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A Generalized Korteweg-de Vries Equation in a Quarter Plane

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Abstract. An initial- and boundary-value problem for the nonlinear wave equation

$$u_t + P(u)_x + u_{xxx} = 0 \qquad (*)$$

is considered in the quarter plane $\{(x,t) : x \geq 0,\ t \geq 0\}$ with initial data and boundary data specified at $t = 0$ and on $x = 0$, respectively. Such problems arise in the modelling of open-channel flows where the waves are generated by a wavemaker mounted at one end of a flume, and in other situations where waves propagate into an undisturbed patch of the dispersive medium. Equation $(*)$, which is a generalized version of the classical Korteweg-de Vries equation, features a general form of nonlinearity in gradient form. With suitable restrictions on P and with conditions imposed on the initial data and boundary data which are quite reasonable with regard to potential applications, the aforementioned initial-boundary-value problem for $(*)$ is shown to be well posed.

Key words: Generalized Korteweg-de Vries equation; quarter-plane problem; initial-boundary-value problem; nonlinear, dispersive, wave equations.

1. Introduction

This paper is concerned with the initial- and boundary-value problem

$$u_t + P(u)_x + u_{xxx} = 0, \qquad \text{for } x,\ t \geq 0, \qquad (1.1a)$$

$$u(x,0) = f(x), \qquad \text{for } x \geq 0, \qquad (1.1b)$$

$$u(0,t) = g(t), \qquad \text{for } t \geq 0, \qquad (1.1c)$$

where $u = u(x,t)$ is real-valued function of the two real variables x and t, P is a real-valued function of a real variable and subscripts adorning a function connote partial differentiation. Equations like (1.1a) are mathematical models for the unidirectional

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propagation of small-amplitude long waves in nonlinear dispersive systems. In such applications, u is typically an amplitude or a velocity, x is often proportional to distance in the direction of propagation and t is proportional to elapsed time.

The well-developed theory for the pure initial-value problem for (1.1a) wherein $u(x,0)$ is specified on the entire real line with zero boundary conditions at $x = \pm\infty$ insures that there exists a unique smooth solution u corresponding to given, smooth initial data f, at least over some time interval $[0, T^*)$, where $T^* = T^*(f) > 0$ [**15, 16, 19, 24, 25, 28-31, 33, 39**]. If $P(u) = u^p$, for $p < 5$, for example, then T^* may be taken to be $+\infty$ because of certain *a priori* bounds that are available in this case [**29**]. However, the question of whether or not T^* can be taken to be $+\infty$ in case $p \geq 5$ is open. Numerical results [**6, 7, 8**] seem to indicate that for $p \geq 5$, solutions of (1.1a) corresponding to significant classes of smooth initial data form singularities in finite time.

The pure initial-value problem is often not practically convenient if one attempts to assess the performance of equations like (1.1a) as models for waves, or to use them predictively. There will usually be difficulty associated with determining the entire wave profile accurately at a given instant of time. Indeed, a much more common situation arises when some sort of wavemaker is used to generate waves at the edge of an undisturbed stretch of the medium in question, which then propagate into the medium. This corresponds to the special case of (1.1) in which $f \equiv 0$. Guided by experimental studies on water waves in channels see [**11, 26, 27, 44**], Bona and Winther [**17, 18**] considered the Korteweg-de Vries-equation (KdV-equation henceforth)

$$u_t + u_x + uu_x + u_{xxx} = 0 \tag{1.2}$$

with initial- and boundary-conditions implemented as in (1.1) and proved that such a quarter-plane problem is well posed (see also [**20, 22**] for theory involving nonlinearities P having the general form in (1.1), but still restricted to grow at most quadratically and with more restricted initial data). Earlier, Bona and Bryant [**4**] had studied the same quarter-plane problem for the regularized long-wave equation

$$u_t + u_x + uu_x - u_{xxt} = 0 \tag{1.3}$$

and proved it to be well posed. (This latter quarter-plane formulation of the wavemaker problem with appropriate dissipative terms appended was later used to test (1.3) against experimentally determined water-wave data in [**11**].)

For physical situations other than wave motion on the surface of a perfect fluid, simple models sometimes yield nonlinearities that are somewhat more complex that the quadratic one appearing in (1.2) or (1.3). Examples include internal wave motion and waves in crystalline lattices [**37, 38, 41**]. This fact gives impetus to the present generalization of the earlier theory.

In this paper, we study the quarter-plane problem (1.1) and show it possesses a unique, global classical solution which depends continuously on variations of the data f and g within their respective function classes. Of course the nonlinearity must be restricted for these results to obtain. Other than being smooth, the nonlinearity P will be required to satisfy a one-sided growth condition of the form $\Lambda(u) \le |u|^\rho$ for all large values of $|u|$ and suitable values of ρ, where $\frac{d\Lambda(u)}{du} = P(u)$. (It is worth note that in our companion paper **[9]** on the quarter-plane problem for a generalized version

$$u_t + u_x + P(u)_x - \nu u_{xx} - \alpha^2 u_{xxt} = 0 \tag{1.4}$$

of (1.3), a similar, but less restrictive condition appears on P. In this latter reference, initial- and two-point boundary-value problems for (1.4) are also studied. Such two-point boundary-value problems seem rather complicated for (1.1a) since they require the imposition of an extra boundary condition **[5, 10, 17]**.)

The paper is organized as follows. Section 2 outlines briefly the notation and terminology to be used subsequently and presents a statement of the principal result. In Section 3 the regularized problem

$$u_t + P(u)_x + u_{xxx} - \epsilon u_{xxt} = 0, \qquad \text{for } x,\, t \ge 0, \tag{1.5a}$$

$$u(x,0) = f(x), \qquad \text{for } x \ge 0, \tag{1.5b}$$

$$u(0,t) = g(t), \qquad \text{for } t \ge 0, \tag{1.5c}$$

is considered, and shown to admit a satisfactory theory when ϵ is fixed and positive. *A priori* adduced, ϵ-independent bounds for solutions of the regularized problem (1.5) are derived in Section 4 and Section 5. Passage to the limit as $\epsilon \downarrow 0$ in the weak-star topology is effected in Section 6, where smooth solutions of the initial- and boundary-value problem (1.1) are shown to exist. In Section 7, these solutions are shown to lie in more restricted spaces and to depend continuously on the initial- and boundary-conditions.

2. Notation and Statement of the Main Results

We begin with a review of terminology and notation. For an arbitrary Banach space X, the associated norm will be denoted $||\cdot||_X$. If $\Omega = (a,b)$ is a bounded open interval in $\mathbb{R}^+ = (0,+\infty)$ and k a non-negative integer, we denote by $C^k(\bar{\Omega}) = C^k(a,b)$ the functions that, along with their first k derivatives, are continuous on $[a,b]$ with the norm

$$||f||_{C^k(\bar{\Omega})} = \sup_{\substack{x\in\bar{\Omega} \\ 0\le j\le k}} |f^{(j)}(x)|. \tag{2.1}$$

If Ω is an unbounded interval, $C_b^k(\bar{\Omega})$ is defined just as when Ω is bounded except that f, f', $\cdots$, $f^{(k)}$ are required to be bounded as well as continuous on $\bar{\Omega}$. The norm is defined as in (2.1). Similar definitions apply if Ω is an open set in $\mathbb{R}^N$. The space $C^\infty(\bar\Omega) = \cap_j C^j(\bar\Omega)$ will appear tangentially, but its Frechet-space topology will not be needed. $\mathcal{D}(\Omega)$ is the usual subspace of $C^\infty(\bar\Omega)$ consisting of functions with compact support in Ω. Its dual space $\mathcal{D}'(\Omega)$ is the space of Schwartz distributions on Ω. For $1 \le p < \infty$, $L_p(\Omega)$ connotes those functions f which are pth-power absolutely integrable on Ω with the usual modification in case $p = \infty$. If $s \ge 0$ is an integer and $1 \le p \le \infty$, let $W^{s,p}(\Omega)$ be the Sobolev space consisting of those $L_p(\Omega)$-functions whose first s generalized derivatives lie in $L_p(\Omega)$, with the usual norm,

$$\|f\|^p_{W^{s,p}(\Omega)} = \sum_{k=0}^{s} \|f^{(k)}\|^p_{L_p(\Omega)}.$$

If $p = 2$ we write $H^s(\Omega)$ for $W^{s,2}(\Omega)$. In the analysis of the quarter-plane problem, the spaces $H^s(\Omega)$ will occur often with s a positive integer and $\Omega = \mathbb{R}^+$ or $\Omega = (0,T)$. Because of their frequent occurrence, it is convenient to abbreviate their norms, thusly;

$$\|\cdot\|_s = \|\cdot\|_{H^s(\mathbb{R}^+)} \qquad \text{and} \qquad |\cdot|_{s,T} = \|\cdot\|_{H^s(0,T)}. \tag{2.2a}$$

If $s = 0$, the subscript s will be omitted altogether, so that

$$\|\cdot\| = \|\cdot\|_{L_2(\mathbb{R}^+)} \qquad \text{and} \qquad |\cdot|_T = \|\cdot\|_{0,T}. \tag{2.2b}$$

Similarly, $C_b^k(\mathbb{R}^+)$ appears frequently and will be denoted simply C_b^k. In case $k = 0$, we will systematically drop the superscript and so the class of bounded continuous functions on $\mathbb{R}^+$ is written C_b. The notation $H^\infty(\Omega) = \cap_j H^j(\Omega)$ will be used for the C^∞-functions on Ω, all of whose derivatives lie in $L_2(\Omega)$. For $s \ge 1$, $H_0^s(\mathbb{R}^+)$ is the closed linear subspace of $H^s(\mathbb{R}^+)$ of functions f such that $f(0) = f'(0) = \cdots = f^{(s-1)}(0) = 0$. $H^s_{loc}(\Omega)$ is the set of real-valued functions f defined on Ω such that, for each $\varphi \in \mathcal{D}(\Omega)$, $\varphi f \in H^s(\Omega)$. This space is equipped with the weakest topology such that all of the mappings $f \to \varphi f$, for $\varphi \in \mathcal{D}(\Omega)$, are continuous from H^s_{loc} into $H^s(\Omega)$. With this topology, $H^s_{loc}(\Omega)$ is a Fréchet space. If X is a Banach space, T a positive real number and $1 \le p \le +\infty$, denote by $L_p(0,T;X)$ the Banach space of all measurable functions $u: (0,T) \to X$, such that $t \to \|u(t)\|_X$ is in $L_p(0,T)$, with norm

$$\|u\|_{L_p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p dt \right)^{\frac{1}{p}}, \qquad \text{if } 1 \le p < +\infty,$$

and if $p = \infty$, then

$$\|u\|_{L_\infty(0,T;X)} = \underset{0<t<T}{\text{essential supremum}} \{\|u(t)\|_X\}.$$

Similarly, if k is a positive integer, then $C^k(0,T;X)$ denotes the space of all continuous functions $u: [0,T] \to X$, such that their derivatives up to the k^{th} order exist and are continuous. The space $L^\infty_{loc}(\bar{\mathbb{R}}^+; X)$ is the class of measurable maps

$u : \bar{\mathbb{R}}^+ \to X$ which are essentially bounded on any compact subset of $\bar{\mathbb{R}}^+$. The abbreviation $\mathcal{B}_T^{k,l}$ will be employed for the functions $u \colon \mathbb{R}^+ \times [0,T] \to \mathbb{R}$ such that $\partial_t^i \partial_x^j u \in C(0,T;C_b)$ for $0 \le j \le k$, and $0 \le i \le l$. This space of functions will carry the norm

$$\|u\|_{\mathcal{B}_T^{k,l}} = \sum_{\substack{0 \le j \le k \\ 0 \le i \le l}} \|\partial_t^i \partial_x^j u\|_{C(0,T;C_b)}.$$

The space $\mathcal{B}_T^{0,0}$ will be abbreviated simply $\mathcal{B}_T$, so that

$$\|u\|_{\mathcal{B}_T} = \sup_{0 \le x,\, 0 \le t \le T} \sup |u(x,t)|.$$

The next few sections are somewhat technical, and it seems useful to state at the outset a sample of one of our principal results so the reader may keep in mind the overall goal of the paper. Throughout the development of our theory, it will be assumed that the nonlinearity P appearing in the differential equation is at least locally Lipschitz. If B is a bounded subset of the real line, then $\gamma(B)$ will denote the Lipschitz constant for P on B, so that $\gamma(B)$ is the smallest number for which

$$|P(z_1) - P(z_2)| \le \gamma(B)|z_1 - z_2|, \qquad \text{for all} \qquad z_1, z_2 \in B. \tag{H1}$$

It will also be presumed that $P(0) = 0$, an assumption that entails no loss of generality since P appears differentiated in the equation.

Main Result. *Let there be given $T > 0$, initial data $f \in H^3(\mathbb{R}^+)$, boundary data $g \in H^2(0,T)$, and assume the compatibility condition $f(0) = g(0)$ to be satisfied. Suppose that in addition to being locally Lipschitz, P satisfies the one-sided growth condition*

$$\limsup_{|s| \to \infty} \frac{\Lambda(s)}{|s|^{10/3}} \le 0, \tag{**}$$

where $\frac{d\Lambda(s)}{ds} = P(s)$ and $\Lambda(0) = 0$. Then the initial-boundary-value problem (1.1) has a unique solution $u \in C(0,T;H^3(\mathbb{R}^+))$ which depends continuously on the auxiliary data f and g. If the auxiliary data (f,g) is further restricted by the requirement $f \in H^4(\mathbb{R}^+)$ and the compatibility conditions

$$f(0) = g(0) \qquad \text{and} \qquad g'(0) + P'(f(0))f'(0) + f'''(0) = 0$$

are satisfied, the solution u lies in $C(0,T;H^4(\mathbb{R}^+))$.

Remark. The appellation *solution* of (1.1) means a distributional solution of (1.1a) for which the auxiliary conditions (1.1b) and (1.1c) can be given a well-defined sense.

3. The Regularized Problem

In this section attention will be given to the following regularized initial- and boundary-value problem:

$$u_t + P(u)_x + u_{xxx} - \epsilon u_{xxt} = 0, \qquad \text{for } x,\, t \geq 0, \tag{3.1a}$$

$$u(x,0) = f(x), \qquad \text{for } x \geq 0, \tag{3.1b}$$

$$u(0,t) = g(t), \qquad \text{for } t \geq 0, \tag{3.1c}$$

with the compatibility condition $u(0,0) = f(0) = g(0)$. The positive parameter ϵ will be treated as fixed in this section. Following the development in [**17**], let

$$v(x,t) = \epsilon^{\frac{1}{p}} u(\epsilon^{\frac{1}{2}}(x-t), \epsilon^{\frac{3}{2}} t),$$

where p is a positive number to be specified later. If $P(u) = cu^{r+1}$, then $p = r$. The function u is a smooth solution of (3.1) if and only if v is a smooth solution of the problem

$$v_t + \epsilon v_x + \epsilon^{\frac{p+1}{p}} P(\epsilon^{-\frac{1}{p}} v)_x - v_{xxt} = 0, \qquad \text{in } \Omega, \tag{3.2a}$$

$$v(x,0) = F(x), \qquad \text{for } x \geq 0, \tag{3.2b}$$

$$v(t,t) = G(t), \qquad \text{for } t \geq 0. \tag{3.2c}$$

Here $\Omega = \{(x,t) : t > 0 \text{ and } x > t\}$, $F(x) = \epsilon^{\frac{1}{p}} f(\epsilon^{\frac{1}{2}} x)$, and $G(t) = \epsilon^{\frac{1}{p}} g(\epsilon^{\frac{3}{2}} t)$. The dependence of F and G on ϵ is suppressed, since ϵ is viewed as fixed for the nonce. The compatibility of f and g at the origin implies and is implied by the relation $F(0) = G(0)$.

By converting the differential equation (3.2a) with initial condition (3.2b) and boundary condition (3.2c) into an integral equation and applying the contraction-mapping theorem to this new equation, a small-time existence theory can be established. The argument closely parallels that worked out in detail in [**9, 17**], and we therefore content ourselves with a sketch. First regard equation (3.2a) as an ordinary differential equation for the independent variable v_t by considering $\epsilon v_x + \epsilon^{\frac{p+1}{p}} P(\epsilon^{\frac{1}{p}} v)_x$ as a given external force. Solving this second-order equation, performing a formal integration by parts, and following that by an integration from 0 to t leads to the equation

$$v(x,t) = F(x) + (G(t) - F(t))e^{-(x-t)} + \mathbb{B}(v)(x,t), \tag{3.3}$$

where

$$\mathbb{B}(v)(x,t) = \int_t^{+\infty} K(x-t, \xi-t) \int_0^t \left[\epsilon^{\frac{p+1}{p}} P(\epsilon^{-\frac{1}{p}} v(\xi,\tau)) + \epsilon v(\xi,\tau) \right] d\tau \, d\xi \tag{3.4}$$

and

$$K(x,\xi) = \frac{1}{2}\left[\exp(-(x+\xi)) + \operatorname{sgn}(x-\xi)\exp(-|x-\xi|)\right]. \tag{3.5}$$

For $v \in \mathcal{B}_T$, define the function $\mathbb{A}v$ by

$$(\mathbb{A}v)(x,t) = F(x) + (G(t) - F(t))e^{-(x-t)} + \mathbb{B}(v)(x,t). \tag{3.6}$$

Assuming that F and G are bounded and continuous, it follows that $\mathbb{A}$ is an operator mapping $v \in \mathcal{B}_T$ into itself since K is integrable. Define the quantity $R(T)$ by

$$\frac{1}{2}R(T) = ||\mathbb{A}\vartheta||_{\mathcal{B}_T} \leq 2||F||_{C_b(\mathbb{R}^+)} + ||G||_{C(0,T)}, \tag{3.7}$$

where $\vartheta(x,t) \equiv 0$, and let

$$\mathbf{B}_T = \{w \in \mathcal{B}_T : ||w||_{\mathcal{B}_T} \leq R(T)\},$$

and

$$\lambda(T) = T(\epsilon + \gamma(R)), \tag{3.8}$$

where $\gamma(R) = \gamma([-R,R])$ is the Lipschitz constant for P on the set $[-R,R]$. Then for u, $v \in \mathbf{B}_T$, it transpires that

$$\begin{aligned} ||\mathbb{A}u - \mathbb{A}v||_{\mathcal{B}_T} &= \sup_{0\leq x} \sup_{0\leq t\leq T} |\mathbb{A}u - \mathbb{A}v| \\ &\leq T(\epsilon + \gamma(R))||u - v||_{\mathcal{B}_T} \\ &\leq \lambda(T)||u - v||_{\mathcal{B}_T}, \end{aligned}$$

and

$$\begin{aligned} ||\mathbb{A}v||_{\mathcal{B}_T} &= ||\mathbb{A}v - \mathbb{A}\vartheta||_{\mathcal{B}_T} + ||\mathbb{A}\vartheta||_{\mathcal{B}_T} \\ &= \lambda(T)||v||_{\mathcal{B}_T} + \frac{1}{2}R(T) \leq (\lambda(T) + \frac{1}{2})R(T). \end{aligned}$$

If T is chosen small enough so $\lambda(T) \leq \frac{1}{2}$, then it follows from the last two inequalities that $\mathbb{A}$ is a contractive mapping of $\mathbf{B}_T$ into itself. These remarks together with the contraction-mapping theorem suffice to establish the following result.

Proposition 3.1. *Let $T > 0$, $F \in C_b(\mathbb{R}^+)$ and $G \in C(0,T)$ be given. Suppose that P is locally Lipschitz continuous. Then there is a positive constant T' depending only on $||F||_{C_b(\mathbb{R}^+)}$, $||G||_{C(0,T)}$ and the Lipschitz constant γ such that if $T_0 = \min(T',T)$, then there is a unique solution of (3.6) in $\mathcal{B}_{T_0}$.*

Remark: Uniqueness follows readily from the sort of inequalities displayed above. A detailed view of the uniqueness may be found in [**17**]. Notice that the size of the time interval T' depends only upon the maximum value of F and G.

It will be important in subsequent sections to have smooth solutions, up to the boundaries, of the regularized problem (3.1) at our disposal. This amounts to the program of relating solutions of the integral equation (3.3) to solutions of the regularized initial-boundary-value problem (3.2). The following result will be sufficient for later developments.

Proposition 3.2. *Suppose that $F \in C_b^k(\bar{\mathbb{R}}^+)$ and $G \in C^m(0,T)$, where $k \geq 2$, $m \geq 1$, and $k \geq m$, and that $F(0) = G(0)$. Further suppose $P \in C^{k+m-1}(\mathbb{R})$. Let v be a solution in $\mathcal{B}_{T_0}$ of the integral equation (3.3), where T_0 lies in the interval $(0,T]$. Then it follows that*

$$\partial_x^i \partial_t^j v \subset \mathcal{B}_{T_0}, \qquad \text{for } 0 \leq j \leq m \text{ and } 0 \leq i \leq k+j. \tag{3.9}$$

Conversely, if v is a classical solution of the transformed problem (3.2) in $\bar{\Omega}_T$, then v is a solution of the integral equation (3.3) over $\bar{\Omega}_T$, and so v satisfies (3.9).

The proof follows from the integral equation (3.3) as in [**4, 16, 17**] and so is omitted here. The partial derivatives in (3.9) may be defined at the boundary of Ω_T via the obvious one-sided difference quotients. In case $j > 0$ in (3.9), the condition $\partial_x^i \partial_t^j v \in \mathcal{B}_T$ connotes that this partial derivative exists classically in $\bar{\Omega}_T \backslash \{(0,0)\}$, is bounded and continuous there, and that it may be extended continuously to $\bar{\Omega}_T$.

Suppose a classical solution v of (3.2), defined on $\bar{\Omega}_T$ for some $T > 0$, is in hand, and suppose the boundary data G is defined at least on $[0, T_1]$, where $T_1 > T$. As soon as an *a priori* bound on the L_∞-norm of a solution defined on $\bar{\Omega}_{T_1}$ is provided, it follows from the Remark below Proposition 3.1 that the solution can be extended to $\bar{\Omega}_{T_1}$ by a finite number of iterations of the local existence result propounded in Proposition 3.1. Moreover, if F and G possess the regularity assumed in Proposition 3.2, it follows that the extended solution does as well.

Provision of the relevant *a priori* bound is now considered. Additional conditions on F and G seem to be needed at this stage, namely that the initial data be suitably evanescent at infinity. This condition is quite reasonable from the point of view of the physical situations for which (1.1) serves as a model.

Lemma 3.3. *Let $F \in C_b^k(\bar{\mathbb{R}}^+)$ and $G \in C^m(0,T)$ with $F(0) = G(0)$, where $k \geq 2$, $m \geq 1$ and $k \geq m$. Let v be a solution of (3.2) in $\mathcal{B}_{T_0}$. Let r lie in the range $0 \leq r \leq k$ and suppose that*

$$\partial_x^j F(x) \to 0 \qquad \text{as } x \to +\infty,$$

for $0 \leq j \leq r$. Then it follows that

$$\partial_x^j \partial_t^i v(x,t) \to 0 \qquad \text{as } x \to +\infty,$$

uniformly for $0 \leq t \leq T_0$, for i, j such that $0 \leq i \leq m$ and $0 \leq j \leq r + i$.

The proof of this technical result follows from the representation (3.3) just as in [**3, 4**] and we may safely skip the details.

Attention is now given to the derivation of the *a priori* bounds needed to guarantee the local solutions provided in Proposition 3.1 admit global extensions to solutions of the initial-boundary-value problem (3.2). According to the remarks above, it suffices to show the following. Suppose to be given suitably restricted initial data F defined on $\mathbb{R}^+$ and boundary data G defined on $[0,T]$ for some positive T. Suppose also that u is a correspondingly smooth solution of (3.2) defined on Ω_{T_0} for some $0 < T_0 \le T$. If it is demonstrated in these circumstances that there is a constant C dependent only on F, G and T such that

$$||u||_{C_b(\bar{\Omega}_T)} \le C, \tag{3.10}$$

then it follows that u can be extended to a solution of (1.1) defined on Ω_T. In particular, if G is given for all $t \ge 0$ and lies in the function class that allows the derivation of (3.10) on any bounded time interval, it will follow that u can be extended as a solution of (3.2) defined on the entire quarter plane $\mathbb{R}^+ \times \mathbb{R}^+$.

A bound on solutions of (3.2) that implies (3.10) is the subject of the next proposition. The proof can be found in [**9**].

Proposition 3.4. *Let Λ be defined by $\Lambda(0) = 0$ and $\Lambda' = P$. Suppose $F \in C_b^3(\bar{\mathbb{R}}^+) \cap H^1(\mathbb{R}^+)$, $G \in C^1(0,T)$ with $F(0) = G(0)$, and that Λ is at least a C^4-function satisfying the one-sided growth condition*

$$\limsup_{|s|\to\infty} |s|^{-4}\Lambda(s) \le 0. \tag{H2}$$

Then for any $T > 0$, the system (3.2) with initial and boundary data F and G has a unique solution $v \in \mathcal{B}_T^{3,1} \cap C(0,T;H^1)$. Moreover, if $F \in C_b^3(\bar{\mathbb{R}}^+) \cap H^2(\mathbb{R}^+)$ and $G \in C^1(0,T)$ with $F(0) = G(0)$, and

$$\limsup_{|s|\to\infty} |s|^{-2}|P''(s)| \le c^*, \tag{H3}$$

for some finite constant c^*, then the initial-boundary-value problem (3.2) has a unique solution $v \in \mathcal{B}_T^{3,1} \cap C(0,T;H^2)$ for arbitrary $T > 0$.

In any case, the solution depends continuously on the initial- and boundary data in the sense that the mapping $(F,G) \longmapsto v$ is continuous from $C_b^3(\bar{\mathbb{R}}^+) \cap H^1(\mathbb{R}^+) \times C^1(0,T)$ to $\mathcal{B}_T^{3,1} \cap C(0,T;H^1)$, or from $C_b^3(\bar{\mathbb{R}}^+) \cap H^2(\mathbb{R}^+) \times C^1(0,T)$ to $\mathcal{B}_T^{3,1} \cap C(0,T;H^2)$.

Corollary 3.5. *Let $f \in C_b^3(\bar{\mathbb{R}}^+) \cap H^2(\mathbb{R}^+)$ and $g \in C^1(0,T)$, where $f(0) = g(0)$. Let u be a classical solution of (3.1), up to the boundary, on $\bar{\mathbb{R}}^+ \times [0,T_0]$. Suppose P satisfies the conditions in Proposition 3.4. Then there exists a constant C dependent on $||f||_2$ and on $|g|_{1,T}$ such that any classical solution u of (3.1) defined on $\bar{\Omega}_{T_0}$, for $T_0 \le T$, satisfies*

$$||u||_{C_b(\bar{\Omega}_{T_0})} \le C.$$

Proof: Let u be a classical solution of (3.1) on $\bar{\Omega}_{T_0}$ for some $T_0 \le T$. Then since

$$v(x,t) = \epsilon^{\frac{1}{p}} u\big(\epsilon^{\frac{1}{2}}(x-t),\ \epsilon^{\frac{3}{2}}t\big), \tag{3.11}$$

v is a classical solution of (3.2a) on $\bar{\mathbb{R}}^+ \times [0, T_0']$, where $T_0' = \epsilon^{\frac{3}{2}} T_0$, which satisfies the auxiliary conditions (3.8) where

$$F(x) = \epsilon^{\frac{1}{p}} f(\epsilon^{\frac{1}{2}} x), \quad \text{and} \quad G(t) = \epsilon^{\frac{1}{p}} g(\epsilon^{\frac{3}{2}} t).$$

Here ϵ is fixed, and so F and G satisfy the hypotheses of Proposition 3.4. Hence the $H^2(\bar{\mathbb{R}}^+)$-norm of v is bounded on $[0, T_0']$ by a constant that depends on $||F||_2$, and on $|G|_{1,T'}$, say, where $T' = \epsilon^{\frac{3}{2}} T$. By an elementary inequality, one has

$$||v(\cdot, t)||_{C_b(\mathbb{R}^+)} \leq \sqrt{2}(||v(\cdot, t)|| \, ||v_x(\cdot, t)||)^{\frac{1}{2}}.$$

It follows that v is bounded on $\bar{\mathbb{R}}^+ \times [0, T_0']$ by a constant C dependent only on $||F||_2$ and $|G|_{1,T'}$. In particular, C does not depend on T_0' for T_0' in the range $[0, T']$.

Since u is defined from v by

$$u(x,t) = \epsilon^{-\frac{1}{p}} v(\epsilon^{-\frac{1}{2}} x + \epsilon^{-\frac{3}{2}} t, \ \epsilon^{-\frac{3}{2}} t), \tag{3.12}$$

the desired result follows. □

Here is one more result about the transformed problem (3.2). The proof follows easily from the integral equation (3.3).

Proposition 3.6. *Let $F \in C_b^k(\bar{\mathbb{R}}^+) \cap H^k(\mathbb{R}^+), G \in C^m(0,T)$, where $F(0) = G(0)$ and $k \geq 3, m \geq 1$ and $k \geq m$. Let v be the solution of (3.2) in $\mathcal{B}_{T_0}(\Omega)$, up to the boundary, where $\Omega = \{(x,t) : t > 0 \text{ and } x > t\}$. Suppose P satisfies (H3) in the Proposition 3.4 and is in $C^{k+m-1}(\mathbb{R})$. Then there exists a constant C, dependent on $||F||_2$ and $|G|_{1,T}$ such that, for each $t \in [0, T_0]$,*

$$||\partial_x^i \partial_t^j v(\cdot, t)||_{L_2((t,\infty))} \leq C$$

provided that $0 \leq j \leq m$ and $0 \leq i \leq k + j$.

It is worth summarizing the accomplishments of the present section. As the transformed problem (3.2) is only of transient interest, the theory is recapitulated in terms of the regularized problem (3.1). Thus the results stated now are consequences of the propositions established above together with the transformation (3.12) taking (3.2) to (3.1).

Theorem 3.7. *Let $f \in C_b^k(\bar{\mathbb{R}}^+)$ and $g \in C^m(0,T)$, with $f(0) = g(0)$, where $k \geq 3$, $m \geq 1$ and $k \geq m$. Let $\epsilon > 0$ and suppose P to lie in $C^{k+m-1}(\mathbb{R})$. Then there exists $T_0 > 0$ and a unique function u in $C_b(\bar{\mathbb{R}}^+ \times [0, T_0])$ which is a classical solution of the regularized problem (3.1). Additionally,*

$$\partial_x^i \partial_t^j u \in C_b(\bar{\mathbb{R}}^+ \times [0, T_0]),$$

for i and j such that $0 \leq j \leq m$, $0 \leq i \leq k$ and $i + j \leq k$. Suppose $f \in H^r(\mathbb{R}^+)$, where P satisfies hypothesis (H2) if $r \geq 1$ or hypothesis (H3) if $r \geq 2$ in Proposition

3.4. Then u may be extended to a solution of (3.1) on $\bar{\mathbb{R}}^+ \times [0,T]$. *In this case, there is a constant* C *which depends on* ϵ *such that, for* $0 \le t \le T$,

$$||\partial_x^i \partial_t^j u(\cdot,t)|| \le C,$$

for i *and* j *such that* $0 \le j \le \min\{r,m\},\ \ 0 \le i \le r$ *and* $i+j \le r$.

Corollary 3.8. *Let* $f \in H^\infty(\mathbb{R}^+)$ *and* $g \in C^\infty(\mathbb{R}^+)$, *with* $f(0) = g(0)$. *Suppose* P *satisfies the growth condition (H3) in the Proposition 3.4 and is a* C^∞*-function. Then there exists a unique solution* u *of (3.1) on the entire quarter-plane* $\bar{\mathbb{R}}^+ \times \bar{\mathbb{R}}^+$ *corresponding to the data* f *and* g. *Moreover the solution* u *is bounded on any finite time interval, lies in* $C^\infty(\bar{\mathbb{R}}^+ \times \bar{\mathbb{R}}^+)$ *and for each* $k \ge 0$,

$$\partial_x^i \partial_t^j u \in C(\bar{\mathbb{R}}^+; H^k(\mathbb{R}^+)),$$

for all $i,j \ge 0$

Proof: The existence of global solutions follows immediately from Theorem 3.7 together with the uniqueness result. Also, for any $i,j \ge 0,\ k > 0$, and $T > 0$, $\partial_x^i \partial_t^j u$ is uniformly bounded in $H^k(\mathbb{R}^+)$, for $0 \le t \le T$. Since $u \in L_\infty(0,T;H^k(\mathbb{R}^+))$ and $u_t \in L_\infty(0,T;H^k(\mathbb{R}^+))$, it follows immediately (cf. [**35**]) that $u \in C(0,T;H^k(\mathbb{R}^+))$. □

4. *A priori* Bounds in H^3 for the Regularized Problem

In Section 3, the bounds obtained in H^1 or H^2 for solutions of (3.2) do not appear to yield ϵ-independent bounds on solutions of (3.1) because the transformation (3.12) that takes solutions of (3.2) to solutions of (3.1) is singular at $\epsilon = 0$. In this and the next section, ϵ-independent *a priori* bounds are derived for solutions of the regularized initial- and boundary-value problem (3.1) which, for any fixed $T > 0$, are independent of $t \in [0,T]$.

Throughout this section it will be assumed that $f \in H^\infty(\mathbb{R}^+)$, $g \in C^\infty(0,T)$, and $f(0) = g(0)$. From Corollary 3.8 it is inferred that there is a classical solution $u = u_\epsilon$ of (3.1) corresponding to the auxiliary data f and g which is such that

$$u \in C^\infty(\bar{\mathbb{R}}^+ \times [0,T]),$$

and, for integers $j,k \ge 0$,

$$\partial_t^j u \in C(0,T;H^k(\mathbb{R}^+)).$$

Some preliminary relations, established via energy-type arguments, will be derived in a sequence of technical lemmas. These prefatory results will be combined to obtain ϵ-independent bounds for u within the function class $C(0,T;H^3(\mathbb{R}^+))$ under the assumption

$$\limsup_{|s|\to\infty} |s|^{-\frac{10}{3}} \Lambda(s) \le 0 \qquad (**)$$

on P, where as before, $\Lambda' = P$ and $\Lambda(0) = 0$. Besides the condition (**), it is also assumed that $P \in C^\infty(\mathbb{R}^+)$, though it will be clear that weaker differentiability suffices for most of the results below. Because of (**) and the fact that $P(0) = 0$, it follows that for any $\delta > 0$, there is a constant $C = C_\delta$ such that

$$\Lambda(s) \le Cs^2 + \delta s^{\frac{10}{3}}$$

for all $s \ge 0$. Note that, because (**) is a one-sided condition, high growth rates at infinity are not excluded. For example, if $P(s) = -(2k+2)s^{2k+1}$, then $\Lambda(s) = -s^{2k+2}$ satisfies (**) no matter how large the positive integer k.

At various times, constants will arise in our considerations that depend only on the data f and g. Many of these will be denoted simply by C, and this symbol's occurence in different formulae is not taken to connote the same constant.

Lemma 4.1. *Let $f \in H^\infty(\mathbb{R}^+)$, $g \in C^\infty(\mathbb{R}^+)$, with $f(0) = g(0)$. Suppose P satisfies the growth condition (**) and suppose $0 < \epsilon \le 1$. There exists a positive constant*

$$a_1 = a_1(||f||_1, |g|_{1,T}), \tag{4.1}$$

such that the solution u of (3.1) corresponding to the data f and g satisfies

$$||u(\cdot,t)||_1^2 + \int_0^t \left[u_x^2(0,s) + (u_{xx}(0,s) - \epsilon u_{xt}(0,s))^2\right] ds \le a_1, \tag{4.2}$$

for $0 \le t \le T$, and uniformly for $\epsilon \in (0,1]$.

Proof: Multiply (3.1a) by $2u$ and integrate the resulting relation over $\mathbb{R}^+ \times (0,t)$. After integrations by parts, in which the fact that u and various of its derivatives vanish at $+\infty$ is used repeatedly, it is verified that

$$\begin{aligned} &||u(\cdot,t)||^2 + \epsilon||u_x(\cdot,t)||^2 + \int_0^t u_x^2(0,s)ds \\ &= \int_0^t [2g(s)(u_{xx}(0,s) - \epsilon u_{xt}(0,s)) + 2Q(g(s))]ds + ||f||^2 + \epsilon||f'||^2 \\ &\le C(||f||_1, |g|_{1,T}) + C(|g|_T)\left(\int_0^t \left(u_{xx}(0,s) - \epsilon u_{xt}(0,s)\right)^2 ds\right)^{\frac{1}{2}}, \end{aligned} \tag{4.3}$$

where $Q'(\lambda) = \lambda P'(\lambda)$ and $Q(0) = 0$. In particular, we have

$$||u(\cdot,t)||^4 \le C(||f||_1, |g|_{1,T}) + C(|g|_T)\int_0^t \left(u_{xx}(0,s) - \epsilon u_{xt}(0,s)\right)^2 ds.$$

Next multiply the regularized equation (3.1a) by the combination $2\epsilon u_{xt} - 2u_{xx} - 2P(u)$ and integrate the result over $\mathbb{R}^+ \times (0,t)$. After several integrations by parts, it is seen that

$$\begin{aligned}
&\|u_x(\cdot,t)\|^2 + \int_0^t \big[u_{xx}(0,s) - \epsilon u_{xt}(0,s) + P(g(s))\big]^2 ds \\
&= \|f'\|^2 + 2\int_0^{+\infty} \Lambda(u(x,t))dx - 2\int_0^{+\infty} \Lambda(f(x))dx \\
&\quad + \int_0^t \big(\epsilon g_t^2(s) - 2g_t(s)u_x(0,s)\big)ds \\
&\le C(||f||_1, |g|_{1,T})\Big(1 + \int_0^t u_x^2(0,s)ds\Big)^{\frac{1}{2}} + \|u\|^2 \tilde{E}(||u||^{\frac{1}{2}}||u_x||^{\frac{1}{2}}),
\end{aligned}$$

where

$$\Lambda'(\lambda) = P(\lambda), \quad \Lambda(\lambda) = \lambda^2 E(\lambda), \quad \text{and} \quad \tilde{E}(r) = \sup_{|\lambda|\le r} E(\lambda).$$

By using (**), the elementary inequality

$$||f||^2_{C_b(\mathbb{R}+)} \le 2||f||\,||f'|| \tag{4.4}$$

and Young's inequality, the last inequality may be put in the form

$$\begin{aligned}
&\|u_x(\cdot,t)\|^2 + \int_0^t \big[u_{xx}(0,s) - \epsilon u_{xt}(0,s) + P(g(s))\big]^2 ds \\
&\le C(||f||_1, |g|_{1,T}, \frac{1}{\delta}) + \delta\Big[||u(\cdot,t)||^{\frac{8}{3}}||u_x(\cdot,t)||^{\frac{2}{3}} + \int_0^t u_x^2(0,s)ds\Big],
\end{aligned} \tag{4.5}$$

for any choice of $\delta > 0$. By a further use of Young's inequality, it is adduced that

$$\begin{aligned}
&\frac{1}{2}\|u_x(\cdot,t)\|^2 + \int_0^t \big[u_{xx}(0,s) - \epsilon u_{xt}(0,s)\big]^2 ds \\
&\le C(||f||_1, |g|_{1,T}, \frac{1}{\delta}) + \delta||u(\cdot,t)||^4.
\end{aligned} \tag{4.6}$$

Substitute (4.3) into (4.6) to obtain

$$\begin{aligned}
&\|u(\cdot,t)\|^4 + \|u_x(\cdot,t)\|^2 + \Big(\int_0^t u_x^2(0,s)ds\Big)^2 \\
&\quad + \int_0^t \big[u_{xx}(0,s) - \epsilon u_{xt}(0,s)\big]^2 ds \le C(||f||_1, |g|_{1,T}, \frac{1}{\delta}),
\end{aligned} \tag{4.7}$$

for a suitable $\delta > 0$. Inequality (4.2) now follows, and the proposition is proved.
□

Remarks: Note that if the boundary data is small in the sense of the norm $|g|_T$, then the condition (**) can be improved. In particular, if the boundary data is zero, then the L_2-bound depends only on the initial data. Hence, following the steps above, if

$$\limsup_{|s|\to\infty} |s|^{-6}\Lambda(s) \le 0, \tag{$**'$}$$

then one derives ϵ-independent H^1-bounds. The restriction ($**'$) is the same restriction imposed on P when pure initial-value problems for equation (1.1a) are considered [**28, 29**]. Note also that if the initial data f and the boundary data g are small enough in $H^1(\mathbb{R}^+)$ and $H^1(0,T)$, respectively, Λ is unrestricted in sign. In fact, from (4.3) and the estimate of $||u_x(\cdot,t)||^2$, one shows that

$$||u(\cdot,t)||_1^2(1-\delta\tilde{E}(||u||_1)) \le C(||f||_1, |g|_{1,T}),$$

where $\delta = ||f||_1^2 + |g|_{1,T}^2$. If δ is small enough relative to $||f||_1$ and $|g|_{1,T}$, one obtains a global H^1-bound from the above estimate. Furthermore, the bound obtained in this way depends only on the auxiliary data and not explicitly on T. In this situation, $||u(\cdot,t)||_1$ grows at most linearly with the energy supplied by the wavemaker.

From (4.4) it follows that

$$||u||^2_{C_b(\bar{\mathbb{R}}^+\times[0,T])} \le 2 \sup_{0\le t\le T}\{||u_x(\cdot,t)||\,||u(\cdot,t)||\} \le a_1, \tag{4.8}$$

for all t in $[0,T]$. Using the differential equation (3.1a) and the fact $\int_0^t u_x^2(0,s)ds \le a_1$, it follows that

$$\int_0^t \left[u_{xxx}(0,s) - \epsilon u_{xxt}(0,s)\right]^2 ds = \int_0^t \left[g_t(s) + a(g(s))u_x(0,s)\right]^2 ds$$
$$\le C(||f||_1, |g|_{1,T}),$$

where $a(\lambda) = P'(\lambda)$. These conclusions are formalized in the following corollary.

Corollary 4.2. *Let f, g and P satisfy the conditions in Lemma 4.1. Then there is a constant a_1 depending only on $||f||_1$ and $|g|_{1,T}$ such that*

$$||u||^2_{C_b(\bar{\mathbb{R}}^+\times[0,T])} \le a_1(||f||_1, |g|_{1,T})$$

and

$$\int_0^t \left(u_{xxx}(0,s) - \epsilon u_{xxt}(0,s)\right)^2 ds \le a_1,$$

for $0 \le t \le T$, and uniformly for $\epsilon \in (0,1]$.

Next we obtain an $H^3(\mathbb{R}^+)$-bound on solutions of (3.1). It will be shown that $||u(\cdot,t)||_3$ is bounded on $[0,T]$, independently of ϵ small enough. First define $A(t)$

and $B(t)$ by

$$A^4(t) = \sup_{0\le s\le t}\{||u(\cdot,s)||_2^4\} + \Big(\int_0^t u_{xxx}^2(0,s)ds\Big)^2 + \Big(\int_0^t u_{xx}^2(0,s)ds\Big)^2 + \Big(\int_0^t \epsilon u_{xt}^2(0,s)ds\Big)^2 \tag{4.9a}$$

and

$$B^2(t) = \sup_{0\le s\le t}\{||u(\cdot,s)||_3^2\} + \epsilon||u_{xxxx}(\cdot,t)||^2 + \int_0^t \epsilon||u_{xt}(\cdot,s)||^2 ds + \int_0^t u_{xxxx}^2(0,s)ds + \int_0^t u_{xt}^2(0,s)ds + \epsilon\int_0^t u_{xxt}^2(0,s)ds. \tag{4.9b}$$

In fact, it will be demonstrated below that $A(t)$ and $B(t)$ are bounded on $[0,T]$, independently of ϵ small enough. The next lemma gives an $H^2(\mathbb{R}^+)$-estimate not directly effective in bounding $||u(\cdot,t)||_2$, independently of ϵ, but which will prove useful later.

Remark: It seems that for the boundary-value problems (1.1), the H^2-bound is difficult to obtain alone, in the same way that the $L_2(\mathbb{R}^+)$-bound was not derived on its own. A similar problem occurs when a two-point, nonhomogeneous boundary-value problem for the KdV equation is considered (see **[10]**).

Lemma 4.3. *Let $T > 0$, $f \in H^\infty(\mathbb{R}^+)$, $g \in H^\infty(0,T)$, with $f(0) = g(0)$. There exist constants a_2, ϵ_1, C_1 and C_2, where*

$$a_2 = a_2(||f||_2 + \epsilon^{\frac12}||f||_3, |g|_{1,T}), \quad \epsilon_1 = \epsilon_1(||f||_1, |g|_{1,T}),$$
$$C_1 = C_1(||f||_1, |g|_{1,T}), \quad \text{and} \quad C_2 = C_2(||f||_1, |g|_{1,T}, T),$$

such that the solution of (3.1) corresponding to the data f and g satisfies

$$A^4(t) \le a_2(||f||_2 + \epsilon^{\frac12}||f||_3, |g|_{1,T}) + C_1\int_0^t u_{xt}^2(0,s)ds + C_2\int_0^t \big[||u_{xx}(\cdot,s)||^4 + \epsilon^2||u_{xxx}(\cdot,s)||^4 + \epsilon||u_{xx}(\cdot,s)||^8\big]ds, \tag{4.10}$$

provided that $t \in [0,T]$ and $\epsilon \in (0,\epsilon_1]$.

Proof: Multiply (3.1a) by $10(u_{xt} - a(u)u_{xx})$ where $a(u) = P'(u)$ as before, differentiate (3.1a) once with respect to x and multiply the result by $2u_{xxx}$, add the equations thus obtained, and integrate their sum over $\mathbb{R}^+ \times (0,t)$. Several integrations by parts and using the relation

$$\int_0^t \int_0^{+\infty} u_{xt}u_x a(u)dxds = \int_0^{+\infty} \big[u_x^2(x,t)a(u(x,t)) - (f'(x))^2 a(f(x))\big]dx + \int_0^t g'(s)a(g(s))u_x(0,s)ds + \int_0^t \int_0^{+\infty} u_{xx}u_t a(u)dxds,$$

yield

$$\begin{aligned}
&6\|u_{xx}(\cdot,t)\|^2 + \epsilon\|u_{xxx}(\cdot,t)\|^2 + \int_0^t u_{xxx}^2(0,s)ds \\
&= 6\|f''\|^2 + \epsilon\|f'''\|^2 + \int_0^{+\infty} \big[10u_x^2(x,t)a(u(x,t)) + 5\epsilon u_{xx}^2(x,t)a(u(x,t)) \\
&\qquad - 10(f'(x))^2a(f(x)) - 5\epsilon(f''(x))^2a(f(x))\big]dx \\
&+ \int_0^t \big[5\epsilon u_{xt}^2(0,s) - 12u_{xt}(0,s)u_{xx}(0,s) + \frac{1}{2}u_x^4(0,s)a''(g(s)) \\
&\qquad - 2a'(g(s))u_x^2(0,s)u_{xx}(0,s) + 4a(g(s))u_{xx}^2(0,s) \\
&\qquad + 10g'(s)a(g(s))u_x(0,s) + 5a^2(g(s))u_x^2(0,s) - 5(g'(s))^2\big]ds \\
&+ \int_0^t \int_0^{+\infty} \big[\frac{1}{2}a'''(u)u_x^5 + 10a(u)a'(u)u_x^3 - 5\epsilon a'(u)u_tu_{xx}^2\big]\,dxds.
\end{aligned} \tag{4.11}$$

Take $\epsilon_1 \le 1$ small enough that

$$5\epsilon\|a(u(x,t))\|_{C_b(\mathbb{R}^+\times[0,T])} \le 3,$$

say, for any $\epsilon \le \epsilon_1$. Then the first six terms on the right-hand side of (4.11) can be controlled by the terms on the left-hand side, the constant a_1 in Lemma 4.1 and a constant depending on $\|f\|_2 + \epsilon^{\frac{1}{2}}\|f\|_3$ and $|g|_{1,T}$. By using (4.2) and choosing ϵ_1 small, say, there obtains

$$\begin{aligned}
\int_0^t \big[u_{xx}^2(0,s) + \epsilon^2u_{xt}^2(0,s)\big]ds &\le a_1 + 2\epsilon\int_0^t u_{xx}(0,s)u_{xt}(0,s)ds \\
&\le C(\|f\|_1, |g|_{1,T}) + \epsilon^{\frac{3}{2}}\int_0^t u_{xt}^2(0,s)ds.
\end{aligned} \tag{4.12}$$

The last three boundary terms in the second integral on the right-hand side of (4.11) may be bounded by a suitable multiple C* of the left-hand side of (4.12). Use of Lemma 4.1 allows the estimation of the first two boundary terms in the second integral on the right-hand side of (4.11) as follows:

$$\begin{aligned}
&\int_0^t \big[5\epsilon u_{xt}^2(0,s) - 12\epsilon u_{xt}(0,s)u_{xx}(0,s)\big]ds \\
&= \int_0^t \big[12u_{xt}(0,s)(\epsilon u_{xt}(0,s) - u_{xx}(0,s))\big]ds - \int_0^t 7\epsilon u_{xt}^2(0,s)ds \\
&\le C(||f||_1, |g|_{1,T})\Big(\int_0^t u_{xt}^2(0,s)ds\Big)^{\frac{1}{2}} - \int_0^t 7\epsilon u_{xt}^2(0,s)ds.
\end{aligned} \tag{4.13}$$

Thus, if it is supposed that ϵ_1 is small enough that $6\epsilon_1 \ge C^*(|g|_{1,T})\epsilon_1^{\frac{3}{2}}$, then (4.12) can be controlled in the form shown on the right-hand side of (4.13), so one has

$$\begin{aligned}
&\int_0^t \big[u_{xx}^2(0,s) + \epsilon u_{xt}^2(0,s)\big]ds \\
&\le C(||f||_1, |g|_{1,T}) + C(||f||_1, |g|_{1,T})\Big(\int_0^t u_{xt}^2(0,s)ds\Big)^{\frac{1}{2}}.
\end{aligned} \tag{4.14}$$

For any $\delta > 0$, apply (4.4) to $u_x(x,t)$ to adduce

$$\begin{aligned}
||u_x||^2_{C_b(\bar{\mathbb{R}}^+\times[0,t])} &\le 2 \sup_{0\le s\le t}\{||u_x(\cdot,s)||\,||u_{xx}(\cdot,s)||\} \\
&\le C\delta^{-1} + \delta \sup_{0\le s\le t}\{||u_{xx}(\cdot,s)||^2\}.
\end{aligned} \tag{4.15}$$

One shows straightforwardly that for any $\delta > 0$,

$$\Big|\int_0^t \frac{1}{2}u_x^4(0,s)a''(g(s))ds\Big| \le C(||f||_1, |g|_{1,T})\delta^{-1} + \delta \sup_{0\le s\le t}\{||u_{xx}(\cdot,s)||^2\}.$$

Similarly, one shows that

$$\begin{aligned}
&\Big|\int_0^t 2a'(g(s))u_x^2(0,s)u_{xx}(0,s)ds\Big| \\
&\le C(|g_{1,T}|)\Big(\int_0^t u_x^4(0,s)ds\Big)^{\frac{1}{2}}\Big(\int_0^t u_{xx}^2(0,s)ds\Big)^{\frac{1}{2}} \\
&\le C(||f||_1, |g|_{1,T})\delta^{-3} + \delta\Big(\sup_{0\le s\le t}\{||u_{xx}(\cdot,s)||^2\} + \int_0^t u_{xx}^2(0,s)ds\Big).
\end{aligned}$$

Thus in summary, the boundary terms in the second integral on the right-hand side of (4.11) are bounded above by

$$C(||f||_1, |g|_{1,T})\delta^{-3} + \delta \sup_{0\le s\le t}\{||u_{xx}(\cdot,s)||^2\} + C(||f||_1, |g|_{1,T})\Big(\int_0^t u_{xt}^2(0,s)ds\Big)^{\frac{1}{2}}$$

for any $\delta > 0$. By using (4.15) and Lemma 4.1, the first two terms of the third integral in (4.11) can be estimated as follows:

$$\Big|\int_0^t\int_0^{+\infty}[\frac{1}{2}a'''(u)u_x^5+10a(u)a'(u)u_x^3]\,dxds\Big|$$
$$\leq C(||f||_1,|g|_{1,T})\Big(\int_0^t||u_{xx}(\cdot,s)||^4ds\Big)^{1/2}.$$

Multiplying (3.1a) by u_t and integrating the resulting expression over $\mathbb{R}^+$, there appears

$$||u_t(\cdot,t)||^2+\epsilon||u_{xt}(\cdot,t)||^2$$
$$=-\epsilon g'(t)u_{xt}(0,t)-\int_0^{+\infty}[u_tu_{xxx}+u_tP(u)_x]dx.$$

Hence it is deduced that

$$\begin{aligned}&||u_t(\cdot,t)||^2+2\epsilon||u_{xt}(\cdot,t)||^2\\&\leq C(||f||_1,|g|_{1,T})+2|\epsilon g'(t)u_{xt}(0,t)|+||u_{xxx}(\cdot,t)||^2.\end{aligned}\tag{4.16}$$

Applying (4.4) to u_t and using (4.16) shows that the last term in the last integral on the right-hand side of (4.11) is bounded above in the following way:

$$\Big|\int_0^t\int_0^{+\infty}5\epsilon a'(u)u_tu_{xx}^2dxds\Big|$$
$$\leq C(||f||_1,|g|_{1,T})\epsilon\int_0^t[||u_{xx}(\cdot,s)||^2||u_t(\cdot,s)||^{\frac{1}{2}}||u_{xt}(\cdot,s)||^{\frac{1}{2}}]ds$$
$$\leq \epsilon^{\frac{1}{2}}C(||f||_1,|g|_{1,T})\int_0^t||u_{xx}(\cdot,s)||^4ds+\epsilon\Big(\int_0^t||u_t(\cdot,s)||^2ds+\epsilon\int_0^t||u_{xt}(\cdot,s)||^2ds\Big)$$
$$\leq \epsilon C(||f||_1,|g|_{1,T},T)+\epsilon^{\frac{1}{2}}C(||f||_1,|g|_{1,T})\int_0^t||u_{xx}(\cdot,s)||^4ds$$
$$+\epsilon^2C(||f||_1,|g|_{1,T})\Big(\int_0^t u_{xt}^2(0,s)ds\Big)^{\frac{1}{2}}+\epsilon\int_0^t||u_{xxx}(\cdot,s)||^2ds.$$

If δ and ϵ_1 are chosen small enough, the above estimates show that (4.11) is reduced to

$$A^2(t)\leq C(||f||_2+\epsilon||f||_3,|g|_{1,T})+C(||f||_1,|g|_{1,T})\Big(\int_0^t u_{xt}^2(0,s)ds\Big)^{\frac{1}{2}}$$
$$+C(||f||_1,|g|_{1,T})\int_0^t[||u_{xx}(\cdot,s)||^2+\epsilon^{\frac{1}{2}}||u_{xx}(\cdot,s)||^4+\epsilon||u_{xxx}(\cdot,s)||^2]ds.$$

The desired inequality now follows. □

The estimate of the $H^2(\mathbb{R}^+)$-norm for the solution u of (3.1) given in Lemma 4.3 will be used in determining the following bound for $A^4(t)+B^2(t)$. When in hand, this bound implies one on $H^3(\mathbb{R}^+)$.

Lemma 4.4. *Let $T > 0$, $f \in H^\infty(\mathbb{R}^+)$, $g \in H^\infty(0,T)$, with $f(0) = g(0)$. There exist a_3 and ϵ_2 where*

$$a_3 = a_3(||f||_3 + \epsilon^{\frac{1}{2}}||f^{(4)}||, |g|_{2,T}, T)$$

and

$$\epsilon_2 = \epsilon_2(||f||_3 + \epsilon^{\frac{1}{2}}||f^{(4)}||, |g|_{2,T})$$

such that the solution of (3.1), corresponding to the data f and g, satisfies

$$A^4(t) + B^2(t) \leq a_3,$$

provided that $t \in [0,T]$ and $\epsilon \in (0, \epsilon_2]$.

Proof: First multiply (3.1a) by $2u_{xxxt} + 2u_{xxx}u_x a'(u) + a'(u)u_{xx}^2 + u_x^2 u_{xx} a''(u)$, differentiate (3.1a) twice with respect to x and multiply this by $u_x^2 a'(u)$, then integrate their sum over $\mathbb{R}^+ \times (0,t)$. Many integrations by parts leads to

$$\begin{aligned}
&||u_{xxx}||^2 + \int_0^t u_{xt}^2(0,s)ds + \int_0^t \epsilon u_{xxt}^2(0,s)ds \\
=&||f'''||^2 + \int_0^{+\infty} \Big[2f'''(x)f'(x)a(f(x)) - 2u_{xxx}(x,t)u_x(x,t)a(u(x,t)) \\
&\quad + (f''(x))^2 a(f(x)) - u_{xx}^2(x,t)a(u(x,t)) + \epsilon u_{xx}^3(x,t)a'(u(x,t)) \\
&\quad - \epsilon(f''(x))^3 a'(f(x)) - u_{xx}(x,t)u_x^2(x,t)a'(u(x,t)) + f''(x)(f'(x))^2 a'(f(x)) \\
&\quad - 3\epsilon(f'(x))^2(f''(x))^2 a''(f(x)) + 3\epsilon u_x^2(x,t)u_{xx}^2(x,t)a''(u(x,t)) \\
&\quad + \epsilon u_x^4(x,t)u_{xx}(x,t)a'''(u(x,t)) - \epsilon(f'(x))^4 f''(x)a'''(f(x))\Big]dx \\
&+ \int_0^t \Big[2u_{xxt}(0,s)g'(s) - 2u_x(0,s)u_{xx}(0,s)a'(g(s))\big[u_{xxx}(0,s) - \epsilon u_{xxt}(0,s)\big] \\
&\quad + u_{xx}^3(0,s)a'(g(s)) + u_x^2(0,s)a'(g(s))\big[u_{xxxx}(0,s) - \epsilon u_{xxxt}(0,s)\big] \\
&\quad - 2u_{xx}(0,s)u_{xt}(0,s)a(g(s)) - u_x^3(0,s)a''(g(s))\big[u_{xxx}(0,s) - \epsilon u_{xxt}(0,s)\big]\Big]ds \\
&+ \int_0^t \int_0^{+\infty} \Big[4\epsilon u_{xxx}u_{xxt}u_x a'(u) - 4u_{xxx}^2 u_x a'(u) - 3u_{xxx}u_x^2 a(u)a'(u) \\
&\quad - u_x^3 u_{xx} a(u)a''(u) + u_x u_{xx}^3 a''(u) - u_x u_{xx}^2 a(u)a'(u) \\
&\quad - u_x^4 u_{xxx} a'''(u) - 3u_x^3 u_{xx}(a'(u))^2 - u_x^5 a'(u)a''(u) \\
&\quad - 6u_{xxx}u_{xx}u_x^2 a''(u) - 6\epsilon u_x u_{xx}^2 u_{xt} a''(u) - 3\epsilon u_x^2 u_{xx}^2 u_t a'''(u) \\
&\quad - \epsilon u_{xx}^3 u_t a''(u) - 4\epsilon u_x^3 u_{xx} u_{xt} a'''(u) - \epsilon u_x^4 u_{xx} u_t a^{(4)}(u)\Big]dxds.
\end{aligned} \tag{4.17}$$

Note that the following integration by parts has been used in deriving (4.17):

$$-\int_0^t\int_0^{+\infty} u_{xxx}u_{xt}a(u)dxds = \int_0^t u_{xx}(0,s)u_{xt}(0,s)a(g(s)) + \int_0^t\int_0^{+\infty}\Big[u_{xx}u_{xt}u_xa'(u) + u_{xx}u_{xxt}a(u)\Big]dxds.$$

Then multiply (3.1a) by $4u_{xxxt}+4u_{xxx}u_xa'(u)$, differentiate (3.1a) once with respect to x and multiply this by $4u_{xxx}a(u)$, then integrate their sum over $\mathbb{R}^+\times(0,t)$. After integrations by parts, one obtains

$$\begin{aligned}
&2||u_{xxx}||^2 + \int_0^t 2u_{xt}^2(0,s)ds + \int_0^t 2\epsilon u_{xxt}^2(0,s)ds \\
&= 2||f'''||^2 + \int_0^{+\infty}\big[2\epsilon u_{xxx}^2(x,t)a(u(x,t)) - 2\epsilon(f'''(x))^2a(f(x)) \\
&\qquad + 4f'''(x)f'(x)a(f(x)) - 4u_{xxx}(x,t)u_x(x,t)a(u(x,t))\big]dx \\
&\quad + \int_0^t\big[4u_{xxt}(0,s)g'(s) + 2u_{xxx}^2(0,s)a(g(s))\big]ds \qquad\qquad (4.18)\\
&\quad + \int_0^t\int_0^{+\infty}\big[4\epsilon u_{xxx}u_xa'(u)u_{xxt} - 8u_{xxx}u_x^2a'(u)a(u) \\
&\qquad - 2u_{xxx}^2u_xa'(u) - 4u_{xxx}u_{xx}a^2(u) - 2\epsilon u_{xxx}^2a'(u)u_t\big]dxds.
\end{aligned}$$

Note also that the relation

$$-u_tu_{xxx}u_xa'(u) = u_{xxx}u_xa'(u)(a(u)u_x + u_{xxx} - \epsilon u_{xxt}),$$

which is obtained by using (1.2a), has been used in deriving (4.18). Finally differentiate (3.1a) twice with respect to x and multiply this by $2u_{xxxx}$, then integrate the result over $\mathbb{R}^+\times(0,t)$. After integrations by parts, there appears

$$\begin{aligned}
&||u_{xxx}||^2 + \epsilon||u_{xxxx}||^2 + \int_0^t u_{xxxx}^2(0,s)ds \\
&= ||f'''||^2 + \epsilon||f^{(4)}||^2 - \int_0^t\Big[6u_x(0,s)u_{xx}(0,s)u_{xxx}(0,s)a'(g(s)) \\
&\qquad + 2u_{xxt}(0,s)u_{xxx}(0,s) + u_{xxx}^2(0,s)a(g(s)) \\
&\qquad + 2u_x^3(0,s)u_{xxx}(0,s)a''(g(s)) - 2u_{xx}^3(0,s)a'(g(s))\Big]ds \qquad\qquad (4.19)\\
&\quad - \int_0^t\int_0^{+\infty}\Big[7u_{xxx}^2u_xa'(u) + 12u_{xxx}u_{xx}u_x^2a''(u) \\
&\qquad + 2u_x^4u_{xxx}a'''(u) - 2u_{xx}^3u_xa''(u)\Big]dxds.
\end{aligned}$$

Subtract (4.17) from (4.18), multiply the result by $\frac{7}{2}$ and then add the result to (4.19). One ends up with

$$
\begin{aligned}
&\frac{9}{2}||u_{xxx}||^2+\epsilon||u_{xxxx}||^2+\int_0^t\Big[\frac{7}{2}u_{xt}^2(0,s)+\frac{7}{2}\epsilon u_{xxt}^2(0,s)+u_{xxxx}^2(0,s)\Big]ds\\
&=\frac{9}{2}||f'''||^2+\epsilon||f^{(4)}||^2+\int_0^{+\infty}\Big[7f'''(x)f'(x)a(f(x))-7u_{xxx}(x,t)u_x(x,t)a(u(x,t))\\
&\quad+7\epsilon u_{xxx}^2(x,t)a(u(x,t))-7\epsilon(f'''(x))^2a(f(x))-\frac{7}{2}(f''(x))^2a(f(x))\\
&\quad+\frac{7}{2}u_{xx}^2(x,t)a(u(x,t))-\frac{7}{2}\epsilon u_{xx}^3(x,t)a'(u(x,t))+\frac{7}{2}\epsilon(f''(x))^3a'(f(x))\\
&\quad+\frac{7}{2}u_{xx}(x,t)u_x^2(x,t)a'(u(x,t))-\frac{7}{2}f''(x)(f'(x))^2a'(f(x))\\
&\quad+\frac{21}{2}\epsilon(f'(x))^2(f''(x))^2a''(f(x))-\frac{21}{2}\epsilon u_x^2(x,t)u_{xx}^2(x,t)a''(u(x,t))\\
&\quad-\frac{7}{2}\epsilon u_x^4(x,t)u_{xx}(x,t)a'''(u(x,t))+\frac{7}{2}\epsilon(f'(x))^4f''(x)a'''(f(x))\Big]dx\\
&+\int_0^t\Big[2u_{xxt}(0,s)\big[\frac{7}{2}g'(s)-u_{xxx}(0,s)\big]+6u_{xxx}^2(0,s)a(g(s))\\
&\quad+7u_{xx}(0,s)u_{xt}(0,s)a(g(s))+\frac{3}{2}u_x^3(0,s)u_{xxx}(0,s)a''(g(s))\\
&\quad-\frac{7}{2}\epsilon u_x^3(0,s)u_{xxt}(0,s)a''(g(s))+u_x(0,s)u_{xx}(0,s)u_{xxx}(0,s)a'(g(s))\\
&\quad-7\epsilon u_x(0,s)u_{xx}(0,s)u_{xxt}(0,s)a'(g(s))-\frac{3}{2}u_{xx}^3(0,s)a'(g(s))\\
&\quad-\frac{7}{2}u_x^2(0,s)a'(g(s))\big[u_{xxxx}(0,s)-\epsilon u_{xxxt}(0,s)\big]\Big]ds\\
&+\int_0^t\int_0^{+\infty}\Big[-14u_{xxx}u_{xx}a^2(u)-\frac{35}{2}u_{xxx}u_x^2a(u)a'(u)+\frac{7}{2}u_x^3u_{xx}a(u)a''(u)\\
&\quad-\frac{3}{2}u_xu_{xx}^3a''(u)+\frac{7}{2}u_xu_{xx}^2a(u)a'(u)+9u_{xxx}u_{xx}u_x^2a''(u)\\
&\quad+\frac{3}{2}u_x^4u_{xxx}a'''(u)+\frac{21}{2}u_x^3u_{xx}\big(a'(u)\big)^2+\frac{7}{2}u_x^5a'(u)a''(u)\\
&\quad+\frac{7}{2}\epsilon u_{xx}^3u_ta''(u)+21\epsilon u_xu_{xx}^2u_{xt}a''(u)+\frac{21}{2}\epsilon u_x^2u_{xx}^2u_ta'''(u)\\
&\quad+14\epsilon u_x^3u_{xx}u_{xt}a'''(u)+\frac{7}{2}\epsilon u_x^4u_{xx}u_ta^{(4)}(u)-7\epsilon u_{xxx}^2u_ta'(u)\Big]dxds.
\end{aligned}
\tag{4.20}
$$

We recall again the convention that constants dependent only on the data f and g will generally be denoted simply by C, and that this symbol's occurrence in different formulae is not taken to connote the same constant.

First, an argument analogous to that leading to (4.15) shows that

$$||u_{xx}||^2_{C_b(\bar{\mathbb{R}}^+\times[0,t])} \le C\delta^{-3} + \delta \sup_{0\le s\le t} \{||u_{xxx}(\cdot,s)||^2\}. \tag{4.21}$$

By (4.12), (4.15) and (4.21), the terms in (4.20) that feature integration with respect to x only are bounded in the terms of a suitable small multiple of $||u_{xxx}(\cdot,t)||^2 + ||u_{xx}(\cdot,t)||^4$ provided ϵ is small. Note that equation (3.1a) implies

$$-\int_0^t u_{xxx}(0,s)u_{xxt}(0,s)ds = \int_0^t \big[u_{xxt}(0,s)[g'(s) + a(g(s))u_x(0,s) - \epsilon u_{xxt}(0,s)]\big]ds.$$

Integration by parts with respect to t yields

$$\begin{aligned}\int_0^t \big[u_{xxt}(0,s)a(g(s))u_x(0,s)\big]ds &= u_{xx}(0,s)a(g(s))u_x(0,s)\big|_{s=0}^{s=t}\\ &\quad - \int_0^t \big[u_{xx}(0,s)[a'(g(s))g'(s)u_x(0,s) + a(g(s))u_{xt}(0,s)]\big]ds.\end{aligned}$$

Similarly, one shows that

$$\int_0^t u_{xxt}(0,s)g'(s)ds = u_{xx}(0,s)g'(s)\big|_{s=0}^{s=t} - \int_0^t u_{xx}(0,s)g''(s)ds.$$

Then due to (4.15) and (4.21), the first boundary term can be estimated as

$$\begin{aligned}&\int_0^t 2u_{xxt}(0,s)\big[\frac{7}{2}g'(s) - u_{xxx}(0,s)\big]ds\\ &= \int_0^t 2u_{xxt}(0,s)\big[\frac{7}{2}g'(s) + g'(s) + a(g(t))u_x(0,s) - \epsilon u_{xxt}(0,s)\big]ds\\ &\le C\delta^{-1} - 2\epsilon\int_0^t u_{xxt}^2(0,s)ds + \delta\int_0^t u_{xt}^2(0,s)ds\\ &\quad + \delta\Big(\int_0^t u_{xx}^2(0,s)ds\Big)^2 + \delta||u_{xxx}(\cdot,t)||^2 + \delta||u_{xx}(\cdot,t)||^4\end{aligned}$$

for any $\delta > 0$, where C depends on $||f||_1$ and $|g|_{2,T}$. Elementary inequalities show that there is a positive constant C depending on $||f||_1$ and $|g|_{2,T}$ such that for any $\delta > 0$,

$$\begin{aligned}&\int_0^t \big[6u_{xxx}^2(0,s)a(g(s)) + 7u_{xx}(0,s)u_{xt}(0,s)a(g(s))\big]ds\\ &\le C\delta^{-3} + \delta\Big(\int_0^t u_{xxx}^2(0,s)ds\Big)^2 + \delta\int_0^t u_{xt}^2(0,s)ds + \delta\Big(\int_0^t u_{xx}^2(0,s)ds\Big)^2.\end{aligned}$$

By using (4.14), there is another constant C depending on $||f||_1$ and $|g|_{1,T}$ such that for any $\delta > 0$,

$$\int_0^t \Big[\frac{3}{2}u_x^3(0,s)u_{xxx}(0,s)a''(g(s)) - \frac{7}{2}\epsilon u_x^3(0,s)u_{xxt}(0,s)a''(g(s))$$

$$+ u_x(0,s)u_{xx}(0,s)u_{xxx}(0,s)a'(g(s)) - 7\epsilon u_x(0,s)u_{xx}(0,s)u_{xxt}(0,s)a'(g(s))\Big]ds$$

$$\leq C||u_x||^2_{C_b(\mathbb{R}^+\times[0,T])}\Big(\int_0^t u_x^2(0,s)ds\Big)^{\frac{1}{2}}\Big[\Big(\int_0^t u_{xxx}^2(0,s)ds\Big)^{\frac{1}{2}} + \epsilon\Big(\int_0^t u_{xxt}^2(0,s)ds\Big)^{\frac{1}{2}}\Big]$$

$$+ C||u_x||_{C_b(\mathbb{R}^+\times[0,T])}\Big(\int_0^t u_{xx}^2(0,s)ds\Big)^{\frac{1}{2}}\Big[\Big(\int_0^t u_{xxx}^2(0,s)ds\Big)^{\frac{1}{2}} + \epsilon\Big(\int_0^t u_{xxt}^2(0,s)ds\Big)^{\frac{1}{2}}\Big]$$

$$\leq C(\delta^{-3}) + \delta\Big(\epsilon\int_0^t u_{xxt}^2(0,s)ds + \Big(\int_0^t u_{xx}^2(0,s)ds\Big)^2 + \sup_{0\leq s\leq t}\{||u_{xx}(\cdot,s)||^4\}\Big)$$

$$+ \delta\Big(\int_0^t u_{xxx}^2(0,s)ds\Big)^2.$$

Similarly, one shows that

$$-\int_0^t \frac{3}{2}u_{xx}^3(0,s)a'(g(s))ds \leq C||u_{xx}||_{C_b(\mathbb{R}^+\times[0,T])}\int_0^t u_{xx}^2(0,s)ds$$

$$\leq C\Big(\sup_{0\leq s\leq t}\{||u_{xx}(\cdot,s)||||u_{xxx}(\cdot,s)||\}\Big)^{\frac{1}{2}}\int_0^t u_{xx}^2(0,s)ds$$

$$\leq C\delta^{-5} + \delta\Big(\int_0^t u_{xx}^2(0,s)ds\Big)^2 + \delta\sup_{0\leq s\leq t}\{||u_{xx}(\cdot,s)||^4 + ||u_{xxx}(\cdot,s)||^2\}.$$

By using (3.1a), the last boundary term in (4.20) is seen to satisfy the inequality

$$-\frac{7}{2}\int_0^t u_x^2(0,s)a'(g(s))\big[u_{xxxx}(0,s) - \epsilon u_{xxxt}(0,s)\big]ds$$

$$= \frac{7}{2}\int_0^t u_x^2(0,s)a'(g(s))\big[u_{xt}(0,s) + u_{xx}(0,s)a(g(s)) + u_x^2(0,s)a'(g(s))\big]ds$$

$$\leq C\delta^{-3} + \delta\Big(\int_0^t u_{xx}^2(0,s)ds\Big)^2 + \delta\int_0^t u_{xt}^2(0,s)ds + \delta\sup_{0\leq s\leq t}\{||u_{xx}(\cdot,s)||^4\}.$$

The use of (4.15), (4.16) and (4.21) shows that the entire set of double integrals in (4.20) except the last one can be controlled by the quantity

$$\int_0^t \big(A^4(s) + B^2(s) + \epsilon A^8(s) + \epsilon B^4(s)\big)ds.$$

Using (4.4) to bound $||u_{xxx}(\cdot,t)||_{C_b(\mathbb{R}^+)}$ and applying (4.16) shows that the last double integral in (4.20) is bounded above by

$$\Big| -7\int_0^t\int_0^{+\infty} \epsilon u_{xxx}^2 u_t a'(u)dxds\Big|$$
$$\leq \int_0^t C\epsilon||u_{xxx}(\cdot,s)||_{C_b(\mathbb{R}^+)}||u_{xxx}(\cdot,s)||||u_t(\cdot,s)||ds$$
$$\leq C\int_0^t \epsilon||u_{xxx}(\cdot,s)||^{\frac{3}{2}}||u_{xxxx}(\cdot,s)||^{\frac{1}{2}}||u_t(\cdot,s)||ds$$
$$\leq C\int_0^t \epsilon||u_{xxx}(\cdot,s)||^2||u_t(\cdot,s)||^{\frac{4}{3}}ds + C\int_0^t \epsilon||u_{xxxx}(\cdot,s)||^2 ds$$
$$\leq C\delta^{-\frac{7}{3}}+\delta\epsilon^{\frac{1}{3}}\Big(\epsilon\int_0^t u_{xt}^2(0,s)ds\Big)^2+C\int_0^t\epsilon||u_{xxx}(\cdot,s)||^4ds+C\int_0^t\epsilon||u_{xxxx}(\cdot,s)||^2ds.$$

From the preceding estimates, it is deduced that there exist positive constants a_3 and C_3, where

$$a_3 = a_3(||f||_3 + \epsilon_1^{\frac{1}{2}}||f^{(4)}||, |g|_{2,T}) \quad \text{and} \quad C_3 = C_3(||f||_1, |g|_{1,T}),$$

such that the solution of (3.1) corresponding to f and g satisfies

$$||u_{xxx}(\cdot,t)||^2 + \epsilon||u_{xxxx}(\cdot,t)||^2 + \int_0^t \big[u_{xt}^2(0,s) + \epsilon u_{xxt}^2(0,s) + u_{xxxx}^2(0,s)\big]ds$$
$$\leq a_3+\delta A^4(t)+C_3\int_0^t \big(A(s)^4+||u_{xxx}(\cdot,s)||^2+\epsilon||u_{xxxx}(\cdot,s)||^2 + \epsilon[||u_{xxx}(\cdot,s)||^2+\epsilon||u_{xxxx}(\cdot,s)||^2]^2\big)ds \tag{4.22}$$

for any $\delta > 0$. By adding an appropriate multiple of (4.22) to (4.10), it is adduced that the functionals $A(t)$ and $B(t)$ associated with the solution u are restricted by the inequality

$$A^4(t) + B^2(t) \leq \alpha + \beta\int_0^t \big[A^4(s) + B^2(s) + \epsilon(A^8(s) + B^4(s))\big]ds,$$

where α and β are positive constants that depend only on initial data f and boundary data g. Define $\bar{A}$ to be the maximal solution of the system

$$\bar{A}(t) = \alpha + \beta\int_0^t [\bar{A}(s) + \epsilon\bar{A}^2(s)]ds.$$

Then, $\bar{A}(t) \geq A^4(t) + B^2(t)$ for all t for which $\bar{A}(t)$ is finite. Moreover, $\bar{A}(t)$ may be determined explicitly as

$$\bar{A}(t) = \frac{\alpha e^{\beta t}}{1+\epsilon\alpha-\epsilon\alpha e^{\beta t}}$$

so long as $\epsilon\alpha e^{\beta t} < 1$, say. The desired result thus follows by choosing ϵ small enough. In fact, if ϵ_2 is chosen so that

$$1 + \epsilon_2\alpha - \epsilon_2\alpha e^{\beta T} \geq \frac{1}{2},$$

then the desired result is established. □

Corollary 4.5. *Let $T > 0$, $f \in H^\infty(\mathbb{R}^+)$, $g \in H^\infty(0,T)$, with $f(0) = g(0)$. There exists a constant a_4 with*

$$a_4 = a_4(||f||_3 + \epsilon^{\frac{1}{2}}||f^{(4)}||, |g|_{2,T})$$

such that the solution of (3.1) corresponding to the data f and g satisfies

$$||u_t(\cdot,t)|| + \epsilon^{\frac{1}{2}}||u_{xt}(\cdot,t)|| \leq a_4$$

provided that $t \in [0,T]$ and $\epsilon \in (0,1]$.

Proof: Using equation (3.1a), one shows that

$$\begin{aligned}||u_t(\cdot,t) - \epsilon u_{xxt}(\cdot,t)||^2 &\leq C(||u(\cdot,t)||_3)\\ &\leq C(||f||_3 + \epsilon^{\frac{1}{2}}||f^{(4)}||, |g|_{2,T}).\end{aligned}$$

Integrating by parts, one derives from the above inequality that

$$\begin{aligned}&||u_t(\cdot,t)||^2 + 2\epsilon||u_{xt}(\cdot,t)||^2 + \epsilon^2||u_{xxt}(\cdot,t)||^2\\ &\leq C(||f||_3 + \epsilon^{\frac{1}{2}}||f^{(4)}||, |g|_{2,T}) + 2\epsilon\left|g'(t)u_{xt}(0,t)\right|\\ &\leq C(||f||_3 + \epsilon^{\frac{1}{2}}||f^{(4)}||, |g|_{2,T}) + 4\epsilon|g|_{2,T}||u_{xt}(\cdot,t)||^{\frac{1}{2}}||u_{xxt}(\cdot,t)||^{\frac{1}{2}}\\ &\leq C(||f||_3 + \epsilon^{\frac{1}{2}}||f^{(4)}||, |g|_{2,T}) + \epsilon||u_{xt}(\cdot,t)||^2 + \frac{1}{2}\epsilon^2||u_{xxt}(\cdot,t)||^2,\end{aligned}$$

or, by elementary means, that

$$\begin{aligned}||u_t(\cdot,t)||^2 + \epsilon||u_{xt}(\cdot,t)||^2 + \frac{1}{2}\epsilon^2||u_{xxt}(\cdot,t)||^2&\\ \leq C(||f||_3 + \epsilon^{\frac{1}{2}}||f^{(4)}||, |g|_{2,T}).&\end{aligned}$$

The corollary is established. □

The bounds established in this section would be sufficient to conclude an existence theory set in the space $L_\infty(\mathbb{R}^+; H^3(\mathbb{R}^+))$ for the quarter-plane problem (3.1). If the further compatibility condition

$$g'(0) + a(f(0))f'(0) + f'''(0) = 0$$

is posited, where $f \in H^4(\mathbb{R}^+)$ and $g \in H^2(\mathbb{R}^+)$, then it will follow from the next two lemmas that the quarter-plane problem has a solution $u \in L_\infty(0,T; H^4(\mathbb{R}^+))$ with $u_t \in L_\infty(0,T; H^1(\mathbb{R}^+))$. These preliminary results will be improved in Section 7 when the issue of continuous dependence is considered.

Lemma 4.6. *Let $T > 0$, $f \in H^\infty(\mathbb{R}^+)$, $g \in H^\infty(0,T)$, with $f(0) = g(0)$. There exists a constant a_5 with*

$$a_5 = a_5(||u_t(\cdot,0)||_1, ||f||_4, |g|_{2,T})$$

such that the solution of (3.1) corresponding to the data f and g satisfies

$$||u_t(\cdot,t)||_1^2 + \int_0^t \left[u_{xxt}(0,s) - \epsilon u_{xtt}\right]^2 ds \leq a_5,$$

provided that $t \in [0,T]$ and $\epsilon \in (0,1]$.

Proof: Let $v(x,t) = u_t(x,t)$ so that v satisfies the partial-differential equation

$$v_t + \left(a(u)v\right)_x + v_{xxx} - \epsilon v_{xxt} = 0, \text{ for } (x,t) \in \bar{\mathbb{R}}^+ \times [0,T]. \tag{4.23}$$

An L_2-bound for $u_t(\cdot,t)$ has been established in Corollary 4.5. Now we derive an H^1-bound for $u_t(\cdot,t)$. Multiplying (4.23) by $2(\epsilon v_{xt} - a(u)v - v_{xx})$, integrating the results over $\mathbb{R}^+ \times (0,t)$ and then using some elementary inequalities including

$$\begin{aligned}\left|\int_0^{+\infty} v^3(x,t)dx\right| &\leq ||v(\cdot,t)||^2 ||v(\cdot,t)||_{C_b(\mathbb{R}^+)} \\ &\leq \sqrt{2}||v(\cdot,t)||^{\frac{5}{2}}||v_x(\cdot,t)||^{\frac{1}{2}} \\ &\leq C||v(\cdot,t)||^{\frac{10}{3}} + ||v_x(\cdot,t)||^3,\end{aligned}$$

one comes to

$$\begin{aligned}&||v_x(\cdot,t)||^2 + \epsilon\int_0^t (g''(s))^2 ds + \int_0^t [v_{xx}(0,s) - \epsilon v_{xt}(0,s)]^2 ds \\ &= ||v_x(\cdot,0)||^2 + \int_0^{+\infty} [a(u(x,t))v^2(x,t) - a(f(x))v^2(x,0)]dx \\ &\quad - \int_0^t a^2(g(s))(g''(s))^2 ds - \int_0^t 2a(g(s))g'(s)[v_{xx}(0,s) - \epsilon v_{xt}(0,s)]ds \\ &\quad - \int_0^t 2g'(s)v_x(0,s)ds - \int_0^t\int_0^{+\infty} a'(u)v^3 dxds \\ &\leq C(||u_t(\cdot,0)||_1, |g|_{2,T}) + C||v(\cdot,t)||^2 + C\int_0^t v_x^2(0,s)ds \\ &\quad + \frac{1}{2}\int_0^t [v_{xx}(0,s) - \epsilon v_{xt}(0,s)]^2 ds + C\int_0^t [||v_x(\cdot,s)||^2 + ||v(\cdot,s)||^{\frac{10}{3}}]ds.\end{aligned} \tag{4.24}$$

The use of Lemma 4.4, Corollary 4.5 and Gronwall's lemma in (4.24) shows that there is a constant a_5 depending on $||u_t(\cdot,0)||_1$ and $|g|_{2,T}$ such that

$$||v(\cdot,t)||_1^2 + \int_0^t [v_{xx}(0,s) - \epsilon v_{xt}(0,s)]^2 ds \leq a_5$$

for all $t \in [0,T]$. □

The constant a_5 in Lemma 4.6 depends on $||u_t(\cdot,0)||_1$, a quantity about which there is currently no information. In order to estimate usefully solutions of (3.1), some control of $||u_t(\cdot,0)||_1$ must be obtained in terms of the data f and g. An appropriate bound is forthcoming if the data satisfies an additional compatibility condition.

Lemma 4.7. *Let $T > 0$, $f \in H^\infty(\mathbb{R}^+)$, $g \in H^\infty(0,T)$, with $f(0) = g(0)$ and*

$$g'(0) = -[a(f(0))f'(0) + f'''(0)]. \tag{4.25}$$

There exists a constant a_6 where

$$a_6 = a_6(||f||_4, |g|_{2,T}),$$

such that the solution of (3.1) corresponding to the data f and g satisfies

$$||u_t(\cdot,0)||_1^2 \leq a_6,$$

for $t \in [0,T]$ and $\epsilon \in (0,\epsilon_2]$.

Proof: First note that by Corollary 4.5, $||u_t(\cdot,0)||^2$ is controlled by a constant of the form $a_4(||f||_3 + \epsilon^{\frac{1}{2}}||f||_4, |g|_{2,T})$. Let

$$\phi(x) = -[a(f(x))f'(x) + f'''(x)].$$

Then $u_t(\cdot,0)$ is a solution of the boundary-value problem

$$u_t(\cdot,0) - \epsilon u_{xxt}(\cdot,0) = \phi, \tag{4.26}$$

$$u_t(0,0) = g'(0), \quad \lim_{x\to+\infty} u_t(x,0) = 0.$$

Differentiate (4.26) with respect to x, multiply the result by $u_{xt}(\cdot,0)$ and integrate the result over $\mathbb{R}^+$. After integrations by parts, there appears the equation

$$||u_{xt}(\cdot,0)||^2 + \epsilon||u_{xxt}(\cdot,0)||^2 = \int_0^{+\infty} u_{xt}(x,0)\phi_x dx,$$

from which one obtains

$$||u_{xt}(\cdot,0)||^2 + \epsilon||u_{xxt}(\cdot,0)||^2 \leq C||\phi_x||^2 \leq C||f||_4^2,$$

where C is a constant independent of ϵ. Note that we have used $u_{xxt}(0,0) = 0$ which is obtained by using the compatibility condition (4.25) and equation (4.26). The proof of the lemma is then finished. □

The following lemmas will be helpful in Section 5 when higher-order estimates are considered.

Lemma 4.8. *Let $T > 0$, $f \in H^\infty(\mathbb{R}^+)$, $g \in H^\infty(0,T)$, with $f(0) = g(0)$. There exists a constant $a_7 = a_7(||u_t(\cdot,0)||_3, |g|_{3,T})$, such that the solution of (3.1) corresponding to the data f and g satisfies*

$$||u_t(\cdot,t)||_3^2 + \int_0^t \{[u_{xxxt}(0,s)]^2 + [u_{xtt}(0,s)]^2 + \epsilon[u_{xxtt}(0,s)]^2\}ds \le a_7,$$

provided that $t \in [0,T]$ and $\epsilon \in (0,\epsilon_2]$.

Proof: Let $v = u_t$, where u is the solution of the regularized initial- and boundary-value problem (3.1) corresponding to the given smooth and compatible data f and g. For t in $[0,T]$, define

$$A^2(t) = ||v(\cdot,t)||_3^2 + \int_0^t \{v_{xxx}^2(0,s) + v_{xt}^2(0,s) + \epsilon v_{xxt}^2(0,s)\}ds.$$

The Lemma 4.4 and Lemma 4.6 imply that

$$\begin{aligned} &||u||_{L_\infty(0,T;H^3(\mathbb{R}^+))}, \qquad ||v||_{L_\infty(0,T;H^1(\mathbb{R}^+))} \le C,\\ &||u||_{L_\infty(0,T;W^{2,\infty}(\mathbb{R}^+))}, \qquad ||v||_{L_\infty(\mathbb{R}^+\times[0,T])} \le C, \\ &\int_0^t v_x^2(0,s)ds \quad \text{and} \quad \int_0^t (v_{xx}(0,s) - \epsilon v_{xt}(0,s))^2 ds \le C, \end{aligned} \tag{4.27}$$

where here, and in the remainder of this proof, C will denote various constants which all depend on the same variables as the constant a_7 in the statement of the lemma, but which will always be independent of ϵ. Note that v satisfies equation (4.23). Differentiate (4.23) once with respect to x, multiply by $-2v_{xxx}$ and integrate the resulting expression over $\mathbb{R}^+ \times (0,t)$. There appears the equation

$$\begin{aligned} &||v_{xx}(\cdot,t)||^2 + \epsilon||v_{xxx}(\cdot,t)||^2 + \int_0^t v_{xxx}^2(0,s)ds \\ &= ||v_{xx}(\cdot,0)||^2 + \epsilon||v_{xxx}(\cdot,0)||^2 - 2\int_0^t v_{xt}(0,s)v_{xx}(0,s)ds \\ &+ 2\int_0^t\int_0^{+\infty} \left(a(u)v\right)_{xx} v_{xxx}dxds. \end{aligned} \tag{4.28}$$

Inequalities (4.27) imply that

$$\int_0^t\int_0^{+\infty} \left(a(u)v\right)_{xx} v_{xxx}dxds \le C\Big(1 + \int_0^t ||v(\cdot,s)||_3^2 ds\Big).$$

Note that

$$\begin{aligned} &-2\int_0^t v_{xt}(0,s)v_{xx}(0,s)ds \\ &= -2\int_0^t v_{xt}(0,s)[v_{xx}(0,s) - \epsilon v_{xt}(0,s)]ds - 2\epsilon\int_0^t v_{xt}^2(0,s)ds. \end{aligned}$$

Then by inequalities (4.27), one shows that for any $\delta > 0$, (4.28) can be estimated in the form

$$
\begin{aligned}
&||v_{xx}(\cdot,t)||^2 + \epsilon||v_{xxx}(\cdot,t)||^2 + \int_0^t v_{xxx}^2(0,s)ds + 2\epsilon\int_0^t v_{xt}^2(0,s)ds \\
&\qquad \le C_\delta + \delta\int_0^t v_{xt}^2(0,s)ds + C\int_0^t A^2(s)ds.
\end{aligned} \tag{4.29}
$$

Next, multiply (4.23) by $2v_{xxxt}$ and integrate the results over $\mathbb{R}^+ \times (0,t)$. After integrations by parts, we obtain the equation

$$
\begin{aligned}
&||v_{xxx}(\cdot,t)||^2 + \int_0^t [v_{xt}^2(0,s) + \epsilon v_{xxt}^2(0,s)]ds \\
&= ||v_{xxx}(\cdot,0)||^2 + \int_0^t 2v_{xxt}(0,s)v_t(0,s)ds \\
&- \int_0^{+\infty}\Big[2v_{xxx}(x,t)\big(a(u(x,t)v(x,t)\big)_x - 2v_{xxx}(x,0)\big(a(f(x)v(x,0)\big)_x\Big]dx \\
&+ \int_0^t\int_0^{+\infty} 2v_{xxx}\Big[\Big(a(u)v_{xt} + a'(u)u_xv_t\Big) + \big(a'(u)v^2\big)_x\Big]dxds.
\end{aligned} \tag{4.30}
$$

Because

$$
||v||_{L_\infty(0,t;W^{2,\infty}(\mathbb{R}^+))} \le C_\delta + \delta||v(\cdot,t)||_3^2, \tag{4.31}
$$

and because of the relations $v_{tt}(0,s) = g'''(s)$ and $v_t(0,s) = g''(s)$, it is adduced that corresponding to any $\delta > 0$ there is a constant C_δ for which

$$
\begin{aligned}
\int_0^t 2v_{xxt}(0,s)v_t(0,s)ds &= 2v_{xx}(0,s)v_t(0,s)\Big|_{s=0}^{s=t} - \int_0^t 2v_{xx}(0,s)v_{tt}(0,s)ds \\
&\le C_\delta + \delta||v(\cdot,t)||_3^2 + C\int_0^t ||v(\cdot,s)||_3^2 ds.
\end{aligned}
$$

One can easily obtain the inequality

$$
\Big|\int_0^{+\infty} v_{xxx}(x,t)\big(a(u(x,t))v(x,t)\big)_x dx\Big| \le C_\delta + \delta||v(\cdot,t)||_3^2,
$$

valid for any $\delta > 0$. Differentiating equation (4.23) with respect to x yields

$$
-v_{xt} = \big(a(u)v\big)_{xx} + v_{xxxx} - \epsilon v_{xxxt}.
$$

Using the above equation leads to the inequality

$$
\begin{aligned}
&\int_0^t \int_0^{+\infty} a(u) v_{xt} v_{xxx} dx ds \\
&= - \int_0^t \int_0^{+\infty} a(u) v_{xxx} \Big[\big(a(u)v\big)_{xx} + v_{xxxx} - \epsilon v_{xxxt} \big) \Big] dx ds \\
&\le C + \epsilon ||a(u)||_{C_b(\bar{\mathbb{R}}^+ \times [0,T])} ||v_{xxx}(\cdot,t)||^2 \\
&\quad + C \int_0^t v_{xxx}^2(0,s) ds + C \int_0^t (||v_{xxx}(\cdot,s)||^2 + ||v_{xx}(\cdot,s)||^2) ds \\
&\le C + C \int_0^t v_{xxx}^2(0,s) ds + C\epsilon ||v_{xxx}(\cdot,t)||^2 + C \int_0^t A^2(s) ds.
\end{aligned}
$$

By using equation (4.23) and (4.27) again, one shows that

$$
||v_t(\cdot,t) - \epsilon v_{xxt}(\cdot,t)||^2 \le ||v_{xxx}(\cdot,t)||^2 + C||v(\cdot,t)||_2^2.
$$

Expanding the norm on the left-hand side of this inequality and integrating by parts the mixed term gives

$$
\begin{aligned}
&||v_t(\cdot,t)||^2 + 2\epsilon ||v_{xt}(\cdot,t)||^2 + \epsilon^2 ||v_{xxt}(\cdot,t)||^2 \\
&\le C||v(\cdot,t)||_3^2 + 2\epsilon \big| v_t(0,t) v_{xt}(0,t) \big| \\
&\le C||v(\cdot,t)||_3^2 + 4\epsilon |g^{(k+1)}(t)| ||v_{xt}(\cdot,t)||^{\frac{1}{2}} ||v_{xxt}(\cdot,t)||^{\frac{1}{2}} \\
&\le C||v(\cdot,t)||_3^2 + C\epsilon^{\frac{1}{2}} |g^{(k+1)}(t)|^2 + \epsilon ||v_{xt}(\cdot,t)||^2 + \frac{1}{2}\epsilon^2 ||v_{xxt}(\cdot,t)||^2,
\end{aligned}
$$

from which there obtains

$$
||v_t(\cdot,t)||^2 + \epsilon ||v_{xt}(\cdot,t)||^2 + \frac{1}{2}\epsilon^2 ||v_{xxt}(\cdot,t)||^2 \le C||v(\cdot,t)||_3^2 + C\epsilon^{\frac{1}{2}} |g^{(k+1)}(t)|^2.
$$

Using this last information yields

$$
\begin{aligned}
\Big| \int_0^t \int_0^{+\infty} a'(u) u_x v_t v_{xxx} dx ds \Big| &\le C \int_0^t ||v_t(\cdot,s)|| \, ||v_{xxx}(\cdot,s)|| ds \\
&\le C + C \int_0^t A(s)^2 ds.
\end{aligned}
$$

Taking recourse to (4.27) again, we see that

$$
\begin{aligned}
\int_0^t \int_0^{+\infty} v_{xxx} \big(a'(u) v^2\big)_x dx ds &\le \int_0^t C ||v_{xxx}(\cdot,s)|| ||v_x(\cdot,s)|| ds \\
&\le C \int_0^t A^2(s) ds.
\end{aligned}
$$

If ϵ is chosen small enough, (4.30) can be estimated as

$$\begin{aligned} &||v_{xxx}(\cdot,t)||^2 + \int_0^t [v_{xt}^2(0,s) + \epsilon v_{xxt}^2(0,s)]ds \\ &\leq C + C\int_0^t v_{xxx}^2(0,s)ds + C\int_0^t A^2(s)ds. \end{aligned} \tag{4.32}$$

Multiply (4.29) by a suitable constant and add the result to (4.32). Then applying Gronwall's lemma shows that $A(t)$ is bounded by a constant a_7, as advertised. □

Lemma 4.9. *Let $T > 0$, $f \in H^\infty(\mathbb{R}^+)$, $g \in H^\infty(0,T)$, with $f(0) = g(0)$ and assume (4.25) also holds. Then there exist constants a_8 and a_9 where*

$$a_8 = a_8(||f||_6, |g|_{2,T}) \qquad \text{and} \qquad a_9 = a_9(||f||_6, |g|_{3,T}),$$

such that the solution of (3.1) corresponding to the data f and g satisfies

$$||u_t(\cdot,0)||_3^2 + \epsilon^2||u_{xxxxt}(\cdot,0)||^2 \leq a_8(||f||_6, |g|_{2,T}),$$

and in consequence,

$$||u_{tt}(\cdot,0)||^2 + \epsilon^2||u_{xtt}(\cdot,0)||^2 \leq a_9(||f||_6, |g|_{3,T}),$$

for $t \in [0,T]$ and $\epsilon \in (0,\epsilon_2]$.

Proof: Differentiate (4.26) with respect to x, multiply by $u_{xxxt}(\cdot,0)$ and integrate the result over $\mathbb{R}^+$. Since $u_{xxt}(0,0) = 0$, there appears

$$||u_{xxt}(\cdot,0)||^2 + \epsilon||u_{xxxt}(\cdot,0)||^2 = \int_0^{+\infty} u_{xxt}(x,0)\phi_{xx}(x)dx. \tag{4.33}$$

Differentiate (4.26) with respect to x twice, multiply the result by $u_{xxxxt}(\cdot,0)$ and integrate the result over $\mathbb{R}^+$ to obtain the equation

$$||u_{xxxt}(\cdot,0)||^2 + \epsilon||u_{xxxxt}(\cdot,0)||^2 = \int_0^{+\infty} u_{xxxt}(x,0)\phi_{xxx}(x)dx, \tag{4.34}$$

by again using the fact $u_{xxt}(0,0) = 0$. Applying some elementary inequalities to (4.33) and (4.34), one immediately obtains the first result in the statement of the lemma.

From equation (3.1a), one shows that u_{tt} is the solution of the boundary-value problem

$$\begin{aligned} u_{tt}(\cdot,0) - \epsilon u_{xxtt}(\cdot,0) &= \psi\big(u_t(\cdot,0), f(x), u_{xt}(\cdot,0), u_{xxxt}(\cdot,0)\big), \\ u_{tt}(0,0) = g''(0) \qquad &\text{and} \qquad \lim_{x\to\infty} u_{tt}(x,0) = 0, \end{aligned} \tag{4.35}$$

where

$$\begin{aligned} &\psi\big(u_t(\cdot,0), f(x), u_{xt}(\cdot,0), u_{xxxt}(\cdot,0)\big) \\ &= -a'(f(x))f(x)u_t(\cdot,0) - a(f(x))u_{xt}(\cdot,0) - u_{xxxt}(\cdot,0). \end{aligned}$$

The use of equation (4.35) shows that

$$||u_{tt}(\cdot,0)-\epsilon u_{xxtt}(\cdot,0)||^2=||\psi||^2.$$

Applying elementary inequalities, the results in Lemma 4.7 and the first part of this lemma, one concludes

$$\begin{aligned}&||u_{tt}(\cdot,0)||^2+2\epsilon||u_{xtt}(\cdot,0)||^2+\epsilon^2||u_{xxtt}(\cdot,0)||^2\\&=2\epsilon u_{tt}(0,0)u_{xtt}(0,0)+||\psi||^2\\&\le a_9(||f||_6,|g|_{3,T})+\frac{1}{2}\big(\epsilon||u_{xtt}(\cdot,0)||^2+\epsilon^2||u_{xxtt}(\cdot,0)||^2\big).\end{aligned}\tag{4.36}$$

The result now follows. □

The main results of this section are collected in the following umbrella theorem.

Theorem 4.10. *Let $T>0$, $f\in H^\infty(\mathbb{R}^+)$, $g\in H^\infty(0,T)$, with*

$$f(0)=g(0).$$

Let u be the solution of the regularized initial- and boundary-value problem (3.1) corresponding to the given data f and g. Then there is a constant a_{10}, depending on $||f||_3+\epsilon^{\frac{1}{2}}||f||_4$ and $|g|_{2,T}$, such that

$$||u(\cdot,t)||_3+||u_t(\cdot,t)||+\epsilon^{\frac{1}{2}}||u(\cdot,t)||_4\le a_{10},$$

for all t in $[0,T]$ and ϵ in $(0,\epsilon_2]$. Here ϵ_2 is the positive constant arising in Lemma 4.4. Moreover if

$$g'(0)+a(f(0))f'(0)+f'''(0)=0$$

holds, then

$$||u(\cdot,t)||_4+||u_t(\cdot,t)||_1\le a_{11}\qquad \text{and}\qquad ||u_t(\cdot,t)||_3\le a_{12},$$

where $a_{11}=a_{11}(||f||_4,|g|_{2,T})$ and $a_{12}=a_{12}(||f||_6,|g|_{3,T})$ for all t in $[0,T]$ and ϵ in $(0,\epsilon_2]$. □

5. Higher-Order Estimates for the Regularized Problem

The derivation of ϵ-independent bounds for solutions of the regularized initial- and boundary-value problem (3.1) is continued in this section. Smoother solutions would be expected to obtain provided the initial and boundary data are smooth enough. A proof of such further regularity, presented in the next section, is based on the additional estimates to be obtained in the present section.

The assumption that $f\in H^\infty(\mathbb{R}^+)$, $g\in H^\infty(0,T)$, and $f(0)=g(0)$ will continue to be enforced throughout this section. This hypothesis will be recalled informally by the stipulation that the data f and g are smooth and compatible. For simplicity, denote

$$u^{(j)}=\partial_t^j u.$$

The next lemma generalizes Lemma 4.6 and Lemma 4.8.

Lemma 5.1. *Let $T > 0$, $f \in H^\infty(\mathbb{R}^+)$, $g \in H^\infty(0,T)$, with $f(0) = g(0)$. There exists a constant b_1 where*

$$b_1 = b_1(\max_{0\le j\le k}\{||u^{(j)}(\cdot,0)||_1\}, |g|_{k+1,T}),$$

such that the solution of (3.1) corresponding to the data f and g satisfies

$$||u^{(k)}(\cdot,t)||_1^2 + \int_0^t [u_{xx}^{(k)}(0,s) - \epsilon u_{xt}^{(k)}(0,s)]^2 ds \le b_1,$$

provided that $t \in [0,T]$ and $\epsilon \in (0,\epsilon_2]$. Moreover there exists a constant b_2 where

$$b_2 = b_2(\max_{0\le j\le k}\{||u^{(j)}(\cdot,0)||_3\}, |g|_{k+2,T}),$$

such that

$$||u^{(k)}(\cdot,t)||_3^2 + \int_0^t \{[u_{xxx}^{(k)}(0,s)]^2 + [u_x^{(k+1)}(0,s)]^2 + \epsilon[u_{xx}^{(k+1)}(0,s)]^2\}ds \le b_2.$$

Proof: First note that for $k = 1$, the desired result is implied by Lemma 4.6 and Lemma 4.8. The proof proceeds by induction on k. Let $k > 1$, and suppose that the stated estimates hold for all nonnegative integers less than or equal to $k-1$. The induction hypothesis implies that

$$\begin{aligned} &||u^{(j)}||_{L_\infty(0,T;H^1(\mathbb{R}^+))}, \quad ||u^{(j)}||_{L_\infty(\mathbb{R}^+\times[0,T])}, \\ &\int_0^t \left(u^{(j)}\right)_x^2(0,s)ds + \int_0^t (u_{xx}^{(j)}(0,s) - \epsilon u_x^{(j+1)}(0,s))^2 ds \le b_1, \end{aligned} \tag{5.1a}$$

and

$$\begin{aligned} &||u^{(j)}||_{L_\infty(0,T;H^3(\mathbb{R}^+))}, \qquad ||u^{(j)}||_{L_\infty(0,T;W^{2,\infty}(\mathbb{R}^+))}, \\ &\int_0^t \{[u_{xxx}^{(j)}(0,s)]^2 + [u_x^{(j+1)}(0,s)]^2 + \epsilon[u_{xx}^{(j+1)}(0,s)]^2\}ds \le b_2, \end{aligned} \tag{5.1b}$$

for $0 \le j \le k-1$. In the remainder of this proof, C will denote various constants which all depend on the same variables as the constant b_1 or b_2 given in the statement of the lemma, but which will always be independent of ϵ. For any integer $j \ge 1$ the function $u^{(j)}$ satisfies the equation

$$u_t^{(j)} + \left(a(u)u^{(j)} + h_j(u)\right)_x + u_{xxx}^{(j)} - \epsilon u_{xxt}^{(j)} = 0, \text{ for } (x,t) \in \bar{\mathbb{R}}^+ \times [0,T], \tag{5.2}$$

where

$$h_j(u) = \sum_{i=1}^{j-1} \binom{j-1}{i} a(u)^{(i)} u^{(j-i)} \quad \text{and} \quad h_1(u) = 0.$$

The induction hypothesis implies that

$$||h_k(u)||_{L_\infty(0,T;W^{2,\infty}(\mathbb{R}^+))} \leq c||h_k(u)||_{L_\infty(0,T;H^3(\mathbb{R}^+))} \leq C. \tag{5.3}$$

Let $v = u^{(k)}$, where u is the solution of the regularized initial- and boundary-value problem (3.1) corresponding to the given smooth and compatible data f and g. Then v satisfies the equation

$$v_t + \big(a(u)v + h_k(u)\big)_x + v_{xxx} - \epsilon v_{xxt} = 0. \tag{5.4}$$

For t in $[0,T]$, define

$$A^2(t) = ||v(\cdot,t)||_1^2 + \int_0^t [v_{xx}(0,s) - \epsilon v_{xt}(0,s)]^2 ds.$$

The function A(t) will be estimated via an energy inequality derived from equation (5.4). Multiply this equation by $2v$ and integrate the result over $\mathbb{R}^+ \times (0,t)$ to obtain

$$\begin{aligned} &||v(\cdot,t)||^2 + \epsilon||v_x(\cdot,t)||^2 + \int_0^t v_x^2(0,s)ds \\ &= ||v(\cdot,0)||^2 + \epsilon||v_x(\cdot,0)||^2 + \int_0^t a(g(s))\big(g^{(k)}(s)\big)^2 ds \\ &+ \int_0^t 2g^{(k)}(s)[v_{xx}(0,s) - \epsilon v_{xt}(0,s)]ds + \int_0^t\int_0^{+\infty} \Big[a'(u)v^2 - 2v\big(h_k\big)_x\Big]dxds. \end{aligned}$$

The induction hypothesis implies that for any $\delta > 0$, there is a constant C_δ such that

$$\begin{aligned} &||v(\cdot,t)||^2 + \epsilon||v_x(\cdot,t)||^2 + \int_0^t v_x^2(0,s)ds \\ &\leq C_\delta\left(1 + \int_0^t ||v(\cdot,s)||_1^2 ds\right) + \delta\int_0^t [v_{xx}(0,s) - \epsilon v_{xt}(0,s)]^2 ds. \end{aligned} \tag{5.5}$$

Multiply (5.4) by $2(\epsilon v_{xt} - a(u)v - v_{xx})$ and integrate the results over $\mathbb{R}^+ \times (0,t)$. After simplifying, one obtains

$$\begin{aligned} &||v_x(\cdot,t)||^2 + \int_0^t \big(v_{xx}(0,s) - \epsilon v_{xt}(0,s) + a(g(s))g^{(k)}(s)\big)^2 ds \\ &= ||v_x(\cdot,0)||^2 + \int_0^{+\infty} a(u(x,t))v^2(x,t)dx - \int_0^{+\infty} a(f(x))v^2(x,0)dx \\ &+ \int_0^t \big(\epsilon(g^{(k+1)}(s))^2 - 2g^{(k+1)}(s)v_x(0,s)\big)ds \\ &+ \int_0^t\int_0^{+\infty} \Big[2\big(h_k(u)\big)_x(v_{xx} + a(u)v - \epsilon v_{xt}) - a'(u)u_t v^2\Big]dxds. \end{aligned} \tag{5.6}$$

Integrations by parts show

$$\int_0^t \int_0^{+\infty} [h_k(u)]_x v_{xx} dx ds$$
$$= -\int_0^t \big(h_k(u(0,s))\big)_x v_x(0,s) ds - \int_0^t \int_0^{+\infty} \big(h_k(u)\big)_{xx} v_x dx ds,$$

and

$$\epsilon \int_0^t \int_0^{+\infty} \big(h_k(u)\big)_x v_{xt} dx ds = \epsilon \int_0^{+\infty} \big(h_k(u(x,s))\big)_x v_x(x,s) dx \Big|_{s=0}^{s=t}$$
$$- \epsilon \int_0^t \int_0^{+\infty} \big(h_k(u)\big)_{xt} v_x dx ds.$$

Note that the leading term of the expression $\big(h_k(u)\big)_{xt}$ is $\big(H(u)v\big)_x$, where $H(u)$ is a function depending on $u, u_t, \cdots, u^{(k-1)}$. By the induction hypothesis (5.2) and (5.3), it transpires that for all $t \in [0,T]$,

$$\Big| \int_0^t \int_0^{+\infty} [h_k(u)]_x [v_{xx} + a(u)v - \epsilon v_{xt}] dx ds \Big|$$
$$\le \epsilon^2 C ||v_x(\cdot,t)||^2 + C\Big(1 + \int_0^t ||v(\cdot,s)||_1^2 ds + \int_0^t v_x^2(0,s) ds\Big).$$

From (5.5) and (5.6), it follows that

$$||v(\cdot,t)||_1^2 + \int_0^t (v_{xx}(0,s) - \epsilon v_{xt}(0,s))^2 ds \le C + C \int_0^t ||v(\cdot,s)||_1^2 ds. \tag{5.7}$$

Gronwall's lemma implies that there is a constant b_1 with

$$b_1 = b_1\big(\max_{0\le j\le k} \{||u^{(j)}(\cdot,0)||_1\}, |g|_{k+1,T}\big),$$

such that

$$A^2(t) = ||v(\cdot,t)||_1^2 + \int_0^t (v_{xx}(0,s) - \epsilon v_{xt}(0,s))^2 ds \le b_1,$$

for all $t \in [0,T]$.

To finish the lemma we need to control

$$||v(\cdot,t)||_3^2 \quad \text{and} \quad \int_0^t \{v_{xxx}^2(0,s) + v_{xt}^2(0,s) + \epsilon v_{xxt}^2(0,s)\} ds.$$

Define the quantity $B(t)$ to be the sum of these two quantities,

$$B^2(t) = ||v(\cdot,t)||_3^2 + \int_0^t \big\{v_{xxx}^2(0,s) + v_{xt}^2(0,s) + \epsilon v_{xxt}^2(0,s)\big\} ds.$$

Differentiate (5.4) once with respect to x, multiply by $-2v_{xxx}$ and integrate the resulting expression over $\mathbb{R}^+ \times (0,t)$ to reach the equation

$$\begin{aligned}
&||v_{xx}(\cdot,t)||^2 + \epsilon||v_{xxx}(\cdot,t)||^2 + \int_0^t v_{xxx}^2(0,s)ds \\
&= ||v_{xx}(\cdot,0)||^2 + \epsilon||v_{xxx}(\cdot,0)||^2 - 2\int_0^t v_{xt}(0,s)v_{xx}(0,s)ds \\
&+ 2\int_0^t\int_0^{+\infty} \big(a(u)v + h_k(u)\big)_{xx} v_{xxx} dxds.
\end{aligned} \tag{5.8}$$

Inequalities (5.1) and (5.3) imply that

$$\int_0^t\int_0^{+\infty} \big(a(u)v + h_k(u)\big)_{xx} v_{xxx} dxds \le C\Big(1 + \int_0^t ||v(\cdot,s)||_3^2 ds\Big).$$

Note that

$$\begin{aligned}
&-2\int_0^t v_{xt}(0,s)v_{xx}(0,s)ds \\
&= -2\int_0^t v_{xt}(0,s)[v_{xx}(0,s) - \epsilon v_{xt}(0,s)]ds - 2\epsilon\int_0^t v_{xt}^2(0,s)ds.
\end{aligned}$$

Then using the induction hypothesis (5.1), one shows that for any $\delta > 0$, (5.8) can be estimated in the form

$$\begin{aligned}
&||v_{xx}(\cdot,t)||^2 + \epsilon||v_{xxx}(\cdot,t)||^2 + \int_0^t v_{xxx}^2(0,s)ds + 2\epsilon\int_0^t v_{xt}^2(0,s)ds \\
&\qquad \le C_\delta + \delta\int_0^t v_{xt}^2(0,s)ds + C\int_0^t B^2(s)ds.
\end{aligned} \tag{5.9}$$

Next, multiply (5.4) by $2v_{xxxt}$ and integrate the results over $\mathbb{R}^+ \times (0,t)$ to come to the equation

$$\begin{aligned}
&||v_{xxx}(\cdot,t)||^2 + \int_0^t [v_{xt}^2(0,s) + \epsilon v_{xxt}^2(0,s)]ds \\
&= ||v_{xxx}(\cdot,0)||^2 + \int_0^t 2v_{xxt}(0,s)v_t(0,s)ds \\
&- \int_0^{+\infty} \Big[2v_{xxx}(x,t)\big(a(u(x,t)v(x,t) + h_k(u(x,t))\big)_x \\
&\qquad + 2v_{xxx}(x,0)\big(a(f(x)v(x,0) + h_k(u(x,0))\big)_x\Big]dx \\
&- \int_0^t\int_0^{+\infty} 2v_{xxx}\Big[\big(a(u)v_{xt} + a'(u)u_x v_t\big) + \big(a'(u)u_t v + h_{k+1}(u)\big)_x\Big]dxds.
\end{aligned} \tag{5.10}$$

Because

$$||v||_{L_\infty(0,t;W^{2,\infty}(\mathbb{R}^+))} \le C_\delta + \delta||v(\cdot,t)||_3^2, \tag{5.11}$$

and because of the relations $v_{tt}(0,s) = g^{(k+2)}(s)$ and $v_t(0,s) = g^{(k+1)}(s)$, it is adduced that corresponding to any $\delta > 0$ there is a constant C_δ for which

$$\begin{aligned}\int_0^t 2v_{xxt}(0,s)v_t(0,s)ds &= 2v_{xx}(0,s)v_t(0,s)\Big|_{s=0}^{s=t} - \int_0^t 2v_{xx}(0,s)v_{tt}(0,s)ds\\ &\le C_\delta + \delta||v(\cdot,t)||_3^2 + C\int_0^t ||v(\cdot,s)||_3^2 ds.\end{aligned}$$

One also easily obtains the inequality

$$\Big|\int_0^{+\infty} v_{xxx}(x,t)\Big(a\big(u(x,t)\big)v(x,t) + h_k\big(u(x,t)\big)\Big)_x dx\Big| \le C_\delta + \delta||v(\cdot,t)||_3^2.$$

Differentiating equation (5.4) with respect to x yields

$$-v_{xt} = \big(a(u)v + h_k(u)\big)_{xx} + v_{xxxx} - \epsilon v_{xxxt}.$$

Using the above equation leads to the inequality

$$\begin{aligned}&\int_0^t\int_0^{+\infty} a(u)v_{xt}v_{xxx}dxds\\ &= -\int_0^t\int_0^{+\infty} a(u)v_{xxx}\Big[\big(a(u)v + h_k(u)\big)_{xx} + v_{xxxx} - \epsilon v_{xxxt})\Big]dxds\\ &\le C + \epsilon||a(u)||_{C_b(\mathbb{R}^+\times[0,T])}||v_{xxx}(\cdot,t)||^2\\ &\quad + C\int_0^t v_{xxx}^2(0,s)ds + C\int_0^t (||v_{xxx}(\cdot,s)||^2 + ||v_{xx}(\cdot,s)||^2)ds\\ &\le C + C\int_0^t v_{xxx}^2(0,s)ds + C\epsilon||v_{xxx}(\cdot,t)||^2 + C\int_0^t B^2(s)ds.\end{aligned}$$

By again using equation (5.4), and the induction hypothesis, one sees that

$$||v_t(\cdot,t) - \epsilon v_{xxt}(\cdot,t)||^2 \le ||v_{xxx}(\cdot,t)||^2 + C||v(\cdot,t)||_2^2.$$

Expanding the norm on the left-hand side of this inequality and integrating by parts the mixed term gives

$$\begin{aligned}&||v_t(\cdot,t)||^2 + 2\epsilon||v_{xt}(\cdot,t)||^2 + \epsilon^2||v_{xxt}(\cdot,t)||^2\\ &\le C||v(\cdot,t)||_3^2 + 2\epsilon\big|v_t(0,t)v_{xt}(0,t)\big|\\ &\le C||v(\cdot,t)||_3^2 + 4\epsilon|g^{(k+1)}(t)|||v_{xt}(\cdot,t)||^{\frac12}||v_{xxt}(\cdot,t)||^{\frac12}\\ &\le C||v(\cdot,t)||_3^2 + C\epsilon^{\frac12}|g^{(k+1)}(t)|^2 + \epsilon||v_{xt}(\cdot,t)||^2 + \frac12\epsilon^2||v_{xxt}(\cdot,t)||^2,\end{aligned}$$

from which one obtains

$$||v_t(\cdot,t)||^2+\epsilon||v_{xt}(\cdot,t)||^2+\frac{1}{2}\epsilon^2||v_{xxt}(\cdot,t)||^2\le C||v(\cdot,t)||_3^2+C\epsilon^{\frac{1}{2}}|g^{(k+1)}(t)|^2.$$

Using this last information yields

$$\begin{aligned}\Big|\int_0^t\int_0^{+\infty}a'(u)u_xv_tv_{xxx}dxds\Big|&\le C\int_0^t||v_t(\cdot,s)||\,||v_{xxx}(\cdot,s)||ds\\&\le C+C\int_0^t B(s)^2ds.\end{aligned}$$

Note also that $|h_{k+1}(u)|\le|H_k(u)v|$, where $H_k(u)$ is a function containing terms in h_k. Using this information and the induction hypothesis shows that

$$\begin{aligned}\int_0^t\int_0^{+\infty}v_{xxx}\big(a'(u)u_tv+h_{k+1}(u)\big)_xdxds&\le\int_0^t C||v_{xxx}(\cdot,s)||||v_x(\cdot,s)||ds\\&\le C\int_0^t B^2(s)ds.\end{aligned}$$

If ϵ is chosen small enough, (5.10) can be estimated as

$$\begin{aligned}&||v_{xxx}(\cdot,t)||^2+\int_0^t[v_{xt}^2(0,s)+\epsilon v_{xxt}^2(0,s)]ds\\&\le C+C\int_0^t v_{xxx}^2(0,s)ds+C\int_0^t B^2(s)ds.\end{aligned}\tag{5.12}$$

Multiply (5.9) by a suitable constant and add the result to (5.12). Then applying Gronwall's lemma shows that $B(t)$ is bounded by a constant b_2, as advertised. This completes the induction argument and hence the proof of Lemma 5.1. □

The bounds established in the last lemma are just what will be needed in Section 6, except that, so far as is known now, not all the arguments of the constant b_1 and b_2 are independent of ϵ. To attain the goal for this section, it will suffice to give conditions on the data f and g which imply that $||u^j(\cdot,0)||_3$ and $||u^{j+1}(\cdot,0)||_1$ are bounded independently of ϵ for $0\le j\le k$.

Let $\phi^{(0)}(x)=f(x)$, and for each integer $j\ge1$ define functions $\phi^{(j)}$ inductively by the recurrence

$$\phi^{(j+1)}=-\Big[\phi_{xxx}^{(j)}+\Big(\sum_{i=0}^{j-1}\binom{j-1}{i}\big(a(\phi)\big)^{(i)}\phi^{(j-i)}\Big)_x\Big].\tag{5.13}$$

Here is the result to which allusion was just made.

Lemma 5.2. *Let $f \in H^\infty(\mathbb{R}^+)$, $g \in H^\infty(0,T)$ be given, with $f(0) = g(0)$. Let $k \geq 1$ be a given integer and suppose additionally that*

$$g^{(j)}(0) = \phi^{(j)}(0), \quad \text{for } j = 1, 2, \cdots, k. \tag{5.14}$$

Then there exists a constant b_3, depending continuously on $||f||_{3k+1}$ and $|g|_{k+1,T}$ such that

$$||u^{(j)}(\cdot,0)||_1 + \epsilon^{\frac{1}{2}}||u^{(j)}(\cdot,0)||_2 \leq b_3,$$

and there exists a constant b_4, depending continuously on $||f||_{3(k+1)}$ and $|g|_{k+2,T}$ such that

$$||u^{(j)}(\cdot,0)||_3 + \epsilon^{\frac{1}{2}}||u^{(j)}(\cdot,0)||_4 \leq b_4,$$

for $0 \leq j \leq k$ and all $\epsilon \in (0,1]$.

Proof: Note that when $k = 1$, the desired result is implied by Lemma 4.7 and Lemma 4.9. The proof proceeds by induction on k. Let $k > 1$, and suppose that the stated estimates hold for all non-negative integers less than or equal to $k - 1$. Let $u^{(k)}(x,0) = v(x,0)$. From equation (5.2), $v(x,0)$ satisfies the boundary-value problem

$$v - \epsilon v_{xx} = \phi^{(k)}, \tag{5.15}$$

with

$$v(0,0) = g^{(k+1)}(0), \quad \text{and} \quad \lim_{x\to+\infty} v(x,0) = 0.$$

By the compatibility conditions (5.14), one has $v_{xx}(0,0) = 0$. Following the line of argument introduced in proving Lemma 4.7, but using (5.15), one shows that there is a constant $b_3 = b_3(||f||_{3k+1}, |g|_{k+1,T})$ such that

$$||u^{(j)}(\cdot,0)||_1 + \epsilon^{\frac{1}{2}}||u^{(j)}(\cdot,0)||_2 \leq b_3$$

for $0 \leq j \leq k$ and all $\epsilon \in (0,1]$. As worked out in Lemma 4.9, one then shows that there is a constant $b_4 = b_4(||f||_{3(k+1)}, |g|_{k+2,T})$, such that

$$||u^{(j)}(\cdot,0)||_3 + \epsilon^{\frac{1}{2}}||u^{(j)}(\cdot,0)||_4 \leq b_4,$$

for $0 \leq j \leq k$ and all $\epsilon \in (0,1]$. □

It is worth summarizing the accomplishments of this section in the following theorem. This is a higher-order analogue of Theorem 4.10. In the statement of the theorem, ϵ_2 is the same positive constant that already appeared in Theorem 4.10.

Theorem 5.3. *Let $T > 0$ and a positive integer k be given. Let $f \in H^\infty(\mathbb{R}^+)$ and $g \in H^\infty(0,T)$ be given with $f(0) = g(0)$. Furthermore, suppose f and g satisfy*

$$g^{(j)}(0) = \phi^{(j)}(0), \quad \text{for } 1 \leq j \leq k,$$

where the functions $\phi^{(j)}$ are related to f as in (5.13). Then there exists a constant $b_5 = b_5(||f||_{3k+1}, |g|_{k+1,T})$, such that

$$||u^{(j)}(\cdot,t)||_1 \leq b_5,$$

holds for $1 \le j \le k$ *and all* $\epsilon \in (0, \epsilon_2]$.

Moreover there exists a constant $b_6 = b_6(||f||_{3(k+1)}, |g|_{k+2,T})$, *depending continuously on its arguments, such that*

$$||u^{(j)}(\cdot,t)||_3 + \epsilon^{\frac{1}{2}}||\partial_x^4 u^{(j)}(\cdot,t)|| \le b_6,$$

holds for $1 \le j \le k$ *and all* $\epsilon \in (0, \epsilon_2]$.

6. Existence and Uniqueness of Solutions

Using the theory developed in Sections 3, 4 and 5, it is comparatively simple to prove existence of smooth solutions of the quarter-plane problem for the equation

$$u_t + P(u)_x + u_{xxx} = 0, \qquad \text{for } x,\ t > 0, \tag{6.1a}$$

subject to the auxiliary conditions,

$$\begin{aligned} u(x,0) &= f(x) && \text{for } x \ge 0, \\ u(0,t) &= g(t) && \text{for } t \ge 0, \end{aligned} \tag{6.1b}$$

where f and g are given functions.

It is useful to first settle uniqueness of solutions of this initial- and boundary-value problem.

Theorem 6.1. *Let* $T > 0$ *and* $s > \frac{3}{2}$. *Then, corresponding to given auxiliary data* f *and* g, *there is at most one solution of (6.1) in the function class* $L_\infty(0,T; H^s(\mathbb{R}^+))$.

Proof: This result is proved as in Theorem 6.1 of **[17]** for the KdV equation (1.2). □

Theorem 6.2. *Let* k *be a positive integer. Suppose* $f \in H^{3k+1}(\mathbb{R}^+)$ *and* $g \in H^{k+1}_{loc}(\mathbb{R}^+)$, *or* $f \in H^{3k+3}(\mathbb{R}^+)$ *and* $g \in H^{k+2}_{loc}(\mathbb{R}^+)$, *and the* $k+1$ *compatibility conditions*

$$g^{(j)}(0) = \phi^{(j)}(0) \quad \text{for } 0 \le j \le k,$$

hold, where $\phi^{(j)}$ *is defined in (5.13). Then, corresponding to the given auxiliary data* f *and* g, *there exists a unique solution* u *of (6.1) in the function class* $L_\infty(0,T; H^s(\mathbb{R}^+))$, *where* $s = 3k+1$, *or* $3k+3$, *respectively. Furthermore there exist a constant* C_{3k+1} *depending on* $||f||_{3k+1}$ *and* $|g|_{k+1,T}$, *and a constant* C_{3k+3} *depending on* $||f||_{3k+3}$ *and* $|g|_{k+2,T}$ *such that for* $0 \le j \le k$,

$$\int_0^T [(u_x^{(j)})^2(0,s) + (u_{xx}^{(j)})^2(0,s)]ds \le C_{3j+1} \tag{6.2}$$

and

$$\int_0^T (u_x^{(k+1)})^2(0,s)ds \le C_{3k+3}. \tag{6.3}$$

If $k > 1$ in the first case, or $k \geq 1$ in the second case, u defines a classical solution, up to the boundary, of (6.1) in the quarter-plane $\mathbb{R}^+ \times \mathbb{R}^+$.

The proof is made in two steps. First, existence of a smooth solution of (6.1) corresponding to smooth initial data and smooth boundary data is established. Then a limit is taken through smooth solutions of (6.1) to infer existence of solutions corresponding to initial data in $H^{3k+1}(\mathbb{R}^+)$, or $H^{3(k+1)}(\mathbb{R}^+)$, and boundary data in $H^{k+1}_{loc}(\mathbb{R}^+)$, or $H^{k+2}_{loc}(\mathbb{R}^+)$, respectively. In pursuing this program, the following technical lemma (see [**16**], Lemma 7) is useful.

Lemma 6.3. *Suppose $u_n \to u$ weak-star in $L_\infty(0,T;H^s(\mathbb{R}^+))$ where $s > \frac{1}{2}$ and $\partial_t u_n \to \partial_t u$ weak-star in $L_2(0,T;H^r(\mathbb{R}^+))$ for some real r. Then there exists a subsequence $\{u_l\}$ of $\{u_n\}$ such that $u_l \to u$ pointwise almost everywhere in $[0,T] \times \mathbb{R}^+$ and $a(u_l)\partial_x u_l \to a(u)u_x$ in $\mathcal{D}'([0,T] \times \mathbb{R}^+)$ in the usual sense of distributions ($a(u)u_x$ is interpreted as $\partial_x P(u)$ in case $s < 1$ and similarly for $a(u_l)\partial_x u_l$).*

First, it is established that solutions of (6.1) exist in case f and g happen to be infinitely smooth.

Proposition 6.4. *Let $T > 0$ and k a positive integer. Let $f \in H^\infty(\mathbb{R}^+)$ and $g \in H^\infty(0,T)$ satisfy the $k+1$ compatibility conditions,*

$$g^{(j)}(0) = \phi^{(j)}(0), \qquad \text{for } 0 \leq j \leq k.$$

Then there exists a solution u of (6.1) in $L_\infty(0,T;H^{3k+3}(\mathbb{R}^+))$ corresponding to the data f and g. Moreover, there exist constants

$$b_{3k+1} = b_{3k+1}(||f||_{3k+1}, |g|_{k+1,T}) \quad \text{and} \quad b_{3k+3} = b_{3k+3}(||f||_{3k+3}, |g|_{k+2,T})$$

such that

$$||u^{(j)}(\cdot,t)||_3 \leq b_{3k+1}, \quad \text{for } 0 \leq j < k, \quad ||u^{(k)}(\cdot,t)||_1 \leq b_{3k+1}, \tag{6.4}$$

and

$$||u^{(j)}(\cdot,t)||_3 \leq b_{3k+3}, \tag{6.5}$$

for $0 \leq j \leq k$. The constants b_s for the various values of s can be chosen to depend continuously on their arguments.

Proof: The argument very closely parallels others appearing in many standard works, so it is presented in outline only. Throughout, $T > 0$ will be fixed, but arbitrary.

According to Corollary 3.8, for any $\epsilon > 0$ there is a smooth solution u_ϵ of the regularized initial- and boundary-value problem (3.1) corresponding to the data f and g. And by Theorem 5.3, there is a constant $b = b(||f||_{3(k+1)}, |g|_{k+2,T})$ depending continuously on its arguments, but independent of ϵ in $(0, \epsilon_2]$, such that

$$||u_\epsilon^{(j)}(\cdot,t)||_3 \leq b,$$

for $0 \le j \le k$. Moreover, for all nonnegative integers i and m,

$$\partial_t^i u_\epsilon \in C(0,T;H^m(\mathbb{R}^+)), \tag{6.6}$$

by Corollary 4.2 and in particular for $0 \le j \le k$, $\{\partial_t^j u_\epsilon\}_{0<\epsilon\le\epsilon_2}$ is bounded in $L_\infty(0,T;H^3(\mathbb{R}^+))$, independently of ϵ.

These bounds together with standard compactness results due to Aubin and Lions (cf. **[34]**, Lemma 6.3), and diagonalization procedures imply that there is a sequence $\{\epsilon_n\}_{n=1}^\infty$ tending to zero as n tends to infinity such that if $u_n = u_{\epsilon_n}$, $n = 1,2,\cdots$, then

$$\partial_t^j u_n \to \partial_t^j u, \quad \text{weak-star in } L_\infty(0,T;H^3(\mathbb{R}^+)) \tag{6.7}$$

for $0 \le j \le k$ and

$$a(u_n)\partial_x u_n \to a(u)\partial_x u \text{ in } \mathcal{D}'(\mathbb{R}^+ \times [0,T]), \tag{6.8}$$

as $n \to +\infty$. Moreover, the function u satisfies the generalized KdV-equation (6.1a) on $\mathbb{R}^+ \times [0,T]$; if $k = 0$, u satisfies the equation in the $L_2(\mathbb{R}^+)$-sense, while if $k \ge 1$, the solution is classical. The initial and boundary conditions are easily inferred to be taken on in appropriate senses. In particular, by standard interpolation results (cf. **[35]**) it is inferred that

$$\partial_t^j u \in C(0,T;H^{3(k-j)+\frac{3}{2}}(\mathbb{R}^+)) \tag{6.9}$$

for $0 \le j < k$. □

With a little change in the details, one shows that the argument of Proposition 6.4 also applies to the case when $s = 3$.

Corollary 6.5. *Let $f \in H^\infty(\mathbb{R}^+)$ and $g \in H^\infty_{loc}(\mathbb{R}^+)$ with $f(0) = g(0)$. Then there exists a unique solution u of (6.1) in $L_\infty(0,T;H^3(\mathbb{R}^+))$ and*

$$||u(\cdot,t)||_3 \le b(||f||_3, |g|_{2,T}).$$

Proof: By following the line of argument adopted in proving Proposition 6.4, and using Lemma 4.4, one shows that u lies in $L_\infty(0,T;H^3(\mathbb{R}^+))$ and $a(u)u_x$ lies in $L_\infty(0,T;H^2(\mathbb{R}^+))$. Hence, from the differential equation (6.1), $u_t \in L_\infty(0,T;L_2(\mathbb{R}^+))$. Moreover, according to Lemma 4.4, If u_ϵ is the approximating solution of the regularized problem corresponding to the value of the perturbation parameter */epsilon*, there is a constant a depending continuously on $||f||_3 + \epsilon^{\frac{1}{2}}||f^{(4)}||$ and $|g|_{2,T}$ such that

$$||u_\epsilon(\cdot,t)||_3 \le a,$$

at least for ϵ sufficiently small. Since the solution u of (6.1) pertains to the weak limit as ϵ tends to zero, it follows that

$$||u(\cdot,t)||_3 \le \limsup_{\epsilon\to 0} a = b,$$

say, where $b = b(||f||_3, |g|_{2,T})$, as advertised. □

Now we pass to the second stage of the proof of Theorem 6.2, where it is supposed that the initial condition f is constrained only to lie in $H^{3k}(\mathbb{R}^+)$ or $H^{3k+1}(\mathbb{R}^+)$, and the boundary condition g to be a member of $H^{k+1}_{loc}(\mathbb{R}^+)$, for some integer $k \geq 1$. The following result from **[17]** will prove to be useful in the present context.

Lemma 6.6. *Let f and g satisfy the conditions in Theorem 6.2. Then there exist sequences $\{f_N\}_{N=1}^{\infty} \subset H^{\infty}(\mathbb{R}^+)$ and $\{g_N\}_{N=1}^{\infty} \subset C^{\infty}(\mathbb{R}^+)$ such that*

$$g_N^{(j)}(0) = \phi_N^{(j)}(0) \quad \text{for } 0 \leq j \leq k$$

and for which, as $N \to \infty$,

$$\begin{array}{llll} (i) & f_N \to f \text{ in } H^{3k+1}(\mathbb{R}^+), & g_N \to g \text{ in } H^{k+1}_{loc}(\mathbb{R}^+) & \text{or} \\ (ii) & f_N \to f \text{ in } H^{3k+3}(\mathbb{R}^+), & g_N \to g \text{ in } H^{k+2}_{loc}(\mathbb{R}^+) & \end{array}$$

where $\phi_N^{(j)}$ is as defined in (5.13) with f_N replacing f, and $g_N^{(j)} = \partial_t^j g_N$.

Proof of Theorem 6.2: Suppose that $f \in H^{3k+3}(\mathbb{R}^+)$ and $g \in H^{k+2}_{loc}(\mathbb{R}^+)$ are fixed, and that f and g satisfy the first $k+1$ compatibility conditions as in the statement of the theorem. For fixed $T > 0$, Lemma 6.6 implies there exist sequences $\{f_N\}_{N=1}^{\infty} \subset H^{\infty}(\mathbb{R}^+)$ and $\{g_N\}_{N=1}^{\infty} \subset H^{\infty}(\mathbb{R}^+)$ such that

$$f_N \to f \quad \text{in} \quad H^{3k+3}(\mathbb{R}^+) \quad \text{and} \quad g_N \to g \quad \text{in} \quad H^{k+2}(0,T) \tag{6.10}$$

as $N \to +\infty$. For each $N > 0$, f_N and g_N satisfy the same $k+1$ compatibility conditions satisfied by f and g. From Proposition 6.4, corresponding to the auxiliary data f_N and g_N there is a solution u_N of (6.1) defined on $\mathbb{R}^+ \times [0,T]$ such that $\partial_t^j u_N \in L_\infty(0,T; H^{(3(k+1)-3j)}(\mathbb{R}^+))$ for $0 \leq j \leq k$, $N = 1, 2, \cdots$. Moreover, there exist constants

$$b_{3k+3}^N = b_{3k+3}(||f_N||_{3k+3}, |g_N|_{k+2,T})$$

such that

$$||\partial_t^j u_N||_{L_\infty(0,T;H^3(\mathbb{R}^+))} \leq b_{3k+3}^N, \quad \text{for } 0 \leq j \leq k. \tag{6.11}$$

Because of (6.10) and the fact that b_{3k+3} is uniformly bounded when its arguments vary over a bounded set, there is a constant B_{3k+3}, independent of N, such that

$$||\partial_t^j u_N||_{L_\infty(0,T;H^3(\mathbb{R}^+))} \leq B_{3k+3}, \tag{6.12}$$

for $0 \leq j \leq k$. Similarly, one shows that if f lies in $H^{3k+1}(\mathbb{R}^+)$ and g lies in $H^{k+1}_{loc}(\mathbb{R}^+)$, there exists a solution u_N of (6.1) on $\mathbb{R}^+ \times [0,T]$ with initial- and boundary-data f_N and g_N, respectively, for which $\partial_t^j u_N \in L_\infty(0,T; H^{(3k+1-3j)}(\mathbb{R}^+))$ for $0 \leq j \leq k$, and a constant B_{3k+1} such that

$$\begin{aligned} &||\partial_t^j u_N||_{L_\infty(0,T;H^3(\mathbb{R}^+))} \leq B_{3k+1}, \quad \text{for } 0 \leq j < k, \text{ and} \\ &||\partial_t^k u_N||_{L_\infty(0,T;H^1(\mathbb{R}^+))} \leq B_{3k+1}. \end{aligned} \tag{6.13}$$

In consequence of the bounds expressed in (6.12) and (6.13), the arguments of Proposition 6.4 may be repeated without essential change (the extra smoothness available during the proof of the proposition was not used, nor was the regularizing term $-\epsilon u_{xxt}$). It is concluded therefore that $\{u_N\}_1^\infty$ converges to a function u_T, say, in the various ways already detailed in the proof of Proposition 6.4. As in the proposition, u_T provides a solution of (6.1) corresponding to the data f and g, in $H^{3k+1}(\mathbb{R}^+)$ and $H_{loc}^{k+1}(\mathbb{R}^+)$, or in $H^{3(k+1)}(\mathbb{R}^+)$, and $H_{loc}^{k+2}(\mathbb{R}^+)$.

The above argument applies for any fixed $T > 0$. Define a function U on $\mathbb{R}^+ \times \mathbb{R}^+$ by,

$$U(x,t) = u_T(x,t),$$

provided that $t < T$. This is well defined because of the uniqueness result. It is clear that U provides the solution whose existence was contemplated in the statement of Theorem 6.2. The fact that U is a classical solution of the problem (6.1), if $f \in H^{3k+1}(\mathbb{R}^+)$ for $k > 1$, or $f \in H^{3k+3}(\mathbb{R}^+)$ for $k \geq 1$, follows exactly as in the proof of the Proposition 6.4. This finishes the proof of Theorem 6.2. □

The above arguments also apply to the case $s = 3$ because of Corollary 6.5.

Corollary 6.7. *Let $f \in H^3(\mathbb{R}^+)$, and $g \in H_{loc}^2(\mathbb{R}^+)$, with $f(0) = g(0)$. Then, corresponding to given auxiliary data f and g, there exists a unique solution u of (6.1) in the function class $L_\infty(0,T;H^3(\mathbb{R}^+))$. Moreover, for any $T > 0$, there is a constant C depending on $||f||_3$ and $|g|_{2,T}$, such that $\int_0^T u_{xt}^2(0,s)ds \leq C$.*

If $f \in H^1(\mathbb{R}^+)$ and $g \in H_{loc}^1(\mathbb{R}^+)$, then Theorem 6.2 also holds because of the ϵ-independent $H^1(\mathbb{R}^+)$-bound established in Lemma 4.1. In this case, the equation will be satisfied in the sense of distributions. However, the uniqueness result does not apply. The proof of existence of these weaker solutions fits more or less directly into the framework exposed in the proof of Proposition 6.4. The outcome is stated here.

Theorem 6.8. *Let $f \in H^1(\mathbb{R}^+)$, and $g \in H_{loc}^1(\mathbb{R}^+)$, with $f(0) = g(0)$. Then for any $T > 0$, there exists a solution of (6.1) in the function class $L_\infty(0,T;H^1(\mathbb{R}^+))$ corresponding to the initial data f and boundary data g.*

7. Solutions in More Restricted Spaces & Continuous-Dependence Results

In Section 6, we obtained a unique solution of the generalized KdV equation posed in a quarter-plane. Thus if $f \in H^s(\mathbb{R}^+)$ where $s = 3k$, or $3k+1$, and $g \in H_{loc}^{k+1}(\mathbb{R}^+)$ satisfy the appropriate compatibility conditions at $(x,t) = (0,0)$, then the quarter-plane problem has a solution in $L_{loc}^\infty(\mathbb{R}^+;H^s(\mathbb{R}^+))$. In this section it will be shown that the solutions lie in $C(\mathbb{R}^+;H^s(\mathbb{R}^+))$, and a result of continuous dependence of solutions on the data in spaces that are as restrictive as the solutions allow will be established.

To prove such a result, we follow the line worked out for the KdV equation (1.2) in **[18]**. Since we deal with a more general nonlinearity and our theory implies that solutions exist corresponding to weaker assumptions on the auxiliary data, details of the proof are provided. We first show that the solution u_n corresponding to smooth initial data f_n and boundary data g_n which approximate f in $H^{3k}(\mathbb{R}^+)$ and g in $H^{k+1}_{loc}(\mathbb{R}^+)$ appropriately are Cauchy in $C(0,T;H^{3k}(\mathbb{R}^+))$ for any fixed $T > 0$. It follows that for any $T > 0$, $u_n \to u$ strongly in $C(0,T;H^{3k}(\mathbb{R}^+))$ for some u. This is accomplished by deriving bounds for the difference between two solutions, say u_1 and u_2, of the initial- and boundary-value problem (6.1). These bounds may be expressed in terms of corresponding differences in the initial and the boundary data for the two solutions. After such bounds are obtained, the result that $\{u_n\}_{n=1}^{\infty}$ is Cauchy in $C(0,T;H^{3k}(\mathbb{R}^+))$ follows by choosing appropriate approximations to the initial and boundary data.

Throughout this section X_{3k} will be the set

$$X_{3k} = \{(f,g) \in H^{3k}(\mathbb{R}^+) \times H^{k+1}_{loc}(\mathbb{R}^+) \colon \phi^{(j)}(0) = g^{(j)}(0) \quad \text{for } 0 \le j \le k-1\}, \tag{7.1}$$

and X_{3k+1} will be the set

$$X_{3k+1} = \{(f,g) \in H^{3k+1}(\mathbb{R}^+) \times H^{k+1}_{loc}(\mathbb{R}^+) \colon \phi^{(j)}(0) = g^{(j)}(0) \quad \text{for } 0 \le j \le k\}, \tag{7.2}$$

where ϕ is expressed in terms of the initial data f as in (5.13) and k is a positive integer. Assume that (f_1, g_1) and (f_2, g_2) are two sets of data for the problem (6.1) which lie in X_{3k}, or X_{3k+1}. By Corollary 6.7 and Theorem 6.2, the corresponding solutions u_1 and u_2 of (6.1) will be elements of $L^{\infty}_{loc}(\mathbb{R}^+; H^{3k}(\mathbb{R}^+))$, or $L^{\infty}_{loc}(\mathbb{R}^+; H^{3k+1}(\mathbb{R}^+))$, respectively. Moreover, there exist constants $C^m_{k,T}$ which depend only on T, $||f_m||_{3k}$, or $||f_m||_{3k+1}$, and $|g|_{k+1,T}$, which bound above

$$||u_m||_{L_\infty(0,T;H^{3k}(\mathbb{R}^+))} \quad \text{or} \quad ||u_m||_{L_\infty(0,T;H^{3k+1}(\mathbb{R}^+))} \tag{7.3}$$

for $m = 1, 2$. As a corollary of Theorem 6.2, one also has that

$$|\partial_x^{3k+1} u_m(0,s)|_{0,T} \quad \text{or} \quad |\partial_x^{3k+2} u_m(0,s)|_{0,T} \tag{7.4}$$

is bounded above by $C^m_{k,T}$, for $m = 1, 2$. The constants $C^m_{k,T}$ will be shown to depend continuously on T, $||f_m||_{3k}$, or $||f_m||_{3k+1}$, and $|g|_{k+1,T}$, $m = 1, 2$. For convenience in writing the difference between datum, denote

$$f \equiv f_1 - f_2, \quad g \equiv g_1 - g_2, \quad \text{and} \quad w \equiv u_1 - u_2.$$

Then from (6.1), the function w satisfies the initial- and boundary-value problem

$$w_t + a(u_1)w_x + [a(u_1) - a(u_2)](u_2)_x + w_{xxx} = 0 \quad \text{for } x, t \in \mathbb{R}^+ \times \mathbb{R}^+, \tag{7.5a}$$

$$w(x,0) = f(x) \quad \text{for } x \ge 0, \tag{7.5b}$$

$$w(0,t) = g(t) \quad \text{for } t \ge 0. \tag{7.5c}$$

As before, for non-negative integers j, denote by $w^{(j)}$ the temporal derivative $\partial_t^j w$. Then $w^{(k)}$ satisfies the partial differential equation

$$\begin{aligned} w_t^{(k)} + (a(u_1))^{(k)} w_x + a(u_1) w_x^{(k)} + [a(u_1) - a(u_2)]^{(k)} (u_2)_x \\ + [a(u_1) - a(u_2)](u_2^{(k)})_x + w_{xxx}^{(k)} + F_k = 0, \end{aligned} \tag{7.6}$$

where $F_1 = 0$ and F_k is defined by

$$F_k = \sum_{j=1}^{k-1} \binom{k}{j} \Big[(a(u_1))^{(k-j)} w_x^{(j)} + [(a(u_1))^{(k-j)} - a((u_2))^{(k-j)}](u_2)_x^{(j)} \Big].$$

Lemma 7.1. *Let $(f_m, g_m) \in X_3$ for $m = 1, 2$. Then for any $T > 0$, there is a constant C_T depending continuously on T, $||f_m||_3$ and $|g_m|_{2,T}$, $m = 1, 2$, such that*

$$||w(\cdot, t)||^2 + \int_0^t w_x^2(0, s) ds \le C_T \{ ||f||^2 + |g|_{1,T}^2 \} \tag{7.7}$$

and

$$||w_x(\cdot, t)||^2 + \int_0^t w_{xx}^2(0, s) ds \le C_T \{ ||f||_1^2 + |g|_{1,T}^2 \} \tag{7.8}$$

for $0 \le t \le T$. If $(f_m, g_m) \in X_4$ for $m = 1, 2$, then there is a constant $C_{1,T}^1$ depending continuously on T, $||f_m||_3$ and $|g_m|_{2,T}$, $m = 1, 2$, such that

$$\begin{aligned} &||w(\cdot, t)||_3^2 + \int_0^t w_{xxxx}^2(0, s) ds \\ &\le C_{1,T}^1 \{ ||f||_3^2 + |g|_{2,T}^2 + ||w||_{L_\infty(\mathbb{R}^+ \times (0,T))}^2 ||u_2||_{L_2(0,T;H^4(\mathbb{R}^+))}^2 \}. \end{aligned} \tag{7.9}$$

Proof: Let $(f_m, g_m) \in X_3$ for $m = 1, 2$. It follows from Corollary 6.7 that u_m lies in $L_\infty(0, T; H^3(\mathbb{R}^+))$ and there is a constant C_T such that

$$||u_m||_{L_\infty(0,T;H^3(\mathbb{R}^+))} \le C_T \tag{7.10}$$

for $m = 1, 2$. Here, and below, C_T will denote different constants possessing the same properties as the constant C_T specified in the statement of the lemma.

First, it is shown that $||w(\cdot, t)||$ is bounded by $C_T\{||f||^2 + |g|_{1,T}^2\}$. Let $U(x, t) = g(t)e^{-x}$ and $y = w - U$. Then y satisfies the initial- and boundary-value problem

$$\begin{aligned} & y_t + [a(y_1 + U_1) - a(y_2 + U_2)](y_2 + U_2)_x \\ & \qquad + a(y_1 + U_1) y_x + y_{xxx} = h, && \text{for } x, t \in \mathbb{R}^+ \times \mathbb{R}^+, \\ & y(x, 0) = f(x) - g(0)e^{-x}, && \text{for } x \ge 0, \\ & y(0, t) = 0, && \text{for } t \ge 0, \end{aligned} \tag{7.11}$$

where $y_m = u_m - g_m(t)e^{-x} \equiv u_m - U_m$, $m = 1, 2$, and $h = U_t + U_{xxx} + a(y_1 + U_1)U_x$. Note that $y = y_1 - y_2$. To establish the advertised bound for $||w(\cdot, t)||$, one need only establish a similar estimate for y. Multiply (7.11) by $2y$ and then integrate the results over $\mathbb{R}^+ \times (0,t)$. After integrations by parts, there appears

$$\begin{aligned} &||y(\cdot,t)||^2 + \int_0^t y_x^2(0,s)ds \\ &= ||f - U(x,0)||^2 + \int_0^t \int_0^\infty 2y\Big(h - a(y_1+U_1)y_x \\ &\qquad - [a(y_1+U_1) - a(y_2+U_2)](y_2+U_2)_x\Big)dxds. \end{aligned} \tag{7.12}$$

By a further integration by parts, one sees that

$$\int_0^\infty 2a(y_1+U_1)yy_x dx = -\int_0^\infty [a(y_1+U_1)]_x y^2 dx.$$

From (7.10) and the definition of U, it therefore follows that

$$||[a(y_1+U_1)]_x||_{L_\infty(\mathbb{R}^+\times(0,T))} \le C_T, \qquad ||h||_{L_2(\mathbb{R}^+\times(0,T))} \le C_T|g|_{1,T}$$
$$\text{and} \quad ||U||_{L_\infty(\mathbb{R}^+\times(0,T))} \le C_T|g|_{1,T}.$$

Because of the Lipschitz-condition satisfied by P, one infers that

$$||(a(y_1+U_1) - a(y_2+U_2))||_{L_2(\mathbb{R}^+\times(0,T))} \le C_T\gamma(\mathcal{B})||y+U||_{L_2(\mathbb{R}^+\times(0,T))}, \tag{7.13}$$

where $\mathcal{B} = [0, C_T]$. Note that this follows since both arguments $y_1 + U_1 = u_1$ and $y_2 + U_2 = u_2$ lie in the set $\{u \colon ||u||_{L_\infty(\mathbb{R}^+\times(0,T))} \le C_T\}$. With the above estimates in hand, it follows from (7.11) and Gronwall's lemma that for $0 \le t \le T$,

$$||y(\cdot,t)||^2 + \int_0^t y_x^2(0,s)ds \le C_T\{||f||^2 + |g|_{1,T}^2\}.$$

To control $||w_x(\cdot,t)||$, multiply equation (7.5a) by $-2w_{xx}$ and integrate the result over $\mathbb{R}^+ \times (0,t)$ to obtain

$$\begin{aligned} ||w_x(\cdot,t)||^2 + \int_0^t w_{xx}^2(0,s)ds &= ||f'||^2 - \int_0^t 2g'(s)w_x(0,s)ds \\ &+ \int_0^t \int_0^\infty 2w_{xx}\big[a(u_1)w_x + (a(u_1) - a(u_2))(u_2)_x\big]dxds. \end{aligned} \tag{7.14}$$

By integrations by parts and using (7.7) and (7.10), one shows that

$$\begin{aligned} &\int_0^t \int_0^\infty 2w_{xx}a(u_1)w_x dxds \\ &= -\int_0^t a(u_1(0,s))w_x^2(0,s)ds - \int_0^t \int_0^\infty [a(u_1)]_x w_x^2 dxds \\ &\le C_T\{||f||_1^2 + |g|_{1,T}^2 + \int_0^t ||w_x(\cdot,s)||^2 ds\}. \end{aligned} \tag{7.15}$$

Also note that

$$\begin{aligned}
&\int_0^t\int_0^\infty 2w_{xx}[a(u_1)-a(u_2)](u_2)_x dxds\\
&=-\int_0^t\int_0^\infty \Big[2w_x[a(u_1)-a(u_2)]_x(u_2)_x+2w_x[a(u_1)-a(u_2)](u_2)_{xx}\Big]dxds\\
&\quad-\int_0^t 2w_x(0,s)[a(u_1(0,s))-a(u_2(0,s))](u_2)_x(0,s)ds \qquad (7.16)\\
&\le C_T\Big(\int_0^t w_x^2(0,s)ds\Big)^{\frac12}\gamma(\mathcal{B})\Big(\int_0^t g^2(s)ds\Big)^{\frac12}+C_T\gamma(\mathcal{B})\int_0^t \|w(\cdot,s)\|_1^2 ds,
\end{aligned}$$

where $\mathcal{B}=[0,C_T]$ again, by using (7.7), (7.10) and (7.13). Hence using (7.15) and (7.16), (7.14) is reduced to

$$\|w_x(\cdot,t)\|^2+\int_0^t w_{xx}^2(0,s)ds\le C_T\{\|f\|_1^2+|g|_{1,T}^2+\int_0^t\|w(\cdot,s)\|_1^2ds\} \qquad (7.17)$$

for $0\le t\le T$. Applying the first result and Gronwall's lemma to the above inequality gives the second result.

To obtain the last result, let $(f_m,g_m)\in X_4$, $m=1,2$. Differentiate equation (7.5a) once and multiply the result by $2w_{xxx}$. Differentiate equation (7.5a) twice and multiply the result by $2w_{xxxx}$. Add the above results together and then integrate over $\mathbb{R}^+\times[0,t]$. After integrations by parts, there appears

$$\begin{aligned}
&\|w_{xx}(\cdot,t)\|^2+\|w_{xxx}(\cdot,t)\|^2+\int_0^t\big[w_{xxx}^2(0,s)+w_{xxxx}^2(0,s)\big]ds\\
&=\|w_{xx}(\cdot,0)\|^2+\|w_{xxx}(\cdot,0)\|^2\\
&\quad-2\int_0^t\Big[w_{xx}(0,s)w_{xs}(0,s)+w_{xxx}(0,s)w_{xxs}(0,s)\\
&\qquad\qquad+w_{xxx}(0,s)\Big(a(u_1)w_x+(a(u_1)-a(u_2))(u_2)_x\Big)_{xx}(0,s)\Big]ds\\
&\quad+2\int_0^t\int_0^{+\infty}\Bigg[\big[a(u_1)w_x+(a(u_1)-a(u_2))(u_2)_x\big]_x \qquad (7.18)\\
&\qquad\qquad-\sum_{j=1}^3\binom{3}{j}\Big[\partial_x^j\big(a(u_1)\big)\partial_x^{(3-j)}(w_x)\\
&\qquad\qquad\qquad+\partial_x^j\big(a(u_1)-a(u_2)\big)\partial_x^{(3-j)}\big((u_2)_x\big)\Big]\\
&\qquad\qquad-a(u_1)w_{xxxx}-(a(u_1)-a(u_2))(u_2)_{xxxx}\Bigg]w_{xxx}dxds.
\end{aligned}$$

By using (7.7), (7.8) and equation (7.5a), the first boundary term in (7.18) is estimated from above as follows:

$$\begin{aligned}\left|2\int_0^t w_{xx}(0,s)w_{xs}(0,s)ds\right| &\leq \frac{1}{\delta}\int_0^t w_{xx}^2(0,s)ds + \delta\int_0^t w_{xs}^2(0,s)ds \\ &\leq C_T(||f||_1,|g|_{1,T}) + \delta\int_0^t w_{xxxx}^2(0,s)ds, \end{aligned} \tag{7.19}$$

for any $\delta > 0$. Similarly, one sees that for any $\delta > 0$,

$$\begin{aligned} &-\int_0^t w_{xxx}(0,s)w_{xxs}(0,s)ds \\ &= \int_0^t \big(g'(s)+a(g_1(s))w_x(0,s)+[a(g_1(s))-a(g_2(s))](u_2)_x(0,s)\big)w_{xxs}(0,s)ds \\ &= \big(g'(s)+a(g_1(s))w_x(0,s)+[a(g_1(s))-a(g_2(s))](u_2)_x(0,s)\big)w_{xx}(0,s)\Big|_0^t \\ &\quad -\int_0^t \Big[g''(s)+a'(g_1(s))g_1'(s)w_x(0,s)+a(g_1(s))w_{xs}(0,s) \\ &\qquad + \big([a(g_1(s))-a(g_2(s))](u_2)_x(0,s)\big)_s\Big]w_{xx}(0,s)ds \\ &\leq C_{1,T}^1(||f||_3,|g|_{2,T}) + \delta\int_0^t w_{xs}^2(0,s)ds \\ &\leq C_{1,T}^1(||f||_3,|g|_{2,T}) + \delta\int_0^t w_{xxxx}^2(0,s)ds. \end{aligned} \tag{7.20}$$

Note that the inequality

$$\int_0^t \big((u_m)_{xs}(0,s)\big)^2 ds \leq C(||f_m||_3,|g_m|_{2,T})$$

has been used in (7.20) for $m = 1, 2$. This fact is obtained via Corollary 6.7. Again by using (7.7) and (7.8), the last boundary integral in (7.18) is bounded above by a constant $C_T = C_T(||f||_1,|g|_{1,T})$.

There is a constant $C_{1,T}^1 = C_{1,T}^1(||f||_3,|g|_{2,T})$ such that all the terms featuring double integrals in (7.18) except the last two terms can be bounded above by

$$C_{1,T}^1(||f||_3,|g|_{2,T})\int_0^t ||w(\cdot,s)||_3^2 ds.$$

By integrations by parts and using once more the hypothesis on P, one shows that

$$\begin{aligned}&\Big|\int_0^t\int_0^{+\infty}2\big[a(u_1)w_{xxxx}+(a(u_1)-a(u_2))(u_2)_{xxxx}\big]w_{xxx}dxds\Big|\\&\le\int_0^t a(g_1(s))w_{xxx}^2(0,s)ds+C_T\int_0^t||w_{xxx}(\cdot,s)||^2ds\\&\quad+C_T(||f||_1,|g|_{1,T})\gamma(\mathcal{B})||w||^2_{L_\infty(\mathbb{R}^+\times(0,T))}||u_2||^2_{L_2(0,T;H^4(\mathbb{R}^+))},\end{aligned}\tag{7.21}$$

where $\mathcal{B}$ is as before. Using the preceding information in (7.18) and choosing δ small enough, one obtains

$$\begin{aligned}||w(\cdot,t)||_3^2\le C^1_{1,T}\{||f||_3+|g|_{2,T}+\int_0^t||w(\cdot,s)||_3^2ds\\+||w||^2_{L_\infty(\mathbb{R}^+\times(0,T))}||u_2||^2_{L_2(0,T;H^4(\mathbb{R}^+))}\}.\end{aligned}$$

Applying Gronwall's lemma to the above inequality shows that

$$||w(\cdot,t)||_3^2\le C^1_{1,T}\{||f||_3+|g|_{2,T}+||w||^2_{L_\infty(\mathbb{R}^+\times(0,T))}||u_2||^2_{L_2(0,T;H^4(\mathbb{R}^+))}\}.$$

The proof is complete. □

Bounds on higher-order Sobolev norms are needed for proving continuous-dependence results in higher-order Sobolev spaces. The next step is to derive such bounds. In the preliminary results stated and proved next, as with the results in Lemma 7.1, bounds on the H^s-norm of solutions are couched in the terms of H^{s+1}-norms of the initial data f if $s=3k$ or $s=3k+1$. This apparent defect in the theory in which smoothness is lost is remedied at a later stage.

Lemma 7.2. *Let $(f_m,g_m)\in X_{3k+1}$ for $m=1,2$. Then for any $T>0$, there exists a constant $C^1_{k,T}$ depending continuously on T, $||f_m||_{3k}$ and $|g_m|_{k+1,T}$, such that*

$$\begin{aligned}&||w^{(k)}(\cdot,t)||^2+\int_0^t(w_x^{(k)}(0,s))^2ds\\&\le C^1_{k,T}\{||f||^2_{3k}+|g|^2_{k+1,T}+||w||^2_{L_\infty(\mathbb{R}^+\times(0,T))}||u_2^{(k)}||^2_{L_2(0,T;H^1(\mathbb{R}^+))}\}.\end{aligned}\tag{7.22}$$

If $(f_m,g_m)\in X_{3k+3}$, then there exists a constant $C^2_{k,T}$ depending continuously on T, $||f_m||_{3k+1}$ and $|g_m|_{k+1,T}$ such that for $0\le t\le T$,

$$\begin{aligned}&||w_x^{(k)}(\cdot,t)||^2+\int_0^t(w_{xx}^{(k)}(0,s))^2ds\\&\le C^2_{k,T}\{||f||^2_{3k+1}+|g|^2_{k+1,T}+||w||^2_{L_\infty(\mathbb{R}^+\times(0,T))}||u_2^{(k)}||^2_{L_2(0,T;H^2(\mathbb{R}^+))}\}.\end{aligned}\tag{7.23}$$

Furthermore if $(f_m, g_m) \in X_{3k+4}$, *then*

$$||w^{(k)}(\cdot,t)||_3^2 + \int_0^t (w^{(k)}_{xxxx}(0,s))^2 ds$$
$$\leq C^1_{k+1,T}\{||f||^2_{3(k+1)} + |g|^2_{k+2,T} + ||w||^2_{L_\infty(\mathbb{R}^+\times(0,T))}||u_2^{(k)}||^2_{L_2(0,T;H^4(\mathbb{R}^+))}\}. \tag{7.24}$$

Proof: The lemma is proved by induction on k. The estimates (7.22), (7.23) and (7.24) are true when $k = 0$ by virtue of Lemma 7.1, and the constants $C^1_{0,T}$, $C^2_{0,T}$ and $C^1_{1,T}$ depend continuously on T, $||f_m||_3$ and $|g_m|_{2,T}$. Assume that the estimates (7.22), (7.23) and (7.24) regarding $w^{(j)}$ hold for $0 \leq j < k$. To obtain (7.22) for $j = k$, let $(f_m, g_m) \in X_{3k+1}$ for $m = 1, 2$. Then Theorem 6.2 implies that $u_m \in L_\infty(0,T; H^{3k+1}(\mathbb{R}^+))$. Hence the following calculations make sense.

First, make a transformation such that the solution vanishes on the boundary. This can be achieved by setting

$$y = w^{(k)} - g^{(k)}(t)e^{-x} \equiv w^{(k)} - U^{(k)}.$$

Then y satisfies the initial- and boundary-value problem

$$y_t + a(y_1 + U_1)y_x + [a(y_1+U_1) - a(y_2+U_2)]^{(k)}(y_2+U_2)_x$$
$$+ [a(y_1+U_1) - a(y_2+U_2)](y_2^{(k)} + U_2^{(k)})_x + y_{xxx} = h_k, \tag{7.25a}$$
$$\text{for } x, t \in \mathbb{R}^+ \times \mathbb{R}^+,$$
$$y(x,0) = \phi^{(k)}(x) - U^{(k)}, \quad \text{for } x \geq 0, \tag{7.25b}$$
$$y(0,t) = 0, \quad \text{for } t \geq 0, \tag{7.25c}$$

where $y_m = u_m^{(k)} - g_m^{(k)}(t)e^{-x} \equiv u_m^{(k)} - U_m^{(k)}$ for $m = 1, 2$,

$$h_k = -F_k + [U_t^{(k)} + U^{(k)}_{xxx} - (a(y_1+U_1))^{(k)}w_x + (a(y_1+U_1))U_x^{(k)}],$$

and $\phi^{(k)}(x)$ takes the form presented in (5.13) with $\phi^{(0)}(x) = f(x)$. After multiplying (7.25a) by $2y$ and integrating over $\mathbb{R}^+ \times (0,t)$, there appears

$$||y(\cdot,t)||^2 + \int_0^t y_x^2(0,s)ds = ||y(\cdot,0)||^2 + \int_0^t\int_0^\infty \Big[2h_k y + (a(y_1+U_1))_x y^2$$
$$- 2y[a(y_1+U_1) - a(y_2+U_2)]^{(k)}(y_2+U_2)_x \tag{7.26}$$
$$- 2y[a(y_1+U_1) - a(y_2+U_2)]\big(y_2^{(k)} + U_2^{(k)}\big)_x\Big]dxds.$$

Taking account of the definition of $U^{(k)}$ and $y(\cdot,0)$, one sees immediately that

$$||y(\cdot,0)|| \leq C^1_{k,T}\{||f||_{3k} + |g|_{k+1,T}\}.$$

By the induction hypothesis, there obtains

$$||h_k||_{L_2(\mathbb{R}^+\times(0,T))} \leq C^1_{k,T}\{||f||_{3k} + |g|_{k+1,T}\}.$$

Hence the first term in the double integral on the right-hand side of (7.26) is bounded above as follows:

$$\int_0^t \int_0^\infty 2yh_k dxds \le C^1_{k,T}\{||f||^2_{3k} + |g|^2_{k+1,T} + \int_0^t ||y(\cdot,s)||^2 ds\}.$$

The assumption about the nonlinearity P implies

$$||[a(u_1) - a(u_2)]^{(j)}||_{L_2(0,T;L_2(\mathbb{R}^+))} \le \gamma(\mathcal{B}) C^1_{k,T} \sum_{n=0}^{j} ||w^{(n)}||_{L_2(0,T;L_2(\mathbb{R}^+))}, \qquad (7.27)$$

where $0 \le j \le k$ and $\mathcal{B} = [0, C_T]$ as before. Note that

$$||(a(y_1 + U_1))_x||_{L_\infty(\mathbb{R}^+\times(0,T))} \le C^1_{k,T}.$$

With this information in hand, it is readily deduced that the second and third terms in the double integral in (7.26) are bounded above; viz.

$$\int_0^t \int_0^\infty \Big[(a(y_1 + U_1))_x y^2 - 2y[a(y_1+U_1) - a(y_2+U_2)]^{(k)}(y_2+U_2)_x\Big] dxds$$
$$\le C^1_{k,T}\{||f||^2_{3k} + ||g||^2_{k+1,T} + \int_0^t ||y(\cdot,s)||^2 ds\}.$$

Finally, note that

$$\Big|2\int_0^t \int_0^\infty y[a(y_1+U_1) - a(y_2+U_2)](y_2^{(k)} + U_2^{(k)})_x dxds\Big|$$
$$\le \int_0^t \int_0^\infty \Big[y^2 + [a(y_1+U_1) - a(y_2+U_2)]^2[(y_2^{(k)} + U_2^{(k)})_x]^2\Big] dxds$$
$$\le C^1_{k,T}\gamma(\mathcal{B})||w||^2_{L_\infty(\mathbb{R}^+\times(0,T))}||u_2^{(k)}||^2_{L_2(0,T;H^1(\mathbb{R}^+))} + \int_0^t ||y(\cdot,s)||^2 ds.$$

Combining the above estimates, (7.26) is reduced to

$$||y(\cdot,t)||^2 + \int_0^t y_x^2(0,s)ds \le C^1_{k,T}\{||f||^2_{3k} + |g|^2_{k+1,T}$$
$$+ ||w||^2_{L_\infty(\mathbb{R}^+\times(0,T))}||u_2^{(k)}||^2_{L_2(0,T;H^1(\mathbb{R}^+))} + \int_0^t ||y(\cdot,s)||^2 ds\}$$

for $0 \le t \le T$. If Gronwall's lemma is applied to this last inequality, one obtains the result (7.22).

To obtain (7.23), let $(f_m, g_m) \in X_{3k+3}$ for $m = 1, 2$. Theorem 6.2 implies that $u_m \in L_\infty(0,T; H^{3k+3}(\mathbb{R}^+))$. Multiply (7.6) by $-2w_{xx}^{(k)}$ and integrate the result over

$\mathbb{R}^+ \times (0,t)$. After integrations by parts, there appears

$$\begin{aligned}&||w_x^{(k)}(\cdot,t)||^2 + \int_0^t (w_{xx}^{(k)}(0,s))^2 ds \\ &= ||w_x^{(k)}(\cdot,0)||^2 - \int_0^t 2g^{(k+1)}(s) w_x^{(k)}(0,s)ds \\ &+ \int_0^t \int_0^\infty 2w_{xx}^{(k)} \Big[a(u_1) w_x^{(k)} + [a(u_1) - a(u_2)]^{(k)} (u_2)_x \\ &\qquad + [a(u_1) - a(u_2)](u_2^{(k)})_x + (a(u_1))^{(k)} w_x + F_k \Big] dxds.\end{aligned} \tag{7.28}$$

First note that

$$\int_0^\infty 2w_{xx}^{(k)} a(u_1) w_x^{(k)} dx = -a(u_1(0,s))[w_x^{(k)}]^2(0,s) - \int_0^\infty a(u_1)_x [w_x^{(k)}]^2 dx.$$

Therefore by using (7.3) and (7.22) one obtains

$$\begin{aligned}&\int_0^t \int_0^\infty 2a(u_1) w_{xx}^{(k)} w_x^{(k)} dxds \\ &\le C_{k,T}^1 \{ ||f||_{3k}^2 + |g|_{k+1,T}^2 + \int_0^t ||w_x^{(k)}(\cdot,s)||^2 ds \},\end{aligned} \tag{7.29}$$

for $0 \le t \le T$. By another integration by parts, one also shows that

$$\begin{aligned}&\int_0^\infty 2w_{xx}^{(k)} [(a(u_1))^{(k)} w_x + F_k] dx \\ &= -2w_x^{(k)}(0,s)[a(u_1(0,s))^{(k)} w_x(0,s) + F_k(0,s)] \\ &- \int_0^\infty 2w_x^{(k)} \Big[\big((a(u_1))^{(k)}\big)_x w_x + (a(u_1))^{(k)} w_{xx} + \big(F_k\big)_x \Big] dx.\end{aligned}$$

Note that (7.3), (7.22) and the induction hypothesis entail the inequality

$$||w^{(j)}||_{L_\infty(0,T;H^3(\mathbb{R}^+))} \le C_{k,T}^1 \{ ||f||_{3k} + |g|_{k+1,T} \}$$

for $0 \le j \le k-1$. This in turn implies that

$$||\partial_x(F_k)||_{L_2(\mathbb{R}^+ \times (0,T))} \le C_{k,T}^1 \{ ||f||_{3k} + |g|_{k+1,T} \}.$$

By the induction hypothesis and the first result of the lemma, one obtains

$$|F_k(0,s)|_T \le C_{k,T}^1 \{ ||f||_{3k} + |g|_{k+1,T} \}.$$

The preceding facts show that

$$\begin{aligned}&\int_0^t\int_0^\infty 2w_{xx}^{(k)}[(a(u_1))^{(k)}w_x+F_k]dxds\\&\le\int_0^t||w_x^{(k)}(\cdot,s)||^2ds+C_{k,T}^2\{||f||_{3k+1}^2+|g|_{k+1,T}^2\}.\end{aligned}\tag{7.30}$$

The inequality (7.27) implies

$$\begin{aligned}&\int_0^t\int_0^\infty 2w_{xx}^{(k)}[a(u_1)-a(u_2)]^{(k)}(u_2)_xdxds\\&=-\int_0^t\int_0^\infty 2w_x^{(k)}\Big[\partial_x\Big([a(u_1)-a(u_2)]^{(k)}\Big)(u_2)_x+[a(u_1)-a(u_2)]^{(k)}(u_2)_{xx}\Big]dxds\\&\quad-\int_0^t\{2w_x^{(k)}[a(u_1)-a(u_2)]^{(k)}(u_2)_x\}(0,s)ds\\&\le C_{k,T}^1\{||f||_{3k}^2+|g|_{k+1,T}^2\}\Big[\int_0^t||w_x^{(k)}(\cdot,s)||^2ds+\gamma\Big].\end{aligned}\tag{7.31}$$

Finally, integration by parts gives

$$\begin{aligned}&\int_0^t\int_0^\infty 2w_{xx}^{(k)}[a(u_1)-a(u_2)](u_2^{(k)})_xdxds\\&=-\int_0^t\{2w_x^{(k)}[a(u_1)-a(u_2)](u_2^{(k)})_x\}(0,s)ds\\&\quad-\int_0^t\int_0^\infty 2w_x^{(k)}\Big[\big(a(u_1)-a(u_2)\big)_x(u_2^{(k)})_x+[a(u_1)-a(u_2)](u_2^{(k)})_{xx}\Big]dxds.\end{aligned}$$

Note that

$$\begin{aligned}&\Big|\int_0^t\int_0^\infty\{2w_x^{(k)}\big(a(u_1)-a(u_2)\big)_x(u_2^{(k)})_x\}(x,s)dxds\Big|\\&\le\int_0^t||w_x^{(k)}(\cdot,s)||^2ds+C_{k,T}^2\{||f||_{3k+1}^2+|g|_{k+1,T}^2\},\end{aligned}\tag{7.32}$$

and that

$$\begin{aligned}&\Big|\int_0^t\int_0^\infty 2w_x^{(k)}[a(u_1)-a(u_2)](u_2^{(k)})_{xx}dxds\Big|\\&\le\int_0^t\int_0^\infty\Big[[w_x^{(k)}]^2+[a(u_1)-a(u_2)]^2[(u_2^{(k)})_{xx}]^2\Big]dxds\\&\le\int_0^t||w_x^{(k)}(\cdot,s)||^2ds+\gamma||w||_{L_\infty(\mathbb{R}^+\times(0,T))}^2||u_2^{(k)}||_{L_2(0,T;H^2(\mathbb{R}^+))}^2.\end{aligned}\tag{7.33}$$

The definition of $w^{(k)}$ together with equation (7.6) entail that

$$||w_x^{(k)}(\cdot,0)|| \le C_{k,T}^2\{||f||_{3k+1} + |g|_{k+1,T}\}. \tag{7.34}$$

Using the preceding relations (7.29), (7.30), (7.31), (7.32), (7.33) and (7.34) in (7.28) and applying Gronwall's lemma, it is confirmed that (7.19) holds.

To finish the induction, let $(f_m, g_m) \in X_{3k+4}$. Differentiate (7.6) once and multiply the result by $2w_{xxx}^{(k)}$. Differentiate (7.6) twice and multiply the result by $2w_{xxxx}^{(k)}$. Add the results together and then integrate over $\mathbb{R}^+ \times [0,t]$. After integrations by parts, we have

$$\begin{aligned}
&||w_{xx}^{(k)}(\cdot,t)||^2 + ||w_{xxx}^{(k)}(\cdot,t)||^2 + \int_0^t \left[(w_{xxx}^{(k)}(0,s))^2 + (w_{xxxx}^{(k)}(0,s))^2\right]ds \\
&= ||w_{xx}^{(k)}(\cdot,0)||^2 + ||w_{xxx}^{(k)}(\cdot,0)||^2 \\
&\quad - 2\int_0^t \Big[w_{xx}^{(k)}(0,s)w_{xs}^{(k)}(0,s) + w_{xxx}^{(k)}(0,s)w_{xxs}^{(k)}(0,s) \\
&\qquad\qquad + w_{xxx}^{(k)}(0,s)\Big(a(u_1)w_x^{(k)} + (a(u_1)-a(u_2))^{(k)}(u_2)_x \\
&\qquad\qquad + \big(a(u_1)\big)^{(k)}w_x + (a(u_1)-a(u_2))(u_2^{(k)})_x + F_k\Big)_{xx}(0,s)\Big]ds \\
&\quad + 2\int_0^t\int_0^{+\infty}\Big[\Big(a(u_1)w_x^{(k)} + (a(u_1)-a(u_2))^{(k)}(u_2)_x \\
&\qquad\qquad + \big(a(u_1)\big)^{(k)}w_x + (a(u_1)-a(u_2))(u_2^{(k)})_x + F_k\Big)_x \\
&\qquad\qquad - \Big((a(u_1)-a(u_2))^{(k)}(u_2)_x + \big(a(u_1)\big)^{(k)}w_x + F_k\Big)_{xxx} \\
&\qquad\qquad - \sum_{j=1}^{3}\binom{3}{j}\Big(\partial_x^j(a(u_1))\partial_x^{(3-j)}(w_x^{(k)}) + \partial_x^j(a(u_1)-a(u_2))\partial_x^{(3-j)}\big((u_2^{(k)})_x\big)\Big) \\
&\qquad\qquad - a(u_1)w_{xxxx}^{(k)} - (a(u_1)-a(u_2))(u_2^{(k)})_{xxxx}\Big]w_{xxx}^{(k)}\,dxds.
\end{aligned} \tag{7.35}$$

Using the first two results of the lemma and equation (7.6) shows that the first boundary term in (7.35) is bounded above, viz.

$$\begin{aligned}
\left|2\int_0^t w_{xx}^{(k)}(0,s)w_{xs}^{(k)}(0,s)ds\right| &\le \frac{1}{\delta}\int_0^t (w_{xx}^{(k)}(0,s))^2ds + \delta\int_0^t (w_{xs}^{(k)}(0,s))^2ds \\
&\le C_{k,T}^1(||f||_{3k+1}, |g|_{k+1,T}) + \delta\int_0^t (w_{xxxx}^{(k)}(0,s))^2ds
\end{aligned} \tag{7.36}$$

for any $\delta > 0$. Note that by Theorem 6.2, there is a constant C depending on $||f_m||_{3k+3}$ and $|g_m|_{k+2,T}$ such that for $m = 1, 2$,

$$\int_0^t \left((u_m^{(k+1)})_x(0,s)\right)^2 ds \le C(||f_m||_{3k+3}, |g_m|_{k+2,T}).$$

These inequalities, when combined with equation (7.6) show that for any $\delta > 0$,

$$\begin{aligned}
&-\int_0^t w_{xxx}^{(k)}(0,s)w_{xxs}^{(k)}(0,s)ds \\
&= \int_0^t \Big[g^{(k+1)}(s) + \big(a(g_1(s))\big)^{(k)} w_x(0,s) + [a(g_1(s)) - a(g_2(s))]^{(k)}(u_2)_x(0,s) \\
&+ a(g_1(s))w_x^{(k)}(0,s) + [a(g_1(s)) - a(g_2(s))](u_2^{(k)})_x(0,s) + F_k(0,s)\Big] w_{xxs}(0,s)ds \\
&= \Big[g^{(k+1)}(s) + \big(a(g_1(s))^{(k)}\big) w_x(0,s) + [a(g_1(s)) - a(g_2(s))]^{(k)}(u_2)_x(0,s) \\
&+ a(g_1(s))w_x^{(k)}(0,s) + [a(g_1(s)) - a(g_2(s))](u_2^{(k)})_x(0,s) + F_k(0,s)\Big] w_{xx}(0,s)\Big|_0^t \qquad (7.37) \\
&- \int_0^t \Big[g^{(k+1)}(s) + (a(g_1(s)))^{(k)} w_x(0,s) + [a(g_1(s)) - a(g_2(s))]^{(k)}(u_2)_x(0,s) \\
&+ a(g_1(s))w_x^{(k)}(0,s) + [a(g_1(s)) - a(g_2(s))](u_2^{(k)})_x(0,s) + F_k(0,s)\Big]_s w_{xx}(0,s)ds \\
&\le C_{k+1,T}^1(||f||_{3k+3}, |g|_{k+2,T}) + \delta \int_0^t \left(w_{xs}^{(k)}(0,s)\right)^2 ds \\
&\le C_{k+1,T}^1(||f||_{3k+3}, |g|_{k+2,T}) + \delta \int_0^t \left(w_{xxxx}^{(k)}(0,s)\right)^2 ds.
\end{aligned}$$

Again using (7.22) and (7.23), the last temporal integral is bounded above by a constant $C_{k+1,T}^1 = C_{k+1,T}^1(||f||_{3k+3}, |g|_{k+2,T})$. There is a constant depending only on $C_{k+1,T}^1$ such that all terms in the double integral in (7.35), except the last two, can be bounded above by

$$C_{k+1,T}^1(||f||_{3k+3}, |g|_{k+2,T}) \int_0^t ||w^{(k)}(\cdot,s)||_3^2 ds.$$

By integrations by parts and the hypothesis on P, one shows that

$$\begin{aligned}
&\Big|\int_0^t \int_0^{+\infty} 2\big[a(u_1)w_{xxxx}^{(k)} + (a(u_1) - a(u_2))(u_2^{(k)})_{xxxx}\big] w_{xxx}^{(k)} dxds\Big| \\
&\le \int_0^t a(g_1(s))(w_{xxx}^{(k)})^2(0,s)ds + C_T \int_0^t ||w_{xxx}^{(k)}(\cdot,s)||^2 ds \qquad (7.38) \\
&+ C_T(||f||_1, |g|_{1,T})\gamma(\mathcal{B})||w||_{L_\infty(\mathbb{R}^+\times(0,T))}^2 ||u_2^{(k)}||_{L_2(0,T;H^4(\mathbb{R}^+))}^2.
\end{aligned}$$

Using this information in (7.35) and choosing δ small enough, one obtains

$$||w^{(k)}(\cdot,t)||_3^2 \le C^1_{k+1,T}\{||f||_{3k+3} + |g|_{k+2,T} + \int_0^t ||w^{(k)}(\cdot,s)||_3^2 ds + ||w||^2_{L_\infty(\mathbb{R}^+\times(0,T))}||u_2^{(k)}||^2_{L_2(0,T;H^4(\mathbb{R}^+))}\}.$$

Applying Gronwall's lemma to the above inequality shows that

$$||w^{(k)}(\cdot,t)||_3^2 \le C^1_{k+1,T}\{||f||_{3k+3} + |g|_{k+2,T} + ||w||^2_{L_\infty(\mathbb{R}^+\times(0,T))}||u_2^{(k)}||^2_{L_2(0,T;H^4(\mathbb{R}^+))}\}.$$

The proof is complete. □

An inductive use of (7.5a) and (7.6), combined with the estimates derived in Lemma 7.2, gives immediately the following estimates for $\|w(\cdot,t)\|_{3k}$ and $\|w(\cdot,t)\|_{3k+1}$.

Lemma 7.3. *Let k be a positive integer. If $(f_m, g_m) \in X_{3k+1}$ for $m = 1,2$, then for any $T > 0$, there are constants $C^1_{k,T}$ depending continuously on T, $||f_m||_{3k}$ and $|g_m|_{k+1,T}$, such that*

$$||w(\cdot,t)||^2_{3k} + \int_0^t (\partial_x^{3k+1} w(0,s))^2 ds \le C^1_{k,T}\{||f||^2_{3k} + |g|^2_{k+1,T} + ||w||^2_{L_\infty(\mathbb{R}^+\times(0,T))}||u_2||^2_{L_2(0,T;H^{3k+1}(\mathbb{R}^+))}\}. \tag{7.39}$$

If $(f_m, g_m) \in X_{3k+3}$, there are constants $C^2_{k,T}$ depending continuously on T, $\|f_m\|_{3k+1}$ and $|g_m|_{k+1,T}$, such that

$$||w(\cdot,t)||^2_{3k+1} + \int_0^t (\partial_x^{3k+2} w(0,s))^2 ds \le C^2_{k,T}\{||f||^2_{3k+1} + |g|^2_{k+1,T} + ||w||^2_{L_\infty(\mathbb{R}^+\times(0,T))}||u_2||^2_{L_2(0,T;H^{3k+2}(\mathbb{R}^+))}\}, \tag{7.40}$$

for $0 \le t \le T$.

Since Lemma 7.3 requires that the corresponding initial- and boundary-data lie in the space X_{3k+1}, or X_{3k+3}, we can not directly obtain that the map $(f,g) \to u$ is continuous from X_s into $C(\mathbb{R}^+; H^s(\mathbb{R}^+))$ where $s = 3k$, or $s = 3k+1$. But by choosing smooth and compatible approximations of the data, this goal can be achieved. Such approximations of the auxiliary data were constructed in [**18**, Proposition 4.1] and used in the study of the initial-boundary-value problem for the KdV-equation. For the reader's convenience, these results are restated in the next Proposition.

Proposition 7.4. *Let $(f,g) \in X_s$ where $s = 3k$, or $s = 3k+1$. Then for any integer $n \ge 0$ and $\epsilon \in (0,1]$, there exist functions*

$$(f_\epsilon,\ g_\epsilon) \in X_{s(n)} \cap (H^\infty(\mathbb{R}^+) \times H^\infty_{loc}(\mathbb{R}^+)),$$

where $s(n) = 3(k+n)$, *or* $s(n) = 3(k+n)+1$ *depending on whether* $s = 3k$ *or* $s = 3k+1$, *such that for any* $T > 0$,

$$(i)\ \|f_\epsilon - f\|_{s-3j},\ |g_\epsilon - g|_{k+1-j,T} = o(\epsilon^j)\ \text{ for } k \geq j \geq 0, \text{ and}$$
$$(ii)\ \|f_\epsilon\|_{s+3j},\ |g_\epsilon|_{k+j+1,T} \leq c\epsilon^{-j}\ \text{ for } j \geq 0$$

as $\epsilon \downarrow 0$, *where the constant* c *depends only on* $\|f\|_s$, $|g|_{k+1,T+1}$, j, n *and* T. *Furthermore, the convergence in (i) depends upon* j, n *and* T, *but is uniform on compact subsets of* $H^s(\mathbb{R}^+) \times H^{k+1}(0,T+1)$. *Finally, for any fixed* $\epsilon \in (0,1]$, *the map* $(f,\, g) \to (f_\epsilon,\, g_\epsilon)$ *is continuous from* X_s *into* $X_{s(n)} \cap (H^\infty(\mathbb{R}^+) \times H^\infty_{loc}(\mathbb{R}^+))$.

Lemma 7.5. *Assume that* $(f,\, g) \in X_s$ *where* $s = 3k$ *or* $3k+1$, k *a positive integer, and let* $(f_\epsilon,\, g_\epsilon) \in X_{s(1)}$ *for* $\epsilon \in (0,1]$ *be approximations to* $(f,\, g)$ *whose existence is guaranteed by Proposition 7.4 with* $n = 1$. *If* u_ϵ *denotes the solution of (6.1) with data* $(f_\epsilon,\, g_\epsilon)$, *then for any* $T > 0$,

$$\|u_\epsilon\|_{L_\infty(0,T;H^{3k+1}(\mathbb{R}^+))} \leq C^1_{k,T}\epsilon^{-\frac{1}{3}} \quad \text{and} \quad \|u_\epsilon\|_{L_\infty(0,T;H^{3k+2}(\mathbb{R}^+))} \leq C^2_{k,T}\epsilon^{-\frac{1}{2}},$$

where the constants $C^i_{k,T}$, $i = 1,2$, *depend only on* T, $\|f\|_s$ *and* $|g|_{k+1,T+1}$.

Proof: Suppose $(f,g) \in X_{3k}$ so that $(f_\epsilon,\, g_\epsilon) \in X_{3k+3}$. Note that from (7.3), (7.4) and Proposition 7.4, it follows that

$$\|u_\epsilon\|_{L_\infty(0,T;\, H^{3k}(\mathbb{R}^+))} + \int_0^t (\partial_x^{(3k+1)} u_\epsilon(0,s))^2 ds \leq C^1_{k,T}. \tag{7.41}$$

The constants $C^i_{k,T}$ will have the same dependence on parameters and data as those specified in the statement of the lemma.

For the moment, denote u_ϵ by simply u. Then for $k \geq 1$ the function $u^{(k)}$ satisfies the equation

$$u_t^{(k)} + a(u)u_x^{(k)} + u_{xxx}^{(k)} = h_k(u), \tag{7.42}$$

where $h_1 = -(a(u))_x u_t$ and for $k > 1$,

$$h_k(u) = -(a(u))_x u^{(k)} - \partial_x \sum_{i=1}^{k-1} \binom{k-1}{i} (a(u))^{(i)} u^{(k-i)}.$$

Multiply (7.42) by $-2u_{xx}^{(k)}$ and integrate the results over $\mathbb{R}^+ \times (0,t)$. After suitable integrations by parts, there appears

$$\|u_x^{(k)}(\cdot,t)\|^2 + 2\int_0^t g_\epsilon^{(k+1)}(s)u_x^{(k)}(0,s)ds + \int_0^t (u_{xx}^{(k)}(0,s))^2 ds$$
$$= \|u_x^{(k)}(\cdot,0)\|^2 + 2\int_0^t \int_0^\infty u_{xx}^{(k)}[(a(u))u_x^{(k)} - h_k(u)]dxds. \tag{7.43}$$

Note that

$$\int_0^\infty 2u_{xx}^{(k)}a(u)u_x^{(k)}dx = -a(u(0,s))[u_x^{(k)}(0,s)]^2 - \int_0^\infty a(u)_x[u_x^{(k)}]^2dx.$$

Therefore by (7.41), it follows that

$$\int_0^t\int_0^\infty 2a(u)u_{xx}^{(k)}u_x^{(k)}dxds \le C_{k,T}^1\{1+\int_0^t ||u_x^{(k)}(\cdot,s)||^2ds\} \tag{7.44}$$

for $0 \le t \le T$. Note also that (7.41) entails as before the inequality

$$||u^{(j)}||_{L_\infty(0,T;H^3(\mathbb{R}+))} \le C_{k,T}^1$$

for $0 \le j \le k-1$. This in turn implies that

$$||\partial_x(h_k)||_{L_2(\mathbb{R}^+\times(0,T))} \le C_{k,T}^1\Big(\int_0^t ||u_x^{(k)}(\cdot,s)||^2ds\Big)^{\frac{1}{2}}.$$

Then since

$$\int_0^\infty 2u_{xx}^{(k)}h_k(u)dx = -2h_k(u(0,s))u_x^{(k)}(0,s) - \int_0^\infty 2u_x^{(k)}(h_k)_xdx,$$

$$|h_k(u(0,s))|_T \le C_{k,T}^1 \quad \text{and} \quad ||\big[h_k(u(\cdot,s))\big]_x|| \le C_{k,T}^1(1+||u_x^{(k)}(\cdot,s)||),$$

one obtains

$$\Big|\int_0^t\int_0^\infty 2u_{xx}^{(k)}h_kdxds\Big| \le \int_0^t ||u_x^{(k)}(\cdot,s)||^2ds + C_{k,T}^1. \tag{7.45}$$

Taking recourse again to (7.41) together with Proposition 7.4 gives

$$\Big|\int_0^t g_\epsilon^{(k+1)}u_x^{(k)}(0,s)ds\Big| \le C_{k,T}^1. \tag{7.46}$$

Finally, from the equation (7.42) and Proposition 7.4, one shows that

$$||u_x^{(k)}(\cdot,0)|| \le C_{k,T}^1||u(\cdot,0)||_{3k+1} \le C_{k,T}^1||f||_{3k+1} \le C_{k,T}^1\epsilon^{-\frac{1}{3}}. \tag{7.47}$$

The above estimates and Gronwall's lemma imply the first-stated inequality in the lemma.

The second inequality will follow if

$$||u_\epsilon||_{L_\infty(0,T;\,H^{3k+3}(\mathbb{R}+))} \le C_{k,T}^2\epsilon^{-1}. \tag{7.48}$$

Let $(f,g) \in X_{3k+1}$, so that $(f_\epsilon,\, g_\epsilon) \in X_{3k+4}$. Then from (7.3), (7.4) and Proposition 7.4, one has

$$||u_\epsilon||_{L_\infty(0,T;\,H^{3k+1}(\mathbb{R}+))} + \int_0^t (\partial_x^{(3k+2)}u_\epsilon(0,s))^2ds \le C_{k,T}^2. \tag{7.49}$$

Replace k by $k+1$ in (7.42) and let

$$y = u_\epsilon^{(k+1)} - g_\epsilon^{(k+1)}(t)e^{-x} \equiv u^{(k+1)} - V^{(k+1)}.$$

Then y satisfies the equation

$$y_t + a(u)y_x + y_{xxx} = h \tag{7.50}$$

with initial- and boundary-values given by

$$y(0,t) = 0, \quad y(x,0) = u^{(k+1)}(x,0) - V^{(k+1)}(x,0),$$

where

$$h = -\Big(V_t^{(k+1)} + a(u)V_x^{(k+1)} + V_{xxx}^{(k+1)} + h_{k+1}(u)\Big).$$

Multiply (7.50) by $2y$ and integrate the results over $\mathbb{R}^+ \times (0,t)$. Suitable integrations by parts lead to

$$\begin{aligned} ||y(\cdot,t)||^2 + \int_0^t y_x^2(0,s)ds + 2\int_0^t\int_0^\infty \{a(u)y_x y\}(x,s)dxds \\ = ||y(\cdot,0)||^2 + 2\int_0^t\int_0^\infty hydxds. \end{aligned} \tag{7.51}$$

By using (7.49), one has

$$\begin{aligned} 2\Big|\int_0^t\int_0^\infty a(u)y_x y dxds\Big| &= \Big| -\int_0^t\int_0^\infty a'(u)u_x y^2 dxds\Big| \\ &\le C_{k,T}^2 \int_0^t ||y(\cdot,s)||^2 ds \end{aligned}$$

for $0 \le t \le T$. The properties of the net (f_ϵ, g_ϵ) mentioned in Proposition 7.4 imply

$$||y(\cdot,0)|| \le C_{k,T}^2 \epsilon^{-\frac{2}{3}}$$

and

$$||h||_{L_2(\mathbb{R}^+\times(0,T))} \le C_{k,T}^2 \epsilon^{-1}.$$

Hence (7.51) and Gronwall's lemma yield

$$||y||_{L_\infty(0,T;\,L_2(\mathbb{R}^+))} \le C_{k,T}^2 \epsilon^{-1}.$$

By the definition of y and referring again to Proposition 7.4, it is adduced that

$$||u_\epsilon||_{L_\infty(0,T;\,H^{3k+2}(\mathbb{R}^+))} \le C_{k,T}^2 \epsilon^{-\frac{1}{2}}.$$

This completes the proof. □

The information derived from Lemmas 7.1 to 7.5 allows us to prove the following proposition.

Proposition 7.6. *Let* $(f,\, g) \in X_s$ *where* $s = 3k$, *or* $s = 3k+1$ *for some integer* $k \geq 1$. *Then there exists a unique solution* u *of (6.1) in* $C(\mathbb{R}^+; H^s(\mathbb{R}^+))$ *corresponding to the given data* f *and* g.

Proof: Denote by s_i the quantity $3k+i-1$ for $i = 1, 2$. Fix a positive value of T, let a net $\{(f_\epsilon,\, g_\epsilon)\}_{\epsilon\in(0,1]} \subset X_{s_i+3}$ of approximations $(f_\epsilon,\, g_\epsilon)$ to the data $(f,\, g)$ be constructed for which the properties delineated in Proposition 7.4 hold and let $\{u_\epsilon\}_{\epsilon\in(0,1]}$ denote the corresponding family of solutions of (6.1). From Theorem 6.2, we have

$$u_\epsilon \in L_\infty(0,T;\, H^{s_i+3}(\mathbb{R}^+)) \quad \text{and} \quad \partial_t u_\epsilon \in L_\infty(0,T;\, H^{s_i}(\mathbb{R}^+)).$$

Hence for all $\epsilon \in (0, 1]$, u_ϵ certainly lies in $C(0,T;\, H^{s_i}(\mathbb{R}^+))$. It will now be shown that $\{u_\epsilon\}$ is Cauchy in $C(0,T;\, H^{s_i}(\mathbb{R}^+))$. Suppose that $0 < \delta < \epsilon \leq 1$. From Lemma 7.3 and Proposition 7.4, there follows the existence of constants $C^i_{k,T}$ depending continuously on $||f||_{s_i}, |g|_{k+1,T+1}$ and T such that for $0 \leq t \leq T$,

$$\begin{aligned} ||u_\epsilon(\cdot,t) - u_\delta(\cdot,t)||_{s_i} \leq C^i_{k,T}\{&||f_\epsilon - f_\delta||_{s_i} + |g_\epsilon - g_\delta|_{k+1,T} \\ &+ ||u_\epsilon - u_\delta||_{L_\infty(\mathbb{R}^+\times(0,T))}||u_\epsilon||_{L_2(0,T;H^{3k+i}(\mathbb{R}^+))}\}. \end{aligned} \tag{7.52}$$

From Lemma 7.1 and Proposition 7.4 follows the inequality

$$||u_\epsilon - u_\delta||_{L_\infty(\mathbb{R}^+\times(0,T))} \leq ||u_\epsilon - u_\delta||_{L_\infty(0,T;H^1(\mathbb{R}^+))} \leq C^i_{k,T}\epsilon^k,$$

and from Lemma 7.5 it is seen that

$$||u_\epsilon||_{L_\infty(0,T;H^{3k+1}(\mathbb{R}^+))} \leq C^1_{k,T}\epsilon^{-\frac{1}{3}} \quad \text{and} \quad ||u_\epsilon||_{L_\infty(0,T;H^{3k+2}(\mathbb{R}^+))} \leq C^2_{k,T}\epsilon^{-\frac{1}{2}}.$$

Moreover, the construction of the regularized data (f_ϵ, g_ϵ) (see again Proposition 7.4) entails

$$||f_\epsilon - f_\delta||_{s_i} \to 0 \quad \text{and} \quad |g_\epsilon - g_\delta|_{k+1,T} \to 0 \quad \text{as } \epsilon \to 0.$$

The last three inequalities imply that $\{u_\epsilon\}_{\epsilon\in(0,1]}$ is Cauchy in $C(0,T; H^{s_i}(\mathbb{R}^+))$. Hence, as $\epsilon \to 0$, $\{u_\epsilon\}_{\epsilon\in(0,1]}$ converges to a function $\bar{u} \in C(0,T; H^{s_i}(\mathbb{R}^+))$. By continuity, it certainly follows that $\bar{u}$ satisfies the differential equation (6.1) in the sense of distributions on $\mathbb{R}^+ \times (0,T)$. Furthermore,

$$||\bar{u}(\cdot,0) - f||_{s_i} \leq ||\bar{u}(\cdot,0) - u_\epsilon(\cdot,0)||_{s_i} + ||f_\epsilon - f||_{s_i} \to 0$$

as $\epsilon \downarrow 0$, and

$$|\bar{u}(0,\cdot) - g|_{k+1,T} \leq |\bar{u}(0,\cdot) - u_\epsilon(0,\cdot)|_{k+1,T} + |g_\epsilon - g|_{k+1,T} \to 0$$

as $\epsilon \downarrow 0$. Hence $\bar{u}$ is a solution of (6.1) with initial and boundary data f and g, respectively. From the uniqueness result of Theorem 6.2 it is therefore implied that $u \equiv \bar{u} \in C(0,T;\, H^{s_i}(\mathbb{R}^+))$. □

Theorem 7.7. *Let $(f,\, g) \in X_s$, where $s = 3k$, or $s = 3k+1$ for some integer $k \geq 1$. Then the map $(f,\, g) \to u$ is continuous from X_s into $C(\mathbb{R}^+; H^s(\mathbb{R}^+))$.*

Proof: Let $\{(f_n,\, g_n)\}_{n=1}^{\infty}$ be a sequence in X_s that converges to $(f,\, g) \in X_s$. Thus for any $T > 0$,

$$||f_n - f||_s + |g_n - g|_{k+1,T} \to 0 \quad \text{as} \quad n \to \infty.$$

Let u_n and u be the solutions of (6.1) corresponding to the data $(f_n,\, g_n)$ and $(f,\, g)$, respectively, $n = 1, 2, \cdots$, and let $T > 0$ be fixed but arbitrary. By Proposition 7.6 it is known that u_n and u lie in $C(0, T;\, H^s(\mathbb{R}^+))$ for all $n \geq 1$. For any $n \geq 1$ and $\epsilon \in (0, 1]$ define approximations $(f_{n,\epsilon},\, g_{n,\epsilon}) \in X_{s+3}$ of (f_n, g_n) for which the properties in Proposition 7.4 hold. Let also (f_ϵ, g_ϵ) be similar approximations of (f, g) and let $u_{n,\epsilon}$ and u_ϵ be the solutions of (6.1) corresponding to the data $(f_{n,\epsilon},\, g_{n,\epsilon})$ and (f_ϵ, g_ϵ), respectively, $n = 1, 2, \cdots$. From the proof of Proposition 7.6, one has

$$||u_\epsilon - u||_{C(0,T;\, H^s(\mathbb{R}^+))} \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{7.53}$$

Next consider the difference $u_{n,\epsilon} - u_n$. Again from Proposition 7.6, one also has

$$||u_{n,\epsilon} - u_n||_{C(0,T;\, H^s(\mathbb{R}^+))} \to 0, \quad \text{as} \quad \epsilon \to 0, \tag{7.54}$$

for each fixed $n \geq 1$. Furthermore, by Proposition 7.4, we know that

$$||f_{n,\epsilon} - f_\epsilon||_s \to 0 \quad \text{and} \quad |g_{n,\epsilon} - g_\epsilon|_{k+1,T} \to 0 \tag{7.55}$$

as $\epsilon \to 0$, uniformly in n. From Lemma 7.1, Lemma 7.3, Lemma 7.5 and Proposition 7.4, one shows that when $s = 3k$,

$$||u_{n,\epsilon} - u_n||_{L_\infty(\mathbb{R}^+ \times (0,T))} ||u_{n,\epsilon}||_{L_\infty(0,T;\, H^{3k+1}(\mathbb{R}^+))} \leq C^1_{k,T} \epsilon^{k-\frac{1}{3}},$$

and when $s = 3k+1$,

$$||u_{n,\epsilon} - u_n||_{L_\infty(\mathbb{R}^+ \times (0,T))} ||u_{n,\epsilon}||_{L_\infty(0,T;\, H^{3k+2}(\mathbb{R}^+))} \leq C^2_{k,T} \epsilon^{k-\frac{1}{2}},$$

where $C^i_{k,T}$ is independent of n for $i = 1, 2$. Because of these inequalities and the relations (7.55) and (7.52), it is seen that the convergence in (7.54) is uniform in n. Let $\gamma > 0$ be arbitrary. Because of the convergence in (7.53) and the uniform convergence in (7.54), there exists an $\epsilon_1 \in (0, 1]$ such that

$$||u_{n,\epsilon} - u_n||_{C(0,T;\, H^s(\mathbb{R}^+))} + ||u_\epsilon - u||_{C(0,T;\, H^s(\mathbb{R}^+))} \leq \gamma, \tag{7.56}$$

for all $\epsilon \in (0, \epsilon_1]$ and all $n \geq 1$. Fix a value of ϵ in the interval $(0, \epsilon_1)$. From Lemma 7.3, one has, when $s = 3k$, that

$$\begin{aligned} ||u_{n,\epsilon} - u_\epsilon||_{C(0,T;\, H^{3k}(\mathbb{R}^+))} \leq C^1_{k,T} \{ &||f_{n,\epsilon} - f_\epsilon||_{3k} + |g_{n,\epsilon} - g_\epsilon|_{k+1,T} \\ &+ ||u_{n,\epsilon} - u_\epsilon||_{L_\infty(\mathbb{R}^+ \times (0,T))} ||u_\epsilon||_{L_\infty(0,T;\, H^{3k+1}(\mathbb{R}^+))} \} \end{aligned}$$

(see (7.39)), and when $s = 3k + 1$,

$$||u_{n,\epsilon} - u_\epsilon||_{C(0,T;\,H^{3k+1}(\mathbb{R}^+))} \leq C^2_{k,T}\{||f_{n,\epsilon} - f_\epsilon||_{3k+1} + |g_{n,\epsilon} - g_\epsilon|_{k+1,T} + ||u_{n,\epsilon} - u_\epsilon||_{L_\infty(\mathbb{R}^+\times(0,T))}||u_\epsilon||_{L_\infty(0,T;\,H^{3k+2}(\mathbb{R}^+))}\}$$

(see (7.40)), where the constants $C^i_{k,T}$ are independent of n for $i = 1, 2$. The continuity of the map $(f,\, g) \to (f_\epsilon, g_\epsilon)$ in $H^s(\mathbb{R}^+) \times H^{k+1}_{loc}(\mathbb{R}^+)$, implies that

$$||f_{n,\epsilon} - f_\epsilon||_s + |g_{n,\epsilon} - g_\epsilon|_{k+1,T} \to 0,$$

as $n \to \infty$. Also by Lemma 7.1 and Proposition 7.4, one has

$$||u_{n,\epsilon} - u_\epsilon||_{L_\infty(\mathbb{R}^+\times(0,T))} \to 0,$$

as $n \to \infty$. It follows that, for fixed ϵ,

$$\lim_{n\to\infty} ||u_{n,\epsilon} - u_\epsilon||_{C(0,T;H^s(\mathbb{R}^+))} = 0 \tag{7.57}$$

if $(f, g) \in X_s$, for $s = 3k$ or $s = 3k + 1$. Thus if we write

$$u_n - u = u_n - u_{n,\epsilon} + u_{n,\epsilon} - u_\epsilon + u_\epsilon - u,$$

then since $\epsilon \in (0, \epsilon_1)$, (7.56) and (7.57) imply that

$$\limsup_{n\to\infty} ||u_n - u||_{C(0,T;H^s(\mathbb{R}^+))} \leq \gamma$$

if $(f, g) \in X_s$. Since $\gamma > 0$ and $T > 0$ were arbitrary, the result follows and the proof is complete. □

Remark. A careful perusal of the preceding arguments indicates that the solution map $(f, g) \to u$ for the initial-boundary-value problem (6.1) is in fact Lipschitz continuous.

8. Conclusion

The well-posedness of the initial- and boundary-value problem (1.1) for the generalized KdV equation has been studied here. Well-posedness locally in time requires only suitable smoothness of the nonlinearity P, while our theory of global well-posedness uses more restrictive assumptions. Precisely stated, if the initial value $f \in H^{3k}(\mathbb{R}^+)$ or $f \in H^{3k+1}(\mathbb{R}^+)$ and the boundary value $g \in H^{k+1}_{loc}(\mathbb{R}^+)$ satisfy the appropriate compatibility conditions at $(x, t) = (0, 0)$ (see Lemma 4.7 and Formula (5.14)) and the growth of the nonlinearity P satisfies the one-sided condition $(**)$ put forward in the beginning of §4, then there corresponds a unique global solution u of the initial-boundary-value problem (1.1) which, for each $T > 0$, lies in $C(0, T; H^{3k}(\mathbb{R}^+))$ or $C(0, T; H^{3k+1}(\mathbb{R}^+))$, respectively. Moreover, u depends continuously in the relevant function classes on the pair $(f,\, g)$. It is worth particular note that the $H^1(\mathbb{R}^+)$-bound obtained in our theory grows roughly linearly with the energy $|g|_{1,T}$ supplied by the wavemaker (see Lemma 4.1). This is a satisfactory aspect of the theory as it corresponds well with what is observed in experiments (see [**11**]). It is also worth note that if $|g|_{1,T}$ is sufficiently small, then the well-posedness theory can proceed under weaker growth conditions on the nonlinearity, namely that $\limsup_{s\to+\infty} \Lambda(s)/s^5 \leq 0$. Thus for small boundary forcing, the theory comes in line to some extent with that available for the pure initial-value problem (see Kato [**29**], Schechter [**40**] and more recent work of Kenig *et al.* [**30, 31**]).

Despite the complexity of the developments presented here, there are many obvious issues left open. Perhaps the foremost is that to which allusion was just made, namely whether or not problem (1.1) is globally well posed for nonlinearities P whose growth at infinity is less than quintic. There is also the slightly unsatisfactory aspect that certain regularity classes for initial data are missing in the results (e.g. $H^{3k+2}(\mathbb{R}^+)$, $k = 0, 1, \cdots$). This is a technical point with little impact on the assessment of (1.1) as a model of real phenomena. However, it presents an interesting analytical challenge.

Other mathematical aspects deserve further attention. The question of smoothing and an associated well-posedness theory set in weak function classes is an interesting and topical issue. As mentioned briefly above, the smoothing established by Kato [**29**] (see also Faminskii [**21**]) holds in the situation envisaged here. However, the more subtal results of Ginibre & Velo [**25**], Kenig *et al.* [**30, 31**] and Bourgain [**19**] have not been considered in the context of initial-boundary-values problems other than with periodic boundary conditions. The issue mentioned parenthetically at the end of Section 7 of smoothness of the mapping that associates the solution to initial- and boundary-data also deserves further study. For the pure initial-value problem, this map is known to be analytic and we expect the same is true for the present initial-boundary-value problem provided the nonlinearity P is entire, say, or analytic in an appropriate neighborhood of the origin in any case.

Finally, since the initial-boundary-value problem is well posed, say for the KdV equation, it would be worthwhile to develop a numerical scheme for this problem along the lines of that put forward for the quarter-plane problem for the regularized long-wave equation in [**11**] and test the model (1.1) quantitatively against experimental data. As in [**11**], damping will need to be incorporated into the model. This in itself presents an interesting challenge, both as regards modelling and from the view of analysis since dissipation may well be a non-local effect at the level of approximation corresponding to that already in effect for nonlinearity and dispersion.

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