STABILIZING, MONETARY INJECTION POLICIES *

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Abstract. The primary concern of this paper is to understand how prices and interest rates respond to open-market operations. It is shown according to a simple mathematical model that a single monetary injection will lead to a temporary fall in both real and nominal interest rates and a gradual rise in prices. A particular focus of the present investigation is the determination of an optimal monetary policy wherein injections take place in more than one time period, and corresponding to which the equilibrium prices are most stable. The existence of such a policy is established by a mathematical analysis and its detailed structure explored by numerical simulation. A feature of this paper is the systematic and interactive application of tools from economic theory, mathematics and modern scientific computation.

1. Introduction. The goal of the present work is to develop theory bearing upon the response of prices and interest rates to government intervention in the money supply. This issue is one of obvious importance that has attracted both descriptive and theoretical commentary. Conclusions have been made from several points of views and based on a variety of models [c.f. Baumol (1952), Tobin (1956), Grandmont and Younes (1973), Bryant and Wallace (1979), Lucas (1980), Stockman (1981), Townsend (1982), Helpman (1982), Polemarchakis (1982), Sargent and Wallace (1982), Grossman and Weiss (1983), Bona and Grossman (1983)]. Analytically, an open-market operation is defined in these papers as a government purchase with money of some other asset together with the associated adjustments in (lump-sum) transfer payments that are needed to keep the path of government consumption unchanged. The main concern is whether the open market operation has any effect on economic quantities such as real and nominal interest rates, level of output and prices. Most of the aforementioned works have concluded that the open-market operations have real economic effects. Not all the studies adhere to this conclusion. Modigliani and Miller suggest that changes in the money supply brought about by open-market operations will have no effect on price level. Wallace (1981) and Chamley and Polemarchakis (1982) have shown that in a model in which money is held only for its rate of return, characteristic open-market operations are neutral.

The approach taken here is to develop a simplified mathematical model which nevertheless maintains the primary relationships between the dependent and independent variables, and to investigate this model using both analytical and numerical tools. The predictions made on the basis of this model are then taken as a potential guide to the actual market response to open-market operations.

The model upon which our study is based was put forward by Grossman and Weiss (1983). This model, described in detail in Section 2, features consumers with perfect foresight, firms owned by the consumers that produce a single, perishable good, and the government. Going beyond this pioneering model where consumers possessed only logarithmic preferences, consideration is given to a broad class of utility functions. In the case considered by Grossman and Weiss, namely consumers having a logarithmic utility function, it was shown that open-market operations can have real

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effects and that a monetary expansion will lead to a temporary reduction in both real and nominal interest rates and an eventual increase in prices. However, current price levels are unaffected by anticipated future monetary injections, a situation contrary to common experience.

The analysis of the class of utility functions considered here, which includes a one-parameter family of the homothetic ones, is considerably more involved than that coming to the fore in Grossman and Weiss (1983). The equilibrium prices as a function of a discrete period variable n are governed by a second-order, implicit, difference equation with but one initial condition, whereas the assumption of logarithmic utility leads to a first-order difference relation. In the second-order case that arises here, the relevant solution is obtained by solving a functional equation of the form $g(f \circ f(x), f(x), x) = 0$ for the unknown function f. The equilibrium price sequence is then realized by iterating the function f on the initial price p_1, so that the price at the n^{th} period is p_n where $p_2 = f(p_1)$, $p_3 = f(p_2)$ and so on. The existence of a price-function f relating the price at period $n+1$ to the price at period n is established in two principal steps. First the stable manifold theorem from dynamical systems theory is employed to prove the existence of a local solution by relating f to the graph of a stable manifold for an auxiliary two-dimensional mapping. The local solution is then shown to be the restriction of a globally defined solution by an extension argument. The analysis just described, which is worked out in detail in Section 3, is telling in the case of homothetic utility functions. In Section 5, an application of the implicit-function theorem allows the results for homothetic utility functions to be extended to a much broader class of utility functions.

The outcome of this analysis is quite satisfactory as regards its economic interpretation. Indeed, the aforementioned drawbacks of the Grossman-Weiss model are seen to be special to the assumption of logarithmic utility. To understand better the detailed structure of solutions for economically realistic values of the model's parameters, numerical simulation seems to be indicated. Accordingly, we report in Section 4 an efficient and accurate scheme for the approximation of solutions of the model which is implemented as a computer code. After suitable testing, the computer program was used in an investigative mode. In addition to providing quantitative information about solutions, the code led to the formulation of an interesting conjecture about monetary policy.

Briefly stated, it is found both analytically and numerically that a one-period expansion of the money supply leads eventually to a new, higher, price level. However, equilibrium prices approach this new steady state in an oscillatory manner. Such oscillation might well seem undesirable, and preliminary numerical simulation indicated it could be suppressed if the injection was spread in a particular manner over two or more periods. This leads to a set of theoretical results about stabilizing the price adjustment the model predicts in response to government monetary injection. These ideas are also developed in Section 4. They have potential implications for government policy which are visited in the concluding Section 6.

2. The Model. In this section, a detailed description is presented of the model which will be the center of attention henceforth.

2.1. *Basic components*

The major components of this class of models are consumers, firms and the government. Consumers are supposed to have perfect foresight and they come equipped initially with a stock of money and government bonds. Firms, owned by consumers, produce an exogenous good and sell it to other consumers for profit. Because consumers need cash to purchase goods, they sell their bonds to the firms and to the government. The firms use the profit from selling goods to purchase bonds from consumers and immediately put them into the owners' interest-bearing accounts, as it is clear that under certainty they will not hold cash if the interest rate is positive. The government uses cash to purchase bonds, thereby injecting money into the market. This is the most common and probably the most flexible way for the government to influence the economy. We assume that the cash will eventually find its way into the hands of consumers. They can only obtain cash from the bond market (which will be referred to as the "bank" henceforth). Consumers use the money from their bank withdrawals to purchase the unique perishable consumption good which is exogenously supplied at a rate of C units per period. Because of transaction costs, consumers tend to bunch cash withdrawals and hence there is a period between consumers' trips to the bank. It is assumed here that the time between two consecutive trips to the bank is fixed, and that consumers will not all go to the bank at the same time. More specifically, we take it that each consumer goes to the bank every other period and withdraws enough cash to finance consumption expenditure over the ensuing two periods. The amount of withdrawal is determined by the possibility of inter-temporal consumption substitution, and thus is influenced by the expected prices and future nominal interest rates. Note that in this simple model, the market could not clear if all consumers simultaneously showed up at the bank to make a withdrawal. For simplicity, we assume that there are two classes, a-type and b-type, consumers. The a-type consumers go to the bank every odd period, whereas b-type consumers appear there every even period. As usual, these two classes of consumers will be lumped, and from now on it will appear that there are exactly two consumers in the economy. The consumers are infinitely lived here, though one can also realize this sort of model in an overlapping-generations context where consumers are participating economic agents for exactly two periods each (see Bona and Grossman [2]).

2.2. *Formulation*

The optimization problem faced by consumers is described in more detail. Let N_a and N_b denote the set of odd and even positive integers, respectively. Each consumer must choose bank withdrawals M_t^i ($t \in N_i$) and consumption plans C_t^i so as to

$$\text{(1)} \qquad \text{maximize} \quad \sum_{t=1}^{\infty} \beta^{t-1} U(C_t^i),$$

$i = a, b$, subject to the constraints that only money can purchase goods and that all the money will be spent before the next bank withdrawal. These constraints may be expressed as

$$\text{(2)} \qquad p_1 C_1^a = M_0^a, \qquad p_1 C_1^b + p_2 C_2^b = M_0^b,$$

and

$$\text{(3)} \qquad p_{t+1} C_{t+1}^i + p_{t+2} C_{t+2}^i = M_t^i, \quad \mathit{for} \quad \mathit{all} \quad t \in N_i, \quad i = a, b.$$

In (1), U is a given utility function that reflects consumer preference and $\beta \in (0,1)$ is a "discount" factor. As usual, we assume that $U'(x) > 0$ and $U''(x) < 0$. The price of the single good in period t is p_t, $t = 0, 1, \cdots$. The two quantities M_0^i are the consumer's initial money holdings and the later money holdings M_t^i must satisfy the overall wealth constraint

$$\sum_{t \in N_i} \frac{M_t^i}{\alpha_t} \leq W^i, \quad i = a, b. \tag{4}$$

In (4), the W^i are the nominal values of the non-monetary wealth of consumer i,

$$\alpha_t = R_1 R_2 \cdots R_{t-1} \quad (\text{with } \alpha_1 = 1)$$

is the nominal growth factor between period 1 and period t, and R_{t-1} is the nominal interest rate on assets held in the t^{th} period (between t and $t+1$).

In this model, the non-monetary wealth W^i of consumer i consists of three components. The first is a claim to a fraction of the firm's revenue which is deposited into an interest-earning account every period. If $s^i \in (0,1)$ denotes the fraction of the firm's revenue claimed by consumer i, then assuming there is no government ownership, we must have

$$s^a + s^b = 1.$$

Thus the current nominal value of the total income of consumer i from the firm's revenue is

$$s^i \sum_{t=1}^{\infty} \frac{p_t C}{\alpha_t},$$

where C is the exogenously given output produced by the firm. The second component of W^i is the one-period maturity government bonds that the consumer currently holds. This component is denoted by B_0^i. The third component of W^i is the per-capita tax that consumer i is obligated to pay to cover the government debt (denoted by B). In this two-consumer economy, the per-capita taxes are taken to be $B/2$, where B equals the value of the government bonds that the consumers currently hold minus the current nominal values of the future government money injections, which is to say

$$B = B_0 - \sum_{t=1}^{\infty} \frac{M_t^s - M_{t-1}^s}{\alpha_t}, \tag{5}$$

where $B_0 = B_0^a + B_0^b$, and M_t^s is the exogenously given money supply. Combining these three components leads to a formula for the non-monetary wealth, namely

$$W^i = s^i \sum_{t=1}^{\infty} \frac{p_t C}{\alpha_t} + B_0^i - \frac{B}{2}. \tag{6}$$

To study the optimization problem (1)-(3), consideration is given to the auxiliary optimization problem that concerns a consumer who has just withdrawn M amount of money from the bank at the end of a given period, and will allocate it between expenditures in the next two periods. This consumer thus faces the problem

$$\max_{C_1, C_2} \quad U(C_1) + \beta U(C_2)$$

subject to

$$p_1C_1 + p_2C_2 = M.$$

Under the assumptions mentioned earlier on the utility function U, it is well-known that the above optimization problem has a unique solution for given p_1, p_2 and M. The solution for the optimal choice C_2 is denoted by

$$C_2 = C_2(X, Y), \quad where \quad X = \frac{p_1}{p_2} \quad and \quad Y = \frac{M}{p_2}. \tag{7}$$

Because of the special structure of the global optimization problem (1)-(3), the just described local optimization problem plays a central role in its resolution. Indeed, it is straightforward to verify the standard transversality condition as in Arrow and Kurz (1970). Then, using the identity

$$\sum_{t=1}^{\infty} \beta^{t-1}U(C_t^i) = U(C_1^a) + \sum_{t \in N_a} \beta^t (U(C_{t+1}^a) + \beta U(C_{t+2}^a))$$

and the fact that the constraints (2) and (3) are decoupled with regard to the period t, it is concluded that the optimal strategy for a consumer is, at each visit to the bank, to withdraw all their available cash and allocate it between the ensuing two periods according to the solution of the local optimization problem. In the notation introduced above, the optimal choices of C_t^a for the problem (1)-(3) are given by

$$p_1C_1^a = M_0^a \tag{8}$$

and

$$p_{t+1}C_{t+1}^a + p_{t+2}C_{t+2}^a = M_t^a, \tag{9}$$

where

$$C_{t+2}^a = C_2\left(\frac{p_{t+1}}{p_{t+2}}, \frac{M_t^a}{p_{t+2}}\right), \quad for \quad t \in N_a.$$

Similarly, the optimal choices of C_t^b satisfy

$$p_{t+1}C_{t+1}^b + p_{t+2}C_{t+2}^b = M_t^b, \tag{10}$$

where

$$C_{t+2}^b = C_2\left(\frac{p_{t+1}}{p_{t+2}}, \frac{M_t^b}{p_{t+2}}\right), \quad for \quad t \in N_b \cup \{0\}.$$

The requirement that the market be in equilibrium implies that the flow of cash into the market at each period equals the relevant consumer's desired withdrawal and that goods demanded by consumers are equal to the goods supplied. These equilibrium conditions are expressed as

$$C_t^a + C_t^b = C, \tag{11}$$
$$M_t^a + M_t^b = M_t^s, \tag{12}$$

which initially satisfies

$$M_0^a + M_0^b = M_0^s.$$

It is assumed that the money supply changes only through open-market operations. For $t > 0$, M_t^i represents the money holding of consumer i, which is, at the start of a banking period $t \in N_i$, equal to the amount just withdrawn from the bank, and on the period following a banking period is equal to the withdrawal in the previous period, but reduced by the previous period's spending, namely

$$M_t^i = M_{t-1}^i - p_t C_t^i \quad \textit{for} \quad t \notin N_i. \tag{13}$$

As a consequence of (13) and (3), it follows that

$$M_{t-1}^i = p_t C_t^i, \quad \textit{for} \quad t \in N_i. \tag{14}$$

Combining (13), (11), (14) and (12), one obtains the formulas

$$M_t^i = p_t C + M_t^s - M_{t-1}^s, \quad \textit{for} \quad t \in N_i, \tag{15}$$

and

$$M_t^i = M_{t-1}^s - p_t C, \quad \textit{for} \quad t \notin N_i. \tag{16}$$

Equation (15) states that the flow of aggregate money withdrawals from the bank must equal the flow of money into the bank. The latter consists of two terms. The first is the aggregate value of firm's receipts $p_t C$ which are assumed to be deposited instantly by firms into the interest-earning accounts. The second term $M_t^s - M_{t-1}^s$ is the change in aggregate nominal money engendered by the government through open-market operations. Thus, under equilibrium conditions, it appears that the bank withdrawals are determined solely by the price levels the consumer will face for the next two periods and the money supply.

Attention is now turned to a specification of the equilibrium prices. Combining (14) with (16) (with $i = b$) gives that, for $t \in N_a$,

$$p_{t+2} C_{t+2}^a = M_{t+1}^a = M_t^s - p_{t+1} C,$$

which, by (9) and (15), can be written as

$$p_{t+1} C + p_{t+2} C_2 \left(\frac{p_{t+1}}{p_{t+2}}, \frac{p_t C + M_t^s - M_{t-1}^s}{p_{t+2}} \right) = M_t^s \quad \textit{for} \quad t \in N_a.$$

A similar argument shows that the above identity also holds for $t \in N_b$. In consequence, for $t \geq 1$, it transpires that

$$p_{t+1} C + p_{t+2} C_2 \left(\frac{p_{t+1}}{p_{t+2}}, \frac{p_t C + M_t^s - M_{t-1}^s}{p_{t+2}} \right) = M_t^s. \tag{17}$$

Combining (8) with (10) at $t = 0$ gives

$$p_1 C + p_2 C_2 \left(\frac{p_1}{p_2}, \frac{M_0^b}{p_2} \right) = M_0^s. \tag{18}$$

The equations in (17) comprise a system of nonlinear, second-order difference equations for the equilibrium prices p_t. It is worth note that once the prices are known, all the other economic quantities are determined by the preceding relations.

2.3. *Homothetic utility function*

To gain more insight into the model, attention is focused on the class of homothetic utility functions. In Grossman and Weiss (1983) where the model originated, a logarithmic utility function was proposed and the model was analyzed on the basis of this special form. The analysis was particularly simple in this case because the equilibrium prices are described by a first-order, linear, difference equation with a single initial condition. However, a serious drawback was that the current price level is independent of anticipated future monetary injections. To better understand the potential of the model, we introduce the one-parameter family of homothetic utility functions $U = U_A$ defined for values of consumption $c \geq 0$ by

$$U(c) = U_A(c) = \frac{c^{1-A}}{1-A},$$

where $A > 0$ is fixed and $A \neq 1$. In this case, it is easy to see that

$$C_2(X,Y) = Y\phi(X),$$

where

$$\phi(X) = \frac{\beta^{1/A}}{\beta^{1/A} + X^{\frac{A-1}{A}}} \tag{19}$$

and $\beta \in (0,1)$ is the aforementioned discount factor. Because of the special form of C_2, equations (17) and (18) reduce to

$$p_{t+1}C + (p_tC + M_t^s - M_{t-1}^s)\phi\left(\frac{p_{t+1}}{p_{t+2}}\right) = M_t^s, \quad for \quad t \geq 1, \tag{20}$$

and

$$p_1C + M_0^b\phi\left(\frac{p_1}{p_2}\right) = M_0^s. \tag{21}$$

Note that ϕ is monotone increasing if $A < 1$, but monotone decreasing if $A > 1$. If $A < 1$, $\phi(0) = 0$ whereas if $A > 1$, $\phi(0) = 1$. In all cases, $\phi(x) \leq 1$ for all $x \geq 0$.

Interest rates. As shown by Grossman and Weiss [4], two-period nominal interest rates can be derived as a function of the price path and path of nominal money withdrawals.

Consider the choice of optimal money withdrawals by an agent at the bank at $t-1$. At an interior optimum the agent is indifferent between withdrawing an extra \$1 and spending it in period t, or letting the \$1 grow to $\$R_{t-1}R_t$ in period $t+1$ and spending this amount in $t+2$. This gives rise to the first-order condition

$$U'(C_{t+1}^i) = \beta^2 U'(C_{t+3}^i)\frac{p_{t+1}}{p_{t+3}}R_tR_{t+1}, \quad t \in N_i. \tag{22}$$

Let $x_t = R_tR_{t+1}$. Then the x_t are uniquely determined by (22) for all $t \geq 1$. Note that

$$\alpha_{2m+1} = \prod_{j=0}^{m-1} x_{2j+1}$$

and

$$\alpha_{2m} = R_1 \prod_{j=1}^{m-1} x_{2j}.$$

To determine R_1, we use the wealth constraints (4). At the first glance, (4) contains two conditions, but in fact (4) with $i = a$ is consistent with (4) with $i = b$. More precisely, we have

LEMMA 2.1. *At equilibrium prices and bank withdrawals,* (4) *holds for* $i = a$ *if and only if it holds for* $i = b$.

Proof. The proof is a straightforward application of identity (15), which holds at the equilibrium prices and bank withdrawals, and the identity (5). □

2.4. *Basic parameters used in the model*

To aid the reader, the basic economic parameters appearing in the model just described are reviewed here.

N_a: the set of odd positive integers;
N_b: the set of even positive integers;
M_0^i: the initial money holdings of consumer i;
B_0^i: the initial nominal holdings of government bonds by consumer i;
M_t^s: the exogenously given money supply;
p_t: the price level during period t;
M_t^i: the money holdings of the consumer i at the end of period t;
R_t: 1 plus the interest earned from the end of period t to the end of period $t+1$;
α_t: $\alpha_t = R_1 \cdot R_2 \cdots R_{t-1}$ with $\alpha_1 = 1$;
C_t^i: the consumption by consumer i during period t;
W^i: the non-monetary wealth of consumer i.

3. Equilibrium Prices. In this section, attention is given to the uniqueness, existence and some qualitative properties of equilibrium prices in the model described in Section 2. The development here is a streamlined version of the account appearing in Bona and Grossman (1983). For simplicity of exposition, and without loss of generality, it will be assumed henceforth that

$$C = 1 \quad and \quad M_0^s = 1.$$

This amounts to a choice of units and it means that the single good is exogenously supplied at a rate of one unit per period and the initial money in the market is one unit. We will concentrate here on the case where $0 < A < 1$. From the construction of the function $C_2(X, Y)$, it is apparent that this represents the situation wherein an increase in the rate of return on savings X, with the level of wealth Y held fixed, results in larger future consumption C_2. That is, the substitution effect dominates the income effect associated with a change in rates of return.

3.1. *Single monetary injection*

For the rest of Section 3, it is assumed that the unannounced monetary policy involves only a $k\%$ increase in the money supply at the end of period 1. Thus money is injected into the market only once, and so

$$M_t^s = (1+k)M_0^s = 1 + k, \quad t \geq 1.$$

In this case (20) can be rewritten as:

$$p_t + \phi(\frac{p_t}{p_{t+1}})p_{t-1} = 1 \quad \text{for} \quad t \geq 2, \tag{23}$$

where $p_1 = \dfrac{p_1 + k}{k+1}$ and $p_t = \dfrac{p_t}{k+1}$ for $t \geq 2$ are scaled prices and the initial condition (18) becomes

$$p_1 + M_0^b \phi\left(\frac{p_1}{p_2}\right) = 1. \tag{24}$$

3.2. *Uniqueness*

It will be demonstrated in this subsection that the equilibrium prices are uniquely determined by the specifications (23)-(24). We start with some general aspects of the situation obtaining for the model with a homothetic utility function and $A < 1$.

Note that the steady-state equilibrium value associated to (23) is

$$\bar{p} = \frac{1}{1+\phi(1)}.$$

In general, as shown below, any solution of (23) will alternate below and above the steady state equilibrium value $\bar{p}$.

LEMMA 3.1. *Let* $\{p_t\}_1^\infty$ *be a sequence satisfying* (23). *Then for any* t, $p_t \leq \bar{p}$ *if and only if* $p_{t+1} \geq \bar{p}$.

Proof. If for some t, $p_t, p_{t+1} < \bar{p}$, then

$$\phi\left(\frac{p_{t+1}}{p_{t+2}}\right) = \frac{1 - p_{t+1}}{p_t} > \frac{1-\bar{p}}{\bar{p}} = \phi(1).$$

Because of the assumption that $A < 1$, ϕ is an increasing function, and it follows that $p_{t+2} < p_{t+1} < \bar{p}$. Continuing this argument inductively shows that

$$p_{s+1} < p_s < p_t < \bar{p}, \quad \text{for all} \quad s > t.$$

Thus, there exists p^* such that $p^* = \lim_{s\to\infty} p_s \leq p_t < \bar{p}$. Taking the limit as the period becomes unboundedly large in equation (23) gives

$$p^* + \phi(1)p^* = 1.$$

This means that $p^* = \bar{p}$, which is a contradiction. Similarly one proves there exists no t such that $p_t, p_{t+1} > \bar{p}$. □

Our next result gives *a priori* bounds for any possible solution of (23).

LEMMA 3.2. *Let* $\{p_t\}_1^\infty$ *be a sequence satisfying* (23). *Then there exist constants* σ_1, $\sigma_2 \in (0,1)$ *so that* $\sigma_1 \leq p_t$ *for* $t \geq 1$ *and* $p_t \leq 1 - \sigma_2$ *for* $t \geq 2$.

Proof. Let p^* (which is less than $\bar{p}$) be the unique solution of the equation

$$p^* + \phi(\frac{p^*}{\bar{p}}) = 1.$$

If $p_t < \bar{p}$, then Lemma 3.1 insures that $p_{t+1} > \bar{p}$. Therefore, again because ϕ is increasing,

$$1 = p_t + \phi(\frac{p_t}{p_{t+1}})p_{t-1} < p_t + \phi(\frac{p_t}{\bar{p}}),$$

which implies that $p_t > p^*$. Thus we may choose $\sigma_1 = \min\{\bar{p}, p^*\} = p^*$ and have $p_t > \sigma_1$ for all $t \geq 1$.

Similarly, if $p_t > \bar{p}$, then $p_{t+1} < \bar{p}$ and hence

$$p_t = 1 - \phi(\frac{p_t}{p_{t+1}})p_{t-1} < 1 - p_{t-1}\phi(1) \leq 1 - \sigma_1\phi(1),$$

provided that $t \geq 2$. Thus, we may take $\sigma_2 = \sigma_1\phi(1)$. □

As regards uniqueness of the price-path $\{p_t\}_1^\infty$, the following result shows the situation to be satisfactory.

LEMMA 3.3. *For any $p_1 \in (0,1]$, there corresponds at most one solution $\{p_t\}_1^\infty$ of* (23).

Proof. Suppose that $\{p_t\}_1^\infty$ and $\{\hat{p}_t\}_1^\infty$ are both solutions of (23) with $p_1 = \hat{p}_1$ and $p_t, \hat{p}_t \in (0,1)$ for all $t \geq 2$. If $p_2 = \hat{p}_2$, then it is obvious that $p_t = \hat{p}_t$ for all t. Without loss of generality, we may assume that $p_2 > \hat{p}_2$. By a direct calculation, using the fact that $\phi^{-1} = \psi$ where

$$\psi(x) = \beta^{1/(A-1)}\left(\frac{1-x}{x}\right)^{\frac{A}{A-1}},$$

it is seen that

$$\text{(25)} \qquad \frac{\hat{p}_{t+1}}{p_{t+1}} = \frac{\hat{p}_t}{p_t}\left(\frac{g(p_t, p_{t-1})}{g(\hat{p}_t, \hat{p}_{t-1})}\right)^{\frac{A}{1-A}}$$

with

$$g(p_t, p_{t-1}) = \frac{1 - p_t}{p_t + p_{t-1} - 1}.$$

It is clear that g is decreasing with respect to each variable. Thus successive applications of (25) yield

$$p_t \geq \hat{p}_t, \quad for \quad all \quad t \geq 1.$$

Furthermore it is elementary to see that $p_t g(p_t, p_{t-1})$ is decreasing with respect to the variable p_t, and thus

$$p_t g(p_t, p_{t-1}) \leq \hat{p}_t g(\hat{p}_t, p_{t-1})) \leq \hat{p}_t g(\hat{p}_t, \hat{p}_{t-1}),$$

which, together with (25) implies that

$$\frac{\hat{p}_{t+1}}{p_{t+1}} \leq \frac{\hat{p}_t}{p_t}\left(\frac{\hat{p}_t}{p_t}\right)^{\frac{A}{1-A}} = \left(\frac{\hat{p}_t}{p_t}\right)^{1+\frac{A}{1-A}}.$$

A recursive application of the above inequality gives

$$\frac{\hat{p}_{t+1}}{p_{t+1}} \leq \left(\frac{\hat{p}_2}{p_2}\right)^{\left(\frac{1}{1-\lambda}\right)^{t-1}} \to 0, \quad \text{as} \quad t \to \infty,$$

which contradicts Lemma 3.2. □

3.3. *Existence*

To study the existence of solutions to the problem (23)-(24), we propose to seek the solution in terms of a price function f such that

$$p_{t+1} = f(p_t), \quad t = 1, 2, 3, \cdots. \tag{26}$$

It is easy to see that the sequence $\{p_t\}_{t=1}^{\infty}$ given by the above formula is a solution of (23) if f satisfies

$$1 - x\phi\left(\frac{f(x)}{f \circ f(x)}\right) = f(x), \quad \text{for} \quad x \in [0, 1]. \tag{27}$$

The function f determined by the above equation will play a crucial role in our study.

LEMMA 3.4. *The functional equation* (27) *has a unique solution* f *defined and real-analytic on* $[0, 1]$. *This solution satisfies the properties*
1. $f(\bar{p}) = \bar{p}$, *and*
2. $-1 < f'(x) < 0$, *for* $x \in [0, 1]$.

The somewhat lengthy proof of this result will be presented in Subsection 3.5 below. Combining it with the previous lemmas, the following results emerge.

THEOREM 3.5. *For any given normalized price* $p_1 \in (0, 1)$, *the sequence* $\{p_t\}_{t=1}^{\infty}$ *satisfies* (23) *if and only if*

$$p_{t+1} = f(p_t), \quad t = 1, 2, 3, \cdots, \tag{28}$$

where f *is the solution of* (27).

THEOREM 3.6. *There exists a unique sequence* $\{p_t\}_{t=1}^{\infty}$ *that satisfies both* (23) *and* (24). *Furthermore,* p_1 *is the unique solution of*

$$p_1 + \phi\left(\frac{p_1}{(1+k)f(\frac{p_1+k}{1+k})}\right) M_0^b = 1. \tag{29}$$

Proof. Consider the function

$$g_1(p) = p + \phi\left(\frac{p}{(1+k)f(\frac{p+k}{1+k})}\right) M_0^b - 1. \tag{30}$$

Note that g_1 is monotonically increasing and $g_1(0) < 0$, $g_1(1) > 0$. The desired result then follows from the intermediate-value theorem for continuous functions. □

An interesting consequence is that the prices are increasing in the first three periods as a result of the unannounced money injection. After this initial increase, prices oscillate around and converge asymptotically to a value that comprises a new equlilibrium for the system.

THEOREM 3.7. *Let $\{p_t\}_1^\infty$ be an equilibrium price path corresponding to a positive monetary injection $k > 0$. Then we have that*

$$p_1 < p_2 < (1+k)\bar{p}, \tag{31}$$

and for $t > 2$,

$$p_t > (1+k)\bar{p}, \quad t \in N_a, \qquad p_t < (1+k)\bar{p}, \quad t \in N_b.$$

Moreover, it must be the case that

$$\lim_{t\to\infty} p_t \doteq (1+k)\bar{p}. \tag{32}$$

Proof. If $q_1 = p_1/(1+k)$, then (29) may be rewritten as

$$H_1(q_1, k) = (1+k)q_1 + \phi\left(\frac{q_1}{f(q_1 + k/(1+k))}\right) M_0^b = 1. \tag{33}$$

Recall that for all $x \in (0,1)$, $\phi'(x) > 0$, and $f'(x) < 0$ according to Lemma (3.4). In consequence, since $q_1 \le 1$, we have

$$\frac{\partial H_1}{\partial q_1} > 0 \text{ and } \frac{\partial H_1}{\partial k} > 0.$$

It follows that if q_1 is viewed as a function of k, then $\frac{d\,q_1}{d\,k} < 0$. When $k = 0$, $q_1 = p_1 = \bar{p}$. Hence for $k > 0$, $q_1 < \bar{p}$, or what is the same, $p_1 < (1+k)\bar{p}$. Now to show $p_2 < (1+k)\bar{p}$, define $q_2 = (p_1 + k)/(1+k)$. It is easily seen that if p_1 is a solution to (29), then $q_2 < 1$. Write (29) as

$$H_2(q_2, k) = q_2(1+k) - k + \phi\left(\frac{q_2 - k/(1+k)}{f(q_2)}\right) M_0^b = 1. \tag{34}$$

Simple considerations establish that

$$\frac{\partial H_2}{\partial q_2} > 0 \text{ and } \frac{\partial H_2}{\partial k} < 0,$$

so that $\frac{d\,q_1}{d\,k} > 0$ Thus q_2 is an increasing function of k, whence $f(q_2)$ is a decreasing function of k. Since $f(q_2) = \frac{p_2}{(1+k)}$, this latter quantity decreases as a function of k. At $k = 0$, $\frac{p_2}{(1+k)} = p_2 = \bar{p}$, and so for $k > 0$ $p_2/(1+k) < \bar{p}$, as advertised.

It remains to prove (32). By Lemma 3.4, we know that

$$\mu = \max_{x\in[0,1]} |f'(x)| < 1.$$

It follows from the mean-value theorem that

$$|f(x) - \bar{p}| = |f(x) - f(\bar{p})| = |f'(\xi)|\,|x - \bar{p}| \le \mu|x - \bar{p}|.$$

Thus, if p_1 is given, then for $j \geq 1$,

$$|p_{j+1} - \bar{p}| \leq \mu^j |p_1 - \bar{p}|,$$

and the conclusion follows. □

3.4. *A numerical illustration*

In this subsection are presented some numerically-generated examples of price functions and equilibrium prices. The numerical methods used to obtain these results are somewhat technical. They are described very briefly here and are then used to demonstrate various interesting aspects of the model. A complete account of the numerical methods together with their theoretical analysis and extensive numerical testing is reported in Li (1993) and Li (1997).

Price function. The price function can be computed, for example, by a finite-element discretization. Let $C([0,1])$ connote the continuous, real-valued functions defined on the closed interval $[0,1]$. Given a positive integer n, consider a partition of $[0,1]$ as follows:

$$0 = x_1 < x_2 < ... < x_i < \cdots < x_n = 1, \quad \text{with} \quad x_i = \frac{i-1}{n-1}, \quad 1 \leq i \leq n.$$

Let V_h connote the subspace

$$V_h = \{v \in C[0,1] : v \text{ is linear on each interval } (x_{i-1}, x_i), \quad i = 0, \ldots n-1\}.$$

Let $w_i \in V_h$ be the unique function satisfying

$$w_i(x_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{w_i\}_{i=1}^n$ forms a basis of V_h which is known as the *nodal basis.* In fact, every $v \in V_h$ can be represented exactly as

$$v(x) = \sum_{i=1}^n v(x_i) w_i(x).$$

The so-called nodal-value interpolant

$$I_h : C[0,1] \rightarrow V_h$$

plays an important role. It is defined by

$$(I_h v)(x) = \sum_{i=1}^n v(x_i) w_i(x).$$

The fully discretized version of the equilibrium equation (27) is the equation

$$f_h(x) = F_h(f)(x) = I_h(F(f))(x), \quad \textit{where} \quad F(x) = 1 - x\phi\left(\frac{f(x)}{f \circ f(x)}\right)$$

for f_h in V_h. A rigorous analysis of the approximated problem provides several useful facts. First, for h small enough, it is inferred that the approximate problem possesses a unique solution, at least for relatively small monetary shocks. Secondly, if f is the

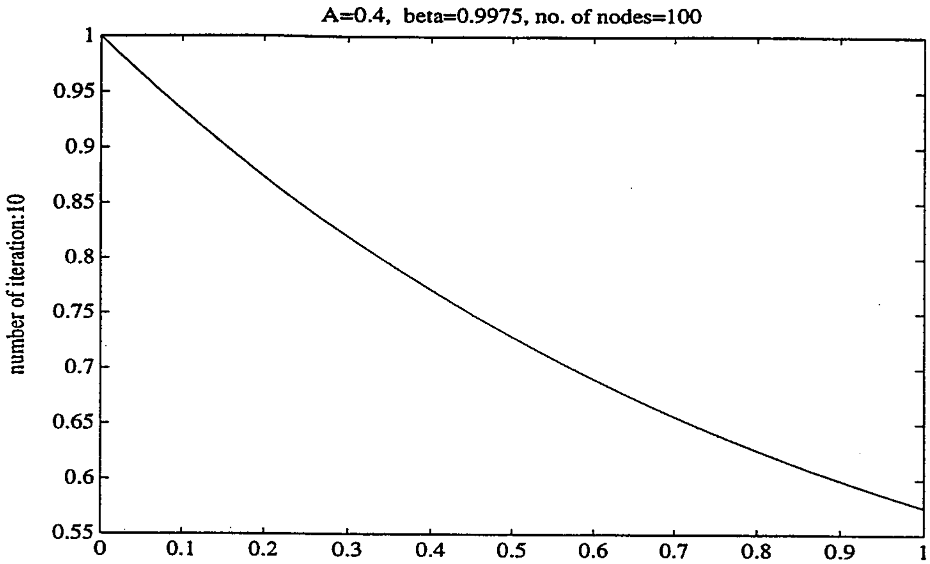

FIG. 1. *The price function f*

solution of the continuous problem and f_h is the solution of the discrete problem, then the difference between them is bounded above as follows:

$$\|f - f_h\|_{C(0,1)} \leq Bh^2\|f''\|_{C(0,1)},$$

where the constant B depends on the size of the shock, but not on h.

Figure 1 shows the profile of a computed price function, denoted by $\tilde{f}$, based on the subspace of piecewise linear functions defined on 29 subinterval of $[0, 1]$ with equal size $1/29$. The particular price function displayed here corresponds to the parameters $A = 0.4$ and $\beta = 0.9975$.

To check the accuracy of the numerical approximation of the price function f, we computed the residual vector $r = (r_i) \in R^{30}$, where

$$r_i = \tilde{f}(x_i) - F(\tilde{f}(x_i)), \quad x_i = (i-1)/29, \ 1 \leq i \leq 30.$$

This residual vector is plotted in Figure 2. We observe that the maximal value of r_i is of the order of 10^{-11}. This shows that the computed price function very accurately satisfies the equation (27) and suggests that it is a good approximation of the exact solution.

Using the computed price-function $\tilde{f}$ and the initial price p_1, the equilibrium prices are readily obtained via the iteration (28) in Theorem 3.5. Figure 3 shows an example of a typical solution. Notice that $p_1 < p_2 < (1+k)\bar{p}$ and that the subsequent prices oscillate around $(1+k)\bar{p}$, a situation in agreement with the conclucions enunciated in Lemma 3.1 and Theorem 3.7. Also evident in Figure 1 is that the price function is decreasing just as described in Lemma 3.4. Thus the qualitative features of the model, established by exact analysis are reproduced in the numerical simulations. This aspect lends further assurance of the veracity of the numerical caculations.

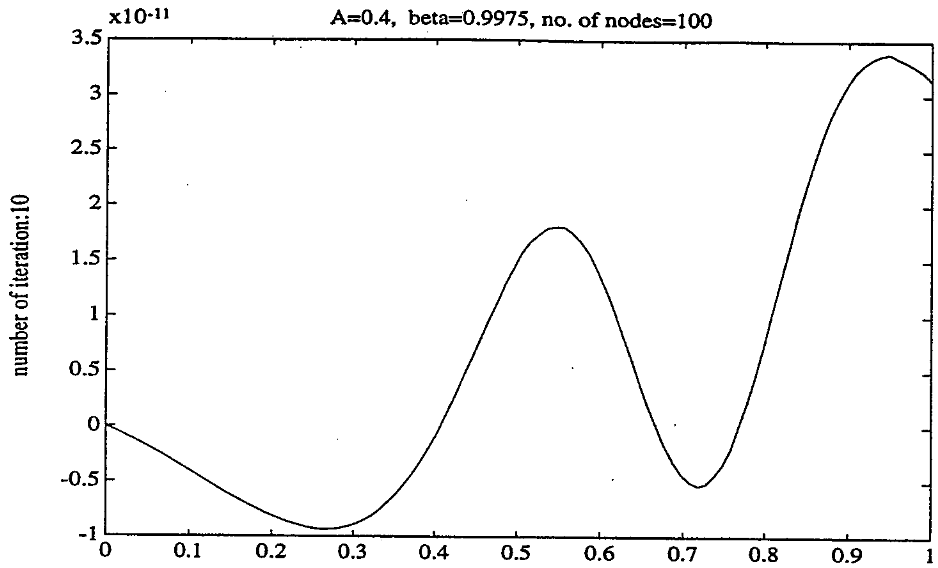

FIG. 2. *Plot of the residual*

3.5. *A proof of Lemma 3.4*

This subsection provides a proof of Lemma 3.4. The proof will be divided into two steps. The first step is to establish a local existence result and the second step is to establish global existence by extending the local solution.

The proof of local existence is based on the following well-known result from dynamical systems theory as expounded in Hirsch & Pugh (1968), for example. Our approach is a slight refinement of that appearing already in Bona & Grossman (1983).

LEMMA 3.8. *Assume that $U \subset R^2$ is an open set and $g : U \mapsto R^2$ is a differentiable mapping satisfying*

1. *$g(\bar{X}) = \bar{X}$ for some $\bar{X} \in U$, and*
2. *the eigenvalues λ_1 and λ_2 of $Dg(\bar{X})$ satisfy*

$$|\lambda_1| > 1 \quad \text{and} \quad |\lambda_2| < 1.$$

Then there exists a unique curve γ defined in a neighborhood of $\bar{X}$, and passing through $\bar{X}$, such that

1. *γ is the stable manifold for g, which is to say that*

$$g(\{\gamma\}) \subset \{\gamma\},$$

2. *the eigenvector of $Dg(\bar{X})$ corresponding to λ_2 is tangent to γ at $\bar{X}$, and*
3. *if $g \in C^k$ for some $1 \leq k \leq \infty$, then $\gamma \in C^k$ and γ is real-analytic if g is real-analytic.*

The above lemma will be applied to prove local existence. It will be shown that there exists a positive constant δ and a real-analytic function f defined on $\Omega_\delta = (\bar{p} - \delta, \bar{p} + \delta)$ such that for all $x \in \Omega_\delta$, $F(f) = f$, $f(\bar{p}) = \bar{p}$ and $-1 < f'(x) < 0$.

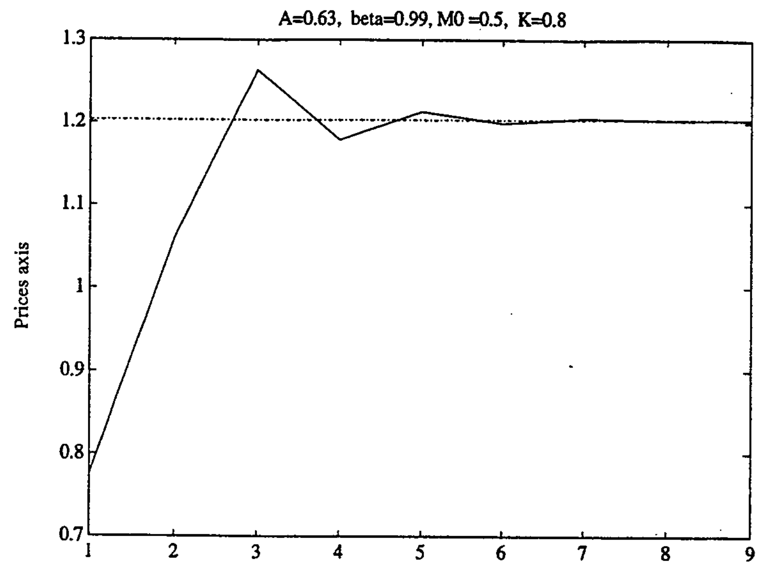

FIG. 3. *Plot of a typical price path*

Let $\psi = \phi^{-1}$ as in the proof of lemma 3.3 and let $g : R^2 \mapsto R^2$ be given by

$$g(p,q) = \left(\frac{p}{\psi\left(\frac{1-p}{q}\right)}, p \right).$$

Obviously g is real-analytic near the point $\bar{X} = (\bar{p},\bar{p})$. Moreover, $g(\bar{X}) = \bar{X}$ and the eigenvalues of $Dg(\bar{X})$ are seen to be

$$\begin{aligned}
\lambda_1 &= \frac{1+\phi'(1)+\sqrt{(1+\phi'(1))^2+4\phi(1)\phi'(1)}}{2\phi'(1)}, \\
\lambda_2 &= \frac{1+\phi'(1)-\sqrt{(1+\phi'(1))^2+4\phi(1)\phi'(1)}}{2\phi'(1)},
\end{aligned}$$

with associated eigenvectors

$$Y_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} \quad \text{and} \quad Y_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix},$$

respectively. It is easily verified that $-1 < \lambda_2 < 0$ and $\lambda_1 > 1$ since $\phi'(1),\ \phi(1) \geq 0$. Hence, according to Lemma 3.8, there is a stable manifold for the mapping g near $(\bar{p},\bar{p})$ given by a curve $\gamma = \gamma(s)$, defined for s near zero, say, such that

$$\gamma(s) = (\gamma_1(s), \gamma_2(s)) \quad \text{with} \quad \gamma_1(0) = \gamma_2(0) = \bar{p} \text{ and } \frac{\gamma_1'(0)}{\gamma_2'(0)} = \lambda_2.$$

Because of these properties and the form of g, the curve γ may be reparameterized in the form

$$\gamma(x) = (f(x), x),$$

for $x \in \Omega_\delta$ for some $\delta > 0$, where f is an analytic function defined on Ω_δ such that $f(\bar{p}) = \bar{p}$ and $f'(\bar{p}) = \lambda_2$. In particular, we see that $0 > f'(\bar{p}) > -1$. Because g maps γ into itself, if x is near $\bar{p}$, then there is a y near $\bar{p}$ such that

$$g(f(x), x) = (f(y), y).$$

By the definition of g, this means that

$$f(y) = \frac{f(x)}{\psi(\frac{1-f(x)}{x})} \quad \text{and} \quad y = f(x).$$

Consequently, it transpires that

$$f(f(x)) = \frac{f(x)}{\psi(\frac{1-f(x)}{x})},$$

or equivalently

$$f(x) = 1 - x\phi\left(\frac{f(x)}{f(f(x))}\right).$$

This proves that $f = F(f)$, at least when f is viewed as a mapping of Ω_δ into itself.

Since $-1 < f'(\bar{p}) < 0$ and f' is continuous, it follows that for sufficiently small δ, $-1 < f'(x) < 0$ if $|x - \bar{p}| \leq \delta$. This result comprises the local existence theory and finishes step 1 of the proof of Lemma 3.4.

Attention is now turned to the proof of existence of the price function on the whole domain $[0, 1]$. The idea is to extend the local solution defined near $\bar{p}$ obtained above.

First note that the uniqueness of f is obvious. In fact, if $\hat{f}$ is another such solution, but $\hat{f}(p_1) \neq f(p_1)$ for some $p_1 \in (0, 1]$, then the sequences generated by iteration, namely $p_{t+1} = f(p_t)$ and $\hat{p}_{t+1} = \hat{f}(\hat{p}_t)$ with $\hat{p}_1 = p_1$ will be two different solutions of (23) with the same initial configuration. This contradicts Lemma 3.3.

Consider the auxiliary function

$$g(t) = \frac{1-t}{\phi\left(\frac{t}{f(t)}\right)} = (1-t)\left(1 + \beta^{-\frac{1}{A}}(\frac{t}{f(t)})^{\frac{1}{A}-1}\right)$$

for all values of t for which f is defined. It is easy to see that f is a solution of (27) if and only if

$$g(f(x)) = x$$

for all x in the interval of the definition of f. Thus g is actually the inverse function of f.

LEMMA 3.9. *Let g be as defined above, where f is a solution of* (27) *on some interval* $\Omega \subset [0, 1]$. *Then it follows that*

$$g'(t) \leq -(1 + \gamma), \text{ for all } t \in \Omega_\delta, \tag{35}$$

where

$$\gamma = \beta^{-\frac{1}{A}} \sigma_1^{\frac{1}{A}} \min(\sigma_1^{-1}, \frac{1}{A} - 1). \tag{36}$$

and β is the discount factor, A is the parameter in the homothetic utility function, while $\sigma_1 \in (0,1)$ is the parameter appearing in Lemma 3.2.

Proof. If $z = \frac{t}{f(t)}$, then by Lemma 3.2,

$$z \leq \frac{t}{\sigma_1} \leq \frac{1}{\sigma_1}.$$

Direct calculation and Lemma 3.2 then imply that

$$\begin{aligned}
-g'(t) &= \frac{1}{\phi^2(z)}\left(\phi(z) + (1-t)\phi'(z)\frac{f(t)-tf'(t)}{f^2(t)}\right) \\
&\geq \frac{1}{\phi(z)} + (1-t)\frac{\phi'(z)}{\phi^2(z)} \\
&= 1 + \beta^{-\frac{1}{A}} z^{1-\frac{1}{A}} + (\frac{1}{A} - 1)\beta^{-\frac{1}{A}} z^{-\frac{1}{A}}(1-t) \\
&\geq 1 + \beta^{-\frac{1}{A}} \sigma_1^{\frac{1}{A}} (\sigma_1^{-1} t + (\frac{1}{A} - 1)(1-t)) \\
&\geq 1 + \gamma.
\end{aligned}$$

Let i_0 be the smallest integer that satisfies

$$(1+\gamma)^{i_0}\delta \geq 1.$$

For $i \leq i_0 + 1$, define $\Omega_i = \Omega_{\delta_i}$ *where* $\delta_i = (1+\gamma)^i \delta$. Let $f_0 = f$ and iteratively define

$$\left\{\begin{array}{ll} g_{i-1}(t) = \dfrac{1-t}{\phi\left(\dfrac{t}{f_{i-1}(t)}\right)}, & \text{for} \quad x \in \Omega_{i-1} \\ f_i(x) = g_{i-1}^{-1}(x), \quad \text{for} \quad x \in \Omega_i & \end{array}\right. \tag{37}$$

for $i = 0, 1, \cdots, i_0 + 1$. By induction, it is easy to see that f_i and g_i are both well defined, the estimate (35) holds for each g_i, and

$$\Omega_{i-1} \subset \Omega_i \subset g(\Omega_{i-1}), \qquad f_i|_{\Omega_{i-1}} = f_{i-1}.$$

By the definition of i_0, $1 \in \Omega_{i_0} \subset \Omega_{i_0+1}$. Thus $0 = g_{i_0}(1) \in \Omega_{i_0+1}$. As Ω_{i_0+1} must be a subinterval of $[0,1]$, we conclude that

$$\Omega_{i_0+1} = [0,1].$$

Consequently f_{i_0+1} is defined on the whole interval $[0,1]$ and satisfies (27). The elementary properties of real analytic functions allow one to adduce the extension given by (37) is analytic and hence f is analytic on $[0,1]$. The proof of Lemma 3.4 is now complete. □

4. Multiple monetary injections and stabilization. This section aims to study how government monetary policy can be designed to control the fluctuation of equilibrium prices that is a consequence of monetary expansion. Assume that the desired monetary policy involves a fraction k increase in the money supply. This amount of money can be injected into the market either at one time or at several, perhaps consecutive times.

It turns out, as we shall show, that multiple injection is preferable to one-time injection in the sense that price and interest-rate oscillation can be eliminated by appropriately chosen policy. As a consequence, we conclude that the government can use the open-market operations as a tool to stimulate the economy while prices are kept relatively stable.

4.1. *The Set-up*

In this subsection, we take it that money is injected into the market at n different, but consecutive times $j = 1, 2, \cdots, n$, the $j-th$ injection in the amount $k_j M_0^s$ takes place at the end of time period j, where

$$k = \sum_{j=1}^{n} k_i,$$

and the money supply at the end of period j is

$$M_j^s = \begin{cases} 1 + \sum_{l=1}^{j} k_l & 1 \le j \le n-1, \\ 1 + k & j \ge n. \end{cases}$$

In this case, (20)-(21) can be rewritten as

$$p_t + p_{t-1}\phi(\frac{p_t}{p_{t+1}}) \quad = \quad 1 \qquad \text{for } t \ge n+1, \textit{ and} \tag{38}$$

$$p_i + (p_{j-1} + k_{j-1})\phi(\frac{p_j}{p_{j+1}}) \quad = \quad 1 + \sum_{l=1}^{j-1} k_l \quad 1 \le j \le n, \tag{39}$$

where $p_n = \dfrac{p_t + k_n}{k+1}$, $p_t = \dfrac{p_t}{k+1}$ for $t \ge n+1$, $p_0 = M_0^b$, and $k_0 = 0$.

Obviously we can solve (39) for p_{i+1} in terms of p_i and p_{i-1}, namely

$$p_{j+1} = S_j(p_j, p_{j-1}), \quad 1 \le j \le n-1, \tag{40}$$

for certain functions S_j. Applying (40) recursively for $j = 1, 2, ..n-1$ gives that

$$p_j = T_j(p_1) \quad for \quad 1 \le j \le n+1,$$

for some functions T_j. The explicit expression for T_i can easily be obtained inductively.

LEMMA 4.1. *Assume that p_1 belonging to $(0,1)$ satisfies*

$$T_n(p_1) + (T_{n-1}(p_1) + k_{n-1})\phi\left(\frac{T_n(p_1)}{(1+k)f(\frac{T_n(p_1)-k_n}{1+k})}\right) = 1 + \sum_{j=1}^{n-1} k_j. \tag{41}$$

Then if p_j, $1 \le j \le n$, is determined by (40) and $p_{t+1} = f(p_t)$, for $t \ge n$, the path $\{p_t\}_1^\infty$ satisfies (39) and (38).

The proof of the above Lemma is straightforward. However whether or not there exists a p_1 belonging to $(0,1)$ satisfying (41) is a little subtle. We shall establish existence of such values p_1 in some special and practically interesting cases.

4.2. *Two injections*

The special case wherein money will be injected into the open market in exactly two consecutive periods will be discussed in this section. Since the total amount kM_0^s is placed in the market in two periods, the money supply in the first two periods are M_0^s and $M_1^s = (1+k_1)M_0^s$, respectively, and the money supply at all other periods $t = 2, 3, \cdots$, is $M_t^s = (1+k)M_0^s$. According to (39) and (38), the equilibrium prices p_1, p_2 and p_t have to satisfy

$$\begin{cases} p_1 + M_0^b\phi(\dfrac{p_1}{p_2}) = 1, \\ \dfrac{p_2}{1+k_1} + \dfrac{p_1+k_1}{1+k_1}\phi(\dfrac{p_2}{p_3}) = 1, \\ p_t + p_{t-1}\phi(\dfrac{p_t}{p_{t+1}}) = 1, \quad for \quad t \geq 3, \end{cases}$$

where $p_2 = \dfrac{p_2 + k - k_1}{k+1}$ and $p_t = \dfrac{p_t}{k+1}$ for $t \geq 3$. Corresponding to (40), we have $p_2 = S_1(p_1)$ where

$$S_1(p_1) = \left[\frac{M_0^b\beta^{1/A}}{1-p_1} - \beta^{1/A}\right]^{\frac{A}{A-1}} p_1$$

with $p_2 = S_1(p_1)$, whereas (41) now becomes

$$S_1(p_1) + p_1 + k_1\phi(\frac{S_1(p_1)}{(1+k)f(\frac{S_1(p_1)+k-k_1}{1+k})}) = 1 + k_1.$$

Define the function H by

$$H(x) = \frac{S_1(x)}{1+k_1} + \frac{x+k_1}{1+k_1}\phi(\frac{S_1(x)}{(1+k)f(\frac{S_1(x)+k-k_1}{1+k})}) - 1.$$

Evidently $H(x)$ is continuous and

$$\lim_{x\to p^*} H(x) = -1 \quad \lim_{x\to 1} H(x) = \infty$$

where $p^* = 1 - M_0^b$. By the intermediate-value theorem, there exists $p_1 \in (p^*, 1)$. satisfying $H(p_1) = 0$, with $p_2 = S_1(p_1)$ and $p_t = f(p_{t-1})$ for $t \geq 3$. The equilibrium price path $\{p_t\}$ is thereby obtained.

Comparison with one injection. Assume that p_t and q_t are the equilibrium prices for one injection and two injections, respectively, that correspond to the same total monetary expansion.

The unscaled equilibrium prices $\{p_t\}$ and $\{q_t\}$ are plotted in Figures 4–7 for a set of values of k_1. As shown in Figure 4, two equilibrium prices coincide for $t \geq 2$ if $k = k_1$. But as k_1 gets smaller, $\{q_t\}$ gets closer to the steady-state equilibrium price. However, after k_1 passes a certain critical value, the prices appear to drift away from the steady state. Such a critical value $k_1 = .532$ (shown in Figure 6) that makes the equilibrium prices most stable will be called a *stabilizing injection* and its existence will be rigorously justified in the next subsection.

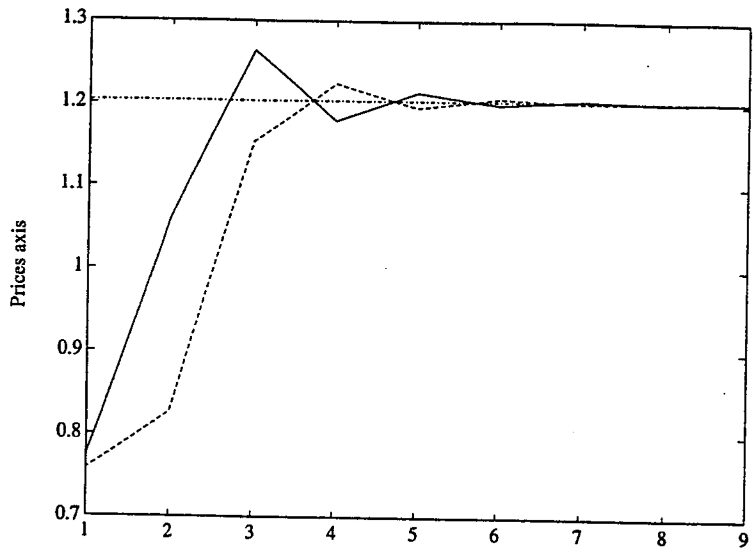

FIG. 4. *First test: Single injection of 0.8 and double injection with* $k_1 = 0.3$, $k_2 = 0.5$. *Here* $A = 0.63$, $\beta = 0.99$, $M_0^a = 0.5$ *and* $K = 0.8$. *In the graph, the solid line corresponds to one injection, the dashed line to two injections and the equilibrium price* $\bar{p}$ *is marked by* $- \cdot -$.

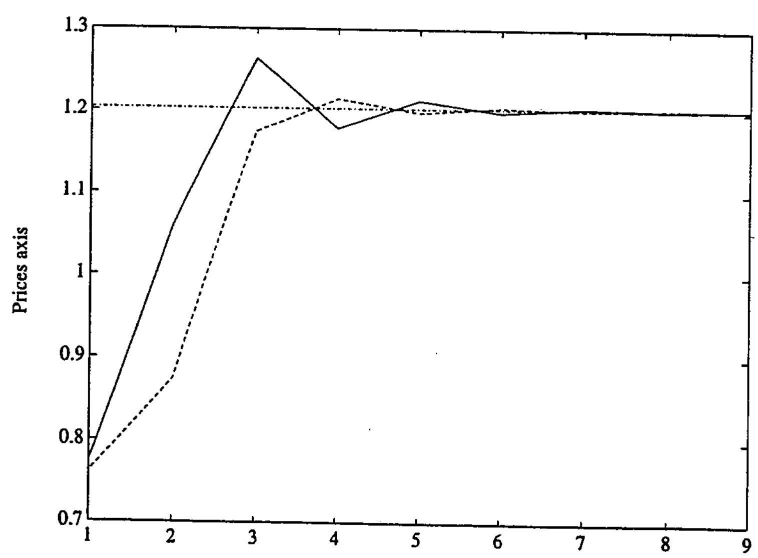

FIG. 5. *Second test: Single injection of 0.8 and double injection with* $k_1 = 0.4$, $k_2 = 0.4$. *Here* $A = 0.63$, $\beta = 0.99$, $M_0^a = 0.5$ *and* $K = 0.8$. *In the graph, the solid line corresponds to one injection, the dashed line to two injections and the equilibrium price* $\bar{p}$ *is marked by* $- \cdot -$.

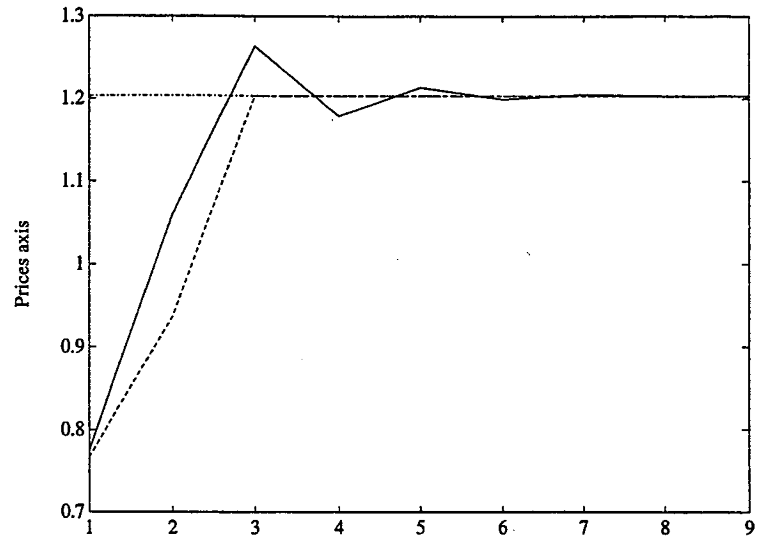

FIG. 6. *Third test: Single injection of 0.8 and double injection with* $k_1 = 0.529$, $k_2 = 0.271$. *Here* $A = 0.63$, $\beta = 0.99$, $M_0^a = 0.5$ *and* $K = 0.8$.*In the graph, the solid line corresponds to one injection, the dashed line to two injections and the equilibrium price* $\bar{p}$ *is marked by* $- \cdot -$.

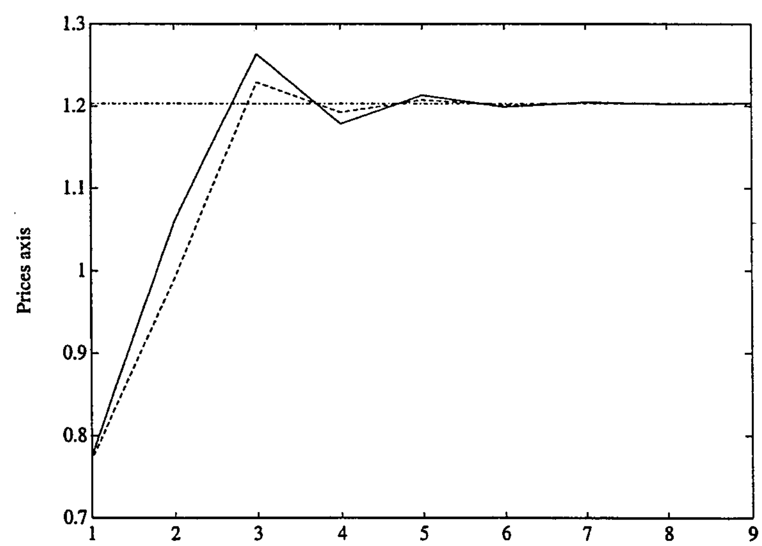

FIG. 7. *Fourth test: Single injection of 0.8 and double injection with* $k_1 = 0.65$, $k_2 = 0.15$. *Here* $A = 0.63$, $\beta = 0.99$, $M_0^a = 0.5$ *and* $K = 0.8$. *In the graph, the solid line corresponds to one injection, the dashed line to two injections and the equilibrium price* $\bar{p}$ *is marked by*$- \cdot -$.

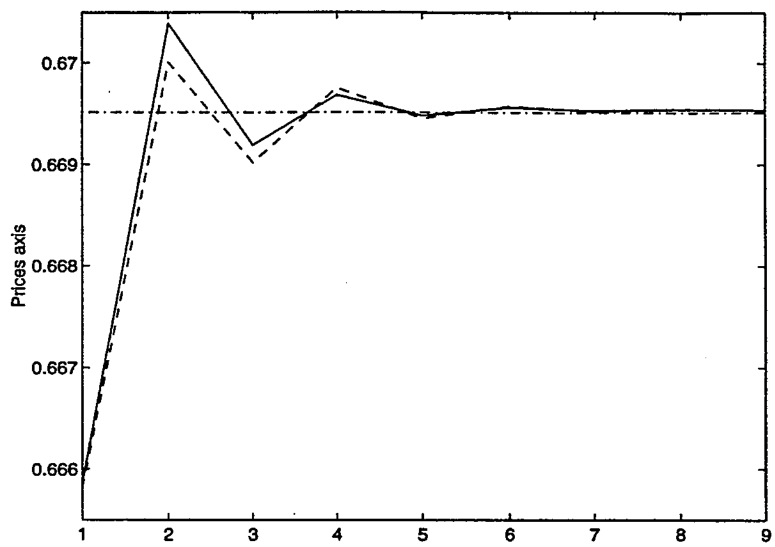

FIG. 8. *Fifth test: A smaller value of K. Single injection of 0.0016 and double injection with* $k_1 = 0.0008$, $k_2 = 0.0008$. *Here* $A = 0.63$, $\beta = 0.99$, $M_0^a = 0.5$ *and* $K = 0.0016$. *In the graph, the solid line corresponds to one injection, the dashed line to two injections, and the equilibrium price* $\bar{p}$ *is marked by* $- \cdot -$.

Existence of stabilizing injections. The stabilizing injection observed in the numerical experiments reported above is of both theoretical and practical interest. This paragraph is devoted to a proof of its existence.

THEOREM 4.2. *Suppose that money is injected into the market at two consecutive times and that* $M_0^b \in [\bar{p}(1 - k/\phi(1)), \bar{p}(1+k)]$. *Then, there exists a* $k_1 \in (0, k)$ *such that,*

$$p_t = \bar{p} \tag{42}$$

for all $t \geq 2$, *where*

$$p_2 = \frac{p_2 + k - k_1}{1+k}, \quad p_t = \frac{p_t}{1+k} \quad \textit{for} \quad t \geq 3$$

Proof. It suffices to show (42) for $t = 2$. Recall that p_2 satisfies $G(p_2, s) = 0$ where

$$G(p_2, s) = \frac{(1+k)p_2 + s - k}{1+s} + \frac{p_1 + s}{1+s}\phi(\frac{(1+k)p_2 + s - k}{(1+k)f(p_2)}) - 1$$

and $p_1 = p_1(p_2)$ is given by

$$p_1 + M_0^b \phi(\frac{p_1}{(1+k)p_2 + s - k}) = 1.$$

We need to show there exists a $k_1 \in (0,k)$ such that $G(\bar{p},k_1)=0$. As G is obviously continuous with respect to s, it suffices to show

$$G(\bar{p},0)<0, \quad G(\bar{p},k)>0.$$

We first claim that if $s=k$, then

$$\frac{p_1+k}{1+k}>\bar{p}.$$

If not, then $p_1 \leq (1+k)\bar{p}-K$ and thus

$$\begin{aligned} 0 &= p_1+M_0^b\phi(\frac{p_1}{(1+k)p_2+s-k})-1 \\ &\leq (1+k)\bar{p}-k+M_0^b\phi(1-\frac{k}{(1+k)\bar{p}})-1 \\ &< (1+k)(\bar{p}-1)+M_0^b\phi(1-\frac{k}{(1+k)\bar{p}}) \\ &\leq (M_0^b-(1+k)\bar{p})\phi(1)<0, \end{aligned}$$

which is a contradiction. Consequently

$$G(\bar{p},k)=\bar{p}+\frac{p_1+k}{1+k}\phi(1)-1=(\frac{p_1+k}{1+k}-\bar{p})\phi(1)\geq 0.$$

Next, it is asserted that if $s=0$, then

$$\frac{p_1}{1+k}\leq \bar{p}.$$

For, otherwise we have $p_1>\bar{p}(1+k)$, and then it transpires that

$$\begin{aligned} 0 &= p_1+M_0^b\phi(\frac{p_1}{(1+k)\bar{p}-k})-1 \\ &> (1+k)\bar{p}+M_0^b\phi(\frac{(1+k)\bar{p}}{(1+k)\bar{p}-k})-1 \\ &> M_0^b\phi(\frac{(1+k)\bar{p}}{(1+k)\bar{p}-k})-(\phi(1)-k)\bar{p} \\ &= \phi(1)[M_0^b-(1-k/\phi(1))\bar{p}]>0, \end{aligned}$$

which is a contradiction. Consequently, it is seen that

$$\begin{aligned} G(\bar{p},0) &= (1+k)\bar{p}-k+p_1\phi(1-\frac{k}{(1+k)\bar{p}})-1 \\ &= \bar{p}(1+k)[\phi(1-\frac{k}{(1+k)\bar{p}})-\phi(1)]<0. \end{aligned}$$

□

Remark It follows from the proof that the restriction on M_0^b can be weakened as follows:

$$\frac{\phi(1)-k}{\phi(1+\frac{k}{(1+k)\bar{p}-k})}\bar{p}\leq M_0^b\leq \frac{(1+k)\phi(1)}{\phi(1-\frac{k}{(1+k)\bar{p}})}\bar{p}.$$

As a consequence, if $k\geq\phi(1)$, then Theorem 4.2 is true for all $M_0^b\in[0,1]$. The above theorem shows that optimal injection exists if k is not too small. In fact, numerical experiments indicate that no optimal k_1 exists if k is very small.

5. More general utility functions. The preceding results were all predicated upon the assumption that the utility function is homothetic. It is natural to inquire what part, if any, of the forging theory remains valid for more general utility functions. A first step in the direction of more generality is taken in this section, where we study a class of utility functions near to the homothetic ones. To fix ideas, consider the situation wherein the economy features unit output and there is a single monetary injection. In this case, the (normalized) equilibrium prices satisfy the equations

$$p_t + p_{t+1} C_2 \left(\frac{p_t}{p_{t+1}}, \frac{p_{t-1}}{p_{t+1}} \right) = 1, \quad t = 2, 3, \ldots, \tag{43}$$

where C_2 is the demand function which is related to the utility function via the local optimization problem (7). When the consumer is equipped with a homothetic utility function, then $C_2(X, Y) = Y\phi(X)$. The next preparatory result shows that if U is 'nearly' homothetic, then C_2 almost has the form just mentioned.

LEMMA 5.1. *Let* $U(C) = \frac{C^{1-A}}{1-A} + v(C)$ *where* v *is such that*

$$v'(C) = (C + g_0(C))^{-A} - C^{-A} \tag{44}$$

for a given function g_0. *Then the demand function* $C_2(X, Y)$ *satisfies*

$$C_2(X, Y) = Y\phi(X) + g(X, Y),$$

where

$$|g(X, Y)| \leq \sup_{0<t<Y} g_0(t) + X \sup_{0<t<Y} g_0(t/X). \tag{45}$$

Proof. By their definitions, the demand functions C_1 and C_2, comprise the solution of the problem

$$\max_{XC_1 + C_2 = Y} U(C_1) + \beta U(C_2)$$

for the appropriate utility functions U. Applying the well-know necessary condition for maximization problems, there appears the relation

$$C_1^{-A} + v'(C_1) = \beta X (C_2^{-A} + v'(C_2)).$$

Since $v'(C) = (c + g_0(C))^{-A} - C^{-A}$, it is seen that

$$C_1 = (\beta X)^{-1/A} (C_2 + g_0(C_2)) - g_0(C_1)$$

and

$$C_2 = Y - XC_1 = Y - X(\beta X)^{-1/A} C_2 - X(\beta X)^{-1/A} g_0(C_2) + X g_0(C_1).$$

It follows that

$$C_2 = Y\phi(X) + g(X, Y),$$

where

$$g(X,Y) = -(\beta^{-1/A}X^{1-1/A})\phi(X)g_0(C_2) + X\phi(X)g_0(C_1).$$

Noting that $\beta^{-1/A}X^{1-1/A}\phi(X) \leq 1$ and $\phi(X) \leq 1$, there obtains

$$\begin{aligned} |g(X,Y)| &\leq \sup_{XC_1+C_2=Y}(|g_0(C_2)| + X|g_0(C_1)|) = \sup_{0<t<Y}(g_0(t) + Xg_0\left(\frac{Y-t}{X}\right)) \\ &\leq \sup_{0<t<Y} g_0(t) + \sup_{0<t<Y} Xg_0(t/X) = \sup_{0<t<Y} g_0(t) + X \sup_{0<t<Y/X} g_0(t). \end{aligned}$$

□

Because of (7) and (45), the equilibrium equation is reduced to

$$p_t + p_{t-1}\phi(\frac{p_t}{p_{t+1}}) + p_{t+1}g\left(\frac{p_t}{p_{t+1}}, \frac{p_{t-1}}{p_{t+1}}\right) = 1.$$

As before, we seek solutions to the above equation in the form $p_{t+1} = f(p_t)$ with the function f necessarily satisfying

$$f(x) + x\phi\left(\frac{f(x)}{f\circ f(x)}\right) + f\circ f(x)g\left(\frac{f(x)}{f\circ f(x)}, \frac{x}{f\circ f(x)}\right) = 1. \tag{46}$$

It is previously established that if $g_0 = 0$ (whence $g = 0$), then there exists an f satisfying the above equation. We will use the Implicit-Function theorem to show that the price function f also exists (hence so do the equilibrium price sequences) if g_0 (as in (44)) is sufficiently small.

LEMMA 5.2. *If a, b, c are constants and $c \neq 0$, then*

$$\frac{a + bt + O(t^2)}{c + dt + O(t^2)} = \frac{1}{c}\left[a + (b - \frac{ad}{c})t\right] + O(t^2),$$

as $t \to 0$.

Proof. The proof is elementary. □

LEMMA 5.3. *Let*

$$F(f,g)(x) = f(x) + x\phi\left(\frac{f(x)}{f\circ f(x)}\right) - 1 + f\circ f(x)g\left(\frac{f(x)}{f\circ f(x)}, \frac{x}{f\circ f(x)}\right).$$

Assuming that f is as in Theorem 1.3, then

$$\left.\frac{\partial F}{\partial f}\right|_{(f,0)} h(x) = a(x)(I+T)h(x)$$

where

$$(Th)(x) = \frac{b(x)}{a(x)} h\circ f(x), \tag{47}$$

and

$$\begin{aligned} a(x) &= 1 - b(x)\frac{f\circ f(x)}{f(x)} + b(x)f'\circ f(x), \\ b(x) &= -x\phi'\left(\frac{f(x)}{f\circ f(x)}\right)\frac{f(x)}{(f\circ f(x))^2}. \end{aligned}$$

Proof. By definition, for any h in $C(\Omega_\delta)$,

$$\left.\frac{\partial F}{\partial f}\right|_{(f,0)} h(x) = \lim_{t\to 0} \frac{F(f+th,0)(x) - F(f,0)(x)}{t} = \lim_{t\to 0} \frac{F(f+th,0)(x)}{t}.$$

Here we have used the fact that $F(f,0)(x) = 0$. Using this fact again yields

$$F(f+th,0)(x) = th(x) + x\left[\phi\left(\frac{f(x)+th(x)}{(f+th)\circ(f+th)(x)}\right) - \phi\left(\frac{f(x)}{f\circ f(x)}\right)\right].$$

By Taylor expansion, it is deduced that

$$\begin{aligned}(f+th)\circ(f+th)(x) &= (f+th)\circ f(x) + (f'+th')\circ f(x)th(x) + O(t^2)\\ &= f\circ f(x) + t(h\circ f(x) + f'\circ f(x)h(x)) + O(t^2),\end{aligned}$$

as $t \to 0$ An application of Lemma 5.2 leads to

$$\begin{aligned}\frac{(f+th)(x)}{(f+th)\circ(f+th)(x)} &= \frac{f(x)}{f\circ f(x)} + \frac{1}{f\circ f(x)}\left[h(x) - \frac{f(x)(h\circ f(x) + f'\circ f(x)h(x))}{f\circ f(x)}\right]t\\ &\quad +O(t^2).\end{aligned}$$

It follows from another Taylor expansion that

$$\begin{aligned}&\phi\left(\frac{f(x)+th(x)}{(f+th)\circ(f+th)(x)}\right)\\ &= \phi\left(\frac{f(x)}{f\circ f(x)}\right) + \frac{1}{f\circ f(x)}\left[h(x) - \frac{f(x)(h\circ f(x) + f'\circ f(x)h(x))}{f\circ f(x)}\right]t + O(t^2))\\ &= \phi\left(\frac{f(x)}{f\circ f(x)}\right) + \phi'\left(\frac{f(x)}{f\circ f(x)}\frac{1}{f\circ f(x)}\right)\left[h(x) - \frac{f(x)(h\circ f(x) + f'\circ f(x)h(x))}{f\circ f(x)}\right]t\\ &\quad +O(t^2),\end{aligned}$$

as $t \to 0$. Therefore, one has

$$\begin{aligned}\left.\frac{\partial F}{\partial f}\right|_{(f,0)} h(x) &= h(x) + x\phi'\left(\frac{f(x)}{f\circ f(x)}\right)\frac{1}{f\circ f(x)}\left[h(x) - \frac{f(x)(h\circ f(x) + f'\circ f(x)h(x))}{f\circ f(x)}\right]\\ &= h(x)\left[1 + x\phi'\left(\frac{f(x)}{f\circ f(x)}\right)\frac{1}{f\circ f} - x\phi'\left(\frac{f(x)}{f\circ f(x)}\right)\frac{f(x)f'\circ f(x)}{(f\circ f(x))^2}\right]\\ &\quad -x\phi'\left(\frac{f(x)}{f\circ f(x)}\right)\frac{f(x)}{(f\circ f(x))^2}h\circ f(x)\\ &= a(x)h(x) + b(x)h\circ f(x) = a(x)(I+T)h(x).\end{aligned}$$

$\square$

LEMMA 5.4. *The operator T defined in (47) satisfies*

$$\|T\|_{B(C(\Omega_\delta),C(\Omega_\delta)} \le \gamma < 1,$$

provided δ is sufficiently small. (If X and Y are Banach spaces, $B(X,Y)$ connotes the bounded linear operators from X to Y)

Proof. Since

$$b(\bar{p}) = -\phi'(1),$$

it follows that

$$a(\bar{p}) = 1 + \phi'(1) - \phi'(1)f'(\bar{p}) \geq 1 + \phi'(1).$$

Hence, if δ is sufficiently small, then

$$\|b/a\|_{C(\Omega_\delta)} \leq \frac{.5 + \phi'(1)}{1 + \phi'(1)} = \gamma < 1.$$

Therefore, we have

$$\|Th\|_{B(C(\Omega_\delta), C(\Omega_\delta)} \leq \|b/a\|_{C(\Omega_\delta)} \|h\|_{C(\Omega_\delta)} \|f\|_{C(\Omega_\delta)} \leq \gamma \|h\|_{C(\Omega_\delta)},$$

whence $\|T\|_{B(C(\Omega_\delta), C(\Omega_\delta)} \leq \gamma$. □

LEMMA 5.5. *Define a mapping A by*

$$A = \left.\frac{\partial F}{\partial f}\right|_{(f,0)} = a(x)(I + T).$$

Then $A : C(\Omega_\delta) \to C(\Omega_\delta)$ is one-to-one, onto and A^{-1} is bounded.

Proof. First remark that $I + T$ has the desired properties since the norm of T viewed as a mapping of $C(\Omega_\delta)$ is less than one. Since the function a is bounded above and away from zero, it follow that multiplication by a is a one-to-one map of $C(\Omega_\delta)$ with a bounded inverse. The result follows since A is the composition of two such maps.

□

LEMMA 5.6. *There are two positive constants δ and η such that for any*

$$g_0 \in C(R^1) \quad \textit{with} \quad \|g_0\|_{C(R^1)} < \eta,$$

there exists an $f \in C(\Omega_\delta)$ that satisfies (46) *for all $x \in (\bar{p} - \delta, \bar{p} + \delta) \cap [0, 1]$.*

Proof. Let

$$F(f, g)(x) = f(x) + x\phi\left(\frac{f(x)}{f\circ f(x)}\right) - 1 + f\circ f(x) g\left(\frac{f(x)}{f\circ f(x)}, \frac{x}{f\circ f(x)}\right).$$

By Lemmas 1.3 and 1.4, $A = \left.\frac{\partial F}{\partial f}\right|_{(f,o)}$ is a one-to-one, onto map having a bounded inverse. The desired result then follows from the Implicit-Function Theorem. □

THEOREM 5.7. *There are two positive constants δ and η such that for any*

$$g_0 \in C(R^1) \quad \textit{with} \quad \|g_0\|_{C(R^1)} < \eta,$$

there exists an $f \in C(\Omega_\delta)$ that satisfies (46) *for all $x \in (\bar{p} - \delta, 1]$.*

Proof. It suffices to show that $|\frac{b(x)}{a(x)}| < 1$ for $x \in [\bar{p}, 1]$. It is straightforward to ascertain that this inequality is equivalent to

$$1 - \frac{b(x)}{f(x)} [f \circ f(x) - f(x) - f' \circ f(x) f(x)] > 0.$$

The latter inequality is valid for $x \in (\bar{p}, 1]$ since $b(x) < 0$, $f' \circ f(x) < 0$ and

$$f \circ f(x) - f(x) \geq 0.$$

$\square$

With the existence of the price function established, the existence of equilibrium price sequence then follows directly.

6. Concluding remarks. The principal conclusions derived on the basis of the foregoing analysis is that a monetary expansion in an economy will lead to a gradual increase in price level. Moreover in the absence of further perturbations, the price path will approach a new steady-state in a oscillatory manner. These general conclusions appear to be in broad agreement with some of what is observed in real economies.

The numerical experiments based on the Grossman-Weiss model show clearly that the size of the price fluctuation depends substantially on the level of government intervention in the money supply. A potentially interesting aspect of the present investigation is the existence of what we have termed 'stabilizing' monetary injections, in which the new equilibrium prices and interest rates are reached monotonically, without oscillation and in a minimal length of time. There is also the prospect of using multiple injections to move the economy from the old equilibrium to the new equilibrium monotonically, and in small increments.

Finally, it deserves remark that the indogeneous inclusion of productions is needed to truly justify the appellation 'optimal' to the multi-step injection polices discussed in Section 4. An investigation of this aspect is currently under way and the outcome will be reported in due course.

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