3}{4}}$. The constant C_N, which is obtained explicitly, depends on the first moment of the initial data about origin and on the double integral of $u^{p+1}(x,t)$ over $\mathbb{R}\times\mathbb{R}^+$. It is also shown that the decay of the difference between the solution of (1.1) or (1.2) and the corresponding linear equation, in L_2-norm, has the form $C_N t^{-\frac{3}{4}}$. This result makes clear that although the L_2-norms of solutions u of the nonlinear equations (1.1) or (1.2) and the solutions w of the corresponding linear equations both decay like $t^{-\frac{3}{4}}$, the limits

$$\lim_{t\to+\infty} t^{\frac{3}{4}}|u(\cdot,t)|_2 \quad \text{and} \quad \lim_{t\to+\infty} t^{\frac{3}{4}}|w(\cdot,t)|_2$$

are different when $\alpha = 1$.

It is thus concluded that nonlinear effects appear in the lowest order temporal asymptotics of solutions of (1.1) and (1.2) when $p = 1$ and, if $p > 1$ when $\alpha \geq 1$. Otherwise, the combined effect of dissipation and dispersion carry the clay at lowest order and it is only in the higher-order aspects of the decay that nonlinearity is felt.

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