Institute of Physics Publishing Nonlinearity

Nonlinearity **15** (2002) 759–786 PII: S0951-7715(02)25087-2

Scaling, stability and singularities for nonlinear, dispersive wave equations: the critical case

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Received 22 May 2001, in final form 20 February 2002
Published 3 April 2002
Online at stacks.iop.org/Non/15/759

Recommended by W J Zakrzewski

Abstract

For a class of generalized Korteweg–de Vries equations of the form

$$u_t + (u^p)_x - D^\beta u_x = 0 \qquad (*)$$

posed in $\mathbb{R}$ and for the focusing nonlinear Schrödinger equations

$$\mathrm{i}u_t + \Delta u + |u|^p u = 0 \qquad (**)$$

posed on $\mathbb{R}^n$, it is well known that the initial-value problem is globally in time well posed provided the exponent p is less than a critical power p_{crit}. For $p \geqq p_{\mathrm{crit}}$, it is known for equation $(**)$ and suspected for equation $(*)$ (known for $p = 5$ and $\beta = 2$) that large initial data need not lead to globally defined solutions. It is our purpose here to investigate the critical case $p = p_{\mathrm{crit}}$ in more detail than heretofore. Building on previous work of Weinstein, Laedke, Spatschek and their collaborators, earlier work of the present authors and others, a stability result is formulated for small perturbations of ground-state solutions of $(**)$ and solitary-wave solutions of $(*)$. This theorem features a scaling that is natural in the critical case. When interpreted in the contexts in view, our general result provides information about singularity formation in the case the solution blows up in finite time and about large-time asymptotics in the case the solution is globally defined.

Mathematics Subject Classification: 35B05, 35B35, 35B40, 35Q35, 35Q53, 35Q55, 76B15, 76B25, 76E30

0. Introduction

Evolution equations that feature a balance between nonlinearity and dispersion have been the focus of intensive investigation for more than three decades. The initial impetus for this scientific activity was the inverse scattering theory (IST) for the Korteweg–de Vries equation. A wide range of theoretical and applied studies are currently centred around nonlinear dispersive wave equations. These works use tools including, but certainly not restricted to, the inverse scattering transform.

One of the lessons learned from the IST is that solitary waves play a central role in the long-time asymptotics of solutions to the initial-value problem (a conclusion buttressed by the present study). Indeed, for the Korteweg–de Vries equation (KdV-equation henceforth)

$$u_t + u_x + (u^2)_x + u_{xxx} = 0$$

itself, general classes of initial disturbances are known to resolve into a finite sequence of solitary waves followed by a dispersive tail. A companion result is that individual solitary waves are orbitally stable solutions of the evolution equation. The exact theory of stability of solitary waves commenced with Benjamin's paper [Be2] (see also [Bo]) and reached a state of some maturity more than a decade ago (e.g. [AB, ABH, Al1, Al2, BBBSS, BS, LBS, W6]). In the mid-1980s in the papers of Strauss and his collaborators and Weinstein [BSS, GSS, W3, W5], it came to light that not all solitary-wave solutions are stable. Both necessary and sufficient conditions for stability of the solitary-wave solutions of a range of nonlinear dispersive evolution equations appear in various of the above references. These results have been supplemented by more recent studies (see [AS, Le1, Le2, Lo1, Lo2, Lo3, M, MM1, MM2, MM3, PW]).

An interesting issue emerges from the fact that there are unstable solitary waves, namely the question of their longer-time behaviour under general perturbations. The answer to this query does not follow from the theory leading to the conclusion of instability. The instability results are based on a local analysis made in a neighbourhood of the solitary wave whose stability is in question. This analysis, which uses in an essential way the Hamiltonian structure of the equation, shows that there are initial data consisting of arbitrarily small perturbations of the solitary wave that leave a small but fixed neighbourhood of the solitary-wave orbit in finite time. Once the solution has left this neighbourhood, there is no reason to expect that calculus based on linearizing around the solitary wave being studied will continue to provide helpful information about the solution.

Numerical simulations have provided a set of conjectures related to the question raised above. The answer appears to depend upon the particular equation which is under study. As an example, consider the class of generalized KdV-equations

$$u_t + u_x + (u^p)_x + u_{xxx} = 0, \tag{0.1}$$

where p is a positive integer. Solitary-wave solutions $u(x,t) = \phi_c(x - ct)$ exist for any $c > 1$ and are unique up to translations in the underlying spatial domain (and sign in the case p is odd). These travelling-wave solutions are stable, and even asymptotically stable, for all values of the speed c of propagation provided that $p < 5$ (see [ABH, BSS, MM2, PW]). For $p \geqslant 5$, solitary waves of all speeds are unstable (see [BSS] for $p > 5$ and [MM3] for $p = 5$). In the unstable case, it appears that initial data for an appropriately perturbed solitary wave lead to a solution that blows up (becomes infinite) in L_∞-norm in finite time (see [BDKM1, BDKM2, BDKM3, BW, DM]). For equation (0.1) in the critical case $p = 5$, this has been rigorously confirmed in a recent, remarkable paper of Martel and Merle [MM4]. On the other hand, for the generalized regularized long-wave equations or BBM-equations of the form

$$u_t + u_x + (u^p)_x - u_{xxt} = 0, \tag{0.2}$$

solitary waves also exist and are unique to within spatial translations (and sign if p is odd) for any value of the speed $c > 1$. These travelling waves are stable, for all values of the speed when $p \leqq 5$, whilst for each $p > 5$, there is a critical value $c_p > 1$ for which the solitary waves with speed in the range $1 < c < c_p$ are unstable, but those with speed $c \geqslant c_p$ are all orbitally stable (see [SS]). In this latter situation, extensive numerical simulation of the equations in (0.2) indicates that solutions move away from the unstable waves and converge to a sequence of one or more stable solitary waves followed by a dispersive structure (see [BMR]). In any event, solutions cannot become infinite in finite time no matter how large is p. Certain, more general versions of the equations in (0.1) in which the nonlinearity has the form $\partial_x f(u)$, where f is not necessarily homogeneous, have also been analysed recently and both stability and instability results are available in this context (see [Lo3]).

This paper aims to add to the discussion outlined above. We formulate and prove a theorem relating to the solitary-wave solutions of the initial-value problem (IVP henceforth) for the KdV-type-equations

$$\begin{cases} u_t + u_x + (u^p)_x - D^\beta u_x = 0, & x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) = u_0(x), \end{cases} \tag{0.3}$$

where $\beta \geqq 1$, $p \geqslant 2$ is an integer, and $u : \mathbb{R} \times [0, T) \to \mathbb{R}$ for some $T > 0$. The homogeneous operator D^β is a Fourier-multiplier operator defined for suitable functions $v : \mathbb{R} \to \mathbb{R}$ by

$$\widehat{D^\beta v}(\xi) = |\xi|^\beta \hat{v}(\xi) \tag{0.4}$$

for $\xi \in \mathbb{R}$. Here and below, a circumflex surmounting a function connotes that function's Fourier transform with regard to the spatial variable x.

The solitary-wave solutions of (0.3) are smooth, travelling-wave solutions of the form $u(x, t) = S(x - \eta t)$, with $\eta > 1$, that are symmetric about their crest and which decay to zero at infinity (see [Al1, Al2, Be2, BBB, BL1, BL2, CB, W6]). After a scaling, they satisfy the elliptic pseudo-differential equation

$$D^\beta S + \eta S - S^p = 0, \tag{0.5}$$

and while they are probably unique to within sign and translation in the underlying spatial domain, they are known to be unique only for special cases (see [AT, AlT]). These solitary-wave solutions are orbitally stable when $p < p_{\text{crit}} = 2\beta + 1$ (see [ABH, Al1, Al2, BBBSS, BSS, W4, W6]). Meanwhile, they are unstable if $p > p_{\text{crit}}$ (see [BSS]). For this class of equations, it is well known (see [ABFSa, AL, BBBSS]) that the IVP is globally well-posed provided the exponent p is less than the same critical power $p_{\text{crit}} = 2\beta + 1$ that arises in the question of orbital stability. The results herein pertain exactly to the critical value of the parameters where $p = 2\beta + 1$.

For nonlinear Schrödinger-type equations, a similar commentary applies. Consider the IVP for the focusing nonlinear Schrödinger equation (NLS-equation in what follows)

$$\begin{cases} iu_t + \Delta u + |u|^p u = 0, & x \in \mathbb{R}^n, \quad t \geqq 0, \\ u(x, 0) = u_0(x), \end{cases} \tag{0.6}$$

where Δ is the Laplace operator on $\mathbb{R}^n$ and $u : \mathbb{R}^n \times [0, T) \to \mathbb{C}$ for some $T > 0$. For any p with $0 < p < 4/(n-2)$ (any $p > 0$ if $n = 1$ or 2), the evolution equation in (0.6) admits standing-wave solutions of the form

$$u_\lambda(x, t) = G(x)e^{i\lambda t}, \tag{0.7}$$

where $\lambda > 0$ and $G = G_\lambda$ satisfies

$$\Delta G - \lambda G + G^{p+1} = 0. \tag{0.8}$$

A *ground state* G_λ is the solution of (0.8) that has minimum energy. These special solutions, which are real-valued, positive and decrease rapidly to zero at infinity, play an important role in the theory of Schrödinger-type equations just as solitary waves do for (0.3). They are unique to within spatial translation and may be taken to be radially symmetric by an appropriate choice of coordinates (see [BL1, BL2, BLP, Kw, S]). When $p < p_{\mathrm{crit}} = 4/n$, these standing waves are orbitally stable in $H^1(\mathbb{R}^n)$ (see [C, CL, W1]), but if $p \geqslant p_{\mathrm{crit}}$ the ground states are known to be unstable (see [BC, GSS]). Regarding the IVP (0.6), global well-posedness obtains for arbitrarily large data provided that $p < p_{\mathrm{crit}}$, just as for the KdV-type equations (0.3). On the other hand, if $p \geqslant p_{\mathrm{crit}}$, then there are solutions that blow up in finite time, although the exact structure of this singularity formation has not been settled (see [Z, G] for early results about singularity formation). The first result describing the asymptotic behaviour of blowing-up solutions of the NLS-equation appears to have been that of Weinstein [W3, W5]. Later, Laedke *et al* [LBSK] studied Weinstein's results and attempted to extend them to a larger class of initial data. In [ABLS], certain details related to the last cited work were provided by the present authors that added precision to the resulting theory. Numerical studies of this blow-up phenomenon were pioneered by Papanicolaou and co-workers [LePSS, LPSS] (see also the recent work of Akrivis *et al* [ADKM1, ADKM2] and the references contained therein). See Weissler [We] for recent results where the nonlinearity $|u|^p u$ is generalized to the form $f(u)$ where f need not be homogeneous.

In this paper, in addition to our analysis of solitary waves, we also extend the previous analysis reported in [ABLS] of ground-state solutions of the IVP for the focusing NLS-equation (0.6). Roughly speaking, our earlier work established that the blow up near unstable ground-state solutions for the case $p = p_{\mathrm{crit}} = 1 + 4/n$ is *stable* in a sense made precise in theorem 5.1 below. We also announced in [ABLS] a result providing a more detailed description of the evolution of the parameters involved in theorem 5.1, modelled on theorem 7 of [BS], which applies directly only in case $p < p_{\mathrm{crit}}$. In this paper, we offer in theorem 5.2 a detailed proof corresponding to our earlier speculations.

The results obtained here for KdV-type and Schrödinger-type equations are particularly interesting because the analysis is formulated exactly for the critical power of the nonlinearity, and so for cases where the solitary waves or the ground states are just unstable. Because there is special structure available for the critical values of the parameters, issues associated with these values have naturally received considerable analytical and numerical attention.

The plan of this paper is as follows. Section 1 is devoted to notation, a more detailed review of prior theory that bears upon the present developments for KdV-type equations, and a statement of our principal results for KdV-type equations. Section 2 is concerned with sharp conditions for global well-posedness of the IVP (0.3). Then, in sections 3 and 4, the stability theory for KdV-type equations, including results about the variation of the stability parameters, is set forth. Section 5 contains the aforementioned results on the variation of the stability parameters for the NLS-equations. Our conclusions are briefly reviewed in the last section.

1. Notation, prior results and statement of main theorems

1.1. Notation

The notation in force is simple. An unadorned norm symbol $\|\cdot\|$ will always denote the $L_2(\mathbb{R}^n)$-norm in the spatial variable x. The norm of a function $f \in L_p(\mathbb{R}^n)$ is written $|f|_p$. Usually, the value of the dimension n will be obvious from the context, and so we may write simply L_p rather than $L_p(\mathbb{R}^n)$. The norm in the function class $H^s = H^s(\mathbb{R}^n)$ is denoted $\|\cdot\|_s$. The multiple notations for the L_2-norm $\|\cdot\| = \|\cdot\|_0 = |\cdot|_2$ do not appear to cause confusion. The L_2-inner product of two functions f and g is denoted $\langle f, g\rangle$. The orthogonality condition $\langle f, g\rangle = 0$ will sometimes be indicated by the annotation $f \perp g$.

If X is a Banach space of functions of the spatial variable x, then the class $C^m(0, T; X)$ is the class of m-times continuously differentiable functions from $[0, T]$ to X with the norm

$$\|u\|_{C^m(0,T;X)} = \sup_{0 \leqslant j \leqslant m} \sup_{0 \leqslant t \leqslant T} \|\partial_t^j u(t)\|_X.$$

When X is a concrete function class such as $H^s(\mathbb{R})$, an element $u \in C^m(0, T; H^s(\mathbb{R}))$ is such that $u(t) \in H^s(\mathbb{R})$ for each $t \in [0, T]$. When it is useful to display the spatial dependence of $u(t)$, we will write $u(x, t)$ rather than $u(t)(x)$. As already mentioned, $\hat{u}$ is the Fourier transform of the function u with respect to the spatial variable x. It will also be convenient on occasion to write $\mathcal{F}u$ or $\mathcal{F}_x u$ if the variable is in doubt, for the same Fourier transform. As noted in (0.4), for $\beta \geqslant 0$, $D^\beta f$ is defined via the Fourier transform as the pseudo-differential operator with symbol $|\xi|^\beta$. Notice that $D = D^1 = \mathcal{H}\partial_x$, where $\mathcal{H}$ is the Hilbert transform.

1.2. Preliminary results for KdV-type equations

The local well-posedness results for the IVP (0.3) in the L_2-based Sobolev classes $H^s(\mathbb{R})$ are as follows. For any integer $p \geqslant 1$, the IVP (0.3) defines a continuous mapping from the class of initial data X to solutions in $C(0, T; X)$ for some positive time interval $[0, T]$

(i) for $\beta \geqslant 2$ and $X = H^s(\mathbb{R})$ for any $s \geqq \frac{1}{2}\beta$ or
(ii) for $\beta > 0$ and $X = H^s(\mathbb{R})$ for any $s > \frac{3}{2}$.

For the first result, see [AL]; to establish the latter result, one may use Kato's theory [K] (see also [ABFSa, Sa]). For more subtle results in the special cases $\beta = 2$ and $\beta = 1$ (the Korteweg–de Vries and Benjamin–Ono cases, respectively), see [GV, I, KPV1, KPV2, KPV3, P, T] and the references contained in these articles. One can also establish the existence of weak solutions in the case $\beta \geqslant 1$ in $H^s(\mathbb{R})$ for $s \geqq \frac{1}{2}\beta$ and $p < p_{\text{crit}}$, for $p = p_{\text{crit}}$ and initial data that are small enough in $L_2(\mathbb{R})$, or for $p > p_{\text{crit}}$ and data that are small enough in $H^{\beta/2}(\mathbb{R})$. This may be accomplished by regularizing the evolution equation and passing to the limit in a standard way using weak-star compactness of bounded sequences in $L_\infty(0, T; H^s)$.

To extend the local results to global ones, the following conserved quantities associated with (0.3) are useful:

$$\begin{aligned} E(v) &= \frac{1}{2}\int_{\mathbb{R}} vD^\beta v - \frac{2}{p+1}v^{p+1}\,\mathrm{d}x, \\ F(v) &= \frac{1}{2}\int_{\mathbb{R}} v^2\,\mathrm{d}x. \end{aligned} \tag{1.1}$$

Provided that $v \in H^{\beta/2}$ where $\beta \geqslant (p-1)/(p+1)$, these integrals both converge (where $\int vD^\beta v$ is interpreted as $\int |D^{\beta/2}v|^2$). They are independent of t when evaluated on solutions of (0.3) that lie in $C(0, T; H^{\beta/2})$. In this latter case, their values are thus determined by the initial data. Regarding global well-posedness in the critical case $p = p_{\text{crit}}$, we are only aware of results in $H^{\beta/2}(\mathbb{R})$ for $\beta \geqq 2$, and with a restriction on the size of the $L_2(\mathbb{R})$-norm of the data (see [AL]). In the next section, as part of our development, there is obtained what is probably a sharp bound on how large the data can be for global existence to obtain in the critical case (theorem 2.1). For $1 \leqq \beta < 2$ and the critical value of p, global well-posedness results seem not to be available even for small initial values.

Concerning the solitary-wave solutions $S = S_\eta$ of equation (0.3), these satisfy (0.5) and may be shown to exist by variational methods (e.g. the concentration compactness method [Al1, Al2, CB, W6] in the case the nonlinearity is homogeneous) and by positive operator methods or the recent ideas of Lopes for more general nonlinearities [BBB, CB, Lo2]. These techniques allow one to deduce the existence of even solutions of (0.5) that decay to zero at infinity. Regarding positivity of solutions, extra conditions are needed to guarantee this property (see the discussions in [BBB, CB], for example, where there is established, under suitable restrictions, existence of even positive travelling waves, strictly monotone decreasing on $\mathbb{R}^+$ and such that S and all its derivatives are bounded, continuous L_1-functions). However, their existence is established and, independently of the question of uniqueness, the spatial asymptotics and analyticity of such solitary-wave solutions was determined in [BLi, LiB].

Explicit solutions for the pseudo-differential equation (0.5) are known only in certain special cases. For the case $\beta = 2$ (the gKdV-equation), solutions which are powers of the hyperbolic secant, suitably normalized, exist for every $p \geqq 2$. On the other hand, for the gBO-equation ($\beta = 1$) the only explicit solution available is

$$S_\eta(\xi) = \frac{2\eta}{\eta^2 + \xi^2}$$

found by Benjamin in the mid-1960s for the case $p = 2$ [Be1].

A theory of orbital stability that applies to the solitary-wave solutions of (0.3) has been developed by a number of authors over the last three decades (see e.g. [Al1, Al2, ABH, Be2, Bo, BSS, W6]). Listed next are sufficient conditions due to Bona *et al* [BSS] for both stability and instability. These conditions are often satisfied by the solitary-wave solutions of equations of type (0.3). Let $\mathcal{L}$ be the linear, self-adjoint, closed, unbounded operator defined on the dense subset H^β of $L_2(\mathbb{R})$, given by

$$\mathcal{L} = D^\beta + \eta - pS_\eta^{p-1}. \tag{1.2}$$

Suppose $\mathcal{L}$ and S_η satisfy the following properties.

(H_1) The operator $\mathcal{L}$ has a single negative eigenvalue which is simple, with eigenfunction $g_\eta > 0$, the zero eigenvalue is simple with eigenfunction S'_η, and the remainder of the spectrum of $\mathcal{L}$ is positive and bounded away from zero.

(H_2) The curve $\eta \to S_\eta$ is C^1 with values in $H^{1+\beta/2}(\mathbb{R})$.

(H_3) The mapping $\eta \to g_\eta$ is continuous with values in $H^{1+\beta/2}(\mathbb{R})$. Moreover, for each $\eta > 0$, $(1+|x|)^{1/2}(\mathrm{d}/\mathrm{d}\eta)S_\eta$ and $(1+|x|)^{1/2}g_\eta$ lie in $L_1(\mathbb{R})$.

For $\eta > 0$ define the function $d : \mathbb{R}^+ \to \mathbb{R}$ by

$$d(\eta) = E(S_\eta) + \eta F(S_\eta).$$

In the case (H_1) and (H_2) hold, then the solitary-wave solution S_η is orbitally stable if d is a convex function of η, which is to say $d''(\eta) > 0$. On the other hand, if $d''(\eta) < 0$ and (H_1), (H_2) and (H_3) all hold, then the solitary wave is orbitally unstable. It was shown in [BSS] (theorem 4.1) that for any $\eta > 1$, S_η is stable if $p < 2\beta + 1$ and unstable if $p > 2\beta + 1$.

Hypotheses (H_1) and (H_2) have been verified for a range of values of β and p. For instance, they are known to hold for the gKdV-equation $\beta = 2$ and $p \geqq 2$ (see [ABH, Be2, Bo, W4]) and for the BO-equation $\beta = 1$ and $p = 2$ (see [Al1, BBBSS]). Moreover, a set of conditions implying the properties of the spectrum of $\mathcal{L}$ required by (H_1) can be found in the works of Albert [Al1] and Albert and Bona [AB].

Some of the more recent work (e.g. [Al2, Le1, Le2, MS]) on stability of solitary waves has focused on weakening the spectral hypotheses listed above. These theories are more satisfactory in this aspect. However, in the absence of an appropiate uniqueness result, they only establish stability of the *set* of travelling-wave solutions, but do not ensure the stability of individual solitary waves.

2. Global solutions

What we believe to be sharp size restrictions on the data for the global well-posedness for the IVP (0.3) in the critical case are established here. A helpful preliminary result is the following sharp inequality of Gagliardo–Nirenberg type.

Theorem 2.1. *Let $\beta \geqq 1$. If $f \in H^{\beta/2}(\mathbb{R})$, then for any $r \geqslant 2$, $f \in L_r(\mathbb{R})$ and there is a constant $C_{r,\beta}$ such that*

$$|f|_r^r \leqq C_{r,\beta} \|D^{\beta/2} f\|^{(r-2)/\beta} \|f\|^{(2+r(\beta-1))/\beta}. \tag{2.1}$$

The smallest constant $C_{r,\beta}$ for which this inequality is valid is

$$C_{r,\beta} = \frac{r\beta}{2+r(\beta-1)} \left[\left(\frac{2+r(\beta-1)}{r-2} \right)^{1/\beta} \frac{1}{\|\Psi\|^2} \right]^{(r-2)/2},$$

where Ψ is a function in $H^{\beta/2}(\mathbb{R})$, satisfying

$$D^\beta \Psi + \Psi - |\Psi|^{r-2}\Psi = 0. \tag{2.2}$$

Proof. For $1 < t < 2$ and $1/r + 1/t = 1$, the Fourier transform $\mathcal{F}$ is a bounded linear operator from $L_t(\mathbb{R})$ into $L_r(\mathbb{R})$. For any $f \in L_r(\mathbb{R})$ and $\lambda > 0$, it follows from the last remark and Hölder's inequality that for $w = 2/(2-t)$, there is a constant M such that

$$\begin{aligned}
|f|_r &\leqq M|\mathcal{F}^{-1}(\bar{f})|_t = M|\mathcal{F}(f)|_t = M \left(\int_{\mathbb{R}} (\lambda + |\xi|^\beta)^{t/2} \frac{|\hat{f}(\xi)|^t}{(\lambda + |\xi|^\beta)^{t/2}} \, \mathrm{d}\xi \right)^{1/t} \\
&\leqq M \left(\int_{\mathbb{R}} (\lambda + |\xi|^\beta) |\hat{f}(\xi)|^2 \, \mathrm{d}\xi \right)^{1/2} \left(\int_{\mathbb{R}} \frac{1}{(\lambda + |\xi|^\beta)^{tw/2}} \, \mathrm{d}\xi \right)^{1/tw} \\
&= \lambda^{(r-2)/2r\beta - 1/2} M \left(\lambda \|f\|^2 + \|D^{\beta/2} f\|^2 \right)^{1/2} \left(\int_{\mathbb{R}} \frac{1}{(1 + |y|^\beta)^{r/(r-2)}} \, \mathrm{d}y \right)^{(r-2)/2r} \\
&= M_1 \lambda^b (\lambda \|f\|^2 + \|D^{\beta/2} f\|^2)^{1/2},
\end{aligned}$$

where M_1 is a constant and $b = (r-2)/2r\beta - \frac{1}{2} < 0$. Choosing $\lambda = \|D^{\beta/2} f\|^2 / \|f\|^2$ in the last inequality gives (2.1).

Concerning the value of the best constant $C_{r,\beta}$, it will suffice to show that the infimum of the functional

$$J(f) = \frac{\|D^{\beta/2} f\|^{(r-2)/\beta} \|f\|^{(2+r(\beta-1))/\beta}}{|f|_r^r}$$

over $f \in H^{\beta/2}(\mathbb{R})$ is attained. If this is the case, then

$$C_{r,\beta} = \frac{1}{J(\phi)},$$

where ϕ is any element for which J takes on its minimum value. It will turn out that any such ϕ must satisfy the equation depicted in (2.2).

To show that the infimum is attained, consider the following two-parameter family of minimization problems:

$$I(\vartheta, \tau) = \inf \left\{ -\int_{\mathbb{R}} |f(x)|^r \, \mathrm{d}x : f \in H^{\beta/2}(\mathbb{R}) \text{ with } \|D^{\beta/2} f\|^2 = \vartheta, \text{ and } \|f\|^2 = \tau \right\},$$

where ϑ, τ are positive constants. It follows from (2.1) that $I(\vartheta, \tau) > -\infty$. By homogeneity, it transpires that $I(\vartheta, \tau) = \vartheta^a \tau^d I(1, 1)$ where $a = (r-2)/2\beta$ and $d = (2 + r(\beta - 1))/2\beta$. Since $I(\vartheta, \tau) < 0$ and $a + d = \frac{1}{2}r > 1$, there follows the sub-additive property

$$I(\vartheta, \tau) < I(\sigma, \zeta) + I(\vartheta - \sigma, \tau - \zeta)$$

for $(\sigma, \zeta) \in (0, \vartheta) \times (0, \tau)$. The sub-additivity of $I(\vartheta, \tau)$ together with the concentration-compactness theory (see [CB, L1, L2]) implies the existence of a $\varphi \in H^{\beta/2}(\mathbb{R}) \cap C^\infty(\mathbb{R})$ such that $\|D^{\beta/2}\varphi\|^2 = \vartheta$, $\|\varphi\|^2 = \tau$ and $|\varphi|_r^r = -I(\vartheta, \tau)$. An appeal to homogeneity then assures that $J(\varphi) = \inf\{J(f) : f \in H^{\beta/2}(\mathbb{R}) - \{0\}\}$, whence $J'(\varphi) = 0$ and so φ satisfies the equation

$$2aD^\beta\varphi + 2bc_1\varphi = c_2|\varphi|^{r-2}\varphi \tag{2.3}$$

for suitable positive constants c_1, c_2. To determine the value of $C_{r,\beta}$, multiply (2.3) by φ and integrate the result over $\mathbb{R}$ to obtain $2a\|D^{\beta/2}\varphi\|^2 = -2bc_1\|\varphi\|^2 + c_2|\varphi|_r^r$. Next, multiply (2.3) by $x\varphi'$ and use the equalities $\mathcal{H}(x\varphi'') = x\mathcal{H}\varphi''$ and $D^{\beta-1}(x\mathcal{H}\varphi'') = (\beta - 1)D^\beta\varphi + xD^\beta\varphi'$, to obtain $a(\beta - 1)\|D^{\beta/2}\varphi\|^2 = bc_1\|\varphi\|^2 - (c_2/r)|\varphi|_r^r$. In consequence,

$$|\varphi|_r^r = \frac{rc_1}{c_2}\|\varphi\|^2 \qquad \text{and} \qquad \|D^{\beta/2}\varphi\|^2 = \frac{(b-1)c_1}{a(\beta-1)}\|\varphi\|^2.$$

If

$$\Psi(x) = \left(\frac{c_2}{2bc_1}\right)^{1/(r-2)} \varphi\left(\sqrt[\beta]{\frac{a}{bc_1}}x\right),$$

then Ψ satisfies (2.2) and $C_{r,\beta} = [J(\Psi)]^{-1}$ as advertised. □

The following additional hypothesis is needed in the proof of global well-posedness for (0.3) for certain initial data in the critical case $p = 2\beta + 1$.

(H_0) The positive solitary-wave solution of (0.3) in $H^{\beta/2}(\mathbb{R})$ is unique up to sign and translations.

As previously mentioned, this hypothesis is known to hold in special cases such as the KdV-equation itself, the Benjamin–Ono equation and the intermediate long-wave equation (see [AT, AIT]). It is a very likely assumption in a wide range of circumstances. Presuming the validity of (H_0), theorem 2.1 implies that for each $\eta > 0$, $\|S_\eta\| = \|\Psi\|$ whenever $r = p + 1$. Moreover, the best value of $C_{p+1,\beta}$ is $(p + 1)/2\|S_1\|^{2\beta}$.

The next theorem results from assuming (H_0) and taking into account the local well-posedness results already mentioned in section 1 (see section 1.2).

Theorem 2.2. *Let $p > 0$ be an integer and suppose that $p = 2\beta + 1$. If $\beta \geqq 2$, then the IVP (0.3) in $H^{\beta/2}(\mathbb{R})$ is globally well-posed in the open ball $B_R(0)$ in $H^{\beta/2}(\mathbb{R})$ for $R = \|S\|$, where S is the solitary-wave profile solving (0.5). Moreover, if $\|u_0\| \in \overline{B_R(0)}$, then $E(u_0) \geqq 0$.*

Proof. Since the $L_2(\mathbb{R})$-norm of a solution u of the IVP (0.3) is conserved, it is only necessary to establish an appropriate *a priori* bound for $\|D^{\beta/2}u(t)\|$. The conserved functional E presents itself as a natural candidate in this endeavour. Observe that

$$\frac{1}{2}\|D^{\beta/2}u(t)\|^2 \leqq E(u_0) + \frac{1}{p+1}|u(t)|_{p+1}^{p+1}.$$

Using theorem 2.1 with $r = p + 1$, a calculation shows that

$$\|D^{\beta/2}u(t)\|^2\left[1 - \left(\frac{\|u_0\|}{\|S\|}\right)^{2\beta}\right] \leqq 2E(u_0),$$

and the result follows. □

Remark. It follows from the last inequality that if $\beta \geqslant 2$ and the initial data u_0 lies in $H^s(\mathbb{R})$ for some $s > \frac{3}{2}$ and satisfies the condition $\|u_0\| \leqq \|S\|$, then $E(u_0) \geqq 0$. The same conclusion is valid if $1 \leqq \beta < 2$. In particular, for $\beta \geqslant 1$, $E(S_\eta) \geqq 0$ for all $\eta > 0$.

3. Stability theorem for KdV-type equations in the critical case

Attention is turned to the behaviour of solutions of equation (0.3) in the critical case $p = 2\beta+1$.

Following [LBSK, W3, W5], introduce the auxiliary functions

$$\phi(x,t) = \mu(t)^{-1/2} u(\mu(t)^{-1} x, t), \tag{3.1}$$

where

$$\mu(t) = \left(\frac{\langle u(t), D^\beta u(t)\rangle}{\langle S, D^\beta S\rangle}\right)^{1/\beta} = \frac{\|D^{\beta/2} u(t)\|^{2/\beta}}{\|D^{\beta/2} S\|^{2/\beta}}, \tag{3.2}$$

$S = S_\eta$, $\mu(0) = 1$ and $0 \leqq t < t^*$ with t^* the maximal time of existence of the solution of (0.3) under consideration (if the solution is global, $t^* = +\infty$). Note that unless u is the zero solution, $\mu(t) \in (0, \infty)$ for $0 < t < t^*$. The normalization $\mu(0) = 1$ is a temporary one made to simplify the presentation of the argument. It will be dispensed with later. It is easy to check that the function ϕ verifies the identities

$$\text{(i) } \|\phi(\cdot,t)\| = \|u(\cdot,t)\| = \|u_0\|, \tag{3.3}$$

$$\text{(ii) } \langle \phi(\cdot,t), D^\beta \phi(\cdot,t)\rangle = \langle S, D^\beta S\rangle, \tag{3.4}$$

$$\text{(iii) } E(\phi(\cdot,t)) = \frac{1}{\mu(t)^\beta} E(u(\cdot,t)). \tag{3.5}$$

Our first lemma states that the function ϕ lies in an appropriate smoothness class.

Lemma 3.1. *If $u \in C([0,t^*); H^{\beta/2}(\mathbb{R}))$, then $\phi \in C([0,t^*); H^{\beta/2}(\mathbb{R}))$.*

Proof. This is a straightforward consequence of the fact that $\mu \in C([0,t^*); \mathbb{R})$ and $0 < \mu(t) < \infty$ for $0 \leqq t < t^*$. □

Since the stability considered here is with respect to form, i.e. up to translation in space, it is propitious to introduce the orbit

$$\mathcal{O}(S_\eta) = \{g \mid g = \tau_r S_\eta \text{ for some } r \in \mathbb{R}\}$$

of S_η. Here τ_r is the translation operator given by $\tau_r f(x) = f(x+r)$ for all $x \in \mathbb{R}$. To measure the deviation of ϕ from $\mathcal{O}(S_\eta)$, introduce the pseudo-metric

$$\rho_\eta(\phi(\cdot,t), S_\eta)^2 = \inf_{r\in\mathbb{R}} \{\|D^{\beta/2}\phi(\cdot+r,t) - D^{\beta/2}S(\cdot)\|^2 + \eta\|\phi(\cdot+r,t) - S(\cdot)\|^2\}$$

on $H^{\beta/2}(\mathbb{R})$ (see [Be2, Bo, CL, W4]). Define the set $\mathcal{K}$ to be either

$$\mathcal{K} = \{u_0 \mid u_0 \in H^{\beta/2}(\mathbb{R}) \text{ and } E(u_0) \leqq 0\} \tag{3.6a}$$

if $\beta \geqq 2$ or, if $1 \leqslant \beta < 2$,

$$\mathcal{K} = \{u_0 \mid u_0 \in H^{s}(\mathbb{R}) \text{ and } E(u_0) \leqq 0\} \tag{3.6b}$$

where $s > \frac{3}{2}$ is fixed, but otherwise arbitrary.

The next theorem is a stability result which pertains to the spatial structure of the solutions of (0.3) in the critical case $p = 2\beta + 1$.

Theorem 3.2. *Let $p \geqslant 2$ be an integer and let $p = 2\beta + 1$. For $\eta > 0$, let $S = S_\eta$ be a solitary-wave solution of (0.3). For any $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that if $u_0 \in \mathcal{K}$ with $\rho_\eta(u_0, S) < \delta$ and u is the solution of (0.3) corresponding to the initial value u_0, then $u \in C([0,t^*); H^{\beta/2}(\mathbb{R}))$ and*

$$\inf_{r\in\mathbb{R}} \{\eta\|u(\cdot,t) - \mu(t)^{1/2} S(\mu(t)(\cdot - r))\|^2 + \frac{1}{\mu(t)^\beta}\|D^{\beta/2}u(\cdot,t) - \mu(t)^{1/2} D^{\beta/2} S(\mu(t)(\cdot - r))\|^2\} < \epsilon \tag{3.7}$$

for all $t \in [0,t^*)$, where t^* is the maximal existence time for the solution u and μ is as in (3.2).

Remark. For $\beta = 2$ which is the generalized KdV-equation, a result similar to our theorem 3.2 was established by Merle [M, lemma 1] making use of a concentration-compactness argument as in Weinstein [W3]. The difference here, other than the broader range of β, is two-fold. In [M], the parameter $\lambda(t)$ which corresponds to our $\mu(t)$ is defined implicitly via a minimization problem or by using the implicit-function theorem. Here, the function $\mu(t)$ is defined directly in term of the solution u. Our proof of theorem 3.2 may thus be made by direct calculation and thereby includes the case $E(u_0) = 0$ as well as the case $E(u_0) < 0$.

Proof. Suppose at the outset that $\mu(0) = 1$. The proof is based on the time-dependent functional (see [LBS, LBSK, LS])

$$B_t[u] = \frac{1}{\mu(t)^\beta} E(u(\cdot, t)) + \frac{\eta}{2} \left(\frac{\|u(\cdot, t)\|}{\|S\|} \right)^{2k} (\|u(\cdot, t)\|^2 - \|S\|^2),$$

where $k \in \mathbb{N}$ will be chosen later. From the definition of B_t, it is clear that if u is a solution of (0.3) then $B_t[u] = B_t[u_0]$. Using (3.1)–(3.3) and (3.5), we may write $B_t[u]$ in terms of ϕ thus:

$$\tilde{B}_t[\phi] = E(\phi(\cdot, t)) + \frac{\eta}{2} \left(\frac{\|\phi(\cdot, t)\|}{\|S\|} \right)^{2k} (\|\phi(\cdot, t)\|^2 - \|S\|^2), \tag{3.8}$$

where the explicit dependence on μ disappears.

As will be argued presently, if it is established that, modulo translations, the inequalities

$$\text{(i) } \Delta\tilde{B}_t \leqq c_0 \|u_0 - S\|, \tag{3.9}$$

$$\text{(ii) } \Delta\tilde{B}_t \geqq c_1 \|\phi(\cdot, t) - S\|_{\beta/2}^2 - c_2 \sum_{j=1}^{p-1} \|\phi(\cdot, t) - S\|_{\beta/2}^{j+2} - \sum_{j=1}^{2k} c_{k,j} \|\phi(\cdot, t) - S\|_{\beta/2}^{j+2} \tag{3.10}$$

hold for $\Delta\tilde{B}_t = \tilde{B}_t[\phi] - \tilde{B}_t[S]$, where c_i, $c_{k,j}$ are fixed constants, then the result in theorem 3.2 follows.

Hence, attention is turned to establishing these bounds. The upper bound (3.9) is a straightforward consequence of $E(u_0) \leqq 0$ and $E(S) \geqq 0$, where the constant c_0 depends on $\|S\|$ (and on an upper bound for the choice of δ).

To prove (3.10), consider the perturbation of the solitary-wave solution S,

$$\phi(x + \gamma, t) = S(x) + a(x, t), \tag{3.11}$$

where a is a real function and $\gamma = \gamma(t)$ will be chosen later so as to minimize the functional

$$\Pi_t(\gamma) = \|D^{\beta/2}\phi(\cdot + \gamma, t) - D^{\beta/2}S(\cdot)\|^2 + \eta\|\phi(\cdot + \gamma, t) - S(\cdot)\|^2.$$

Using the representation (3.11), one calculates that

$$\begin{aligned} \Delta\tilde{B}_t &= \tilde{B}_t[S + a] - \tilde{B}_t[S] \\ &= E(S + a) - E(S) + \frac{\eta}{2} \left(\frac{\|S + a\|}{\|S\|} \right)^{2k} (\|S + a\|^2 - \|S\|^2) \\ &\geqq \frac{1}{2}\langle \mathcal{L}a, a\rangle + \frac{2k\eta}{\|S\|^2}\langle a, S\rangle^2 - c_2(\eta) \sum_{j=1}^{p-1} \|a\|_{\beta/2}^{j+2} - \sum_{j=1}^{2k} c_{k,j}(\eta)\|a\|_{\beta/2}^{j+2}. \end{aligned} \tag{3.12}$$

The inequality in (3.12) is obtained using the definition (1.2) of $\mathcal{L}$, the Cauchy–Schwarz inequality and the interpolation estimate (2.1).

A suitable lower bound on the quadratic form Π_t is the next order of business.

Lemma 3.3. *Suppose that, for some $t \in [0, t^*)$ and some $\tilde{\gamma} \in \mathbb{R}$, it is the case that*

$$\Pi_t(\tilde{\gamma}) < \|(D^\beta + \eta)^{1/2} S\|^2. \tag{3.13}$$

Then, it follows that

$$\inf\{\Pi_t(\gamma) \mid \gamma \in \mathbb{R}\} \tag{3.14}$$

is attained at least once in $\mathbb{R}$.

Proof. It is immediate that $\Pi_t(\gamma)$ is a continuous function of $\gamma \in \mathbb{R}$. Moreover,

$$\begin{aligned}\lim_{|\gamma|\to\infty} \Pi_t(\gamma) &= \|D^{\beta/2}\phi(t)\|^2 + \|D^{\beta/2} S\|^2 + \eta(\|\phi(t)\|^2 + \|S\|^2)\\ &= \|(D^\beta + \eta)^{1/2} S\|^2 + \|D^{\beta/2} S\|^2 + \eta\|u_0\|^2. \end{aligned}\tag{3.15}$$

Hypothesis (3.13), the continuity of Π_t and (3.15) imply the indicated result. □

Next, it is established that the minimum in (3.14) is achieved at finite values of γ at least for all t in some interval $[0, \tilde{T}]$. To this end, it is sufficient to obtain the condition (3.13) in some such interval. Let $\epsilon > 0$ be such that

$$\epsilon^2 < \tfrac{1}{2}\max\{1, \eta\}\|(D^\beta + \eta)^{1/2} S\|^2.$$

The solitary-wave solution $u(x, t) = S(x - \eta t)$ is globally defined, and hence, by the continuous dependence theory for (0.3) (see [AL] for $\beta \geqq 2$ and [ABFSa] for $1 \leqq \beta < 2$) for the value of ϵ just specified and $T > 0$, there exists a $\delta > 0$ such that if $\|u_0 - S\|_{\beta/2} < \delta$ then the solution u of (0.3) corresponding to appropriately smooth initial data u_0 exists at least for $0 \leqq t \leqq T$ and, in addition,

$$\|u(\cdot, t) - S(\cdot - \eta t)\|_{\beta/2} < \tfrac{1}{2}\epsilon$$

for all $t \in [0, T]$. Because both ϕ and u are continuous mappings of the time axis into $H^{\beta/2}(\mathbb{R})$ over the interval $[0, t^*)$, it follows that there is a $T_1 > 0$ such that, for all $t \in [0, T_1]$,

$$\|\phi(\cdot, t) - \phi(\cdot, 0)\|_{\beta/2} \leqq \tfrac{1}{4}\epsilon \qquad \text{and} \qquad \|u(\cdot, t) - u(\cdot, 0)\|_{\beta/2} \leqq \tfrac{1}{4}\epsilon.$$

Then, for $0 \leqq t \leqq \tilde{T} = \min\{T, T_1\}$, we have

$$\begin{aligned}\|\phi(\cdot, t) - S(\cdot - \eta t)\|_{\beta/2} &\leqq \|\phi(\cdot, t) - u(\cdot, t)\|_{\beta/2} + \|u(\cdot, t) - S(\cdot - \eta t)\|_{\beta/2}\\ &\leqq \|\phi(\cdot, t) - \phi(\cdot, 0)\|_{\beta/2} + \|u(\cdot, 0) - u(\cdot, t)\|_{\beta/2} + \tfrac{1}{2}\epsilon \leqq \epsilon.\end{aligned}$$

Thus, the infimum (3.14) is taken on at values $\gamma(t)$ throughout the time interval $[0, \tilde{T}]$. We take such values of γ as providing a meaning for the definition of a in (3.11), at least for $t \in [0, \tilde{T}]$.

The result of lemma 3.3 together with (1.2) provide us with a compatibility relation on a, namely

$$\int_{\mathbb{R}} S^{p-1}(x) S'(x) a(x, t)\,\mathrm{d}x = 0 \tag{3.16}$$

for all $t \in [0, \tilde{T}]$. This relation is obtained by differentiating Π_t with respect to γ and evaluating at values that minimize Π_t.

The issue of obtaining the lower bound (3.10) for the right-hand side of inequality (3.12) is addressed in the next several lemmas. The first one is an abstract result from spectral theory.

Lemma 3.4. *Let A be a self-adjoint operator on $L_2(\mathbb{R})$ having exactly one negative eigenvalue λ with corresponding ground-state eigenfunction $f_\lambda \geqq 0$ and let $\tilde{g} \in \mathcal{N}^\perp(A)$. Assume $\langle \tilde{g}, f_\lambda\rangle \neq 0$ and that*

$$-\infty < \alpha \equiv \min_{\substack{\|f\|=1\\ \langle f, \tilde{g}\rangle = 0}} \langle Af, f\rangle.$$

If $\langle A^{-1}\tilde{g}, \tilde{g}\rangle \leqq 0$, then it must be the case that $\alpha \geqq 0$.

Proof. See lemma E.1 in [W2]. □

Corollary 3.5. *Let $\beta \geqslant 1$ and let $\mathcal{L} = D^\beta + \eta - pS^{p-1}$ satisfy hypothesis (H_1). Then there exists $\sigma < 0$ such that if $\tilde{h} = S - \sigma D^\beta S$, then*

$$\min_{\substack{\|f\|=1 \\ \langle f, \tilde{g}\rangle = 0}} \langle \mathcal{L}f, f\rangle = 0.$$

Proof. For any given value σ, define the function f_0 by

$$f_0(x) = -\frac{1}{2\beta\eta}S(x) - \frac{1+\eta\sigma}{\beta\eta}xS'(x).$$

Then using the relations $\mathcal{H}(xS'') = x\mathcal{H}S''$, $D^{\beta-1}(x\mathcal{H}S'') = (\beta-1)D^\beta S + xD^\beta S'$ and (1.2), we obtain that $\mathcal{L}f_0 = S - \sigma D^\beta S = \tilde{h}$ and, consequently, that

$$\langle f_0, \mathcal{L}f_0\rangle = \langle f_0, \tilde{h}\rangle = \left(\frac{1}{2\beta}\|S\|^2 + \frac{1}{2\eta}\|D^{\beta/2}S\|^2\right)\sigma + \left(\frac{\beta-1}{2\beta}\|D^{\beta/2}S\|^2\right)\sigma^2.$$

It is thus obvious that for small negative values of σ, it is possible to have both $\langle\tilde{h}, g_\eta\rangle = \int g_\eta S\,\mathrm{d}x - \sigma\int g_\eta D^\beta S\,\mathrm{d}x \neq 0$ and $\langle\mathcal{L}^{-1}\tilde{h}, \tilde{h}\rangle = \langle f_0, \tilde{h}\rangle < 0$.

Since $\mathcal{N}(\mathcal{L}) = \mathrm{Span}\{S'\}$ and $\tilde{h} \in \mathcal{N}^\perp(\mathcal{L})$, it follows immediately from lemma 3.4 that

$$\theta = \min\{\langle\mathcal{L}f, f\rangle : \|f\| = 1 \text{ and } \langle f, \tilde{h}\rangle = 0\} = 0.$$

This completes the proof of the corollary. □

Lemma 3.6. *If $\tilde{h} \equiv S - \sigma D^\beta S$ with $\sigma < 0$ chosen as in the last corollary, then*

$$\inf\{\langle\mathcal{L}f, f\rangle : \|f\| = 1, \langle f, \tilde{h}\rangle = 0, f \perp S^{p-1}S'\} \equiv \nu > 0. \tag{3.17}$$

Proof. Because of corollary 3.5, it is inferred that $\nu \geqq 0$. Suppose that $\nu = 0$. Let $\{f_j\}$ be a sequence of $H^{\beta/2}(\mathbb{R})$-functions with $\|f_j\| = 1$, $f_j \perp \tilde{h}$, $f_j \perp S^{p-1}S'$ and

$$\lim_{j\to\infty}\langle\mathcal{L}f_j, f_j\rangle = 0.$$

Then, for any $\epsilon > 0$, there is a J such that, for $j > J$,

$$0 < \eta \leqq \|D^{\beta/2}f_j\|^2 + \eta\|f_j\|^2 \leqq p\int S^{p-1}(f_j)^2\,\mathrm{d}x + \epsilon. \tag{3.18}$$

Since $|S|_\infty < \infty$, (3.18) implies $\|f_j\|_{\beta/2}$ to be uniformly bounded as j varies. Therefore, there exists a subsequence of the $\{f_j\}$, which we denote again by $\{f_j\}$, and an $f^* \in H^{\beta/2}(\mathbb{R})$ such that $f_j \rightharpoonup f^*$ weakly in $H^{\beta/2}(\mathbb{R})$ and, by compact embedding and a Cantor diagonalization argument, strongly in $L_{2,\mathrm{loc}}(\mathbb{R})$. The function f^* satisfies the conditions $f^* \perp \tilde{h}$ and $f^* \perp S^{p-1}S'$. A consequence of the just-mentioned properties of the sequence f_j and the decay of S to 0 as $|x| \to \infty$ is that

$$\int S^{p-1}(f_j)^2\,\mathrm{d}x \to \int S^{p-1}(f^*)^2\,\mathrm{d}x$$

as $j \to \infty$. Taking the limit in (3.16) as $j \to \infty$ yields

$$0 < \eta \leqq p\int S^{p-1}(f^*)^2\,\mathrm{d}x + \epsilon.$$

As ϵ is arbitrary, it must be the case that $f^* \neq 0$.

It is now shown that the infimum is achieved. Indeed, weak convergence is lower semi-continuous, so

$$\|D^{\beta/2}f^*\| \leqq \liminf_{j\to\infty}\|D^{\beta/2}f_j\|.$$

Since $\langle S^{p-1}f_j, f_j\rangle \to \langle S^{p-1}f^*, f^*\rangle$ as $j \to \infty$, as noted above, it follows that

$$0 \leqq \langle \mathcal{L}f^*, f^*\rangle \leqq \liminf_{j\to\infty}\langle \mathcal{L}f_j, f_j\rangle = 0.$$

Since $f^* \neq 0$, define $g^* = f^*/\|f^*\|$. Then, we have $\|g^*\| = 1$, $g^* \perp \tilde{h}$, $g^* \perp S^{p-1}S'$ and $\langle \mathcal{L}g^*, g^*\rangle = 0$. A consequence of the reasoning just put forward is that there exists at least one non-trivial critical point $(g^*, \alpha, \theta, \nu)$ for the Lagrange multiplier problem

$$\begin{cases} \mathcal{L}f = \alpha f + \theta\tilde{h} + \nu S'S^{p-1}, & \text{subject to} \\ \|f\| = 1, \quad f \perp S'S^{p-1} \quad \text{and} & \langle f, \tilde{h}\rangle = 0. \end{cases} \tag{3.19}$$

Using the fact that $\langle \mathcal{L}g^*, g^*\rangle = 0$, it is easily seen that (3.19) implies $\alpha = 0$. Moreover, since $\mathcal{L}S' = 0$, we have that $\langle \mathcal{L}g^*, S'\rangle = \langle g^*, \mathcal{L}S'\rangle = \nu\int (S')^2 S^{p-1}\,dx = 0$, which implies $\nu = 0$. It is thereby concluded that

$$\mathcal{L}f = \theta\tilde{h} \tag{3.20}$$

has non-trivial solutions (g^*, θ) satisfying the side constraints. But, if f_0 is the auxiliary function arising in the proof of corollary 3.5, $\mathcal{L}f_0 = \tilde{h}$, whence $\mathcal{L}(g^* - \theta f_0) = 0$, and therefore $g^* - \theta f_0 \in \mathcal{N}(\mathcal{L})$. From the property $\langle f_0, \tilde{h}\rangle \neq 0$ established in corollary 3.5, it follows from the preceding that $\theta = 0$. Therefore, for some non-zero $\lambda \in \mathbb{R}$, it is true that $g^* = \lambda S'$, which is a contradiction since such a function cannot be orthogonal to $S^{p-1}S'$. Therefore, the minimum in (3.17) is positive and the proof of the lemma is complete. □

Attention is now turned to estimating the term $\frac{1}{2}\langle \mathcal{L}a, a\rangle + (2k\eta/\|S\|^2)\langle a, S\rangle^2$ in (3.12), where a satisfies the compatibility relation (3.16). We continue to carry over the notation from corollary 3.5 and lemma 3.6. In particular, σ is chosen so that the conclusions of corollary 3.5 are valid. Define $a_\parallel$ and $a_\perp$ to be

$$a_\parallel = \frac{\langle a, \tilde{h}\rangle}{\|\tilde{h}\|^2}\tilde{h} \qquad \text{and} \qquad a_\perp = a - a_\parallel.$$

It follows from the properties of a and $\tilde{h} = S - \sigma D^\beta S$ that $\langle a_\perp, \tilde{h}\rangle = 0$ and $\int S^{p-1}S'a_\perp\,dx = 0$. Without loss of generality, take it that $\langle a, \tilde{h}\rangle < 0$. Thus, from lemma 3.6, the Cauchy–Schwarz inequality and the properties of a, $a_\perp$, $a_\parallel$ and $\tilde{h}$, it follows that

$$\begin{aligned} \langle \mathcal{L}a_\perp, a_\perp\rangle &\geqq D_1\|a_\perp\|^2, \\ \langle \mathcal{L}a_\parallel, a_\parallel\rangle &= \frac{\|a_\parallel\|^2}{\|\tilde{h}\|^2}\langle \tilde{h}, \mathcal{L}\tilde{h}\rangle, \\ \langle \mathcal{L}a_\parallel, a_\perp\rangle &= \frac{\langle a, \tilde{h}\rangle}{\|\tilde{h}\|^2}\langle \mathcal{L}\tilde{h}, a_\perp\rangle \geqq -D_2\|a_\perp\|\,\|a_\parallel\| \end{aligned} \tag{3.21}$$

for some positive constants D_1 and D_2. Identity (3.4) and elementary properties of Hilbert spaces imply that

$$-2\langle a, D^\beta S\rangle = \|D^{\beta/2}a\|^2.$$

Thus, from the Cauchy–Schwarz inequality we obtain (recall σ and $\langle a, \tilde{h}\rangle$ are both negative)

$$\begin{aligned} \frac{2k\eta}{\|S\|^2}\langle a, S\rangle^2 &\geqq \frac{2k\eta}{\|S\|^2}(\langle a, \tilde{h}\rangle^2 - \sigma\langle a, \tilde{h}\rangle\|D^{\beta/2}a\|^2) \\ &\geqq \frac{2k\eta}{\|S\|^2}(\|\tilde{h}\|^2\|a_\parallel\|^2 + \sigma\|\tilde{h}\|\,\|a\|\,\|D^{\beta/2}a\|^2) \\ &\geqq \frac{2k\eta}{\|S\|^2}\|\tilde{h}\|^2\|a_\parallel\|^2 + 2k\eta\sigma D_3\|a\|^3_{\beta/2}, \end{aligned} \tag{3.22}$$

with $D_3 > 0$ also. Choose $\theta > 0$ so that $D_1 - \theta D_2 \equiv D_4 > 0$. By Young's inequality, $\|a_\perp\|\|a_\parallel\| \leqq \theta\|a_\perp\|^2 + (1/\theta)\|a_\parallel\|^2$. Finally, fix k in such a way that

$$\frac{2k\eta}{\|S\|^2}\|\tilde{h}\|^2 + \frac{\langle\tilde{h}, \mathcal{L}\tilde{h}\rangle}{\|\tilde{h}\|^2} - \frac{D_2}{\theta} \equiv D_5 > 0.$$

With these choices, it follows from (3.21) and (3.22) that

$$\begin{aligned}\frac{1}{2}\langle\mathcal{L}a, a\rangle + \frac{2k\eta}{\|S\|^2}\langle a, S\rangle^2 &\geqq D_5\|a_\parallel\|^2 + D_4\|a_\perp\|^2 + 2k\eta\sigma D_3\|a\|^3_{\beta/2}\\ &\geqq D'\|a\|^2 - D''\|a\|^3_{\beta/2}\end{aligned} \tag{3.23}$$

for some positive constants D', D''. With (3.23) in hand, it follows easily from the specific form of the operator $\mathcal{L}$ that

$$\frac{1}{2}\langle\mathcal{L}a, a\rangle + \frac{2k\eta}{\|S\|^2}\langle a, S\rangle^2 \geqq \tilde{D}_1\|a\|^2_{\beta/2} - \tilde{D}_2\|a\|^3_{\beta/2}, \tag{3.24}$$

with $\tilde{D}_1, \tilde{D}_2 > 0$. Finally, using (3.24) in conjunction with (3.12), there obtains

$$\begin{aligned}\Delta\tilde{B}_t &\geqq \tilde{D}_1\|a\|^2_{\beta/2} - \tilde{D}_2\|a\|^3_{\beta/2} - c_2(\eta)\sum_{j=1}^{p-1}\|a\|^{j+2}_{\beta/2} - \sum_{j=1}^{2k} c_{k,j}(\eta)\|a\|^{j+2}_{\beta/2}\\ &\geqq c_1\|a\|^2_{\beta/2} - c_2\sum_{j=1}^{p-1}\|a\|^{j+2}_{\beta/2} - \sum_{j=1}^{2k} c_{k,j}\|a\|^{j+2}_{\beta/2},\end{aligned}$$

where $c_1, c_2, c_{k,j}$, are positive constants that depend only on η.

Now we are in a position to prove theorem 3.2. Suppose first that u_0 lies in the set $\mathcal{K}$ of 'negative-energy' initial values defined in (3.6) and suppose $\|u_0 - S\|_{\beta/2} = \delta$. Then at least for $t \in [0, T]$, it follows from (3.9) and (3.10) that

$$q(\rho_\eta(\phi(\cdot, t), S)) \leqq \Delta\tilde{B}_t \leqq c_0\delta, \tag{3.25}$$

where $q(x) = c_1x^2 - c_2\sum_{j=1}^{p-1} x^{j+2} - \sum_{j=1}^{2k} c_{k,j}x^{j+2}$. Since $\|a(\cdot, t)\|^2_{\beta/2} = \rho_\eta(\phi(\cdot, t), S)^2$ is a continuous function of $t \in [0, t^*)$ (see lemma 2 in [Bo]), it follows from the inequality

$$q(\rho_\eta(\phi(\cdot, 0), S)) \leqq c_0\delta \tag{3.26}$$

and (3.25) that given $\epsilon \geqslant 0$, then for all $t \in [0, \tilde{T}]$,

$$\rho_\eta(\phi(\cdot, t), S) \leqq \epsilon, \tag{3.27}$$

provided that δ is chosen small enough at the outset.

To finish the proof, we show that the inequality (3.27) is still true for $t \in [0, t^*)$. Following [Bo], let

$$\mathcal{A} = \{t : \text{the infimum in (3.14) is attained at finite values of } \gamma\}.$$

As shown above, $[0, \tilde{T}] \subset \mathcal{A}$. Let T_1 be the largest value such that $[0, T_1) \subset \mathcal{A}$ and suppose that $T_1 < t^*$. Then from (3.25) we obtain that

$$\inf \Pi_t = \rho_\eta(\phi(\cdot, t), S)^2 \leqq \epsilon^2 \leqq \tfrac{1}{2}\|(D^\beta + \eta)^{1/2}S\|^2.$$

Since $\inf \Pi_t$ is a continuous function of t for all $t \in [0, t^*)$, there is a $T > 0$ such that

$$\inf \Pi_t < \|(D^\beta + \eta)^{1/2}S\|^2$$

for $t \in [T_1, T_1 + T]$. But then lemma 3.3 implies that the infimum in (3.14) is taken at finite values of γ and this contradicts the choice of T_1. Therefore, $T_1 = t^*$ and the stability theorem 3.2 is established if $\mu(0) = 1$.

Now we discuss the general case wherein the initial data are not necessarily such that $\mu(0) = 1$. First, note that if S_η is a solution of (0.5) for $\eta > 0$, then $R(x) = \eta^{1/(1-p)} S_\eta(\eta^{-1/\beta} x)$ satisfies

$$D^\beta R + R - R^p = 0.$$

By the uniqueness hypothesis, this means $S(x) = S_\eta(x) = \eta^{1/p-1} R(\eta^{1/\beta} x)$, up to translations and sign. Moreover, since $p-1 = 2\beta$ we have that $\|R\| = \|S_\eta\|$ and $\|D^{\beta/2} S_\eta\|^2 = \eta \|D^{\beta/2} R\|^2$.

Let $u_0 \in \mathcal{K}$ obey the inequality

$$\eta \|u_0 - S_\eta\|^2 + \|D^{\beta/2} u_0 - D^{\beta/2} S_\eta\|^2 < \delta^2, \tag{3.28}$$

where δ will be determined presently. Corresponding to $\eta_1 > 0$, a solution S_{η_1} of (0.5) has $\|D^{\beta/2} S_{\eta_1}\|^2 = \eta_1 \|D^{\beta/2} R\|^2$. It is therefore possible to choose η_1 such that $\|D^{\beta/2} S_{\eta_1}\|^2 = \|D^{\beta/2} u_0\|^2$. Then if

$$\mu_{\eta_1}(t) = \left(\frac{\|D^{\beta/2} u(t)\|^2}{\|D^{\beta/2} S_{\eta_1}\|^2} \right)^{1/\beta},$$

it is obviously the case that $\mu_{\eta_1}(0) = 1$.

The idea is to apply the preceding theory to the case $\eta = \eta_1$ and then use the triangle inequality to conclude the desired result for the given value of η and u_0 (see [ABLS, Be1, Bo]).

An estimate of the quantity

$$\begin{aligned} I_\eta(u(\cdot,t), S_\eta, \mu) \equiv\ & \eta\, \|u(\cdot,t) - \mu(t)^{1/2} S_\eta(\mu(t)\,\cdot)\|^2 \\ & + \mu(t)^{-\beta}\, \|D^{\beta/2} u(\cdot,t) - \mu(t)^{1/2} D^{\beta/2} S_\eta(\mu(t)\,\cdot)\|^2 \end{aligned}$$

will be helpful. Denoting S_η by S as before and S_{η_1} and μ_{η_1} by S_1 and μ_1, respectively, it follows from the definitions of μ and μ_1 that

$$\begin{aligned} I_\eta(u(\cdot,t), S_\eta, \mu) \leqq\ & \max\left\{ \frac{\eta}{\eta_1}, \frac{\eta_1}{\eta} \right\} I_{\eta_1}(u(\cdot,t), S_1, \mu_1) + \eta\, \|\mu^{1/2} S(\mu\,\cdot) - \mu_1^{1/2} S_1(\mu_1\,\cdot)\|^2 \\ & + \frac{1}{\mu^\beta} \|\mu^{1/2} D^{\beta/2} S(\mu\,\cdot) - \mu_1^{1/2} D^{\beta/2} S_1(\mu_1\,\cdot)\|^2. \end{aligned} \tag{3.29}$$

The right-hand side of (3.29) may be bounded above as follows. First observe that

$$\mu^{1/2} S(\mu x) \equiv \mu_1^{1/2} S_1(\mu_1 x)$$

and consequently the second and third terms on the right-hand side of (3.29) vanish.

Therefore, it is only necessary to estimate the term $I_{\eta_1}(u(\cdot,t), S_1, \mu_1)$ in (3.29). For this, it suffices to show that $\rho_{\eta_1}(u_0, S_1) \leqq \tilde{C}\delta$, where $\tilde{C} = \tilde{C}(\eta, R)$, and then apply the foregoing theory for the special case $\mu_1(0) = 1$. Because $\rho_{\eta_1}(u_0, S_1) \leqq \rho_{\eta_1}(u_0, S) + \rho_{\eta_1}(S, S_1)$, we may estimate $\rho_{\eta_1}(u_0, S)$ and $\rho_{\eta_1}(S, S_1)$ separately and still reach the desired inequality. First, consider the term $\rho_{\eta_1}(S, S_1)$:

$$\begin{aligned} [\rho_{\eta_1}(S, S_1)]^2 \leqq\ & \eta_1 \|S - S_1\|^2 + \|D^{\beta/2} S - D^{\beta/2} S_1\|^2 \\ =\ & \eta_1 \int_{\mathbb{R}} \left| R(x) - \left(\frac{\eta_1}{\eta}\right)^{1/2\beta} R\left(\left(\frac{\eta_1}{\eta}\right)^{1/\beta} x\right) \right|^2 \mathrm{d}x \\ & + \eta \int_{\mathbb{R}} \left| D^{\beta/2} R(x) - \left(\frac{\eta_1}{\eta}\right)^{1/2\beta} D^{\beta/2} R\left(\left(\frac{\eta_1}{\eta}\right)^{1/\beta} x\right) \right|^2 \mathrm{d}x. \end{aligned} \tag{3.30}$$

The first integral in (3.30) can be bounded above as follows:

$$\eta_1 \int_{\mathbb{R}} \left| R(x) - \left(\frac{\eta_1}{\eta}\right)^{1/2\beta} R\left(\left(\frac{\eta_1}{\eta}\right)^{1/\beta} x\right)\right|^2 \mathrm{d}x$$

$$\leqq 2\eta_1 \left(\frac{\eta_1}{\eta}\right)^{1/\beta} \int_{\mathbb{R}} \left| R(x) - R\left(\left(\frac{\eta_1}{\eta}\right)^{1/2\beta} x\right)\right|^2 \mathrm{d}x + 2\frac{\eta_1 |\eta_1^{1/2\beta} - \eta^{1/2\beta}|^2}{\eta^{1/\beta}} \|R\|^2.$$

Thus, the fundamental theorem of calculus together with Minkowski's inequality yields

$$\int_{\mathbb{R}} \left| R(x) - R\left(\left(\frac{\eta_1}{\eta}\right)^{1/\beta} x\right)\right|^2 \mathrm{d}x \leqq \int_{\mathbb{R}} \left(\int_{(\eta_1/\eta)^{1/\beta}}^1 \left|\frac{\mathrm{d}}{\mathrm{d}t} R(tx)\right| \mathrm{d}t\right)^2 \mathrm{d}x$$

$$\leqq \left(\int_{(\eta_1/\eta)^{1/\beta}}^1 \left(\int_{\mathbb{R}} \left|\frac{\mathrm{d}}{\mathrm{d}t} R(tx)\right|^2 \mathrm{d}x\right)^{1/2} \mathrm{d}t\right)^2$$

$$= \|x R'\|^2 \left(\int_{(\eta_1/\eta)^{1/\beta}}^1 t^{-3/2}\, \mathrm{d}t\right)^2 = \frac{4|\eta_1^{1/2\beta} - \eta^{1/2\beta}|^2}{\eta_1^{1/\beta}} \|x R'\|^2.$$

Consequently, it transpires that

$$\eta_1 \int_{\mathbb{R}} \left| R(x) - \left(\frac{\eta_1}{\eta}\right)^{1/2\beta} R\left(\left(\frac{\eta_1}{\eta}\right)^{1/\beta} x\right)\right|^2 \mathrm{d}x \leqq \frac{2\eta_1 |\eta_1^{1/2\beta} - \eta^{1/2\beta}|^2}{\eta^{1/\beta}} [\|R\|^2 + 4\|x R'\|^2]. \tag{3.31}$$

Similarly, the second integral on the right-hand side of (3.30) may be bounded above thus:

$$\eta \int_{\mathbb{R}} \left| D^{\beta/2} R(x) - \left(\frac{\eta_1}{\eta}\right)^{1/2\beta} D^{\beta/2} R\left(\left(\frac{\eta_1}{\eta}\right)^{1/\beta} x\right)\right|^2 \mathrm{d}x$$

$$\leqq \frac{2\eta |\eta_1^{1/2\beta} - \eta^{1/2\beta}|^2}{\eta^{1/\beta}} [\|D^{\beta/2} R\|^2 + 4\|x D^{\beta/2} R'\|^2]. \tag{3.32}$$

The inequalities (3.31) and (3.32) imply

$$[\rho_{\eta_1}(S, S_1)]^2 \leqq \frac{2\eta_1 |\eta_1^{1/2\beta} - \eta^{1/2\beta}|^2}{\eta^{1/\beta}} [\|R\|^2 + 4\|x R'\|^2]$$

$$+ \frac{2\eta |\eta_1^{1/2\beta} - \eta^{1/2\beta}|^2}{\eta^{1/\beta}} [\|D^{\beta/2} R\|^2 + 4\|x D^{\beta/2} R'\|^2]$$

$$\leqq C_1(R) \frac{\eta + \eta_1}{\eta^{1/\beta}} |\eta_1^{1/2\beta} - \eta^{1/2\beta}|^2. \tag{3.33}$$

It is now determined that there is a positive constant $C = C(\eta, R)$ such that

$$|\eta_1 - \eta| \leqq C\delta \tag{3.34}$$

at least for small values of δ. Indeed, from Young's inequality and (3.28) there follows

$$|\eta_1 - \eta| = \frac{1}{\|D^{\beta/2} R\|^2} |\|D^{\beta/2} u_0\|^2 - \|D^{\beta/2} S\|^2|$$

$$\leqq \frac{2\delta}{\|D^{\beta/2} R\|^2} \|D^{\beta/2} S\|^2 + \frac{1}{\|D^{\beta/2} R\|^2} \left(1 + \frac{1}{2\delta}\right) \|D^{\beta/2} u_0 - D^{\beta/2} S\|^2$$

$$\leqq \frac{1}{\|D^{\beta/2} R\|^2} \left(2\eta \|D^{\beta/2} R\|^2 + \frac{3}{2}\right) \delta.$$

The inequality (3.34) certainly implies that $|\eta/\eta_1 - 1| \leqq 1$ and $|\eta_1^{1/2\beta} - \eta^{1/2\beta}| \leqq C_2\delta$, where $C_2 = C_2(\eta, R)$. From (3.33) it then follows that

$$\rho_{\eta_1}(S, S_1) \leqq C_3(R)\delta. \tag{3.35}$$

From (3.34) it is concluded that $\eta_1/\eta \leqq 1 + (C/\eta)\delta$. Therefore, the assumption (3.28) implies

$$[\rho_{\eta_1}(u_0, G)]^2 \leqq \max\left\{\frac{\eta_1}{\eta}, 1\right\}\left[\eta\|u_0 - G\|^2 + \|D^{\beta/2}u_0 - D^{\beta/2}S\|^2\right] \leqq C_4\delta^2, \tag{3.36}$$

where $C_4 = C_4(\eta, R)$. Inequalities (3.35) and (3.36) imply that $\rho_{\eta_1}(u_0, S_1) \leqq \tilde{C}\delta$, and therefore that

$$I_{\eta_1}(u(\cdot, t), S_1, \mu_1) \leqq \epsilon^2.$$

Theorem 3.2 is now established. □

4. Behaviour of the stability parameters: KdV-type equations

In the proof of theorem 3.2, it was actually shown that there is a concrete choice of $\gamma = \gamma(t)$ (lemma 3.3) for which

$$(\|D^{\beta/2}\phi(\cdot + \gamma, t) - D^{\beta/2}S(\cdot)\|^2 + \eta\|\phi(\cdot + \gamma, t) - S(\cdot)\|^2)^{1/2} = \rho_\eta(\phi(\cdot, t), S) \leqq \epsilon \tag{4.1}$$

for all $t < t^*$, where ϕ is the rescaled version of the solution u of (0.3) defined in (3.1) and (3.2). In fact, a choice of γ for which (4.1) holds may be determined via the orthogonality condition (see (3.16))

$$\int_{\mathbb{R}} \phi(x + \gamma, t)S^{p-1}(x)S'(x)\,dx = 0. \tag{4.2}$$

By an application of the implicit-function theorem as in [BS], it will be shown that as long as ϕ satisfies (4.1), there is a unique, continuously differentiable choice of the value $\gamma(t)$ that achieves (4.2).

The principal result regarding the behaviour of the parameter γ is stated in theorem 4.3. First, we need the following lemmas.

Lemma 4.1. *Let $u_0 \in H^s(\mathbb{R})$ for s sufficiently large. Then the function μ defined in (3.1) belongs to the class $C^1([0, t^*); \mathbb{R})$, where t^* is the maximal time of existence of the solution u of (0.3) with initial value u_0.*

Proof. If $s > \max\{\beta, \frac{3}{2}\}$, say, it follows from the theory in section 2 and the differential equation that $u \in C^1([0, t^*); L_2(\mathbb{R})) \cap C([0, t^*); H^s(\mathbb{R}))$. Now, from the relation

$$\mu(t)^\beta = \frac{2}{\|D^{\beta/2}S\|^2}\left[E(u_0) + \frac{1}{p+1}\int_{\mathbb{R}} u^{p+1}(x, t)\,dx\right]$$

for $t \in [0, t^*)$, it is adduced that

$$\beta\mu(t)^{\beta-1}\mu'(t) = \frac{2}{\|D^{\beta/2}S\|^2}\int_{\mathbb{R}} u^p(x, t)u_t(x, t)\,dx. \tag{4.3}$$

Therefore, from lemma 3.1, the Cauchy–Schwarz inequality and the Sobolev embedding theorem, we see that $\mu \in C^1([0, t^*); \mathbb{R})$. □

Lemma 4.2. *Let $S = S_\eta$ be a solitary-wave solution of (0.3) stable in the sense of theorem 3.2. Let $\epsilon_0 = \min\{\|(D^\beta+\eta)^{1/2}S\|, \|(D^\beta+\eta)^{1/2}S'\|^2/\|(D^\beta+\eta)S''\|\}$. If ϵ and u_0 are chosen such that $\epsilon < \epsilon_0$ and $u_0 \in H^s(\mathbb{R})$ for s sufficiently large, then there is a unique function $\gamma : [0, t^*) \to \mathbb{R}$ such that $\gamma(t)$ satisfies (4.2) for all t in $[0, t^*)$. Moreover, the function γ is continuously differentiable.*

Proof. First, note that, for each $t < t^*$, there is an $r \in \mathbb{R}$ such that the function

$$G(t,r) = -\int_{\mathbb{R}} \phi(x+r,t)S^{p-1}(x)S'(x)\,\mathrm{d}x \tag{4.4}$$

vanishes at (t,r). Indeed, since $\epsilon^2 < \|(D^\beta+\eta)^{1/2}S\|^2$, lemma 3.3 shows that there is $r \in \mathbb{R}$ such that $\Pi_t(r) \leqq \Pi_t(s)$ for every $s \in \mathbb{R}$. Therefore, $\Pi_t'(r) = 0$ and consequently $G(t,r) = 0$.

Next, it is shown that the point r corresponding to the time t is unique. First, from lemma 4.1 μ is of class C^1 and, as u_0 is sufficiently smooth, $u_t \in C([0,t^*); L_2(\mathbb{R}))$. Since S is an $H^\infty(\mathbb{R})$-function, it follows that G is a C^1-function. Let (t,r) be a point where $G(t,r) = 0$. Then from theorem 3.2, relation (4.1) holds with $\gamma(t) = r$, where $\epsilon < \epsilon_0$. Now, calculate $\partial G/\partial r$ at (t,r). Since $u_z(z,t) = \mu(t)^{3/2}\phi_\omega(\omega,t)$, where $\omega = \mu z$, it is easily deduced that

$$\begin{aligned}
\left.\frac{\partial G}{\partial r}\right|_{(t,r)} &= -\frac{1}{p\mu^{3/2}}\int_{\mathbb{R}} \mu^{3/2}\phi_\omega(\omega,t)(D^\beta+\eta)S'(\omega-r)\,\mathrm{d}\omega \\
&= -\frac{1}{p}\int_{\mathbb{R}} [S'(y)+a_y(y,t)](D^\beta+\eta)S'(y)\,\mathrm{d}y \\
&= -\frac{1}{p}\|(D^\beta+\eta)^{1/2}S'\|^2 + \frac{1}{p}\int_{\mathbb{R}} a(y,t)(D^\beta+\eta)S''(y)\,\mathrm{d}y \\
&\leqq -\frac{1}{p}\|(D^\beta+\eta)^{1/2}S'\|^2 + \frac{\epsilon}{p}\|(D^\beta+\eta)S''\| \\
&\leqq -\|(D^\beta+\eta)S''\|(\epsilon_0-\epsilon) < 0.
\end{aligned} \tag{4.5}$$

Thus G is a strictly decreasing function of r and hence, for each t, can take the value 0 at most once.

Finally, to show that the correspondence $t \to \gamma = \gamma(t)$ with $G(t,\gamma(t)) = 0$ is a C^1-function, it suffices by the implicit-function theorem to verify the transversality condition at each point $(t,\gamma(t))$. But this condition is simply that $(\partial G/\partial r)|_{(t,\gamma)} \neq 0$, and this is indeed provided by (4.5) because $\epsilon < \epsilon_0$. The proof of lemma 4.2 is completed. □

The relation between the translation and dilation parameters involved in our stability result is now stated.

Theorem 4.3. *Let $S = S_\eta$ be an even solitary-wave solution of (0.5). For any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that if $u_0 \in H^s(\mathbb{R}) \cap \mathcal{K}$, with s sufficiently large and $\|u_0 - S\|_{\beta/2} < \delta$, then there exists a C^1-mapping $\gamma : [0,t^*) \to \mathbb{R}$ such that*

(i) $\|\phi(\cdot+\gamma(t),t) - S\|_{\beta/2} \leqq \epsilon$ *for* $t \in [0,t^*)$,

(ii) $\left|\gamma(t) - \eta\mu(t)\int_0^t \mu^\beta(s)\,\mathrm{d}s\right| \leqq C\epsilon\mu(t)\left(\int_0^t \mu^\beta(s)\,\mathrm{d}s + \int_0^t \frac{|\mu'(s)|}{\mu^2(s)}\,\mathrm{d}s\right)$ *for* $t \in [0,t^*)$,

where C depends only on S. If $\mu' > 0$ and $\mu(0) = 1$, then

$$\left|\gamma(t) - \eta\mu(t)\int_0^t \mu^\beta(s)\,\mathrm{d}s\right| \leqq C\epsilon\left(\mu(t)\int_0^t \mu^\beta(s)\,\mathrm{d}s + \mu(t) - 1\right).$$

Proof. A consequence of lemma 4.2 is that there is a unique C^1-function γ satisfying (i) which is determined by the relationship

$$G(t,\gamma(t)) = 0, \tag{4.6}$$

where G is defined in (4.4). Using the definition of ϕ in (3.1) and differentiating relation (4.6) with respect to t gives the equation

$$0 = \frac{\mu'(t)}{2\mu(t)}\int_{\mathbb{R}} \phi(x+\gamma(t),t)S^{p-1}(x)S'(x)\,\mathrm{d}x - \frac{\gamma'(t)}{p}\int_{\mathbb{R}} \phi_x(x+\gamma(t),t)(D^\beta+\eta)S'(x)\,\mathrm{d}x$$

$$+\frac{\mu'(t)}{p\mu(t)^{1/2}}\int_{\mathbb{R}} zu_z(z,t)(D^\beta+\eta)S'(\mu(t)z-\gamma(t))\,\mathrm{d}z$$
$$-\frac{\mu(t)^{1/2}}{p}\int_{\mathbb{R}} u_t(z,t)(D^\beta+\eta)S'(\mu(t)z-\gamma(t))\,\mathrm{d}z. \tag{4.7}$$

The terms on the right-hand side of (4.7) are estimated separately. It follows immediately from (4.6) that the first term is equal to zero. Since $\phi(x+\gamma(t),t)=S(x)+a(x,t)$, the second term may be written as

$$\frac{\gamma'}{p}\int_{\mathbb{R}}\phi_x(x+\gamma,t)(D^\beta+\eta)S'(x)\,\mathrm{d}x=\frac{\gamma'}{p}\left[\|(D^\beta+\eta)^{1/2}S'\|^2-\int_{\mathbb{R}}a(x,t)(D^\beta+\eta)S''(x)\,\mathrm{d}x\right]$$
$$=\frac{\gamma'}{p}[\|(D^\beta+\eta)^{1/2}S'\|^2-A(t)], \tag{4.8}$$

where $A(t)=\mathrm{O}(\epsilon)$ as $\epsilon\downarrow 0$, uniformly in time, provided that the value of ϵ in (4.1) is smaller than the ϵ_0 of lemma 4.2. It is then seen that (4.8) is non-zero for such values of ϵ.

Now, from relation (4.6) together with $u_z(z,t)=\mu(t)^{3/2}\phi_\omega(\omega,t)$ where $\omega=\mu z$ and the assumption that S is even, the third term may be calculated as follows:

$$\frac{\mu'}{p\mu^{1/2}}\int_{\mathbb{R}} zu_z(z,t)(D^\beta+\eta)S'(\mu z-\gamma)\,\mathrm{d}z$$
$$=\frac{\mu'}{p\mu}\int_{\mathbb{R}}\omega\phi_\omega(\omega,t)(D^\beta+\eta)S'(\omega-\gamma)\,\mathrm{d}\omega$$
$$=\frac{\mu'}{p\mu}\int_{\mathbb{R}}(x+\gamma)[S(x)+a(x,t)]_x(D^\beta+\eta)S'(x)\,\mathrm{d}x$$
$$=\frac{\gamma\mu'}{p\mu}\left[\|(D^\beta+\eta)^{1/2}S'\|^2-\int_{\mathbb{R}}a(x,t)(D^\beta+\eta)S''(x)\,\mathrm{d}x\right]$$
$$-\frac{\mu'}{p\mu}\int_{\mathbb{R}}xa(x,t)(D^\beta+\eta)S''(x)\,\mathrm{d}x$$
$$=\frac{\gamma\mu'}{p\mu}\left[\|(D^\beta+\eta)^{1/2}S'\|^2-A(t)\right]+\frac{\mu'}{p\mu}B(t). \tag{4.9}$$

The Cauchy–Schwarz inequality and (4.1) imply that

$$|B(t)|\leqq M\epsilon,$$

where M depends only on S.

The final term in (4.7) may be analysed in a like manner as follows. From (0.3), (0.5) and the condition $p=2\beta+1$, it is deduced that

$$\frac{\mu^{1/2}}{p}\int_{\mathbb{R}}u_t(z,t)(D^\beta+\eta)S'(\mu z-\gamma)\,\mathrm{d}z$$
$$=\frac{\mu^{1+\beta}}{p}\int_{\mathbb{R}}[D^\beta\phi(x+\gamma,t)-(\phi(x+\gamma,t))^p]_x(D^\beta+\eta)S'(x)\,\mathrm{d}x$$
$$=-\frac{\mu^{1+\beta}}{p}\left[\eta\|(D^\beta+\eta)^{1/2}S'\|^2+\int_{\mathbb{R}}aD^\beta(D^\beta+\eta)S''\right.$$
$$\left.-([S+a]^p-S^p)(D^\beta+\eta)S''\,\mathrm{d}x\right]$$
$$=-\eta\frac{\mu^{1+\beta}}{p}\|(D^\beta+\eta)^{1/2}S'\|^2-\frac{\mu^{1+\beta}}{p}C(t), \tag{4.10}$$

where

$$C(t) = \int_{\mathbb{R}} a[D^{\beta}(D^{\beta}+\eta)S'' - g(a,S)(D^{\beta}+\eta)S'']\,dx$$

and

$$g(a,S) = \begin{cases} \dfrac{(S+a)^p - S^p}{a}, & a \neq 0 \\ pS^{p-1}, & a = 0 \end{cases}$$
$$= \int_0^1 f'(S+ra)\,dr,$$

where $f(r) = r^p$. According to theorem 3.2 and the Sobolev embedding theorem, a is bounded uniformly in x and t and hence $|g(a,S)|_\infty$ is bounded, independently of t. Thus the Cauchy–Schwarz inequality implies that

$$C(t) \leqq \|a\|(\|D^{\beta}(D^{\beta}+\eta)S''\| + |g(a,S)|_\infty\|(D^{\beta}+\eta)S''\|)$$
$$\leqq M\epsilon,$$

where M depends only on S.

Substituting formulae (4.8)–(4.10) into (4.7) leads to the relation

$$\left(\gamma'(t) - \frac{\mu'(t)}{\mu(t)}\gamma(t)\right) = \eta\mu(t)^{1+\beta} + \mu(t)^{1+\beta}\frac{C(t)+\eta A(t)}{\|(D^{\beta}+\eta)^{1/2}S'\|^2 - A(t)}$$
$$+\frac{\mu'(t)}{\mu(t)}\frac{B(t)}{A(t) - \|(D^{\beta}+\eta)^{1/2}S'\|^2}$$
$$= \eta\mu(t)^{1+\beta} + \mu(t)^{1+\beta}D(t) + \frac{\mu'(t)}{\mu(t)}E(t), \tag{4.11}$$

where, for ϵ sufficiently small, $D(t) = O(\epsilon)$ and $E(t) = O(\epsilon)$. Part (ii) of the theorem follows from (4.11) and the formula for the solution of a first-order ordinary differential equation. This completes the proof. □

5. Stability theory and behaviour of the stability parameters for the NLS in the critical case $p = 4/n$

The aim of this section is to establish a stability result similar to theorem 3.2 for solutions of the nonlinear Schrödinger equation (0.6) in the critical case. Henceforth, it is assumed in (0.6) that $p = 4/n$ and $u_0 \in H^1(\mathbb{R}^n)$. It is further assumed that either

(a) $n = 1$ and $H(u_0) < 0$, or

(b) $n \geqq 2$, $H(u_0) < 0$, and u_0 is radially symmetric, or (5.1)

(c) $n \geqq 1$, $|x|u_0 \in L_2(\mathbb{R}^n)$ and $H(u_0) \leqq 0$.

In these cases it is known that

$$\lim_{t\uparrow t^*} \|\nabla u(\cdot,t)\| = +\infty$$

holds for the corresponding solution u of (0.6) (see [N, OT]).

Just as for the KdV-type equations, in [LBSK], following Weinstein [W3], consideration was given to the functions

$$\begin{cases} \psi(x,t) = \mu(t)^{-n/2}u\left(\frac{x}{\mu(t)},t\right), \text{ with} \\ \mu(t) = \frac{\|\nabla u(\cdot,t)\|}{\|\nabla G\|}, \quad 0 \leqq t < t^*, \quad \text{and} \quad \mu(0) = 1. \end{cases} \tag{5.2}$$

Here t^* is the maximal time of existence of the solution of (0.6) under consideration. Note that, unless u is the zero solution, $0 < \mu(t) < \infty$ for $0 < t < t^*$. It is easy to check that the function ψ verifies the identities

$$\begin{aligned} &\text{(i) } \|\psi(\cdot,t)\| = \|u(\cdot,t)\| = \|u_0\|,\\ &\text{(ii) } \|\nabla\psi(\cdot,t)\| = \|\nabla G\|,\\ &\text{(iii) } H(\psi(\cdot,t)) = \|\nabla\psi(\cdot,t)\|^2 - \frac{2}{p+2}|\psi(\cdot,t)|_{p+2}^{p+2} = \frac{1}{\mu^2(t)}H(u(\cdot,t)). \end{aligned} \tag{5.3}$$

Since the stability considered here is with respect to form, which is to say, up to translation in space and phase, it is useful, as with KdV-type equations, to introduce the orbit

$$\mathcal{O}(G_\lambda) \equiv \{g \mid g(x) = G_\lambda(x+\alpha_0)\mathrm{e}^{\mathrm{i}\alpha_1}, (\alpha_0, \alpha_1) \in \mathbb{R}^n \times [0, 2\pi)\}$$

of G_λ. An induced metric on the space $H^1(\mathbb{R}^n)$ factored by the closed subset $\mathcal{O}(G_\lambda)$ provides a pseudo-metric on $H^1(\mathbb{R}^n)$ (cf [Be1, Bo, CL, W4]), namely

$$\begin{aligned} \rho_\lambda(\psi(\cdot,t), G_\lambda)^2 \equiv \inf_{\substack{\alpha_0 \in \mathbb{R}^n\\ \alpha_1 \in [0,2\pi)}} &\{\|\nabla\psi(\cdot+\alpha_0,t)\mathrm{e}^{\mathrm{i}\alpha_1} - \nabla G(\cdot)\|^2\\ &+\lambda\|\psi(\cdot+\alpha_0,t)\mathrm{e}^{i\alpha_1} - G(\cdot)\|^2\}. \end{aligned} \tag{5.4}$$

Define the set $\mathcal{S}$ to be

$$\mathcal{S} = \{u_0 \mid u_0 \in H^1(\mathbb{R}^n) \text{ and one of the conditions in (5.1) holds for } u_0\}. \tag{5.5}$$

Theorem 5.1. *Let $p = 4/n$, $\lambda > 0$ and let $G = G_\lambda$ be a ground-state solution of (0.8). For any $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that if $u_0 \in \mathcal{S}$ with $\rho_\lambda(u_0, G) < \delta$ and u is the solution of (0.6) corresponding to u_0 whose blow-up time is t^*, say, then $u \in C([0,t^*); H^1(\mathbb{R}^n))$ and*

$$\begin{aligned} \inf_{\substack{\alpha_0 \in \mathbb{R}^n\\ \alpha_1 \in [0,2\pi)}} &\{\lambda\|u(\cdot,t) - \mu(t)^{n/2}G(\mu(t)(\cdot+\alpha_0))\mathrm{e}^{-\mathrm{i}\alpha_1}\|^2\\ &+\mu(t)^{-2}\|\nabla u(\cdot,t) - \mu(t)^{n/2}\nabla G(\mu(t)(\cdot+\alpha_0))\mathrm{e}^{-\mathrm{i}\alpha_1}\|^2\} < \epsilon \end{aligned} \tag{5.6}$$

for all $t \in [0, t^*)$, where $\mu(t)$ is as in (5.2).

Proof. The proof may be made by the same arguments as those appearing in the proof of theorem 3.2. For more details, see [ABLS]. □

In the proof of theorem 5.1, it is actually shown that there is a choice of $\alpha_0 = \alpha_0(t)$ and $\alpha_1 = \alpha_1(t)$ for which

$$\begin{aligned} \rho_\lambda(\psi(t), G) &= (\|\nabla\psi(\cdot+\alpha_0,t)\mathrm{e}^{\mathrm{i}\alpha_1} - \nabla G(\cdot)\|^2 + \lambda\|\psi(\cdot+\alpha_0,t)\mathrm{e}^{\mathrm{i}\alpha_1} - G(\cdot)\|^2)^{1/2}\\ &= \rho_\lambda(\psi(t), G) \leqq \epsilon \end{aligned} \tag{5.7}$$

for all $t < t^*$, and that a choice of α_0 and α_1 for which (5.7) holds may be determined via the orthogonality relations (see theorem 2.2 in [ABLS])

$$\begin{aligned} &\mathrm{Im}\int_{\mathbb{R}^n} G^{p+1}(x)[\mathrm{e}^{\mathrm{i}\alpha_1(t)}\psi(x+\alpha_0(t),t)]\,\mathrm{d}x = 0,\\ &\mathrm{Re}\int_{\mathbb{R}^n} G^{p}(x)G_{x_i}(x)[\mathrm{e}^{\mathrm{i}\alpha_1(t)}\psi(x+\alpha_0(t),t)]\,\mathrm{d}x = 0 \end{aligned} \tag{5.8}$$

for $i = 1, \ldots, n$ and $\psi(x,t) = \mu(t)^{-n/2}u(\mu(t)^{-1}x, t)$.

Just as for the KdV-type equations, the function μ defined in (5.2) is easily determined to be of class C^1 on $[0, t^*)$ provided the initial data are smooth enough (see lemma 4.1). Therefore, an application of the implicit-function theorem combined with the relations in (5.8) allows one to establish the existence of unique C^1-functions $\alpha_0(t)$ and $\alpha_1(t)$ satisfying (5.8) for all $t \in [0, t^*)$.

The next theorem is the main result concerning the behaviour of the parameters α_0 and α_1 for the NLS-equation.

Theorem 5.2. *Let $p = 4/n$ and $G = G_\lambda$ be a ground-state solution of (0.8). For any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that if $u_0 \in \mathcal{S}$ with $\|u_0 - G\|_1 < \delta$ and t^* is the existence time for the solution emanating from u_0, then there are C^1-mappings $\alpha_0 : [0, t^*) \to \mathbb{R}^n$ and $\alpha_1 : [0, t^*) \to \mathbb{R}$ such that*

(i) $\|\psi(\cdot + \alpha_0(t), t)\mathrm{e}^{\mathrm{i}\alpha_1(t)} - G(\cdot)\|_1 \leqq \epsilon$ for $t \in [0, t^*)$, and

(ii) for $\alpha_0(t) = (\alpha_{0,1}(t), \ldots, \alpha_{0,n}(t))$ and $t \in [0, t^*)$,

$$\left|\alpha_1(t) + \lambda \int_0^t \mu^2(s)\,\mathrm{d}s\right| \leqq C\epsilon\left[\int_0^t \mu^2(s)\,\mathrm{d}s + \int_0^t \frac{|\mu'(s)|}{\mu(s)}\,\mathrm{d}s\right],$$

$$|\alpha_{0,i}(t)| \leqq C\epsilon\mu(t)\left[\int_0^t \mu(s)\,\mathrm{d}s + \int_0^t \frac{|\mu'(s)|}{\mu^2(s)}\,\mathrm{d}s\right], \qquad i = 1, \ldots, n,$$

where C depends only on G.

Proof. To show part (i), use is made of theorem 5.1. Differentiating the first equation in (5.8) with respect to t and using the definition of ψ in (5.2) and the fact that $p = 4/n$ leads to the equation

$$\begin{aligned}\mathrm{Im}\Bigg\{\int_{\mathbb{R}^n} G^{p+1}(x)\mathrm{e}^{\mathrm{i}\alpha_1(t)}\Bigg[&\mathrm{i}\alpha_1'(t)\psi(x+\alpha_0(t),t) - \frac{n\mu'(t)}{2\mu(t)}\psi(x+\alpha_0(t),t)\\ &+\mu(t)\nabla_x\psi(x+\alpha_0(t),t)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x+\alpha_0(t)}{\mu(t)}\right) + \mathrm{i}\mu(t)^2\Delta_x\psi(x+\alpha_0(t),t)\\ &+\mathrm{i}\mu(t)^2|\psi(x+\alpha_0(t),t)|^p\psi(x+\alpha_0(t),t)\Bigg]\mathrm{d}x\Bigg\} = 0.\end{aligned} \tag{5.9}$$

Writing ψ in the form

$$\psi(x+\alpha_0,t)\mathrm{e}^{\mathrm{i}\alpha_1} = G(x) + a(x,t) + \mathrm{i}b(x,t),$$

it follows readily from (0.8), (5.8) and (5.9) that

$$\begin{aligned}\alpha_1'(t)\int_{\mathbb{R}^n} G^{p+2}(x)\,\mathrm{d}x &+ \alpha_1'(t)\int_{\mathbb{R}^n} G^{p+1}(x)a(x,t)\,\mathrm{d}x + \frac{\mu'(t)}{\mu(t)}\int_{\mathbb{R}^n} x\cdot\nabla(G^{p+1})(x)b(x,t)\,\mathrm{d}x\\ &-\sum_{j=1}^n\left[\alpha_{0,j}'(t) - \frac{\mu'(t)}{\mu(t)}\alpha_{0,j}(t)\right]\int_{\mathbb{R}^n} G_{x_j}^{p+1}(x)b(x,t)\,\mathrm{d}x\\ &= -\lambda\mu^2(t)\int_{\mathbb{R}^n} G^{p+2}(x)\,\mathrm{d}x - \mu^2(t)\int_{\mathbb{R}^n} a(x,t)\Delta G^{p+1}(x)\,\mathrm{d}x\\ &\quad-\mu^2(t)\int_{\mathbb{R}^n} G^{p+1}[|G+a+\mathrm{i}b|^p(G+a) - G^{p+1}]\,\mathrm{d}x.\end{aligned} \tag{5.10}$$

Similarly, there obtains from the second equation in (5.8) that

$$\begin{aligned}-\alpha_1'(t)\int_{\mathbb{R}^n} G^p(x)G_{x_i}(x)b(x,t)\,\mathrm{d}x &+ \frac{\mu'(t)}{\mu(t)}\int_{\mathbb{R}^n} x\cdot\nabla(G^pG_{x_i})(x)a(x,t)\,\mathrm{d}x\\ &-\sum_{j=1}^n\left[\alpha_{0,j}'(t) - \frac{\mu'(t)}{\mu(t)}\alpha_{0,j}(t)\right]\int_{\mathbb{R}^n} G^p(x)(G_{x_i})_{x_j}(x)a(x,t)\,\mathrm{d}x\end{aligned}$$

$$+\sum_{j=1}^{n}\left[\alpha_{0,j}'(t)-\frac{\mu'(t)}{\mu(t)}\alpha_{0,j}(t)\right]\int_{\mathbb{R}^n} G^p(x)G_{x_i}(x)G_{x_j}(x)\,\mathrm{d}x$$
$$+\frac{\mu'(t)}{\mu(t)}\int_{\mathbb{R}^n}\left[x\cdot\nabla(G^pG_{x_i})(x)G(x)+G^{p+1}(x)G_{x_i}(x)\right]\mathrm{d}x$$
$$=\mu(t)^2\int_{\mathbb{R}^n} b(x,t)\Delta(G^pG_{x_i})(x)\,\mathrm{d}x+\mu(t)^2\int_{\mathbb{R}^n} G^pG_{x_i}|G+a+\mathrm{i}b|^p\,\mathrm{d}x. \tag{5.11}$$

Because G is spherically symmetric, it follows that the fourth and fifth terms on the left-hand side of (5.11) are zero. Using the notation

$$
\begin{aligned}
g_{00} &= \int_{\mathbb{R}^n} G^{p+2}(x)\,\mathrm{d}x, \qquad g_{ii}=\int_{\mathbb{R}^n} G^p(x)(G_{x_i})^2(x)\,\mathrm{d}x,\\
A_{00}(t) &= \int_{\mathbb{R}^n} a(x,t)G^{p+1}(x)\,\mathrm{d}x,\\
A_{0j}(t) &= -\int_{\mathbb{R}^n} b(x,t)(G^{p+1})_{x_j}(x)\,\mathrm{d}x,\\
A_{i0}(t) &= -\int_{\mathbb{R}^n} b(x,t)G^p(x)G_{x_i}(x)\,\mathrm{d}x,\\
A_{ij}(t) &= -\int_{\mathbb{R}^n} a(x,t)(G^pG_{x_i})_{x_j}(x)\,\mathrm{d}x,\\
B_0(t) &= -\int_{\mathbb{R}^n} x\cdot\nabla(G^{p+1})(x)b(x,t)\,\mathrm{d}x,\\
B_i(t) &= -\int_{\mathbb{R}^n} x\cdot\nabla(G^pG_{x_i})(x)a(x,t)\,\mathrm{d}x,\\
C_0(t) &= -\int_{\mathbb{R}^n} a(x,t)\Delta G^{p+1}(x)-\int_{\mathbb{R}^n} G^{p+1}\left[|G+a(x,t)+\mathrm{i}b(x,t)|^p(G+a)-G^{p+1}\right],\\
C_i(t) &= \int_{\mathbb{R}^n} b(x,t)\Delta(G^pG_{x_i})(x)\,\mathrm{d}x+\int_{\mathbb{R}^n} G^pG_{x_i}|G+a(x,t)+\mathrm{i}b(x,t)|^p\,b(x,t)\,\mathrm{d}x
\end{aligned}
\tag{5.12}
$$

for $1\leqq i,\,j\leqq n$, the $n+1$ equations appearing in (5.10) and (5.11) may be written in the form

$$
\begin{pmatrix}
g_{00}+A_{00}(t) & A_{01}(t) & \dots & A_{0j}(t)\\
A_{10}(t) & g_{11}+A_{11}(t) & \dots & A_{1n}(t)\\
\vdots & \vdots & \vdots & \vdots\\
A_{n0}(t) & A_{n1}(t) & \dots & g_{nn}+A_{nn}(t)
\end{pmatrix}\cdot
\begin{pmatrix}
\alpha_1'(t)\\
\alpha_{0,1}'(t)-\dfrac{\mu'(t)}{\mu(t)}\alpha_{0,1}(t)\\
\vdots\\
\alpha_{0,n}'(t)-\dfrac{\mu'(t)}{\mu(t)}\alpha_{0,n}(t)
\end{pmatrix}
$$
$$
=-\lambda\mu(t)^2\begin{pmatrix} g_{00}\\ 0\\ \vdots\\ 0\end{pmatrix}+\frac{\mu'(t)}{\mu(t)}\begin{pmatrix} B_0(t)\\ B_1(t)\\ \vdots\\ B_n(t)\end{pmatrix}+\mu(t)^2\begin{pmatrix} C_0(t)\\ C_1(t)\\ \vdots\\ C_n(t)\end{pmatrix}. \tag{5.13}
$$

Because the ground state is stable in the sense provided by theorem 5.1, the inequality (5.7) holds provided that $\|u_0-G\|_1$ is sufficiently small. Therefore, it follows that $\|a(\cdot,t)\|_1\leqq\epsilon$ and $\|b(\cdot,t)\|_1\leqq\epsilon$ for all t. Moreover, as $\epsilon\downarrow 0$, there obtains from (5.12) the relations

$$A_{ij}(t)=\mathrm{O}(\epsilon),\qquad g_{ii}+A_{ij}(t)>0,\qquad 0\leqq i,j\leqq n, \tag{5.14}$$
$$B_i(t)=\mathrm{O}(\epsilon),\qquad C_i(t)=\mathrm{O}(\epsilon),\qquad 0\leqq i\leqq n, \tag{5.15}$$

uniformly in t. Denoting by $A_t = (a_{i,j}(t))$ the $(n+1) \times (n+1)$ matrix on the left-hand side of (5.13), it follows from (5.14) that $\det A_t \neq 0$ for all t. Thus, A_t is invertible and (5.13) can therefore be written in the form

$$\begin{pmatrix} \alpha_1'(t) \\ \alpha_{0,1}'(t) - \dfrac{\mu'(t)}{\mu(t)}\alpha_{0,1}(t) \\ \vdots \\ \alpha_{0,n}'(t) - \dfrac{\mu'(t)}{\mu(t)}\alpha_{0,n}(t) \end{pmatrix} = -\frac{\lambda g_{00}\,\mu(t)^2}{\det A_t} \begin{pmatrix} \det A_t(1|1) \\ -\det A_t(1|2) \\ \vdots \\ (-1)^{i+1}\det A_t(1|i) \\ \vdots \\ (-1)^n \det A_t(1|n+1) \end{pmatrix} + \frac{\mu'(t)}{\mu(t)} \begin{pmatrix} \tilde{B}_0(t) \\ \tilde{B}_1(t) \\ \vdots \\ \tilde{B}_n(t) \end{pmatrix} + \mu(t)^2 \begin{pmatrix} \tilde{C}_0(t) \\ \tilde{C}_1(t) \\ \vdots \\ \tilde{C}_n(t) \end{pmatrix}, \tag{5.16}$$

where $\det A_t(1|i)$ is the determinant of the $n \times n$ matrix that is obtained by omitting the first row and the ith column of A_t. Relations (5.14) and (5.15) yield that

$$\begin{aligned} &\det A_t(1|i) = O(\epsilon), \qquad i = 2, 3, \ldots, n+1, \\ &\tilde{B}_i(t) = O(\epsilon), \qquad \tilde{C}_i(t) = O(\epsilon), \qquad i = 0, 1, \ldots, n, \end{aligned} \tag{5.17}$$

as $\epsilon \to 0$, uniformly in t. Now since

$$\begin{aligned} -\lambda g_{00}\,\mu(t)^2 \frac{\det A_t(1|1)}{\det A_t} &= -\lambda\mu(t)^2 + \mu(t)^2 \frac{\lambda \det A_t - \lambda g_{00} \det A_t(1|1)}{\det A_t} \\ &\equiv -\lambda\mu(t)^2 + \mu(t)^2 \tilde{A}_0(t) \end{aligned}$$

with $\tilde{A}_0(t) = O(\epsilon)$, uniformly in t, (5.16) can finally be written in the form

$$\begin{pmatrix} \alpha_1'(t) \\ \alpha_{0,1}'(t) - \dfrac{\mu'(t)}{\mu(t)}\alpha_{0,1}(t) \\ \vdots \\ \alpha_{0,n}'(t) - \dfrac{\mu'(t)}{\mu(t)}\alpha_{0,n}(t) \end{pmatrix} = -\lambda\mu(t)^2 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \mu(t)^2 \begin{pmatrix} \tilde{D}_0(t) \\ \tilde{D}_1(t) \\ \vdots \\ \tilde{D}_n(t) \end{pmatrix} + \frac{\mu'(t)}{\mu(t)} \begin{pmatrix} \tilde{B}_0(t) \\ \tilde{B}_1(t) \\ \vdots \\ \tilde{B}_n(t) \end{pmatrix} \tag{5.18}$$

with $\tilde{D}_i(t) = O(\epsilon)$ as $\epsilon \to 0$, $i = 0, 1, \ldots, n$. Thus, part (ii) of the theorem follows from (5.18) and the formula for the solution of a first-order ordinary differential equation. This completes the proof. □

6. Conclusions

Our investigation has been concerned with solitary-wave solutions of a class of generalized KdV-type equations and the focusing nonlinear Schrödinger equation. Interest has been given to the critical cases where global existence of solutions corresponding to large initial data just fails, apparently because the dispersion is not quite strong enough to overcome nonlinear effects in the face of large values of the dependent variable. In these cases, the solitary waves are known at least in some cases to lose stability (e.g. for the focusing NLS-equation and for the generalized KdV-equation).

Making use of a natural scaling under which the family of solitary waves or ground states is invariant, an analysis is made of the outcome of 'negative energy' perturbations of a given solitary wave or ground state, S, say. Were S to be stable, we would infer that a solution u emanating from suitably close initial data stays forever close to S, modulo translations of space (and time). What is demonstrated instead is that u remains forever close in the energy norm to the branch $\mathcal{B}$ of solitary-wave or ground-state orbits obtained by applying translation in space (and time) and the aforementioned scaling transformation to S. The analysis posits a solution form $u = G + a$, where G lies on $\mathcal{B}$ and a is an associated remainder. It is shown that by choosing the element $G = G(t) \in \mathcal{B}$ appropriately, a can be controlled in norm, remaining always small if it begins that way. It is shown that this selection can be made in a natural and smooth way.

While interesting and suggestive, these results should be viewed as preliminary because the choice of G is made implicitly based on the gross behaviour of u rather than in some explicit way.

In particular, our results show that if the solution corresponding to an appropriately perturbed solitary wave is to become infinite in finite time, then it must do so by essentially running along the solitary-wave or ground-state branch. This result is in the same direction as the early weak-convergence results of Weinstein [W3] and the recent weak stability of a blow-up profile obtained by Martel and Merle [MM1].

Acknowledgments

The authors are grateful for support during the period this research was being undertaken, from CNPq (Brazil) (grant number 300654/96-0, JA and number 300964/88-9, MS), NSF (USA) and the Texas Institute for Computational and Applied Mathematics at the University of Texas at Austin.

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