Blow-up of Spatially Periodic Complex-Valued Solutions of Nonlinear Dispersive Equations

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ABSTRACT. Considered here are spatially-periodic, complex-valued solutions of a class of nonlinear, dispersive evolution equations. Some easily checked criteria are presented that imply these solutions lose regularity in finite time.

1. INTRODUCTION

Studied here are complex-valued solutions $u(t,x)$ of nonlinear, dispersive evolution equations of the form

$$iu_t + R\left(\frac{1}{i}\frac{\partial}{\partial x}\right)u - Q\left(\frac{1}{i}\frac{\partial}{\partial x}\right)(u^{p+1}) = 0, \tag{1.1}$$

where $p \geq 1$ is an integer. When realized as particular evolution equations arising in physics, mathematics, mechanics, etc., the functions R and Q are typically polynomials in one variable with real coefficients, but our theory will countenance considerably more general functions. This formulation is like that of Dix [19], [20], who also considered dissipative as well as nonlinear and dispersive effects (see also Craig, Kappeler and Strauss [18]). The class delineated in (1.1) includes a wide range of interesting partial differential equations including nonlinear Schrödinger-type equations and Korteweg-de Vries-type equations. In the case of Schrödinger equations, complex-valued solutions are natural, whereas for Korteweg-de Vries-type equations, real-valued solutions are normally considered. (On the other hand, see Birnir [6], [7], who used the inverse scattering transform to study singularities of complex-valued solutions of the Korteweg-de Vries equation itself. See also Weinstein [44] where the KdV-equation with a derivative Schrödinger-type nonlinearity $\partial_x(|u|^p u)$ and complex-valued solutions are studied, and Martel [30] where the linear KdV-equation with the nonlinearity $iu|u|^{p-1}$ and complex solutions is considered with $1 \leq p < 7$.)

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Attention will be given to solutions of (1.1) that are spatially periodic, and therefore representable in terms of Fourier series in the spatial variable x. Of special interest will be solutions whose Fourier series contain only positive modes, by which is meant the coefficients of e^{ikx} vanish for all $k \le 0$.

Consideration of equations of the form (1.1) arose in our study of the interaction between nonlinear and dispersive effects in evolution equations, and especially when addressing the possible loss of regularity and singularity formation. This is a subject that has received considerable attention in the last decade or so. Perhaps the first major step was the virial-type relation derived by Zakharov [46] and, independently but later, by Glassey [21], which they used to show blow-up of certain solutions of the nonlinear Schrödinger equation

$$iu_t + \Delta u + |u|^{p-1}u = 0 \qquad (1.2)$$

in $\mathbb{R}^n$, for $p \ge 1 + (4/n)$. Since Zakharov's and Glassey's work, there has been considerable effort to improve and extend their results (e.g. Ogawa and Tsutsumi [37], [38], [39], Martel [31], and Kavian [22]). With the exception of [22] and [39], the known blow-up results are set on the whole of $\mathbb{R}^n$. Kavian considers a bounded domain with homogeneous Dirichlet boundary conditions, while Ogawa and Tsutsumi consider periodic boundary conditions as we do in the present paper. More precisely, they establish existence of blowing-up solutions for the case $p = 5$ (the so called pseudo-conformal case when $n = 1$), first by adapting their own work on $\mathbb{R}^n$ [38] and then by using the ideas of Merle [34]. In addition to energy-type arguments, there is another argument relying on the pseudo-conformal transformation of the equation for the particular power $p = 1 + (4/n)$ that also establishes blow-up (see Weinstein [43], [45], Cazenave and Weissler [17], and Merle [34], [35]). In this case, there are solutions given explicitly in terms of the ground state for the equation that blow up in finite time, and, as Merle has shown, solutions exist that blow up at an arbitrary finite number of pre-assigned points in $\mathbb{R}^n$. It is worth to remark that numerical simulations (e.g. Le Mesurier et al. [28], [29], Landman et al. [27], and Akrivis el al. [1]) indicate that the blow-up in the pseudo-conformal case is fundamentally different from that obtaining for other values of p. In the critical, or pseudo-conformal case, there are also some interesting results about the stability of the blow-up. These were initially developed in a series of papers by various combinations of Blaha, Laedke, Spatschek, Stenflo and Kuznetsov (see [23], [24], [25], [26]). A careful appraisal of these ideas may be found in [3]. Helpful general references for the nonlinear Schrödinger equation include Cazenave [15], [16] and Sulem and Sulem [40].

Another interesting class of equations where blow-up results have been in the foreground are Korteweg-de Vries-type equations. For such equations, the situation is less clear than in the Schrödinger case. Formal calculations and theoretical considerations (cf. Albert et al. [2], Bona and Weissler [14], Angulo et al. [4], Merle [36], Martel and Merle [32]), and detailed numerical simulations (see

[8], [9], [10], [11]) indicate that some real-valued solutions of the generalized Korteweg-de Vries equation

$$u_t + u^p u_x + u_{xxx} = 0 \tag{1.3}$$

become infinite at one or more points in finite time provided that $p \geq p_{\text{crit}} = 4$. Indeed, computer approximations of solutions to (1.3) and rigorous analysis indicate for example, that blow-up occurs when solitary waves are perturbed appropriately, which corresponds to the fact that solitary waves are unstable if $p \geq 4$ (see [13], [33]).

Under assumptions on R and Q to be made precise in the next two sections, it will be shown by a method that appears different from those just mentioned, that certain smooth, periodic initial data yield solutions of equation (1.1) which form singularities in finite time. As will be apparent later, our results do not depend upon a critical value of the exponent in the nonlinearity, and hence they are unlike the blow-up results for the Schrödinger and Korteweg-de Vries-type equations discussed above, where global existence of smooth solutions is known for values of the exponent in the nonlinearity less than the critical value. Furthermore, the present theory is not like the dispersive blow-up results of Bona and Saut [12] because they do in general depend upon the size of the initial data. Because of this aspect, and as is otherwise obvious from a study of our detailed development, the singularities obtained here depend upon nonlinearity in an essential way.

Examples where our theory is effective include the following equations:

Example 1.1. The generalized Korteweg-de Vries equation ($R(y) = y^3$ and $Q(y) = y$):

$$u_t + u_{xxx} + (u^{p+1})_x = 0. \tag{GKdV}$$

Example 1.2. The generalized Korteweg-de Vries equation in laboratory coordinates ($R(y) = y^3 - y$ and $Q(y) = y$):

$$u_t + u_x + u_{xxx} + (u^{p+1})_x = 0. \tag{L-GKdV}$$

Example 1.3. The nonlinear Schrödinger equation with power nonlinearity ($R(y) = -y^2$ and $Q(y) = -1$):

$$iu_t + u_{xx} + u^{p+1} = 0. \tag{NLS}$$

Example 1.4. The following derivative nonlinear Schrödinger equation ($R(y) = -y^2$ and $Q(y) = -y$):

$$iu_t + u_{xx} - i(u^{p+1})_x = 0. \tag{d-NLS}$$

Example 1.5. The "mixed" Korteweg-de Vries-Schrödinger equation ($R(y) = y^3 \pm y^2$ and $Q(y) = y$):

$$\text{(KdV-NLS)} \qquad u_t + u_{xxx} \pm iu_{xx} + (u^{p+1})_x = 0.$$

It is worth noting that none of the other methods available to establish blow-up of solutions to the nonlinear Schrödinger equation are effective on Examples 1.3 and 1.4 above. Moreover, since $f(u) = u^{p+1}$ (p a positive integer) is Lipschitz on bounded subsets of $H^1(\mathbb{S}^1)$, it follows that the initial-value problem for Example 1.3 is locally well posed in $H^1(\mathbb{S}^1)$. Thus the results of this paper imply the existence of $H^1(\mathbb{S}^1)$-solutions which form singularities in finite time in the H^1-norm. A more detailed view of the sense in which blow-up is established is presented in Section 3.

Here and below, $\mathbb{S}^1$ connotes $\mathbb{R}/(2\pi)$, the one-torus. When convenient, we will think of functions defined on $\mathbb{S}^1$ as functions defined on $\mathbb{R}$ that are periodic of period 2π.

The present theory is developed in two stages. In Section 2, equation (1.1) is considered as an infinite system of ordinary differential equations (in time) for the associated Fourier coefficients. A great simplification of this infinitely coupled system is effected by the restriction to solutions whose nonpositive Fourier modes all vanish. Indeed, the infinitely coupled system reduces to a sequence of linear inhomogeneous ordinary differential equations, which are immediately seen to admit global solutions, and whose structure can be studied in some detail. In Section 3, the properties of these Fourier coefficients, which exist globally in time, are used to show that certain solutions themselves given by the corresponding Fourier series, cannot remain regular for all time.

2. A Class of Formal Solutions

In this section equation (1.1) is transformed into a sequence of ordinary differential equations and the solutions of these equations are investigated. More precisely, consideration is given to solutions to (1.1) of the form

$$u(t,x) = \sum_{k=1}^{\infty} a_k(t)e^{ikx}, \tag{2.1}$$

where the time-dependent coefficients $a_k(t)$ may be complex-valued. In particular, nontrivial real-valued solutions of (1.1) are excluded by the form (2.1). The initial value, which may be located at $t = 0$ since the equation is time invariant, is

$$u(0,x) = \sum_{k=1}^{\infty} a_k(0)e^{ikx}. \tag{2.2}$$

If the function u is given by (2.1), then u^{p+1} is given (at least formally) by

$$u(t,x)^{p+1} = \sum_{k=1}^{\infty} b_k(t)e^{ikx}, \tag{2.3}$$

where the series (2.3) is determined from (2.1) by series multiplication. Because the series (2.1) features only positive modes,

$$b_1(t) = b_2(t) = \cdots = b_p(t) = 0, \quad \text{for all } t \in \mathbb{R}. \tag{2.4}$$

If R and Q are polynomials, then

$$R\left(\frac{1}{i}\frac{\partial}{\partial x}\right)e^{ikx} = R(k)e^{ikx} \quad \text{and} \quad Q\left(\frac{1}{i}\frac{\partial}{\partial x}\right)e^{ikx} = Q(k)e^{ikx}. \tag{2.5}$$

For more general functions R and Q, (2.5) is taken as the definition of $R(-i\partial_x)$ and $Q(-i\partial_x)$ in the present, periodic context.

Equation (1.1) is equivalent (at least formally) to the system of ordinary differential equations

$$ia_k'(t) + R(k)a_k(t) - Q(k)b_k(t) = 0, \quad k = 1, 2, 3, \ldots. \tag{2.6}$$

In particular,

$$a_k(t) = a_k(0)\exp[iR(k)t], \quad \text{for } k = 1, \ldots, p. \tag{2.7}$$

Each coefficient $a_k(t)$ is determined only by the first k equations in (2.6), i.e., by a finite system. Moreover, the nonlinear term $b_k(t)$ is determined by $a_1(t), \ldots, a_{k-1}(t)$, and so each equation in (2.6) for $k \geq 2$ is in fact a nonhomogeneous linear equation. It follows by a simple induction argument that the initial-value problem associated with (2.6), where the coefficients $a_k(t)$, $k = 1, 2, 3, \ldots$, are specified at $t = 0$, say, is globally well posed in the sense that each coefficient $a_k(t)$ is uniquely determined, defined for all $t \in \mathbb{R}$, and depends smoothly on $a_1(0), \ldots, a_k(0)$ for $k = 1, 2, \ldots$.

It is also clear that any sufficiently regular solution of (1.1), having the form (2.1), will also satisfy the system (2.6). It seems therefore propitious to study the infinite system (2.6) in its own right. Thus, consider the system of ordinary differential equations depicted in (2.6) for a family of functions $a_k : \mathbb{R} \to \mathbb{C}$, $k \geq 1$, where the $b_k(t)$'s are given by (2.3). Specifically, attention is given to the initial-value problem associated with data $a_k(0)$, $k \geq 1$. This initial-value problem is equivalent to the system of integral equations

$$\begin{aligned} a_k(t) &= \exp[iR(k)t]a_k(0) \\ &\quad - iQ(k)\exp[iR(k)t]\int_0^t \exp[-iR(k)s]b_k(s)\,ds, \quad k \geq 1. \end{aligned} \tag{2.8}$$

Definition. *We say that the series for* $u(t,x)$ *as in* (2.1), *with* $u(t,x)^{p+1}$ *given by* (2.3), *is a formal solution of* (1.1) *if the coefficients* $a_k(t)$ *and* $b_k(t)$ *satisfy* (2.6), *or equivalently,* (2.8).

Henceforth, the following structural assumption on the function R will be in force.

Condition (S).

(a) *The function R is real-valued and maps integers to integers,*
(b) *either* $R(y) \geq 0$ *for all* $y \geq 1$, *or* $R(y) \leq 0$, *for all* $y \geq 1$, *and*
(c) *for all* $y \geq 1$ *and* $z \geq 1$, $|R(y)| + |R(z)| < |R(y+z)|$.

The following proposition singles out another, and perhaps more transparent property that implies Condition (S).

Proposition 2.1. *Let* $0 \leq y_0 \leq 1$ *and let R be a real-valued function defined on* $[y_0, \infty)$ *such that*

(a) $R(y_0) = 0$;
(b) *R is strictly increasing on* $[y_0, \infty)$;
(c) *for* $y_0 \neq 0$, *R is convex on* $[y_0, \infty)$; *if* $y_0 = 0$, *it is demanded that R be strictly convex on* $[0, \infty)$.

If in addition $R : \mathbb{N} \to \mathbb{Z}$, *then R satisfies Condition (S).*
Similarly, if

(a) $R(y_0) = 0$;
(b′) *R is strictly decreasing on* $[y_0, \infty)$;
(c′) *for* $y_0 \neq 0$, *R is concave on* $[y_0, \infty)$; *if* $y_0 = 0$, *the function R is strictly concave;*

then R satisfies Condition (S) if it maps $\mathbb{N}$ *to* $\mathbb{Z}$.

Proof. The proof is given for the first statement only. Suppose $y \geq 1$ and $z \geq 1$. By convexity, assuming $y \neq y_0$,

$$\frac{R(z+y) - R(z)}{y} \geq \frac{R(y) - R(y_0)}{y - y_0} \geq \frac{R(y)}{y}.$$

Since one of these inequalities is strict (the second one if $y_0 \neq 0$, and the first one if $y_0 = 0$), it follows that $R(z+y) > R(y) + R(z)$.

If $y = y_0$ (in which case $y_0 = 1$) then $R(y) + R(z) = R(y_0) + R(z) = R(z) < R(z+y)$, since R is strictly increasing. ◻

Proposition 2.2. *Suppose that R satisfies Condition (S). Let u be a formal solution of* (1.1) *of the form displayed in* (2.1). *It follows that the functions* a_k, $k \geq 1$, *are periodic with period* 2π. *More precisely, the time-dependent coefficients* $a_k(t)$ *and* $b_k(t)$ *are of the following form.*

If $R(y) \geq 0$ for all $y \geq 1$, then

$$a_k(t) = \sum_{h=kR(1)}^{R(k)} \alpha_{k,h} e^{iht}, \tag{2.9}$$

$$b_k(t) = \sum_{h=kR(1)}^{R(k)-1} \beta_{k,h} e^{iht}, \tag{2.10}$$

whereas if $R(y) \leq 0$ for all $y \geq 1$, then

$$a_k(t) = \sum_{h=k|R(1)|}^{|R(k)|} \alpha_{k,h} e^{-iht}, \tag{2.11}$$

$$b_k(t) = \sum_{h=k|R(1)|}^{|R(k)|-1} \beta_{k,h} e^{-iht}, \tag{2.12}$$

where the $\alpha_{k,h}$ and $\beta_{k,h}$ are numerical constants.

Remarks. It is not necessary that $R(k) \in \mathbb{Z}$ for $k \in \mathbb{N}$. What is really needed to obtain a result of this nature is that for all $k \in \mathbb{N}$, $R(k) \in \mathbb{Z}\tau$ for some fixed $\tau \neq 0$. In this case, the Fourier coefficients $a_k(t)$ will be periodic in t with period $2\pi/\tau$. For example, if R is a polynomial whose coefficients are rational multiples of a real number ν, say, then this latter condition will be satisfied with $\tau = \nu/\kappa$, where κ is the least common multiple of the denominators of the rational numbers appearing in the coefficients.

The upper limits in (2.9) and (2.10) make it clear that the coefficients $a_k(t)$ and $b_k(t)$ are finite sums of exponentials.

If R satisfies Condition (S) and $R(y) \geq 0$ for all $y \geq 1$, then for $k > 1$, $0 \leq kR(1) = R(1) + \cdots + R(1) < R(k)$, and so the sums in (2.9) and (2.10) include at least one value of h. If $k = 1$, the sum in (2.10) is empty. Not all the coefficients in the above sums are nonzero. Indeed, if $1 \leq k \leq p$, then $\beta_{k,h} = 0$ for all possible h. Moreover, $\beta_{k,R(k)} = 0$ for all values of k. If R satisfies Condition (S) but $R(y) \leq 0$ for all $y \geq 1$, then $0 \geq kR(1) = R(1) + \cdots + R(1) > R(k)$, if $k > 1$, and similar remarks hold.

Proof. Consideration is given to the case where $R(y) \geq 0$ for all $y \geq 0$, the other case being very similar. An argument in favor of the result is made by induction on k. For $k = 1, \ldots, p$, formulas (2.9) and (2.10) are obvious since

$$a_k(t) = a_k(0)\exp[iR(k)t] \qquad \text{for } k = 1, \ldots, p, \text{ and}$$
$$b_1(t) = b_2(t) = \cdots = b_p(t) = 0 \quad \text{for all } t \in \mathbb{R},$$

and $R(k)$ is an integer for each k by assumption.

Suppose now that $k \geq p+1$ and that the formulas (2.9) and (2.10) have been shown to hold through $k-1$. By definition, the coefficient $b_k(t)$ is the sum of all products of the form

$$a_{k_1}(t)a_{k_2}(t)\cdots a_{k_{p+1}}(t),$$

where $k_1 + \cdots + k_{p+1} = k$, and $k_j \geq 1$. Moreover, by the induction hypothesis, each such product is a linear combination of products of exponentials of the form

$$\exp[ih_1 t]\exp[ih_2 t]\cdots\exp[ih_{p+1}t] = \exp\Big[i\Big(\sum_{j=1}^{p+1} h_j\Big)t\Big],$$

where each integer h_j satisfies

$$k_j R(1) \leq h_j \leq R(k_j)$$

for $j = 1, \ldots, p+1$. It follows that

$$kR(1) = \Big(\sum_{j=1}^{p+1} k_j\Big)R(1) \leq \sum_{j=1}^{p+1} h_j \leq \sum_{j=1}^{p+1} R(k_j) < R\Big(\sum_{j=1}^{p+1} k_j\Big) = R(k).$$

In other words, $b_k(t)$ is a linear combination of exponential functions of the form e^{iht}, where $kR(1) \leq h < R(k)$. This proves formula (2.10). Formula (2.9) is an immediate consequence of (2.10) and the integral representation (2.8) for the $a_k(t)$. ◻

For future reference, note that

$$a_1(t) = \alpha_{1,|R(1)|}e^{iR(1)t}, \tag{2.13}$$

and so $a_1(0) = \alpha_{1,|R(1)|}$, in the notation of the last proposition.

Corollary 2.3. *Under the same hypotheses as those appearing in Proposition 2.2, if u given by (2.1) is a formal solution of equation (1.1), then u and u^{p+1} are obtained by the following (formal) double series. If $R(y) \geq 0$ for all $y \geq 0$, then*

$$u(t,x) = \sum_{k=1}^{\infty}\sum_{h=kR(1)}^{R(k)} \alpha_{k,h}e^{iht}e^{ikx} \quad \text{and} \tag{2.14}$$

$$u(t,x)^{p+1} = \sum_{k=1}^{\infty}\sum_{h=kR(1)}^{R(k)-1} \beta_{k,h}e^{iht}e^{ikx}, \tag{2.15}$$

whereas if $R(y) \le 0$ *for all* $y \ge 1$, *then*

$$(2.16) \qquad u(t,x) = \sum_{k=1}^{\infty} \sum_{h=k|R(1)|}^{|R(k)|} \alpha_{k,h} e^{-iht} e^{ikx} \quad \text{and}$$

$$(2.17) \qquad u(t,x)^{p+1} = \sum_{k=1}^{\infty} \sum_{h=k|R(1)|}^{|R(k)|-1} \beta_{k,h} e^{-iht} e^{ikx}.$$

The coefficients $\alpha_{k,h}$ *and* $\beta_{k,h}$ *satisfy, in the first case,*

$$(2.18) \qquad \alpha_{k,h} = \frac{Q(k)}{R(k)-h} \beta_{k,h}, \quad kR(1) \le h < R(k),\ k \ge 2,$$

and in the second case,

$$(2.19) \qquad \alpha_{k,h} = \frac{-Q(k)}{|R(k)|-h} \beta_{k,h}, \quad k|R(1)| \le h < |R(k)|,\ k \ge 2.$$

Finally, the relationships (2.18) *and* (2.19) *are, in their respective cases, necessary and sufficient for the series given respectively by* (2.14) *and* (2.16) *to be a formal solution of equation* (1.1).

Remark. The following special cases of (2.18) and (2.19) are of particular interest:

$$(2.20) \qquad \alpha_{k,k|R(1)|} = \frac{Q(k)}{R(k)-kR(1)} \beta_{k,k|R(1)|}, \quad \text{for } k \ge 2.$$

Note that this one formula includes the two cases $h = kR(1)$ and $h = k|R(1)|$ in (2.18) and (2.19), respectively.

Lemma 2.4. *The coefficients* $\beta_{k,kR(1)}$ *in* (2.15) *are determined uniquely by and can be calculated in terms of the coefficients* $\alpha_{k',k'R(1)}$ *with* $k' < k$. *A similar statement holds for the coefficients* $\beta_{k,k|R(1)|}$ *in* (2.17).

Proof. The proof is presented only for the case (2.15), the other case being entirely analogous. By definition of the coefficients, each $\beta_{k,kR(1)}$ is a sum of all products of the form $\alpha_{k_1,h_1}\alpha_{k_2,h_2}\cdots\alpha_{k_{p+1},h_{p+1}}$, where $k_1 + k_2 + \cdots + k_{p+1} = k$, $h_1 + h_2 + \cdots + h_{p+1} = kR(1)$, and $k_jR(1) \le h_j \le R(k_j)$, for all $j = 1, \ldots, p+1$.

The only way all three of these conditions can be met is if $h_j = k_jR(1)$, for $j = 1, \ldots, p+1$. This proves the lemma. □

The following result is an immediate consequence of this lemma.

Proposition 2.5. *Suppose R satisfies Condition (S), and let u given by* (2.14), *respectively* (2.16), *be a formal solution of* (1.1). *Let v be the formal series obtained from u by deleting all terms except those whose coefficients are of the form $\alpha_{k,k|R(1)|}$, so that*

$$v(t,x) = \sum_{k=1}^{\infty} \alpha_{k,k|R(1)|} e^{ik(x+R(1)t)}. \tag{2.21}$$

Then the series v is also a formal solution of (1.1) *and*

$$v(t,x)^{p+1} = \sum_{k=1}^{\infty} \beta_{k,k|R(1)|} e^{ik(x+R(1)t)}, \tag{2.22}$$

where the coefficients $\beta_{k,kR(1)}$ are the same as in (2.15), *respectively* (2.17).

Remark. While the above formulas are valid whether $R(y) \geq 0$ for all $y \geq 1$ or $R(y) \leq 0$ for all $y \geq 1$, in the latter case $R(1) \leq 0$ and so they are more conveniently written in the following way:

$$v(t,x) = \sum_{k=1}^{\infty} \alpha_{k,k|R(1)|} e^{ik(x-|R(1)|t)}, \tag{2.23}$$

$$v(t,x)^{p+1} = \sum_{k=1}^{\infty} \beta_{k,k|R(1)|} e^{ik(x-|R(1)|t)}. \tag{2.24}$$

Proposition 2.6. *Suppose that R satisfies Condition (S) and let u given by* (2.14), *respectively* (2.16), *be a formal solution of* (1.1). *Suppose further that $p \geq 2$. It follows that*

$$\alpha_{k,k|R(1)|} = \beta_{k,k|R(1)|} = 0, \quad \text{if } k \not\equiv 1 \pmod{p}. \tag{2.25}$$

Proof. The proof is presented in case $R(y) \geq 0$ for all $y \geq 1$. By (2.7) and the fact that $kR(1) < R(k)$, for $k \geq 2$, it follows that $\alpha_{k,kR(1)} = 0$ for $k = 2, \ldots, p$. Also, $\beta_{k,h} = 0$, for $k = 1, \ldots, p$, as noted already in (2.4). Thus, (2.25) is correct if $k \leq p + 1$.

The strategy now is to prove (2.25) by induction. Suppose that it has been proved through $k - 1$, for some $k > p + 1$. Following the arguments in the proof of Lemma 2.4, it is observed that $\beta_{k,kR(1)}$ is the sum of all products of the form $\alpha_{k_1,k_1R(1)}\alpha_{k_2,k_2R(1)} \cdots \alpha_{k_{p+1},k_{p+1}R(1)}$, where $k_1 + k_2 + \cdots + k_{p+1} = k$, $k_j \geq 1$, $j = 1, \ldots, p + 1$. By the induction hypothesis, each such product can be nonzero only if $k_j \equiv 1 \pmod{p}$, $j = 1, \ldots, p + 1$. If this is the case, then $k = k_1 + k_2 + \cdots + k_{p+1} \equiv 1 \pmod{p}$. This proves (2.25) for $\beta_{k,kR(1)}$. The assertion for $\alpha_{k,kR(1)}$ now follows from (2.20). ☐

Proposition 2.7. *Suppose that R satisfies Condition (S). There exist two sequences* $\{c_k : k = 1, 2, \ldots,$ *and* $k \equiv 1 \pmod p\}$ *and* $\{d_k : k = 1, 2, \ldots,$ *and* $k \equiv 1 \pmod p\}$ *with* $c_1 = 1$ *and* $d_1 = 0$*, depending only on the functions R and Q and the integer p, such that if u given by* (2.14)*, respectively* (2.16)*, is a formal solution of* (1.1) *with* $\alpha_{1,|R(1)|} = a \in \mathbb{C}$*, then*

$$\alpha_{k,k|R(1)|} = c_k a^k, \ \beta_{k,k|R(1)|} = d_k a^k, \quad \text{for all } k \geq 1,\ k \equiv 1 \pmod p. \tag{2.26}$$

In particular,

$$c_k = \frac{Q(k)}{R(k) - kR(1)} d_k, \quad \text{for } k \geq 2,\ k \equiv 1 \pmod p. \tag{2.27}$$

Also, if

$$f(z) = \sum_{k=1}^{\infty} c_k z^k, \tag{2.28}$$

then

$$f(z)^{p+1} = \sum_{k=2}^{\infty} d_k z^k, \tag{2.29}$$

both in the ring of formal power series and as analytic functions inside their joint radius of convergence.

Remarks. The radius of convergence ρ of the power series in (2.28) will play a key role in the blow-up result described in Section 3. Notice that ρ depends only on the functions R and Q and the integer p. Throughout the remainder of the paper, in case $p = 1$, the condition $k \equiv 1 \pmod p$ is taken to be a vacuous restriction.

Proof. As before, the proof is presented in case $R(y) \geq 0$ for all $y \geq 1$. The statement (2.26) is clearly true for $k = 1$. We construct the sequences

$$\{c_k : k = 1, 2, \ldots, \text{ and } k \equiv 1 \pmod p\},$$
$$\{d_k : k = 1, 2, \ldots, \text{ and } k \equiv 1 \pmod p\}$$

iteratively, taking care to observe at each step that the construction depends only on R, Q, and p.

Suppose the sequences have been constructed up through $k - 1$ in such a way that (2.26) holds. The term $\beta_{k,kR(1)}$ is the sum of all products of the form $\alpha_{k_1,k_1R(1)}\alpha_{k_2,k_2R(1)} \cdots \alpha_{k_{p+1},k_{p+1}R(1)}$, where $k_1 + k_1 + \cdots + k_{p+1} = k$, $k_j \geq 1$, $j = 1, \ldots, p+1$, and $k_j \equiv 1 \pmod p$. Since (2.26) holds up through $k - 1$, we know that $\alpha_{k_j,k_jR(1)} = c_{k_j}a^{k_j}$, $j = 1, \dots, p+1$, and so $\beta_{k,kR(1)}$ is the sum of all products of the form $c_{k_1}c_{k_2}\cdots c_{k_{p+1}}a^k$, with the same conditions on the indices k_j. This sum produces the appropriate value of d_k, and the corresponding value of c_k is derived from (2.20). The formula (2.29) is a consequence of Lemma 2.4. The proposition is established. ◻

The proof of the following proposition is now obvious.

Proposition 2.8. *Suppose that R satisfies Condition (S). For some function Q and positive integer p, let*

$$\{c_k : k = 1, 2, \dots, \text{ and } k \equiv 1 \pmod p\},$$
$$\{d_k : k = 1, 2, \dots, \text{ and } k \equiv 1 \pmod p\}$$

be the two sequences constructed in Proposition 2.7. Let ρ denote the radius of convergence of the power series $\sum c_k z^k$ in (2.28). If $a \in \mathbb{C}$ is such that $|a| < \rho$, then the function

$$v(t,x) = \sum_{m=0}^{\infty} c_{mp+1}a^{mp+1}e^{i(mp+1)(x+R(1)t)}$$

is an infinitely differentiable traveling-wave solution of (1.1).

Remark. As before, in the case $R(y) \le 0$ for all $y \ge 1$, it is more revealing to express $v(t,x)$ as $v(t,x) = \sum_{m=0}^{\infty} c_{mp+1}a^{mp+1}e^{i(mp+1)(x-|R(1)|t)}$.

3. Finite Time Blow-Up of Regular Solutions

In this section, it is shown that certain solutions to (1.1) must lose regularity in finite time. In outline, this result is established as follows. First, a class of weak solutions u of (1.1) is delineated, which include the condition that $u : [0,T] \to L^{2p+2}(\mathbb{S}^1)$ is continuous at least for some $T > 0$. (The symbol $\mathbb{S}^1$ is the one-torus as mentioned in Section 1.) It will be clear from the definition of these weak solutions that they have associated Fourier coefficients $a_k(t)$, $k \in \mathbb{Z}$. In case $a_k(t) = 0$ for $0 \le t \le T$ and $k \le 0$, these Fourier coefficients give rise to what was earlier termed a formal solution of (1.1). It then follows from Proposition 2.2 that the coefficients $\{a_k(t)\}_{k=1}^{\infty}$ may be extended to the entire t-axis and that, so extended, they are periodic in t with period 2π. In case $T \ge 2\pi$, the $L^2([0,2\pi] \times [0,2\pi])$-norm of the solution u can be calculated by Fourier analysis. On the other hand, under hypotheses to be made precise presently, the radius of convergence of the power series constructed in Proposition 2.7 is finite, say equal to $\rho < +\infty$. A consequence of these facts is that, if $|a_1(0)| > \rho$, then the norm of u in $L^2([0,2\pi] \times [0,2\pi])$ is infinite, contradicting the presumption that $u : [0,T] \to L^{2p+2}(\mathbb{S}^1)$ is continuous.

We now embark upon the program just outlined. A few basic facts about the function classes of periodic distributions $\mathcal{D}'(\mathbb{S}^1)$ and $H^m(\mathbb{S}^1)$ for $m \in \mathbb{R}$ will find use. If $u \in \mathcal{D}'(\mathbb{S}^1)$, then its Fourier coefficients are defined to be

$$a_k(u) = \frac{1}{2\pi}\langle u, e^{-ikx}\rangle,$$

where $\langle , \rangle$ connotes the usual dual pairing of $\mathcal{D}'(\mathbb{S}^1)$ with $\mathcal{D}(\mathbb{S}^1) = C^\infty(\mathbb{S}^1)$. The Fourier coefficients $\{a_k(\varphi)\}_{k\in\mathbb{Z}}$ corresponding to $\varphi \in \mathcal{D}(\mathbb{S}^1)$ form a sequence of rapid decrease, meaning that $\sum_{k\in\mathbb{Z}}(1+k^2)^N|a_k(\varphi)| < +\infty$ for all real N, while the Fourier coefficients $\{a_k(u)\}_{k\in\mathbb{Z}}$ of $u \in \mathcal{D}'(\mathbb{S}^1)$ are a sequence of slow increase, meaning that that for large enough values of M, $\sup_{k\in\mathbb{Z}}(1+k^2)^{-M}|a_k(u)| < +\infty$. For $s \in \mathbb{R}$,

$$H^s(\mathbb{S}^1) = \Big\{u \in \mathcal{D}'(\mathbb{S}^1) : \sum_{k\in\mathbb{Z}} |a_k(u)|^2(1+k^2)^s < \infty\Big\},$$

with the obvious inner product and induced norm; thus, $\mathcal{D}'(\mathbb{S}^1) = \bigcup_{s\in\mathbb{R}} H^s(\mathbb{S}^1)$. Moreover, any sequence $\{a_k\}_{k\in\mathbb{Z}}$ which is rapidly decreasing determines a unique element φ in $\mathcal{D}(\mathbb{S}^1)$ whose Fourier coefficients $a_k(\varphi) = a_k$ for all k. Similarly, any sequence $\{a_k\}_{k\in\mathbb{Z}}$ such that $\sum_{k\in\mathbb{Z}}|a_k|^2(1+k^2)^s$ is finite determines a unique $u \in H^s(\mathbb{S}^1)$ whose Fourier coefficients $a_k(u) = a_k$ are precisely the specified sequence. In particular, any element $u \in \mathcal{D}'(\mathbb{S}^1)$ is determined uniquely by its Fourier coefficients (see e.g. Treves [42] for an account of these facts about periodic distributions).

Henceforth, the functions Q and R will both be assumed to have polynomial growth. This is expressed in the following formal requirement.

Condition (G). *There is an $m \in \mathbb{R}$ and a constant C such that*

$$|R(k)| + |Q(k)| \le C(1+k^2)^{m/2} \quad \text{for all } k \in \mathbb{R}.$$

The definitions of the operators $R(-i\partial/\partial x)$ and $Q(-i\partial/\partial x)$ in (2.5) clearly make sense when applied to $u \in H^s(\mathbb{S}^1)$. Indeed, if $u \in H^s(\mathbb{S}^1)$ has Fourier coefficients $\{a_k(u)\}_{k\in\mathbb{Z}}$, then $R(-i\partial/\partial x)u$ is the periodic distribution whose Fourier coefficients are $\{R(k)a_k(u)\}_{k\in\mathbb{Z}}$. Because R satisfies Condition (G) with an index $m \in \mathbb{R}$, it is easily deduced that $R(-i\partial/\partial x) : H^s(\mathbb{S}^1) \to H^{s-m}(\mathbb{S}^1)$ is a continuous linear operator. The same remarks apply to Q.

In fact, the bound in Condition (G) will only be used for $k \ge 1$ in our principal developments.

The following notion of solutions in L_2-based Sobolev spaces of negative order provides a broad context in which to discuss initial-value problems for (1.1).

Definition. *Let $m \ge 0$. A function u is a strong H^{-m}-solution of* (1.1) *on* $[0,T]$ *if* $u \in C^1([0,T]; H^{-m}(\mathbb{S}^1))$,

$$Q\left(\frac{1}{i}\frac{\partial}{\partial x}\right)(u^{p+1}) \quad \text{and} \quad R\left(\frac{1}{i}\frac{\partial}{\partial x}\right)u \tag{3.1}$$

lie in $C([0,T];H^{-m}(\mathbb{S}^1))$, *and the combination of terms on the left-hand side of* (1.1) *is the zero function in* $C([0,T];H^{-m}(\mathbb{S}^1))$.

If Q and R satisfy Condition (G) and $u \in C([0,T];L^{2p+2}(\mathbb{S}^1))$ is a distributional solution of (1.1) in $\mathcal{D}'((0,T)\times\mathbb{S}^1)$, then it follows that u is a strong H^{-m}-solution of (1.1), where m is the index appearing in Condition (G). Indeed, both u and u^{p+1} lie in $C([0,T];L^2(\mathbb{S}^1))$, and hence by the continuity of the operators Q and R mentioned above, both the terms in (3.1) lie in $C([0,T];H^{-m}(\mathbb{S}^1))$. It follows from (1.1), therefore, that $u_t \in C(0,T;H^{-m}(\mathbb{S}^1))$. But $u \in C([0,T];L^2(\mathbb{S}^1)) \subset C([0,T];H^{-m}(\mathbb{S}^1))$, and so $u \in C^1([0,T];H^{-m}(\mathbb{S}^1))$. Since all three terms in (1.1) are thus known to be continuous functions with values in $H^{-m}(\mathbb{S}^1)$, it follows, since the equation is satisfied distributionally, that it is satisfied in the sense of $H^{-m}(\mathbb{S}^1)$, identically in t.

Notice that many standard solution classes arising in the theory of differential equations lie in the class of strong H^{-m}-solutions. Certainly a classical solution, by which is meant a function $u(t,x)$ such that each term in (1.1) can be interpreted as a continuous function on $[0,T]\times\mathbb{S}^1$, is a strong H^{-m}-solution with $m = 0$, say. Similarly, a solution in the sense that every term in (1.1) can be interpreted as an L^2-function of space and time also qualifies as a strong H^{-m}-solution. In particular, suppose that R satisfies Condition (S), so, in particular, it is real-valued, and consider writing a solution of (1.1) formally as an integral equation by use of Duhamel's principle thusly:

$$u(t) = e^{iRt}u_0 - iQ\left(\frac{1}{i}\frac{\partial}{\partial x}\right)\int_0^t e^{i(t-s)R}u(s)^{p+1}\,ds, \tag{3.2}$$

where the spatial variable x has been suppressed, $u_0(x) = u(0,x)$ is the initial data, and it is supposed as above that $u : [0,T] \to L^{2p+2}(\mathbb{S}^1)$ is continuous. On account of the last presumption, $u_0 \in L^2(\mathbb{S}^1)$, as is $u(s)^{p+1}$ for each $s \in [0,T]$. The C_0-group e^{iRt} is defined via its action on the Fourier coefficients of u, viz. $(e^{iRt}u)_k = e^{iR(k)t}a_k$, where $a_k = a_k(u)$ is the k^{th} Fourier coefficient of u. As R is real-valued, this group acts unitarily on all the spaces $H^s(\mathbb{S}^1)$, and in particular on $L^2(\mathbb{S}^1)$. Thus, because of the polynomial growth assumptions in Condition (G) on R and Q, all the terms in the integral equation (3.2) lie in $H^{-m}(\mathbb{S}^1)$ for each t. It is routine to verify that, if $u \in C([0,T];L^{2p+2}(\mathbb{S}^1))$ is a solution of (3.2), then it is in fact a strong H^{-m}-solution of (1.1) as previously defined. The key point needed to ascertain this fact is that $L^2(\mathbb{S}^1)$ is contained in the domain of the generator iR of the unitary group e^{iRt} on $H^{-m}(\mathbb{S}^1)$.

Theorem 3.1. *Suppose for some* $m \geq 0$, *the functions Q and R satisfy Condition (G). Suppose also that R satisfies Condition (S). Let p be a positive integer and let* $u \in C([0,T];L^{2p+2}(\mathbb{S}^1))$ *be a strong* H^{-m}*-solution of* (1.1) *whose Fourier coefficients* $a_k(t)$ *vanish on* $[0,T]$ *for all* $k \leq 0$. *Let* ρ *be the radius of convergence of the power series* (2.28) *in Proposition* 2.7. *If* $|a_1(0)| > \rho$, *then it must be the case that* $T < 2\pi$.

Remarks. This theorem provides a blow-up result in the following sense. Suppose that equation (1.1) is well posed as an initial-value problem in some Banach space X embedded continuously in $L^{2p+2}(\mathbb{S}^1)$. For example, X might be $H^s(\mathbb{S}^1)$ for $s \geq p/(2p+2)$. Thus for $u_0 \in X$, there is a $T > 0$ and an associated solution $u \in C([0,T];X)$ of (1.1) at least in the sense of distributions. Moreover, we presume that the correspondence between initial data and the solution is continuous and that the value of T can be chosen uniformly on subsets of initial data that are bounded in X. Since $L^{2p+2}(\mathbb{S}^1) \supset X$, u comprises a strong H^{-m}-solution of (1.1) as remarked above, where m is the value appearing in Condition (G). Suppose that the solution u has only positive Fourier modes as in (2.1). Our theorem together with the just posited local well-posedness result in X implies that the solution u cannot remain finite in the X-norm on the entire temporal interval $[0, 2\pi]$. These remarks apply to the GKdV equation introduced in Examples 1.1 and 1.2 and to the Schrödinger-type equations appearing in Example 1.3.

It is worth to remark that the Weierstrass $\wp$-function can be used in the case of the KdV-equation itself, Example 1.1 with $p = 1$, to provide complex solutions that form singularities in finite time. If we consider $\wp$ with real and imaginary half-periods K and iK', respectively, then $u(t,x) = -6\wp(x - it + it_0 \mid K, iK') + i/2$ defines a solution of (GKdV) with $p = 1$. If we choose t_0 to be a small positive number, say, then u is infinitely smooth (analytic in fact) as a function of x and periodic in x with period $2K$ at $t = 0$. However, when t reaches t_0, u exhibits a singularity (pole of order 2) at $x = 0$. More elaborate explicit examples can be constructed using the finite-gap solutions of the KdV-equation (see e.g. the monograph of Belokolos et al. [5] for an exposition of these). So far as we know, these are unrelated to the solutions constructed here when our arguments are specialized to the KdV-equation.

Proof. It is clear that the H^{-m}-solutions posited to exist have Fourier coefficients that define formal solutions in the sense of Section 2. In consequence, the theoretical considerations developed in Section 2 are available to us.

Suppose to the contrary that $T \geq 2\pi$. Then the solution u is a member of the function class $L^2([0,2\pi]\times\mathbb{S}^1)$. Because R satisfies Condition (S), Proposition 2.2 implies the Fourier coefficients $\{a_k(t)\}_{k=1}^{\infty}$ of u are given by (2.9). In consequence of the orthogonality properties of the functions $\{e^{i(kx+ht)}\}_{k,h\in\mathbb{Z}}$ in $L^2([0,2\pi] \times [0,2\pi])$, it follows from (2.14) that

$$||u||^2_{L^2([0,2\pi]\times[0,2\pi])} = 4\pi^2 \sum_{k=1}^{\infty} \sum_{h=kR(1)}^{R(k)} |\alpha_{k,h}|^2 \geq 4\pi^2 \sum_{k=1}^{\infty} |\alpha_{k,k|R(1)|}|^2.$$

On the other hand, according to Proposition 2.7 there is a sequence $\{c_k\}_{k=1}^{\infty}$ such that $\alpha_{k,kR(1)} = c_k a^k$ where $a = \alpha_{1,|R(1)|} = a_1(0)$. In consequence, the last inequality may be continued thusly:

$$\|u\|^2_{L^2([0,2\pi]\times[0,2\pi])} \geq 4\pi^2 \sum_{k=1}^{\infty} |\alpha_{k,k|R(1)|}|^2 = 4\pi^2 \sum |c_k|^2 \, |a_1(0)|^{2k} = +\infty,$$

since $|a_1(0)| > \rho$, where ρ is the radius of convergence of the series $\sum_{k=1}^{\infty} c_k z^k$. □

The remainder of this section is devoted to obtaining a better understanding of the parameter ρ that figured so prominently in the last result. Conditions on Q and R will be given which imply that $\rho < \infty$. In such cases, the preceding theorem is not vacuous. Conditions are also presented that distinguish between the two possibilities $\rho > 0$ and $\rho = 0$. The essential assumptions are connected with the behavior of the quantity $Q(k)/[R(k) - kR(1)]$ that has arisen in earlier considerations.

Proposition 3.2. *Let $p \geq 1$ be a fixed integer. Suppose that R satisfies Condition (S). For some function Q, let*

$$\{c_k : k = 1, 2, \ldots, \text{ and } k \equiv 1 \ (\mathrm{mod}\, p)\},$$
$$\{d_k : k = 1, 2, \ldots, \text{ and } k \equiv 1 \ (\mathrm{mod}\, p)\}$$

be the two sequences constructed in Proposition 2.7, and let ρ denote the radius of convergence of the power series $\sum c_k z^k$ in (2.28).

(a) *If there exist $C > 0$ and $\alpha > 0$ such that*

$$\frac{Q(k)}{R(k) - kR(1)} \geq Ck^{\alpha}, \quad \text{for all } k \geq 2, \ k \equiv 1 \ (\mathrm{mod}\, p), \tag{3.3}$$

then $\rho = 0$.

(b) *If there exist $C > 0$ and a nonnegative integer m such that*

$$\frac{Q(k)}{R(k) - kR(1)} \geq \frac{C}{k^m}, \quad \text{for all } k \geq 2, \ k \equiv 1 \ (\mathrm{mod}\, p), \tag{3.4}$$

then ρ is finite.

Remark. The condition (3.4) is clearly satisfied if R and Q are polynomials such that $R(k)Q(k) > 0$, for all $k \geq 2$, $k \equiv 1 \ (\mathrm{mod}\, p)$. If, in addition, $\deg Q > \deg R$, then condition (3.3) is valid with $\alpha = 1$.

Proof of part (a). Recall that by Proposition 2.7, c_k and d_k are nonzero only if $k \equiv 1 \ (\mathrm{mod}\, p)$. Let $\alpha \geq 0$ be given and suppose that (3.3) holds for this value of α. Then there exists $L > 0$ such that

$$c_k \geq L\left(\frac{k-1}{p}\right)^{\alpha} d_k, \quad \text{for all } k \geq 2, \ k \equiv 1 \ (\mathrm{mod}\, p).$$

It will be shown by induction that if $\gamma \le (p+1)L$, then

$$c_k \ge \gamma^{(k-1)/p}\left[\left(\frac{k-1}{p}\right)!\right]^{\alpha}, \quad \text{for all } k \ge 1,\ k \equiv 1 \pmod p. \tag{3.5}$$

It is clear that (3.5) is true for $k = 1$, since $c_1 = 1$. Suppose (3.5) has been proved through $k-1$. As observed several times, d_k is the sum of all products of the form $c_{k_1}c_{k_2}\cdots c_{k_{p+1}}$, where $k_1 + \cdots + k_{p+1} = k$, and $k_j \equiv 1 \pmod p$, $j = 1, 2, \ldots, p+1$. In particular, the product $c_1 c_1 \cdots c_1 c_{k-p}$ is equal to c_{k-p} since $c_1 = 1$. Moreover, the latter product occurs $p+1$ times in the sum. Since all the c_k are nonnegative, it follows that $d_k \ge (p+1)c_{k-p}$. Thus, by the induction hypothesis,

$$\begin{aligned} c_k &\ge L\left(\frac{k-1}{p}\right)^{\alpha} d_k \ge L\left(\frac{k-1}{p}\right)^{\alpha}(p+1)c_{k-p} \\ &\ge L\left(\frac{k-1}{p}\right)^{\alpha}(p+1)\gamma^{(k-p-1)/p}\left[\left(\frac{k-p-1}{p}\right)!\right]^{\alpha} \\ &= L\gamma^{-1}(p+1)\gamma^{(k-1)/p}\left[\left(\frac{k-1}{p}\right)!\right]^{\alpha} \ge \gamma^{(k-1)/p}\left[\left(\frac{k-1}{p}\right)!\right]^{\alpha}. \end{aligned}$$

This proves (3.5).

From (3.5), it is adduced that if $\alpha > 0$, then the radius of convergence is zero as asserted in statement (a). If $\alpha = 0$, then the radius of convergence is finite, which is the assertion in statement (b) when $m = 0$. □

Proof of part (b). The case $m = 0$ is already in hand, but notice that if (3.4) is valid for a value m_0, say, then it is valid for any $m \le m_0$. Thus in establishing statement (b), it suffices to demonstrate the conclusion is valid for m sufficiently large.

Fix an integer $m \ge 1$ and a real number $r > 0$. Consider the two analytic functions

$$f(z) = \frac{z}{(1-(rz)^p)^{m/p}} \quad \text{and} \quad f(z)^{p+1} = \frac{z^{p+1}}{(1-(rz)^p)^{m(p+1)/p}}.$$

For an integer $n \ge 0$, denote by A_{np+1} and B_{np+1}, respectively, the coefficient of z^{np+1} in the Taylor series expansion of $f(z)$ and $f(z)^{p+1}$ around 0. Note that both of these series have radius of convergence equal to $1/r$. Elementary calculations establish that $A_1 = 1$, $B_1 = 0$, and

$$\frac{A_{np+1}}{B_{np+1}} = \frac{r^p}{T_m(n)}, \quad n \ge 1,$$

where T_m is the polynomial

$$T_m(n) = \frac{n(n+m/p)(n+m/p+1)\cdots(n+m/p+(m-2))}{(m/p+m-1)(m/p+m-2)\cdots(m/p)}$$

of degree m with positive coefficients depending on m and p. In other words, for all $k \geq 2$, $k \equiv 1 \pmod{p}$, one has

$$A_k = \frac{r^p}{T_m((k-1)/p)} B_k.$$

Let R and Q be as in the statement of the proposition and suppose (3.4) to be valid for some positive integer m and some positive constant C. By choosing r small enough, it can be assured that

$$\frac{Q(k)}{R(k) - kR(1)} \geq \frac{r^p}{T_m((k-1)/p)}, \quad \text{for all } k \geq 2,\ k \equiv 1 \pmod{p},$$

and therefore that

$$c_k \geq \frac{r^p}{T_m((k-1)/p)} d_k, \quad \text{for all } k \geq 2,\ k \equiv 1 \pmod{p}.$$

Since $c_1 = A_1 = 1$ and $d_1 = B_1 = 0$, a straightforward induction argument shows that for all $k \geq 1$, $c_k \geq A_k$ and $d_k \geq B_k$. This proves statement (b) and the proposition is established. □

Proposition 3.3. *Fix an integer $p \geq 1$. Suppose that R satisfies Condition (S). For some function Q, let*

$$\{c_k : k = 1, 2, \ldots, \text{ and } k \equiv 1 \pmod{p}\},$$
$$\{d_k : k = 1, 2, \ldots, \text{ and } k \equiv 1 \pmod{p}\}$$

be the two sequences constructed in Proposition 2.7, and let ρ denote the radius of convergence of the power series (2.28) based on the sequence $\{c_k\}_{k=1}^{\infty}$. Let

$$\{C_k : k = 1, 2, \ldots, \text{ and } k \equiv 1 \pmod{p}\},$$
$$\{D_k : k = 1, 2, \ldots, \text{ and } k \equiv 1 \pmod{p}\}$$

be the two sequences constructed via Proposition 2.7 relative to R and $-Q$. Then the radius of convergence of the power series $\sum C_k z^k$ is also ρ. Thus the radius of convergence, and hence the blow-up results of Theorem 3.1 do not depend on the sign of the nonlinear term.

Proof. Let $\{c_k\}$ and $\{d_k\}$ be as in the statement of the proposition, the coefficients obtained from the polynomials R and Q via Proposition 2.7, so that

$$c_k = \frac{Q(k)}{R(k) - kR(1)} d_k, \quad \text{for } k \geq 2,\ k \equiv 1 \pmod{p}.$$

Let $\varepsilon \in \mathbb{C}$ be such that $\varepsilon^p = -1$. Set $C_k = \varepsilon^{k-1} c_k$, for all $k \geq 1$ with $k \equiv 1 \pmod p$. (It will turn out that $\{C_k\}_{k=1}^{\infty}$ is exactly the sequence defined via Proposition 2.7 for the functions R and $-Q$.) Note that $C_1 = 1$ and that $\varepsilon^{k-1} \in \mathbb{R}$, for all $k \geq 1$ with $k \equiv 1 \pmod p$. Let D_k be the coefficients obtained from the C_k by forming the product

$$\Big(\sum_{k=1}^{\infty} C_k z^k\Big)^{p+1} = \sum_{k=p+1}^{\infty} D_k z^k$$

in the ring of formal power series. In other words, D_k is the sum of all products of the form $C_{k_1} C_{k_2} \cdots C_{k_{p+1}} = \varepsilon^{k-p-1} c_{k_1} c_{k_2} \cdots c_{k_{p+1}}$, where $k_1 + \cdots + k_{p+1} = k$ and $k_j \equiv 1 \pmod p$, $j = 1, 2, \ldots, p+1$. It follows by inspection that $D_k = \varepsilon^{k-p-1} d_k$, for all $k \geq p+1$, $k \equiv 1 \pmod p$, whence

$$C_k = \frac{-Q(k)}{R(k) - kR(1)} D_k, \quad k \geq 2,\ k \equiv 1 \pmod p.$$

Thus the sequences $\{C_k : k = 1, 2, \ldots, \text{ and } k \equiv 1 \pmod p\}$ and $\{D_k : k = 1, 2, \ldots, \text{ and } k \equiv 1 \pmod p\}$ are determined by the same R, but with $-Q$ instead of Q. This proves the proposition. □

Remark. If u is a formal solution of (1.1) of the form (2.1), it may be that $a_1(t)$ is identically zero. In this case, blow-up results analogous to those enunciated in the preceding propositions may be obtained by studying the lowest nonvanishing Fourier mode. If we denote this mode by κ, then the analogue of Proposition 2.7 is the existence of two sequences $\{c_k\}$ and $\{d_k\}$ such that $c_1 = 1$, $d_1 = 0$, and $\alpha_{k\kappa, k|R(\kappa)|} = c_k a^k$, $\beta_{k\kappa, k|R(\kappa)|} = d_k a^k$, for all $k \geq 1$, $k \equiv 1 \pmod p$, with $a = \alpha_{\kappa, |R(\kappa)|}$. Moreover, the relationship

$$c_k = \frac{Q(k\kappa)}{R(k\kappa) - kR(\kappa)} d_k, \quad k \geq 2,\ k \equiv 1 \pmod p,$$

still holds. With this observation in hand, analogues of the previous propositions (as well as those which follow) are readily established in the context where $a_j = 0$ for $j < \kappa$.

In particular, suppose that R and Q are polynomials such that R satisfies Condition (S), the product $R(k)Q(k)$ is either everywhere positive or everywhere negative for $k \geq 2$, $k \equiv 1 \pmod p$, and $\deg Q > \deg R$. Then any nontrivial regular solution of (1.1) of the form (2.1) must blow up in finite time.

Attention is now turned to specifying conditions under which $\rho > 0$, which then implies the existence of regular solutions of (1.1).

Proposition 3.4. *Let $p \geq 1$ be an integer. Suppose that R satisfies Condition (S). For some function Q, let*

$$\{c_k : k = 1, 2, \ldots, \text{ and } k \equiv 1 \pmod{p}\},$$
$$\{d_k : k = 1, 2, \ldots, \text{ and } k \equiv 1 \pmod{p}\}$$

be the two sequences constructed in Proposition 2.7, and let ρ denote the radius of convergence of the power series in (2.28). *If additionally there exist $C > 0$ and $m \geq 1$ such that*

$$\frac{|Q(k)|}{|R(k) - kR(1)|} \leq \frac{C}{k^m}, \quad \text{for all } k \geq 2,\ k \equiv 1 \pmod{p}, \tag{3.6}$$

then $\rho > 0$. In particular, in this case (1.1) *has an uncountable family of smooth traveling-wave solutions as described in Proposition 2.8.*

Remark. The restriction in (3.6) is satisfied if, in addition to R satisfying Condition (S), R and Q are such that $\deg Q < \deg R$.

Proof. We use the same notation as in the proof of Proposition 3.2. By (3.6), there exist $m \geq 1$ and $r > 0$ such that

$$\frac{|Q(k)|}{|R(k) - kR(1)|} \leq \frac{r^p}{T_m((k-1)/p)}, \qquad \text{for all } k \geq 2,\ k \equiv 1 \pmod{p},$$

and so

$$|c_k| \leq \frac{r^p}{T_m((k-1)/p)} |d_k|, \quad \text{for all } k \geq 2,\ k \equiv 1 \pmod{p}.$$

Since $c_1 = A_1 = 1$ and $d_1 = B_1 = 0$, an easy induction argument shows that $|c_k| \leq A_k$ and $|d_k| \leq B_k$, and these inequalities establish the proposition. □

In the special, but frequently occurring case $p = 1$, a different proof gives the same result assuming only $\deg Q \leq \deg R$. To show this, the following preparatory lemma is useful.

Lemma 3.5. *For any real number $r > 1$, the inequality*

$$\sum_{j=1}^{k-1} \frac{1}{(k-j)^r j^r} \leq \frac{2^{3r+1}}{(r-1)(k+1)^r}$$

is valid for any integer $k > 1$.

Proof.

$$\begin{aligned}\sum_{j=1}^{k-1} \frac{1}{(k-j)^r j^r} &\leq \int_1^k \frac{4^r}{(k+1-x)^r x^r}\,dx \\ &= 2\int_1^{(k+1)/2} \frac{4^r}{(k+1-x)^r x^r}\,dx \\ &\leq \frac{2\cdot 8^r}{(k+1)^r}\int_1^{(k+1)/2} \frac{1}{x^r}\,dx \\ &\leq \frac{2\cdot 8^r}{(r-1)(k+1)^r}.\end{aligned}$$

□

Proposition 3.6. *Suppose that R satisfies Condition (S) and that $p = 1$. For some function Q, let $\{c_k : k = 1,2,\ldots\}$ and $\{d_k : k = 1,2,\ldots\}$ be the two sequences constructed in Proposition 2.7 from Q and R and let ρ denote the radius of convergence of the power series defined via the c_k's as in* (2.28)*. If there exists $C > 0$ such that*

$$\frac{|Q(k)|}{|R(k) - kR(1)|} \leq C, \quad \text{for all } k \geq 2, \tag{3.7}$$

then $\rho > 0$. In particular, (1.1) *has an uncountable family of smooth traveling-wave solutions as described in Proposition 2.8.*

Proof. By (3.7), there exists $L > 0$ such that $|c_k| \leq L|d_k|$, for all $k \geq 2$. Fix $r > 1$, and choose $M > 0$ and $\gamma > 0$ such that

$$LM\frac{2^{3r+1}}{r-1} \leq 1 \quad \text{and} \quad M\gamma \geq 1.$$

(First choose M small enough, then choose γ large enough.)

We prove by induction that for all k,

$$|c_k| \leq \frac{M\gamma^k}{k^r}. \tag{3.8}$$

Clearly (3.8) is correct for $k = 1$, since $c_1 = 1 \leq M\gamma$. Suppose that (3.8) is true through $k - 1$, for some $k \geq 2$. From the induction hypothesis, the previous lemma, and the fact that (since $p = 1$)

$$d_k = \sum_{j=1}^{k-1} c_{k-j}c_j, \quad \text{for each } k \geq 2,$$

it follows that

$$|c_k| \leq L|d_k| \leq L \sum_{j=1}^{k-1} |c_{k-j}c_j| \leq LM^2\gamma^k \sum_{j=1}^{k-1} \frac{1}{(k-j)^r j^r}$$

$$\leq LM^2\gamma^k \frac{2^{3r+1}}{(r-1)(k+1)^r} \leq \frac{M\gamma^k}{(k+1)^r} \leq \frac{M\gamma^k}{k^r}.$$

This concludes the proof. □

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