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COMMUNICATIONS IN PARTIAL DIFFERENTIAL EQUATIONS
Vol. 28, Nos. 7 & 8, pp. 1391–1436, 2003

A Nonhomogeneous Boundary-Value Problem for the Korteweg–de Vries Equation Posed on a Finite Domain

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ABSTRACT

Studied here is an initial- and boundary-value problem for the Korteweg–de Vries equation posed on a bounded interval with nonhomogeneous boundary conditions. This particular problem arises naturally in certain circumstances when the equation is used as a model for waves and a numerical scheme is needed. It is shown here that this initial-boundary-value problem is globally well-posed in the L_2-based Sobolev space $H^s(0,1)$ for any $s \geq 0$. In addition, the mapping that associates to appropriate initial- and boundary-data the corresponding solution is shown to be analytic as a function between appropriate Banach spaces.

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DOI: 10.1081/PDE-120024373
0360-5302 (Print); 1532-4133 (Online)

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1. INTRODUCTION

This article is concerned with the Korteweg–de Vries equation (KdV-equation henceforth)

$$u_t + u_x + uu_x + u_{xxx} = 0 \tag{1.1}$$

posed as an initial- and boundary-value problem. In the conception pursued here, one asks for a solution of (1.1) for $(x, t) \in \Omega \times R^+$ where Ω is an interval in R, subject to an initial condition

$$u(x, 0) = \phi(x), \qquad \text{for } x \in \Omega, \tag{1.2}$$

and appropriate boundary conditions at the ends of the interval. In applications to physical problems, the independent variable x is often a coordinate representing position in the medium of propagation, t is proportional to elapsed time, and $u(x,t)$ is a velocity or an amplitude at the point x at time t. Here and below, if $f = f(x, t)$ is a function of x and t, then f_x is shorthand for $\partial_x f$ and similarly $f_t = \partial_t f$. When $\Omega = R$, the entire real line, this is the classical problem whose study was initiated by Gardner et al. (1974) and Lax (1968) in the middle 1960's by way of the inverse scattering theory and by Sjöberg (1970) and Temam (1969) in the late 1960's using the then new methods for the analysis of nonlinear partial differential equations, and by many others since. As will be described presently, this pure initial-value problem continues to attract attention and its mathematical theory has proved to be subtle.

Another configuration that arises naturally in making predictions of waves is to take $\Omega = R^+ = \{x \mid x > 0\}$ and specify $u(0, t)$ for $t > 0$ and $u(x, 0) = 0$, say, for $x > 0$. This corresponds to a known wavetrain generated at one end and propagating into a quiescent region of the medium of propagation. If $u(0, t)$ is of small amplitude (small compared to one in this scaling) and has primarily low frequency content, then the waves generated by the boundary disturbance will satisfy the assumptions underlying the derivation of the KdV-equation. The semi-infinite aspect of the domain mirrors the fact that the KdV-equation written in the form (1.1) is an approximation only for waves moving in the direction of increasing values of x. Once the incoming waves encounter a boundary, reflection will come into play, and the KdV-equation is no longer expected provide an accurate rendition of reality. The problem of imposition of boundary data at the right-hand end of the domain does not arise when the KdV-equation is posed on R^+ with zero initial data, say, and input from the left-hand boundary. Indeed, the zero boundary conditions at $x = +\infty$ implicit in the formulation may be imposed by function-class restrictions (e.g., $u(\cdot, t) \in L_2(R^+)$ for all relevant values of t). This initial-boundary-value problem fits well with laboratory studies wherein waves are generated by a wavemaker at the left-hand end and these are monitored as they propagate down the channel, with the experiment ceasing as soon as the waves reach the other end of the channel and reflected components intrude (see Bona et al., 1981; Hammack, 1973; Hammack and Segur, 1974; Zabusky and Galvin, 1971). Similarly, when modeling surface waves arriving from deep water into near-shore zones or large-scale internal waves propagating from the deep ocean onto the continental shelf, reflection may sometimes be safely ignored and one encounters a variable-coefficient version of Eq. (1.1) posed on $R^+ \times (0, T)$ with a time-dependent Dirichlet boundary condition at $x = 0$ (see Boczar-Karakiewicz et al., 1991; Boczar-Karakiewicz et al., (submitted) for example). The quarter-plane problem just outlined has been considered recently by the present authors (see Bona et al., 2001) and by several others (see the references in the last-quoted article) and there is a satisfactory theory of well-posedness for this problem.

However, if one is interested in implementing a numerical scheme to approximate solutions of the quarter-plane problem, there arises the issue of cutting off the spatial domain. Once this is done, two more boundary conditions are needed to specify the solution completely. Because the model cannot countenance waves moving to the left, it is usual, as suggested above, to apply the model only on a time scale T short enough that significant wave motion has not reached the right-hand boundary. If the right-hand boundary is located at $x = r$, say, then it is therefore natural in regard to the physical problem to impose $u(r, t) = u_x(r, t) = 0$ for $0 \le t \le T$ to obtain a complete set of boundary conditions. Of course, one might also imagine imposing $u_x(0, t)$ rather than $u_x(r, t)$, but in practical situations, one does not normally have information that warrants the imposition of a second boundary condition at the left-hand, or wavemaker end of the medium of propagation. As far as mathematical analysis is concerned, it makes relatively little difference whether or not the boundary conditions are homogeneous. In consequence, consideration is given here to (1.1)–(1.2) completed by the general nonhomogeneous boundary conditions

$$u(0, t) = h_1(t), \qquad u(r, t) = h_2(t), \qquad u_x(r, t) = h_3(t), \quad \text{for } t \ge 0, \tag{1.3}$$

where the initial value ϕ and the boundary data $h_j, j = 1, 2, 3$ are given functions. The principal concern of the present essay is the *well-posedness* of the initial-boundary-value problem (IBVP henceforth) (1.1)–(1.3). That is, we aim to establish existence, uniqueness, and persistence properties of solutions corresponding to reasonable auxiliary data, together with continuous dependence of the solution upon the auxiliary data. A brief review of the mathematical theory currently available is now presented. The pure initial-value problem (IVP) for (1.1) and its relatives where the initial datum ϕ is specified on the entire real axis R has received a lot of attention in the last three decades, both in case ϕ lies in an $L_2(R)$-based Sobolev space and in case ϕ is periodic (see Bona and Scott, 1976; Bona and Smith, 1978; Bourgain, 1993a; Bourgain, 1993b; Constantin and Saut, 1988; Hammack, 1973; Hammack and Segur, 1974; Kato, 1975; Kato, 1979; Kato, 1983; Kenig et al., 1991a; Kenig et al., 1991b; Kenig et al., 1993a; Kenig et al., 1993b; Kenig et al., 1996; Lax, 1968; Miura, 1976; Russell and Zhang, 1993; Russell and Zhang, 1995; Russell and Zhang, 1996; Saut and Temam, 1976; Sun, 1996; Temam, 1969; Zhang, 1995a, 1995b, 1995c). In particular, various smoothing properties have been discovered for solutions of the (1.1) when posed on the whole line R or on a periodic domain S (e.g., the unit circle in the plane). It is those smoothing properties that enable one to prove that the IVP (1.1)–(1.2) is well-posed in the space $H^s(R)$ for $s > -3/4$ when posed on R and is well-posed in the space $H^s(S)$ for $s > -1/2$ when posed on the periodic domain S (Bourgain, 1993a;

Bourgain, 1993b; Kenig et al., 1993b; Kenig et al., 1996). By contrast, the study of the KdV-equation posed on the half line R^+ or on a finite interval has received much less attention and the results available thus far appear to be not as sharp as those for the IVP on R. For the initial-boundary-value problem (IBVP henceforth) for the KdV-equation posed on the half line R^+,

$$\left.\begin{aligned} &u_t + u_x + uu_x + u_{xxx} = 0, \qquad u(x,0) = \phi(x), \\ &u(0,t) = h(t), \end{aligned}\right\} \tag{1.4}$$

for x, $t \in R^+$, we have provided a review in our recent article Bona et al., (2001) (see the earlier work of Bona and Dougalis, 1980; Bona and Scott, 1974; Bona and Winther, 1989; the related article of Benjamin et al., 1972, on the BBM equation, and the recent article by Colliander and Kenig, 2002). In Bona et al. (2001), we pointed out that for the linear problem obtained from (1.4) by omitting the quadratic term, there are smoothing properties similar to those established by Kenig et al. (1991b) for (1.1)–(1.2) posed on all of R. Consequently, we were able to show the IBVP (1.4) to be well-posed in the space $C([0,T]; H^s(R^+))$ for any $s > 3/4$ provided the data (ϕ, h) is drawn from $H^s(R^+) \times H^{(s+1)/3}(0,T)$, by applying the contraction-mapping principle. The corresponding solution map was shown to be analytic. In their recent work, Colliander and Kenig (2002) showed that (1.4) is well-posed for $s \geq 0$.

For the KdV-equation posed on a finite interval, Bubnov (1979, 1980) studied the general two-point boundary-value problem

$$\begin{cases} u_t + uu_x + u_{xxx} = f(x,t), \qquad u(x,0) = 0, \\ \alpha_1 u_{xx}(0,t) + \alpha_2 u_x(0,t) + \alpha_3 u(0,t) = 0, \\ \beta_1 u_{xx}(1,t) + \beta_2 u_x(1,t) + \beta_3 u(1,t) = 0, \\ \xi_1 u_x(1,t) + \xi_2 u(1,t) = 0, \end{cases} \tag{1.5}$$

posed on the interval $(0,1)$ (see also the related work Bona and Dougalis (1980) on the BBM-equation). Here, $\alpha_i, \beta_i, \xi_j \in R$, $i = 1,2,3$, $j = 1,2$ are real constants and assumptions are imposed so that the L_2−norm of the solutions of the linear version of (1.5) (obtained by dropping the nonlinear term uu_x) is decreasing. It was shown in Bubnov (1979) that for given $T > 0$ and $f \in H^1([0,T]; L_2(0,1))$, there exists a $T^* > 0$ depending on $\|f\|_{H^1([0,T]; L_2(0,1))}$ such that (1.5) admits a unique solution

$$u \in L_2([0,T^*]; H^3(0,1)), \quad u_t \in L_\infty([0,T^*]; L_2(0,1)) \cap L_2([0,T^*]; H^1(0,1)).$$

In Zhang (1994), Zhang considered boundary control of the KdV-equation posed on a finite interval $(0,1)$ with Dirichlet boundary conditions. A feedback control law was introduced to stabilize the system, leading to the initial-boundary-value problem

$$\left.\begin{aligned} &u_t + uu_x + u_{xxx} = 0, \qquad u(x,0) = \phi(x), \quad x \in (0,1), t \geq 0, \\ &u(0,t) = 0, \quad u(1,t) = 0, \qquad u_x(1,t) = \gamma u_x(0,t), \quad t \geq 0, \end{aligned}\right\} \tag{1.6}$$

with $0 \leq |\gamma| < 1$. Note that when $\gamma = 0$, the system (1.6) is (1.1)–(1.3) with homogeneous boundary conditions. It was shown in Zhang (1994) that Eq. (1.6) is globally well-posed in the space $H^{3k+1}(0, 1)$ for $k = 0, 1, \ldots$. In a recent article Colin and Ghidaglia (2001), the authors considered the following initial-boundary value problem

$$\left.\begin{array}{l} u_t + uu_x + u_{xxx} = 0, \qquad u(x, 0) = \phi(x), \quad x \in (0, 1), \quad t \geq 0, \\ u(0, t) = h_1(t), \quad u_x(1, t) = h_2(t), \quad u_{xx}(1, t) = h_3(t), \quad t \geq 0, \end{array}\right\}$$

and showed it to be locally well-posed in the space $H^1(0,1)$ with the initial data ϕ drawn from $H^1(0, 1)$ and the boundary data (h_1, h_2, h_3) taken from the product space $C^1[0, T) \times C^1[0, T) \times C^1[0, T)$. In addition, Rosier (1997) studied the control problem for the system

$$\left.\begin{array}{l} u_t + uu_x + u_{xxx} = 0, \qquad u(x, 0) = \phi(x), \quad x \in (0, 1), \quad t \geq 0, \\ u(0, t) = 0, \quad u(1, t) = 0 \quad u_x(1, t) = h(t), \quad t \geq 0, \end{array}\right\}$$

where the boundary function h is considered as a control input. Rosier showed the system is (locally) exactly controllable in the space $L_2(0,1)$. A similar problem was also considered by Zhang (1999) for the system (1.1)–(1.3) where the boundary value functions $h_j(t)$, $j = 1, 2, 3$ are all taken to be control inputs. This system is shown to be exactly controllable in the space $H^s(0, 1)$ for any $s \geq 0$ in a neighborhood of any smooth solution of the KdV-equation. (Exact controllability means, roughly, that for a given time $T > 0$ and a given pair of functions ϕ and ψ in the space $H^s(0, 1)$, there exist appropriate controls such that the corresponding system possesses a solution u which exactly equals ϕ at $t = 0$ and equals ψ at $t = T$. Put colloquially, given two states ϕ and ψ, there is a control h that will drive the system from ϕ to ψ in time T. Of course, there are obvious approximate controllability analogs of this concept. Readers who are interested in control issues are referred to the excellent review article of Russell (1978) for commentary on controllability and stabilizability of linear partial differential equations and to Russell and Zhang (1993, 1995, 1996) for theory of controllability and stabilizability of the KdV-equation.

In this article, the nonhomogeneous boundary-value problem (1.1)–(1.3) is considered. The aim is to establish the well-posedness of (1.1)–(1.3) in the space $H^s(0, r)$ when the initial data is drawn from $H^s(0, r)$ and the boundary data $(h_1, h_2, h_3,)$ lies in the product space $H^{s_1}(0,T) \times H^{s_2}(0,T) \times H^{s_3}(0,T)$ for some appropriate indices s_1, s_2, and s_3 that depend on s. As we will see later, the natural choices of s_1, s_2 and s_3 are $s_1 = s_2 = (s + 1)/3$ and $s_3 = s/3$. For convenience of writing, we take the underlying spatial domain $(0, r)$ to be $(0, 1)$ throughout. This is a restriction of no consequence as far as the theory is concerned. The well-posedness result for the IBVP (1.1)–(1.3) we establish in this article appears to require some compatibility conditions relating the initial datum $\phi(x)$ and the boundary data $h_j(t)$, $j = 1, 2, 3$. A simple computation shows that if u is a C^∞-smooth solution of the IBVP (1.1)–(1.3), then its initial data $u(x, 0) = \phi(x)$ and its boundary values $h_j(t)$, $j = 1, 2, 3$ must satisfy the following compatibility conditions:

$$\phi_k(0) = h_1^{(k)}(0), \quad \phi_k(1) = h_2^{(k)}(0), \quad \phi_k'(1) = h_3^{(k)}(0) \tag{1.7}$$

for $k = 0, 1, \ldots,$ where $h_j^{(k)}(t)$ is the kth order derivative of h_j and

$$\begin{cases} \phi_0(x) = \phi(x) \\ \phi_k(x) = -\left(\phi_{k-1}'''(x) + \phi_{k-1}'(x) + \sum_{j=0}^{k-1} (\phi_j(x)\phi_{k-j-1}(x))'\right) \end{cases} \tag{1.8}$$

for $k = 1, 2, \ldots.$ When the well-posedness of (1.1)–(1.3) is considered in the space $H^s(0,1)$ for some finite value $s \geq 0$, the following s-compatibility conditions thus arise naturally.

Definition 1.1. **(s-compatibility)** Let $T > 0$ and $s \geq 0$ be given. A four-tuple $(\phi, \vec{h}) = (\phi, h_1, h_2, h_3) \in H^s(0,1) \times H^{(s+1)/3}(0,T) \times H^{(s+1)/3}(0,T) \times H^{s/3}(0,T)$ is said to be s-compatible if

$$\phi_k(0) = h_1^{(k)}(0), \qquad \phi_k(1) = h_2^{(k)}(0) \tag{1.9}$$

holds for $k = 0, 1, \ldots, [s/3] - 1$ when $s - 3[s/3] \leq 1/2$, or (1.9) holds for $k = 0, 1, \ldots, [s/3]$ when $1/2 < s - 3[s/3] \leq 3/2$ and

$$\phi_k(0) = h_1^{(k)}(0), \qquad \phi_k(1) = h_2^{(k)}(0), \qquad \phi_k'(1) = h_3^{(k)}(0)$$

holds for $k = 0, 1, \ldots, [s/3]$ when $s - 3[s/3] > 3/2$. We adopt the convention that Eq. (1.9) is vacuous if $[s/3] - 1 < 0$.

With this compatibility notation, we may state the following two theorems, which comprise the main results of this article.

Theorem 1.2. (Local well-posedness) *Let $T > 0$ and $s \geq 0$ be given. Suppose that*

$$(\phi, \vec{h}) \in H^s(0,1) \times H^{(s+1)/3}(0,T) \times H^{(s+1)/3}(0,T) \times H^{s/3}(0,T)$$

is s-compatible. Then there exists a $T^* \in (0,T]$ depending only on the norm of $(\phi, \vec{h})$ in the space $H^s(0,1) \times H^{(s+1)/3}(0,T) \times H^{(s+1)/3}(0,T) \times H^{s/3}(0,T)$ such that Eqs. (1.1)–(1.3) admits a unique solution

$$u \in C([0,T^*]; H^s(0,1)) \cap L_2([0,T^*]; H^{s+1}(0,1)).$$

Moreover, the solution depends continuously in this latter space on variations of the auxiliary data in their respective function classes.

Theorem 1.3. (Global well-posedness) *Let $T > 0$ be arbitrary and $s \geq 0$. For any s-compatible*

$$(\phi, \vec{h}) \in H^s(0,1) \times H^{\mu_1(s)}(0,T) \times H^{\mu_1(s)}(0,T) \times H^{\mu_2(s)}(0,T),$$

where

$$\mu_1(s) = \begin{cases} \epsilon + (5s+9)/18 & \text{if } 0 \leq s < 3, \\ (s+1)/3 & \text{if } s \geq 3; \end{cases}$$

$$\mu_2(s) = \begin{cases} \epsilon + (5s+3)/18 & \text{if } 0 \leq s < 3, \\ s/3 & \text{if } s \geq 3 \end{cases}$$

and ϵ is any positive constant, the IBVP (1.1)–(1.3) *admits a unique solution*

$$u \in C([0,T]; H^s(0,1)) \cap L_2([0,T]; H^{s+1}(0,1)).$$

Moreover, the solution depends continuously on variations of the auxiliary data in their respective function classes.

Remark 1.4. The global well-posedness result presented in Theorem 1.3 requires slightly stronger regularity of the boundary values h_j, $j = 1,2,3$ in case $0 \le s < 3$ when compared with the local well-posedness result in Theorem 1.2. The same situation appears in the global well-posedness theory for the KdV-equation posed in a quarter plane in Bona et al. (2001).

The proof of our well-posedness result for (1.1)–(1.3) relies on the smoothing properties of the associated linear problem

$$\left.\begin{array}{l} u_t + u_x + u_{xxx} = f, \qquad u(x,0) = \phi(x), \\ u(0,t) = h_1(t), \qquad u(1,t) = h_2(t), \quad u_x(1,t) = h_3(t). \end{array}\right\} \tag{1.10}$$

There are three types of smoothing associated with solving (1.10); these are the smoothing effects of the solution u with respect to the forcing f, the initial value ϕ and the boundary data $h_j = 0$, $j = 1,2,3$, respectively. It will be demonstrated that

(i) For $\phi \in L_2(0,1)$ with $f = 0, h_j = 0$, $j = 1,2,3$, the solution u of (1.10) belongs to the space $C(R^+; L_2(0,1)) \cap L_2(R^+; H^1(0,1))$ and $u_x \in C([0,1], L_{2,t}(R^+))$;

(ii) For $f \in L_1(R^+; L_2(0,1))$ with $\phi = 0, h_j = 0$, $j = 1,2,3$, the solution u of Eq. (1.10) belongs to the space $C(R^+; L_2(0,1)) \cap L_2(R^+; H^1(0,1))$ and $u_x \in C([0,1], L_{2,t}(R^+))$;

(iii) For h_1, $h_2 \in H^{1/3}_{loc}(R^+)$, $h_3 \in L_{2,loc}(R^+)$ with $f = 0$ and $\phi = 0$, the solution u of (1.10) belongs to the space $C(R^+; L_2(0,1)) \cap L_{2,loc}(R^+; H^1(0,1))$ and $u_x \in C([0,1], L_{2,t}(R^+))$.

Various other related linear estimates will also be derived. Once these linear estimates are in hand, a local well-posedness result for (1.1)–(1.3) may be established using the contraction-mapping principle. The long-time results are obtained by finding global a priori estimates for smooth solutions of (1.1)–(1.3). It is interesting to note that while an L_2-estimate of solutions is relatively straightforward to establish, the global H^1- and H^2-bounds on solutions seem difficult to obtain by the usual energy-type methods. The approach used here is to obtain an L_2-estimate of the time derivative u_t of solutions, which, in turn, provides a global H^3-estimate. Nonlinear interpolation theory (Bona and Scott, 1976; Tartar, 1972) is then used to obtain the global H^s-estimates for $0 < s < 3$. Global a priori H^s-estimates for $s > 3$ are established by obtaining a priori bounds on $\partial_t^k u$ for $k = 1,2,\ldots,[s/3]$.

Because of its well-posedness, the IBVP (1.1)–(1.3) defines a continuous nonlinear map $\mathcal{K}_{s,T}$ from the space

$$X_{s,T} = H^s(0,1) \times H^{(s+1)/3}(0,T) \times H^{(s+1)/3}(0,T) \times H^{s/3}(0,T) \tag{1.11}$$

to the space $C([0,T];H^s(0,1))\cap L_2([0,T];H^{s+1}(0,1))$ for given $T>0$ and $s\geq 0$. It follows readily from the proof presented here that $\mathcal{K}_{s,T}$ is locally Lipschitz continuous. In fact, the map $\mathcal{K}_{s,T}$ is much smoother than just Lipschitz. According to the local existence theory, for a given $(\phi,\vec{h})\in X_{s,T}$ which is s-compatible, $\vec{h}=(h_1,h_2,h_3)$, there is a unique local solution u of (1.1)–(1.3). Of course, the existence time T^* for this solution need not be T. Let $\mathcal{D}(\mathcal{K}_{s,T})$ connote those elements of $X_{s,T}$ for which the solution exists on $[0,T)$. As will appear from our detailed theory, $\mathcal{D}(\mathcal{K}_{s,T})$ is an open neighborhood of the zero element in $X_{s,T}$ if $0\leq s\leq 7/2$. In this case, the mapping $\mathcal{K}_{s,T}$ is analytic from $\mathcal{D}(\mathcal{K}_{s,T})$ to the space $C([0,T];H^s(0,1))\cap L_2([0,T];H^{s+1}(0,1))$. That is to say, for given $(\phi,\vec{h})$ in $\mathcal{D}(\mathcal{K}_{s,T})$, there exists a $\delta>0$ such that for any $(\phi_1,\vec{h}_1)\in X_{s,T}$ with $(\phi,\vec{h})+(\phi_1,\vec{h}_1)\in\mathcal{D}(\mathcal{K}_{s,T})$ and $\|(\phi_1,\vec{h}_1)\|_{X_{s,T}}\leq\delta$, then $\mathcal{K}_{s,T}(\phi+\phi_1,\vec{h}+\vec{h}_1)$ has a Taylor series expansion which is uniformly convergent in the space $C([0,T];H^s(0,1))\cap L_2([0,T];H^{s+1}(0,1))$. Each term in the Taylor series is determined by the solution of a forced linear KdV-equation. Thus one obtains the attractive result that solutions of the nonlinear problem (1.1)–(1.3) can be obtained by solving an infinite sequence of linear problems. When $s>7/2$, because of the compatibility conditions, $\mathcal{D}(\mathcal{K}_{s,T})$ is no longer a neighborhood of zero in $X_{s,T}$. However, we can view solutions of the IBVP (1.1)–(1.3) as a special class of solutions of IBVP's for a system of nonlinear equations. Viewed this way, it may be shown that the IBVP for this nonlinear system is well-posed and the corresponding nonlinear map is again analytic.

The approach developed in this article can also be used to obtain similar results for the following general nonhomogeneous boundary-value problem for the KdV-equation:

$$\begin{cases} u_t+u_x+uu_x+u_{xxx}=f(x,t), \qquad u(x,0)=\phi(x), \text{ the}\\ \alpha_1u_{xx}(0,t)+\alpha_2u_x(0,t)+\alpha_3u(0,t)=h_1(t),\\ \beta_1u_{xx}(1,t)+\beta_2u_x(1,t)+\beta_3u(1,t)=h_2(t),\\ \xi_1u_x(1,t)+\xi_2u(1,t)=h_3(t),\end{cases}\tag{1.12}$$

with $x\in(0,1)$ and $t\geq 0$ (cf. Bubnov, 1979, 1980). Roughly speaking, if the parameters α_j, β_j, and ξ_j, $j=1,2,3$, are chosen such that the solution u of the associated homogeneous linear problem (obtained by dropping uu_x and setting $f=0$ and $h_j=0$ for $j=1,2,3$ in (1.12)) satisfies

$$\frac{d}{dt}\int_0^1|u(x,t)|^2\,dx\leq 0\tag{1.13}$$

for any $t>0$, the detailed techniques developed in the remainder of the article apply and one may establish that the IBVP (1.12) is well-posed in the space $H^s(0,1)$ for any $s\geq 0$. In case (1.13) is not valid, the issue of local well-posedness may be more challenging. As far as global existence is concerned, we can give conditions on (1.12) for this to hold. Indeed, because of the strong smoothing resulting from the boundedness of the domain, all that is required is to keep the $L_2(0,1)$—norm bounded on bounded time intervals. We will not enter into the details of this development here.

With global well-posedness results in hand, a natural further question arises about the solutions of the IBVP (1.1)–(1.3), namely their long time, asymptotic behavior. Because the imposition of boundary conditions may exert a weak dissipative mechanism, it is expected that the solutions of the nonlinear system (1.1)–(1.3) will decay as $t \to +\infty$, at least in case the initial value ϕ is small and the boundary data $h_j(t)$, $j = 1, 2, 3$ decay to zero as $t \to \infty$. A special situation occurs when the boundary data are all periodic of some period τ, say. Experiments Bona et al., (1981) suggest that in this case the solution will eventually become time periodic of period τ. This has been rigorously established in Bona et al., (2003) for the quarter-plane problem (1.4) with a damping term included. It would be interesting and useful to have similar results for the finite domain problem considered here. A related question is whether the (1.1) possesses a strictly time-periodic solution if its boundary forcing h_1, h_2, and h_3 are time-periodic functions defined on all of R (cf. Bona et al., 2003; Wayne, 1990, 1997). Those issues will be addressed in our subsequent articles.

The article is organized as follows. In Sec. 2, several estimates pertaining to solutions of the linear problem (1.10) are established which display the smoothing properties mentioned earlier. In Sec. 3, the linear estimates are used to prove that (1.1)–(1.3) is locally well-posed. The global well-posedness of (1.1)–(1.3) is established in Sec. 4. Analyticity of the nonlinear map $\mathcal{K}_{s,T}$ defined by the IBVP (1.1)–(1.3) is discussed in Sec. 5.

2. LINEAR ESTIMATES AND SMOOTHING PROPERTIES

In this section, various smoothing properties that accrue to the linear system Eq. (1.10) will be discussed. As (1.10) is linear, it is convenient to break up the analysis. Considered first is the problem

$$\left.\begin{array}{l} u_t + u_x + u_{xxx} = 0, \quad u(x,0) = \phi(x), \\ u(0,t) = 0, \quad u(1,t) = 0, \quad u_x(1,t) = 0 \end{array}\right\} \tag{2.1}$$

with homogeneous boundary conditions and no forcing. Then we will consider problem (1.10) with non-trivial forcing f but with all three boundary conditions set to zero. The outcome of the analysis of these problems are recorded in Propositions 2.1 and 2.4. Next, problem (1.10) with zero forcing, but non-trivial boundary conditions is taken up. We use the Laplace transform in t to obtain a solution formula. Whilst a little complicated, the representation formulas (2.14) and following are completely explicit. Consequently, their analysis may be carried out in detail. The outcome is recorded in a sequence of propositions that conclude the section.

Let A be the linear operator defined by

$$Af = -f''' - f'.$$

Consider A as an unbounded operator on $L_2(0,1)$ with the domain

$$\mathcal{D}(A) = \{f \in H^3(0,1),\ f(0) = f(1) = f'(1) = 0\}.$$

The IBVP (2.1) can be written as the initial-value problem of an abstract evolution equation in the space $L_2(0,1)$, viz

$$\frac{du}{dt} = Au, \qquad u(0) = \phi, \tag{2.2}$$

where the spatial variable is suppressed. It is easily verified that both A and its adjoint A^* are dissipative, which is to say

$$\langle Af, f\rangle_{L_2(0,1)} \le 0, \qquad \langle A^*g, g\rangle_{L_2(0,1)} \le 0$$

for any $f \in \mathcal{D}(A)$ and $g \in \mathcal{D}(A^*)$, where $A^*g = g''' + g'$ and

$$\mathcal{D}(A^*) = \{f \in H^3(0,1);\ f(0) = f'(0) = f(1) = 0\}.$$

Thus the operator A is the infinitesimal generator of a C_0-semigroup $W_0(t)$ in the space $L_2(0,1)$ (see Pazy, 1983). By standard semigroup theory applied in the overlying space $L_2(0,1)$, for any $\phi \in L_2(0,1)$,

$$u(t) = W_0(t)\phi$$

belongs to the space $C_b(R^+; L_2(0,1))$. The function u thus defined is called a *mild solution* of (2.1). Such solutions certainly solve (2.1) in the sense of distributions (cf. Bona and Winther (1983), Sec. 2). If $\phi \in \mathcal{D}(A)$, then $u(t) = W_0(t)\phi$ belongs to the much smaller space $C(0,\infty; H^3(0,1)) \cap C^1(0,\infty; L_2(0,1))$ and $u(t) \in \mathcal{D}(A)$ for all $t \ge 0$. Moreover, the equation is satisfied in the sense of $C(0,\infty; L_2(0,1))$, and in particular, pointwise almost everywhere. Such solutions are called *strong solutions.* For strong solutions, the boundary values are taken on pointwise. In what follows, a solution of (2.1) is either a mild solution or strong solution in the semigroup context.

Proposition 2.1. *For any $\phi \in L_2(0,1)$, $u(t) = W_0(t)\phi$ satisfies*

$$\|u(\cdot,t)\|^2_{L_2(0,1)} + \int_0^t u_x^2(0,\tau)\,d\tau = \|\phi\|^2_{L_2(0,1)} \tag{2.3}$$

and

$$\int_0^1 xu^2(x,t)\,dx + 3\int_0^t\int_0^1 u_x^2(x,\tau)\,dx\,d\tau \le (1+t)\int_0^1 \phi^2(x)\,dx \tag{2.4}$$

for any $t \ge 0$.

Remark 2.2. The relation (2.3) provides a trace result at $x=0$ which reveals a boundary smoothing effect of the system (2.1).

Remark 2.3. Combining inequalities (2.3) and (2.4) gives

$$\|u\|_{L_2(0,t;H^1(0,1))} \le C(1+t)^{1/2}\|\phi\|_{L_2(0,1)}, \tag{2.5}$$

which is a Kato-type smoothing effect. Note that the original Kato smoothing effect for solutions of (1.1)–(1.2) posed on the whole real line R is local, which is to say,

$\phi \in L_2(R)$ implies that $u \in L_2(0, T; H^1_{loc}(R))$. In contrast, the smoothing effect (2.5) is global. As will be seen later, this global Kato-smoothing effect alone is enough to establish the well-posedness of (1.1)–(1.3) in the space $H^s(0,1)$ for $s \geq 0$. This is in sharp contrast to the problem (1.1)–(1.2) posed on the unbounded domain R or the IBVP (1.4) posed on the unbounded domain R^+, where both the Kato smoothing and the Strichartz smoothing or the Bourgain smoothing are used to establish their well-posedness in weak spaces.

Proof. Assume first that $\phi \in \mathcal{D}(A)$. Then $u(t) \in \mathcal{D}(A)$ for any $t \geq 0$ and $u \in C^1(0,\infty; L_2(0,1))$. To obtain (2.3), multiply both sides of the differential equation in (2.1) by $2u$ and integrate over $(0,1)$ with respect to x and over $(0,t)$ with respect to t. Integration by parts then leads to (2.3). For inequality (2.4), multiply both sides of the equation in (2.1) by $2xu$, integrate the result over $[0,1] \times [0,1]$, and integrate by parts to reach the relation

$$\int_0^1 xu^2(x,t)\,dx + 3\int_0^t\int_0^1 u_x^2(x,\tau)\,dx\,d\tau = \int_0^1 x\phi^2(x)\,dx + \int_0^t\int_0^1 u^2(x,\tau)\,dx\,d\tau$$

from which (2.4) follows on account of (2.3). If, instead, $\phi \in L_2(0,1)$, choose a sequence $\{\phi_n\}$ from $\mathcal{D}(A)$ such that ϕ_n converges to ϕ in $L_2(0,1)$ as $n \to \infty$. Define u_n to be

$$u_n = W_0(t)\phi_n, \qquad n = 1, 2, \ldots.$$

As we have just shown, both (2.3) and (2.1) hold with u replaced by u_n and ϕ replaced by ϕ_n. Let $T > 0$ be fixed. Then the sequence $\{u_n\}$ is bounded in the spaces $C([0,T]; L_2(0,1))$, $L_2(0,T; H^1(0,1))$ and $C([0,T]; L_2(0,1; xdx))$. Here $L_2(0,1;xdx)$ is the weighted L_2-space with the weight x. Moreover $u_{n,x}(0,t) \equiv \partial_x u_n(0,t)$ is a bounded sequence in the space $L_2(0,T)$. Thus there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a

$$u^* \in L_2(0,T; H^1(0,1)) \cap L_\infty(0,T; L_2(0,1)) \cap L_\infty(0,T; L_2(0,1;xdx))$$

with $u_x^*(0,t) \in L_2(0,T)$ such that $\{u_{n_k}\}$ is convergent to u^* weakly in $L_2(0,T; H^1(0,1))$ and weak-star in both spaces $L_\infty(0,T; L_2(0,1))$ and $L_\infty(0,T; L_2(0,1;xdx))$. Furthermore, $\{\partial_x u_{n_k}(0,t)\}$ is weakly convergent to $u_x^*(0,t)$ in the space $L_2(0,T)$. On account of the lower semi-continuity of the various norms with regard to weak convergence, it is adduced that (2.3) and (2.4) hold for u^*. On the other hand u_n converges strongly to $u = W_0(t)\phi$ in $L_\infty(0,T; L_2(0,1))$. We conclude that

$$u \in L_2(0,T; H^1(0,1)) \cap L_\infty(0,T; L_2(0,1)) \cap L_\infty(0,T; L_2(0,1;xdx)),$$

$u_x(0,t) \in L_2(0,T)$ and (2.3) and (2.4) hold for u. □

Next, attention is turned to the inhomogeneous linear problem

$$\left.\begin{array}{l} u_t + u_x + u_{xxx} = f(x,t), \quad u(x,0) = 0, \\ u(0,t) = 0, \quad u(1,t) = 0, \quad u_x(1,t) = 0. \end{array}\right\} \tag{2.6}$$

In terms of the operator A defined above, one may write (2.6) as an initial-value problem for an abstract nonhomogeneous evolution equation, viz.

$$\frac{du}{dt} = Au + f, \qquad u(0) = 0. \tag{2.7}$$

By standard semigroup theory (see again Pazy, 1983), for any $f \in L_{1,\mathrm{loc}}(R^+; L_2(0,1))$,

$$u(t) = \int_0^t W_0(t-\tau) f(\tau)\, d\tau \tag{2.8}$$

belongs to the space $C(R^+; L_2(0,1))$ and is called a mild solution of (2.7). It is a weak solution of (2.6) in the sense of distribution. In addition, if $f(t) \in \mathcal{D}(A)$ for $t > 0$ and $Af \in L_{1,\mathrm{loc}}(R^+; L_2(0,1))$, then $u(t)$ given by (2.8) solves (2.7) a.e. on $[0, T)$ and is called a strong solution of (2.7).

Proposition 2.4. *There exists a constant C such that for any* $f \in L_{1,\mathrm{loc}}(R^+; L_2(0,1))$, *the solution u of Eq.* (2.6) *satisfies*

$$\|u(\cdot,t)\|_{L_2(0,1))} + \|u_x(0,\cdot)\|_{L_2(0,t)} \le C\|f\|_{L_1(0,t;L_2(0,1))} \tag{2.9}$$

and

$$\int_0^1 xu^2(x,t)\,dx + \int_0^t\int_0^1 u_x^2(x,\tau)\,dx\,d\tau \le 2(1+t)\|f\|^2_{L_1(0,t;L_2(0,1))} \tag{2.10}$$

for any $t \ge 0$.

Proof. Without loss of generality, we assume that u is a strong solution. The general case follows using a limiting procedure similar to that appearing in the proof of Proposition 2.1. Multiply the equation in (2.6) by $2u$ and integrate over $(0, 1)$ with respect to x. Integration by parts leads to

$$\frac{d}{dt}\int_0^1 u^2(x,t)\,dx + u_x^2(0,t) \le 2\|f(\cdot,t)\|_{L_2(0,1)}\|u\|_{L_2(0,1)}$$

from which (2.9) follows. To prove (2.10), multiply both sides of the equation in (2.6) by $2xu$ and integrate over the rectangle $(0, 1) \times (0, t)$ in space-time. After integrations by parts, it is seen that

$$\begin{aligned}
&\int_0^1 xu^2(x,t)\,dx + 3\int_0^t\int_0^1 u_x^2(x,\tau)\,dx\,d\tau \\
&\quad = 2\int_0^t\int_0^1 xf(x,\tau)u(x,\tau)\,dx\,d\tau + \int_0^t\int_0^1 u^2(x,\tau)\,dx\,d\tau \\
&\quad \le \int_0^t \|x^{1/2}f(\cdot,\tau)\|_{L_2(0,1)}\|x^{1/2}u(\cdot,\tau)\|_{L_2(0,1)}\,d\tau + \int_0^t\int_0^1 u^2(x,\tau)\,dx\,d\tau
\end{aligned}$$

$$\le \sup_{0\le\tau\le t} \|x^{1/2}u(\cdot,\tau)\|_{L_2(0,1)} \int_0^t \|x^{1/2}f(\cdot,\tau)\|_{L_2(0,1)}\,d\tau + \int_0^t\int_0^1 u^2(x,\tau)\,dx\,d\tau$$
$$\le \frac{1}{2}\sup_{0\le\tau\le t} \|x^{1/2}u(\cdot,\tau)\|^2_{L_2(0,1)} + \frac{1}{2}\left(\int_0^t \|x^{1/2}f(\cdot,\tau)\|_{L_2(0,1)}\,d\tau\right)^2$$
$$+ \int_0^t\int_0^1 u^2(x,\tau)\,dx\,d\tau,$$

which yields the inequality (2.10). □

Next, consider the non-homogeneous boundary-value problem

$$\left.\begin{aligned} &u_t + u_x + u_{xxx} = 0, \qquad u(x,0) = 0,\\ &u(0,t) = h_1(t), \quad u(1,t) = h_2(t), \quad u_x(1,t) = h_3(t). \end{aligned}\right\} \tag{2.11}$$

A common approach to (2.11) is to render its boundary conditions homogeneous as follows. The solution u of (2.11) can be written as

$$u(x,t) = w(x,t) + v(x,t)$$

with

$$v(x,t) = (1-x)h_1(t) + xh_2(t) + x(1-x)(h_3(t) - h_2(t) + h_1(t))$$

and w satisfying

$$\left.\begin{aligned} &w_t + w_x + w_{xxx} = -v_t - v_x, \qquad w(x,0) = 0,\\ &w(0,t) = 0, \quad w(1,t) = 0, \quad w_x(1,t) = 0. \end{aligned}\right\} \tag{2.12}$$

Thus to solve (2.11), one only need solve (2.12), which can be done by applying Proposition 2.2. Here we assume $h_j(0) = 0$ for $j = 1,2,3$. However there is a serious drawback to this approach; it is required that $h_j' \in L_1(R)$ for $j = 1,2,3$ to obtain even a mild solution u of (2.12) in the space $C_b(0,T; L_2(0,1)) \cap L_2(0,T; H^1(0,1))$. Furthermore, for such a mild solution u, although both $u(0,t)$ and $u(1,t)$ are defined thanks to the Kato smoothing, it seems that the trace of $u_x(x,t)$ at $x = 1$ does not make sense since u_x is only known to be in the space $L_2(0,T; L_2(0,1))$. This suggests that a stronger boundary smoothing property of (2.11) is needed if one wants to solve (2.11) in the space $C_b(0,T; L_2(0,1)) \cap L_2(0,T; H^1(0,1))$.

Our approach to solve (2.11) is to seek an explicit solution formula in terms of its boundary values via the Laplace transform as we did in Bona et al., (2001) for the KdV-equation in a quarter plane.

Applying the Laplace transform with respect to t, (2.11) is converted to

$$\left.\begin{aligned} &s\hat{u}(x,s) + \hat{u}_x(x,s) + \hat{u}_{xxx}(x,s) = 0,\\ &\hat{u}(0,s) = \hat{h}_1(s),\ \hat{u}(1,s) = \hat{h}_2(s),\ \hat{u}_x(1,s) = \hat{h}_3(s), \end{aligned}\right\} \tag{2.13}$$

where

$$\hat{u}(x,s) = \int_0^{+\infty} e^{-st}u(x,t)\,dt$$

and

$$\hat{h}_j(s) = \int_0^{+\infty} e^{-st} h_j(t)\, dt, \quad j = 1, 2, 3.$$

The solution $\hat{u}(x, s)$ of (2.13) can be written in the form

$$\hat{u}(x, s) = \sum_{j=1}^{3} c_j(s) e^{\lambda_j(s)x},$$

where $\lambda_j(s)$, $j = 1, 2, 3$, are the three solutions of the characteristic equation

$$s + \lambda + \lambda^3 = 0$$

and $c_j = c_j(s)$, $j = 1, 2, 3$, solve the linear system

$$\begin{cases} c_1 + c_2 + c_3 = \hat{h}_1(s), \\ c_1 e^{\lambda_1(s)} + c_2 e^{\lambda_2(s)} + c_3 e^{\lambda_3(s)} = \hat{h}_2(s), \\ c_1 \lambda_1(s) e^{\lambda_1(s)} + c_2 \lambda_2(s) e^{\lambda_2(s)} + c_3 \lambda_3(s) e^{\lambda_3(s)} = \hat{h}_3(s). \end{cases}$$

Let $\Delta(s)$ be the determinant of the coefficient matrix and $\Delta_i(s)$ the determinants of the matrices that are obtained by replacing the ith-column of $\Delta(s)$ by the column vector $(\hat{h}_1(s), \hat{h}_2(s), \hat{h}_3(s))^T$, $i = 1, 2, 3$. Cramer's rule implies that

$$c_j = \frac{\Delta_j(s)}{\Delta(s)}, \quad j = 1, 2, 3.$$

Taking the inverse Laplace transform of $\hat{u}$ yields

$$u(x, t) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{st} \hat{u}(x, s)\, ds = \sum_{j=1}^{3} \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{st} \frac{\Delta_j(s)}{\Delta(s)} e^{\lambda_j(s)x}\, ds$$

for any $r > 0$. The solution u of (2.11) may also be written in the form

$$u(x, t) = u_1(x, t) + u_2(x, t) + u_3(x, t)$$

where $u_m(x, t)$ solves (2.11) with $h_j \equiv 0$ when $j \neq m$, $m, j = 1, 2, 3$; thus u_m has the representation

$$u_m(x, t) = \sum_{j=1}^{3} \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{st} \frac{\Delta_{j,m}(s)}{\Delta(s)} e^{\lambda_j(s)x} \hat{h}_m(s)\, ds \equiv W_m(t) h_m \tag{2.14}$$

for $m = 1, 2, 3$. Here $\Delta_{j,m}(s)$ is obtained from $\Delta_j(s)$ by letting $\hat{h}_m(t) = 1$ and $h_k(t) \equiv 0$ for $k \neq m$, $k, m = 1, 2, 3$. It is straightforward to determine that in the last two formulas, the right-hand sides are continuous with respect to r for $r \geq 0$. As the left-hand sides do not depend on r, it follows that we may take $r = 0$ in these formulas and in those appearing below. Write u_m in the form

$$\begin{aligned} u_m(x,t) &= \sum_{j=1}^{3} \frac{1}{2\pi i} \int_0^{+i\infty} e^{st} \frac{\Delta_{j,m}(s)}{\Delta(s)} e^{\lambda_j(s)x} \hat{h}_m(s)\,ds \\ &\quad + \sum_{j=1}^{3} \frac{1}{2\pi i} \int_{-i\infty}^{0} e^{st} \frac{\Delta_{j,m}(s)}{\Delta(s)} e^{\lambda_j(s)x} \hat{h}_m(s)\,ds \\ &\equiv I_m(x,t) + II_m(x,t), \end{aligned}$$

for $m = 1, 2, 3$. Letting $s = i(\rho^3 - \rho)$ with $1 \le \rho < +\infty$ in the characteristic equation

$$s + \lambda + \lambda^3 = 0, \tag{2.15}$$

the three roots are given in terms of ρ by

$$\lambda_1^+(\rho) = i\rho, \quad \lambda_2^+(\rho) = \frac{\sqrt{3\rho^2 - 4} - i\rho}{2}, \quad \lambda_3^+(\rho) = \frac{-\sqrt{3\rho^2 - 4} - i\rho}{2}$$

and thus $I_m(x,t)$ and $II_m(x,t)$ may be written in the form

$$I_m = \sum_{j=1}^{3} \frac{1}{2\pi} \int_1^{+\infty} e^{i(\rho^3-\rho)t} e^{\lambda_j^+(\rho)x} \frac{\Delta_{j,m}^+(\rho)}{\Delta^+(\rho)} (3\rho^2 - 1)\hat{h}_m^+(\rho)\,d\rho$$

and

$$II_m = \sum_{j=1}^{3} \frac{1}{2\pi} \int_1^{+\infty} e^{-i(\rho^3-\rho)t} e^{\lambda_j^-(\rho)x} \frac{\Delta_{j,m}^-(\rho)}{\Delta^-(\rho)} (3\rho^2 - 1)\hat{h}_m^-(\rho)\,d\rho$$

where $\hat{h}_m^+(\rho) = \hat{h}_m(i(\rho^3 - \rho))$, $\Delta^+(\rho)$ and $\Delta_{j,m}^+(\rho)$ are obtained from $\Delta(s)$ and $\Delta_{j,m}(s)$, respectively, by replacing s with $i(\rho^3 - \rho)$ and $\lambda_j(s)$ with $\lambda_j^+(\rho)$, for $j = 1, 2, 3$. Notice that, with an obvious notation, $\Delta^-(\rho) = \overline{\Delta^+(\rho)}$ and $\Delta_{j,m}^-(\rho) = \overline{\Delta_{j,m}^+(\rho)}$ for $j = 1, 2, 3$, and $\hat{h}_m^-(\rho) = \overline{\hat{h}_m^+(\rho)}$.

The next result is a technical lemma that will find frequent use in this section. It plays the same role in dealing with the finite-interval problem as does Plancherel's theorem for the pure initial-value problem posed on the line.

Lemma 2.5. *For any $f \in L_2(0 + \infty)$, let Kf be the function defined by*

$$Kf(x) = \int_0^{+\infty} e^{\gamma(\mu)x} f(\mu)\,d\mu$$

where $\gamma(\mu)$ is a continuous complex-valued function defined on $(0, \infty)$ satisfying the following two conditions:

(i) *There exist $\delta > 0$ and $b > 0$ such that*

$$\sup_{0<\mu<\delta} \frac{|\mathrm{Re}\,\gamma(\mu)|}{\mu} \ge b;$$

(*ii*) *There exists a complex number* $\alpha + i\beta$ *such that*

$$\lim_{\mu\to+\infty}\frac{\gamma(\mu)}{\mu} = \alpha + i\beta.$$

Then there exists a constant C *such that for all* $f \in L_2(0,\infty)$,

$$\|Kf\|_{L_2(0,1)} \le C(\|e^{\operatorname{Re}\gamma(\cdot)}f(\cdot)\|_{L_2(R^+)} + \|f(\cdot)\|_{L_2(R^+)}).$$

Proof. Observe that

$$\begin{aligned}
\|Kf\|^2_{L_2(0,1)} &\le \int_0^1\int_0^{+\infty} e^{\operatorname{Re}(\gamma(s)x)}|f(s)|\,ds\int_0^{+\infty} e^{\operatorname{Re}(\gamma(y)x)}|f(y)|\,dy\,dx\\
&= \int_0^{+\infty}\int_0^{+\infty}\int_0^1 e^{\operatorname{Re}(\gamma(s)+\gamma(y))x}\,dx\,|f(s)||f(y)|\,ds\,dy\\
&\le \int_0^{+\infty}\int_0^{+\infty}\left(e^{\operatorname{Re}(\gamma(s)+\gamma(y))}+1\right)\frac{|f(s)f(y)|}{|\operatorname{Re}(\gamma(s)+\gamma(y))|}\,ds\,dy\\
&\le \left\|\int_0^{+\infty}\frac{e^{\operatorname{Re}\gamma(s)}|f(s)|\,ds}{|\operatorname{Re}(\gamma(s)+\gamma(y))|}\right\|_{L_2(R^+)}\|e^{\operatorname{Re}\gamma(\cdot)}f(\cdot)\|_{L_2(R^+)}\\
&\quad+\left\|\int_0^{+\infty}\frac{|f(s)|\,ds}{|\operatorname{Re}(\gamma(s)+\gamma(y))|}\right\|_{L_2(R^+)}\|f\|_{L_2(R^+)}.
\end{aligned}$$

Notice also that

$$\|e^{\operatorname{Re}\gamma(\mu y)}f(\mu y)\|_{L_2(R^+)} \le \mu^{-1/2}\|e^{\operatorname{Re}\gamma(\cdot)}f(\cdot)\|_{L_2(R^+)}$$

and for any $y \in (0, +\infty)$,

$$\frac{y}{|\operatorname{Re}(\gamma(\mu y)+\gamma(y))|} \le \frac{C}{\mu+1}.$$

Using the integral version of Minkowski's inequality yields

$$\begin{aligned}
\left\|\int_0^{+\infty}\frac{e^{\operatorname{Re}\gamma(s)}|f(s)|\,ds}{|\operatorname{Re}(\gamma(s)+\gamma(y))|}\right\|_{L_2(R^+)} &= \left\|\int_0^{+\infty}\frac{e^{\operatorname{Re}\gamma(\mu y)\beta}|f(\mu y)|y\,d\mu}{|\operatorname{Re}(\gamma(\mu y)+\gamma(y))|}\right\|_{L_2(R^+)}\\
&\le \int_0^{+\infty}\left\|\frac{e^{\operatorname{Re}\gamma(\mu y)}f(\mu y)y}{\operatorname{Re}(\gamma(\mu y)+\gamma(y))}\right\|_{L_2(R^+)}d\mu\\
&\le C\int_0^{+\infty}\frac{1}{\sqrt{\mu}(1+\mu)}\,d\mu\|e^{\operatorname{Re}\gamma(\cdot)}f(\cdot)\|_{L_2(R^+)}\\
&\le C\|e^{\operatorname{Re}\gamma(\cdot)}f\|_{L_2(R^+)}
\end{aligned}$$

for some absolute constant C. The same argument also gives

$$\left\| \int_0^{+\infty} \frac{|f(s)|\,ds}{|\mathrm{Re}(\gamma(s)+\gamma(y))|} \right\|_{L_2(R^+)} \leq C\|f\|_{L_2(R^+)}.$$

The proof is complete. □

Lemma 2.6. *Let $a > 0$ be given. For any $f \in L_2(0,a)$, let Gf be the function defined by*

$$Gf(x) = \int_0^a e^{i\xi(\mu)x} f(\mu)\,d\mu$$

where ξ is a continuous real-valued function defined on the interval $[0,a]$ which is C^1 on the open interval $(0,a)$ and such that there is a constant C_1 for which $(1/|\xi'(\mu)|) \leq C_1$ for $0 < \mu < a$. Then there exists a constant C such that for all $f \in L_2(0,a)$,

$$\|Gf\|_{L_2(0,1)} \leq C\|f\|_{L_2(0,a)}.$$

Proof. Let $\omega = \xi(\mu)$. Since $\xi'(\mu) \neq 0$ for $\mu \in (0,a)$, ξ is strictly monotone and so is invertible. Let $\mu = \xi^{-1}(\omega)$ denote its inverse. Note that $d\omega = \xi'(\mu)d\mu$ and so by a change of variables, we may write Gf in terms of ω thusly:

$$Gf(x) = \int_{\xi(0)}^{\xi(a)} e^{i\omega x} f(\xi^{-1}(\omega)) \frac{1}{\xi'(\xi^{-1}(\omega))}\,d\omega.$$

It follows from Plancherel's theorem that

$$\begin{aligned}
\|Gf\|^2_{L_2(0,1)} &= \frac{1}{2\pi}\int_{\xi(0)}^{\xi(a)} (f(\xi^{-1}(\omega)))^2 \left(\frac{1}{\xi'(\xi^{-1}(\omega))}\right)^2 d\omega \\
&= \frac{1}{2\pi}\int_0^a |f(\mu)|^2 \frac{1}{|\xi'(\mu)|}\,d\mu \\
&\leq \frac{C_1}{2\pi}\int_0^a |f(\mu)|^2\,d\mu
\end{aligned}$$

which is the advertised inequality. □

The following three propositions provide estimates for u_1, u_2, and u_3, respectively. They show clearly various smoothing properties that accrue through implementation of the boundary conditions for the linear system (2.7) (cf. Remark 2.3).

Proposition 2.7. *There exists a constant C such that*

$$\|u_1\|_{L_2(R^+;H^1(0,1))} + \sup_{0\leq t<+\infty} \|u_1(\cdot,t)\|_{L_2(0,1)} \leq C\|h_1\|_{H^{1/3}(R^+)} \tag{2.16}$$

and $\partial_x u_1 \in C_b([0,1]; L_2(R^+))$ with

$$\sup_{x\in(0,1)} \|\partial_x u_1(x,\cdot)\|_{L_2(R^+)} \leq C\|h_1\|_{H^{1/3}(R^+)} \tag{2.17}$$

for all $h_1 \in H^{1/3}(R^+)$.

Proof. Since

$$\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \equiv 0,$$

it is readily seen that

$$\begin{aligned}
\Delta_{1,1}(s) &= (\lambda_3(s) - \lambda_2(s))e^{-\lambda_1(s)},\\
\Delta_{2,1}(s) &= (\lambda_1(s) - \lambda_3(s))e^{-\lambda_2(s)},\\
\Delta_{3,1}(s) &= (\lambda_2(s) - \lambda_1(s))e^{-\lambda_3(s)}
\end{aligned}$$

and thus

$$\Delta(s) = (\lambda_3(s) - \lambda_2(s))e^{-\lambda_1(s)} + (\lambda_1(s) - \lambda_3(s))e^{-\lambda_2(s)} + (\lambda_2(s) - \lambda_1(s))e^{-\lambda_3(s)}.$$

In consequence, it follows readily that as a function of the variable ρ introduced above and defined by the relation $s = i(\rho^3 - \rho)$,

$$\frac{\Delta_{1,1}^+(\rho)}{\Delta^+(\rho)} \sim e^{\frac{-\sqrt{3}}{2}\rho}, \qquad \frac{\Delta_{2,1}^+(\rho)}{\Delta^+(\rho)} \sim e^{-\sqrt{3}\rho}, \qquad \frac{\Delta_{3,1}^+(\rho)}{\Delta^+(\rho)} \sim 1$$

as $\rho \to +\infty$. An application of Lemma 2.5 produces a constant C such that

$$\begin{aligned}
\|I_1(\cdot, t)\|_{L_2(0,1)}^2 &\le \sum_{j=1}^{3} \int_1^{+\infty} \left|\frac{\Delta_{j,1}^+(\rho)}{\Delta^+(\rho)}\right|^2 \left(e^{\mathrm{Re}\,\lambda_j^+(\rho)} + 1\right)^2 \left|\hat{h}_1^+(\rho)(3\rho^2 - 1)\right|^2 d\rho\\
&\le C \int_1^{+\infty} |\hat{h}_1^+(\rho)|^2 (3\rho^2 - 1)^2\, d\rho\\
&\le C \int_0^{+\infty} (1 + \mu)^{2/3} \left|\int_0^{+\infty} e^{-i\mu} h_1(\tau)\, d\tau\right|^2 d\mu \le C\|h_1\|_{H^{1/3}(R^+)}^2.
\end{aligned}$$

The same argument applied to $II_1(x, t)$ gives

$$\|II_1(\cdot, t)\|_{L_2(0,1)} \le C\|h_1\|_{H^{1/3}(R^+)}.$$

Thus (2.16) holds. To prove (2.17), observe that

$$\begin{aligned}
\partial_x I_1(x, t) &= \sum_{j=1}^{3} \frac{1}{2\pi} \int_1^{+\infty} e^{i(\rho^3 - \rho)t} \lambda_j^+(\rho) e^{\lambda_j^+(\rho)x} \frac{\Delta_{1,j}^+(\rho)}{\Delta^+(\rho)} (3\rho^2 - 1)\hat{h}_1^+(\rho)\, d\rho\\
&= \sum_{j=1}^{3} \frac{1}{2\pi} \int_0^{+\infty} e^{i\mu t} \lambda_j^+(\theta(\mu)) e^{\lambda_j^+(\theta(\mu))x} \frac{\Delta_{j,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))} \hat{h}_1(i\mu)\, d\mu
\end{aligned}$$

where $\theta(\mu)$ is the real solution of $\mu = \rho^3 - \rho$ for $\rho \ge 1$. Using the Plancherel Theorem (with respect to t) yields that for any $x \in (0, 1)$,

$$\|\partial_x I_1(x, \cdot)\|_{L_2(R^+)}^2 \le \sum_{j=1}^{3} \frac{1}{2\pi} \int_0^{+\infty} \left|\lambda_j^+(\theta(\mu)) e^{\lambda_j^+(\theta(\mu))x} \frac{\Delta_{j,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}\right|^2 |\hat{h}_1(i\mu)|^2\, d\mu.$$

Thus, one finds there is a constant C such that

$$\begin{aligned}
&\int_0^1 \|\partial_x I_1(x,\cdot)\|^2_{L_2(R^+)}\,dx \\
&\quad \le \sup_{x\in(0,1)} \|\partial_x I_1(x,\cdot)\|^2_{L_2(R^+)} \\
&\quad \le C\sum_{j=1}^{3}\int_0^{+\infty} |\lambda_j^+(\theta(\mu))|^2 \sup_{x\in(0,1)} |e^{\lambda_j^+(\theta(\mu))x}|^2 \left|\frac{\Delta_{j,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}\right|^2 |\hat{h}_1(i\mu)|^2\,d\mu \\
&\quad \le C\sum_{j=1}^{3}\int_0^{+\infty} (1+\mu)^{2/3}|\hat{h}_1(i\mu)|^2\,d\mu \le C\|h_1\|^2_{H^{1/3}(R^+)}.
\end{aligned}$$

The following estimates were used in obtaining the last inequality:

$$\sup_{x\in(0,1)} |e^{\lambda_1(\rho)x}|^2 \le C, \qquad \sup_{x\in(0,1)} |e^{\lambda_2(\rho)x}|^2 \le C\left(e^{\sqrt{3}\rho}+1\right),$$

$$\sup_{x\in(0,1)} |e^{\lambda_3(\rho)x}|^2 \le C\rho^{-1}\left(e^{-\sqrt{3}\rho}+1\right).$$

To see $\partial_x I_1$ is continuous from $[0,1]$ to the space $L_2(R^+)$, choose any $x_0 \in [0,1]$ and $x \in (0,1)$ and observe that

$$\begin{aligned}
&\partial_x I_1(x,t) - \partial_x I_1(x_0,t) \\
&\quad = \sum_{j=1}^{3}\frac{1}{2\pi}\int_0^{+\infty} e^{i\mu t}\lambda_j^+(\theta(\mu))\left(e^{\lambda_j^+(\theta(\mu))x} - e^{\lambda_j^+(\theta(\mu))x_0}\right)\frac{\Delta_{j,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}\hat{h}_1(i\mu)\,d\mu.
\end{aligned}$$

Using the Plancherel theorem with respect to t as above yields

$$\begin{aligned}
&\|\partial_x I_1(x,\cdot) - \partial_x I_1(x_0,\cdot)\|^2_{L_2(R^+)} \\
&\quad \le \sum_{j=1}^{3}\frac{1}{2\pi}\int_0^{+\infty}\left|\lambda_j^+(\theta(\mu))\left(e^{\lambda_j^+(\theta(\mu))x} - e^{\lambda_j^+(\theta(\mu))x_0}\right)\frac{\Delta_{j,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}\right|^2 |\hat{h}_1(i\mu)|^2\,d\mu \\
&\quad \le C\sum_{j=1}^{3}\int_0^{+\infty} (1+\mu)^{2/3}|\hat{h}_1(i\mu)|^2\,d\mu.
\end{aligned}$$

An application of Fatou's lemma gives

$$\begin{aligned}
&\lim_{x\to x_0} \|\partial_x I_1(x,\cdot) - \partial_x I_1(x_0,\cdot)\|^2_{L_2(R^+)} \\
&\quad \le \sum_{j=1}^{3}\frac{1}{2\pi}\int_0^{+\infty}\left|\lambda_j^+(\theta(\mu))\lim_{x\to x_0}\left(e^{\lambda_j^+(\theta(\mu))x} - e^{\lambda_j^+(\theta(\mu))x_0}\right)\frac{\Delta_{j,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}\right|^2 |\hat{h}_1(i\mu)|^2\,d\mu \\
&\quad = 0
\end{aligned}$$

Similar considerations establish that

$$\int_0^1 \|\partial_x I_2(x,\cdot)\|^2_{L_2(R^+)}dx \leq \sup_{x\in(0,1)} \|\partial_x I_2(x,\cdot)\|^2_{L_2(R^+)} \leq C\|h_1\|^2_{H^{1/3}(R^+)}$$

and $I_2(x,\cdot) \in C_b([0,1]; L_2(R^+))$. The proof is complete. □

Proposition 2.8. *There exists a constant C such that*

$$\|u_2\|_{L_2(R^+;H^1(0,1))} + \sup_{0\leq t<+\infty} \|u_2(\cdot,t)\|_{L_2(0,1)} \leq C\|h_2\|_{H^{1/3}(R^+)} \tag{2.18}$$

and $\partial_x u_2 \in C_b([0,1]; L_{2,t}(R^+))$ *with*

$$\sup_{x\in(0,1)} \|\partial_x u_2(x,\cdot)\|_{L_2(R^+)} \leq C\|h_2\|_{H^{1/3}(R^+)} \tag{2.19}$$

for all $h_2 \in H^{1/3}(R^+)$.

Proof. Let $u_2(x,t) = I_2(x,t) + II_2(x,t)$ with

$$I_2(x,t) = \sum_{j=1}^{3} \frac{1}{2\pi i} \int_0^{+i\infty} e^{st} \frac{\Delta_{j,2}(s)}{\Delta(s)} e^{\lambda_j(s)x} \hat{h}_2(s)\,ds$$

and

$$II_2(x,t) = \sum_{j=1}^{3} \frac{1}{2\pi i} \int_{-i\infty}^{0} e^{st} \frac{\Delta_{j,2}(s)}{\Delta(s)} e^{\lambda_j(s)x} \hat{h}_2(s)\,ds.$$

As in the proof of Proposition 2.7, one has

$$I_2(x,t) = \sum_{j=1}^{3} \frac{1}{2\pi} \int_1^{+\infty} e^{i(\rho^3-\rho)t} e^{\lambda_j^+(\rho)x} \frac{\Delta_{j,2}^+(\rho)}{\Delta^+(\rho)} (3\rho^2-1)\hat{h}_2^+(\rho)\,d\rho$$

where $\hat{h}_2^+(\rho) = \hat{h}_2(i(\rho^3-\rho))$ and $\Delta_{j,2}^+(\rho) = \Delta_{j,2}(i(\rho^3-\rho))$ for $j = 1,2,3$. Note that

$$\Delta_{1,2}(s) = \lambda_2 e^{\lambda_2} - \lambda_3 e^{\lambda_3}, \quad \Delta_{2,2}(s) = \lambda_3 e^{\lambda_3} - \lambda_1 e^{\lambda_1},$$
$$\Delta_{3,2}(s) = \lambda_1 e^{\lambda_1} - \lambda_3 e^{\lambda_2}.$$

One readily obtains that, as $\rho \to +\infty$,

$$\frac{\Delta_{1,2}^+(\rho)}{\Delta^+(\rho)} \sim 1, \qquad \frac{\Delta_{2,2}^+(\rho)}{\Delta^+(\rho)} \sim e^{\frac{-\sqrt{3}}{2}\rho}, \qquad \frac{\Delta_{3,2}^+(\rho)}{\Delta^+(\rho)} \sim 1.$$

Using Lemma 2.5, it is adduced that there is a constant C for which

$$\|I_2(\cdot,t)\|^2_{L_2(0,1)} \leq \sum_{j=1}^{3} \int_1^{+\infty} \left|\frac{\Delta_{j,2}^+(\rho)}{\Delta^+(\rho)}\right|^2 (e^{\mathrm{Re}\,\lambda_j^+(\rho)} + 1big)^2 |\hat{h}_2^+(\rho)(3\rho^2-1)|^2\,d\rho$$
$$\leq C\int_1^{+\infty} |\hat{h}_2^+(\rho)(3\rho^2-1)|^2\,d\rho \leq C\|h_2\|^2_{H^{1/3}(R^+)}.$$

The same estimate holds for $II_2(x,t)$. To prove (2.19), note that

$$\begin{aligned}\partial_x I_2(x,t) &= \sum_{j=1}^{3}\frac{1}{2\pi}\int_1^{+\infty} e^{i(\rho^3-\rho)t}\lambda_j^+(\rho)e^{\lambda_j^+(\rho)x}\frac{\Delta_{j,2}^+(\rho)}{\Delta^+(\rho)}(3\rho^2-1)\hat{h}_2^+(\rho)\,d\rho \\ &= \sum_{j=1}^{3}\frac{1}{2\pi}\int_0^{+\infty} e^{i\mu t}\lambda_j^+(\theta(\mu))e^{\lambda_j^+(\theta(\mu))x}\frac{\Delta_{j,2}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}\hat{h}_2(i\mu)\,d\mu.\end{aligned}$$

Using the Plancherel theorem (with respect to t),

$$\|\partial_x I_2(x,\cdot)\|^2_{L_2(R^+)} \le \sum_{j=1}^{3}\frac{1}{2\pi}\int_0^{+\infty}\left|\lambda_j^+(\theta(\mu))e^{\lambda_j^+(\theta(\mu))x}\frac{\Delta_{j,2}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}\right|^2|\hat{h}_2^+(i\mu)|^2 d\mu,$$

from which follows

$$\begin{aligned}&\|\partial_x I_1(x,t)\|^2_{L_2(0,1;L_2(R^+))} \\ &\quad\le \sup_{x\in(0,1)}\|\partial_x I_1(x,\cdot)\|^2_{L_2(R^+)} \\ &\quad\le C\sum_{j=1}^{3}\int_0^{+\infty}|\lambda_j^+(\theta(\mu))|^2\sup_{x\in(0,1)}|e^{\lambda_j^+(\theta(\mu))x}|^2\left|\frac{\Delta_{j,2}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}\right|^2|\hat{h}_2(i\mu)|^2\,d\mu \\ &\quad\le C\sum_{j=1}^{3}\int_0^{+\infty}(1+\mu)^{2/3}|\hat{h}_2^+(i\mu)|^2\,d\mu \le C\|h_2\|^2_{H^{1/3}(R^+)}.\end{aligned}$$

The same estimate holds for $II_2(x,t)$. Moreover, a similar argument as that used in the proof of Proposition 2.7 shows that both I_2 and II_2 are continuous from $[0,1]$ to the space $L_2(R^+)$. The proof is complete. □

Proposition 2.9. *There exists a constant C such that*

$$\|u_3\|_{L_2(R^+;H^1(0,1))} + \sup_{0\le t<+\infty}\|u_3(\cdot,t)\|_{L_2(0,1)} \le C\|h_3\|_{L_2(R^+)} \tag{2.20}$$

and $\partial_x u_3 \in C_b([0,1];L_2(R^+))$ *with*

$$\sup_{x\in(0,1)}\|\partial_x u_3(x,\cdot)\|_{L_2(R^+)} \le C\|h_3\|_{L_2(R^+)} \tag{2.21}$$

for all $h_3 \in L_2(R^+)$.

Proof. The function $u_3(x,t)$ can be written in the form $u_3(x,t) = I_3(x,t) + II_3(x,t)$ with

$$I_3(x,t) = \sum_{j=1}^{3}\frac{1}{2\pi i}\int_0^{+i\infty} e^{st}\frac{\Delta_{j,3}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_3(s)\,ds$$

and

$$II_3(x,t) = \sum_{j=1}^{3}\frac{1}{2\pi i}\int_{-i\infty}^{0} e^{st}\frac{\Delta_{j,3}(s)}{\Delta(s)}e^{\lambda_j(s)x}\hat{h}_3(s)\,ds.$$

As in the proof of Proposition 2.7, one has

$$I_3(x,t) = \sum_{j=1}^{3} \frac{1}{2\pi} \int_1^{+\infty} e^{i(\rho^3-\rho)t} e^{\lambda_j^+(\rho)x} \frac{\Delta_{j,3}^+(\rho)}{\Delta^+(\rho)} (3\rho^2 - 1)\hat{h}_2^+(\rho)\, d\rho$$

where $\hat{h}_3^+(\rho) = \hat{h}_3(i(\rho^3-\rho))$ and $\Delta_{j,3}^+(\rho) = \Delta_{j,3}(i(\rho^3-\rho))$, $j = 1, 2, 3$. Since

$$\Delta_{1,3}(s) = e^{\lambda_3(s)} - e^{\lambda_2(s)}, \qquad \Delta_{2,3}(s) = e^{\lambda_1(s)} - e^{\lambda_3(s)}$$

and

$$\Delta_{3,3}(s) = e^{\lambda_2(s)} - e^{\lambda_1(s)},$$

it follows that

$$\frac{\Delta_{1,3}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}, \qquad \frac{\Delta_{2,3}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1} e^{-(\sqrt{3}/2)\rho}, \qquad \frac{\Delta_{3,2}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}$$

as $\rho \to +\infty$. The remainder of the proof follows the lines developed above. □

Let $\vec{h}(t) = (h_1(t), h_2(t), h_2(t))$ and write the solution u of (2.11) as

$$u(t) = W_b(t)\vec{h} = \sum_{j=1}^{3} W_j(t)h_j \tag{2.22}$$

where the spatial variable x is suppressed and the W_j are as defined in (2.14). For $s \geq 0$ and $T > 0$, let

$$\mathcal{H}_{s,T} = H^{(s+1)/3}(0,T) \times H^{(s+1)/3}(0,T) \times H^{s/3}(0,T).$$

For any $\vec{h} \in \mathcal{H}_{s,T}$,

$$\|\vec{h}\|_{\mathcal{H}_{s,T}} \equiv \left(\|h_1\|^2_{H^{(s+1)/3}(0,T)} + \|h_2\|^2_{H^{(s+1)/3}(0,T)} + \|h_3\|^2_{H^{s/3}(0,T)}\right)^{1/2}.$$

If $T = \infty$, denote $\mathcal{H}_{s,T}$ by $\mathcal{H}_s$. Combining Propositions 2.7–2.9 yields the following theorem about the linear IBVP (2.11).

Theorem 2.10. *For any $\vec{h} \in \mathcal{H}_0$, the IBVP Eq. (2.11) admits a unique solution*

$$u(x,t) = [W_b(t)\vec{h}(t)](x)$$

which belongs to the space $C_b(R^+; L_2(0,1)) \cap L_2(R^+; H^1(0,1))$ with $u_x \in C_b([0,1]; L_2(R^+))$. Moreover there exists a constant C such that

$$\|u\|_{L_2(R^+;H^1(0,1))} + \sup_{0 \leq t < +\infty} \|u(\cdot,t)\|_{L_2(0,1)} \leq C\|\vec{h}\|_{\mathcal{H}_0} \tag{2.23}$$

and

$$\sup_{x \in (0,1)} \|u_x(x,\cdot)\|_{L_2(R^+)} \leq C\|\vec{h}\|_{\mathcal{H}_0} \tag{2.24}$$

for all $\vec{h} \in \mathcal{H}_0$.

Remark 2.11. The estimate (2.23) reveals a Kato-type smoothing effect of the system (2.11) while (2.24) shows that the system (2.11) possesses a stronger smoothing effect, the so-called sharp Kato smoothing (see Kato (1983), Kenig et al. (1991a); Vega (1988)). It is this smoothing property that provides the rationale for being able to impose in a strong sense the boundary condition $u_x(1,t) = h_3(t)$ in (2.11).

Remark 2.12. The condition imposed on $\vec{h}$ in Theorem 2.10 appears to be sharp in the context of our approach to the analysis. From the explicit solution formula for (2.11), using arguments similar to those appearing in the proof of Propositions 2.7–2.9, one shows that $u \in C_b([0,1]; H_t^{1/3}(R^+))$ and

$$\sup_{x\in(0,1)} \|u(x,\cdot)\|_{H^{1/3}(R^+)} \le C\|\vec{h}\|_{\mathcal{H}_0}.$$

Of course, this does not mean that results with $s < 0$ are not possible, mearly that the present argument would not be adequate to the task.

Finally we return to the homogeneous IBVP (2.1) to show that it possesses the sharp Kato-smoothing property. Let a function ϕ be defined on the interval $(0,1)$ and let ϕ^* be its extension by zero to the whole line R. Assume that $v = v(x,t)$ is the solution of

$$v_t + v_x + v_{xxx} = 0, \qquad v(x,0) = \phi^*(x)$$

for $x \in R, t \ge 0$. If

$$g_1(t) = v(0,t), \quad g_2(t) = v(1,t), \quad g_3(t) = v_x(1,t),$$

then in terms of $W_b(t)$ defined in (2.22),

$$v_{\vec{g}} = v_{\vec{g}}(x,t) \equiv W_b(t)\vec{g}$$

is the corresponding solution of the nonhomogeneous boundary-value problem Eq. (2.1) with boundary conditions $h_j(t) = g_j(t), j = 1,2,3$, for $t \ge 0$. It is clear that for $x \in (0,1)$, the function $v(x,t) - v_{\vec{g}}(x,t)$ solves the IBVP (2.1), and this in turn leads to a representation of the semigroup $W_0(t)$ in terms of $W_b(t)$ and $W_R(t)$, where $W_R(t)$ is the C_0-semigroup in the space $L_2(R)$ generated by the operator A_R defined by

$$A_R f = -f' - f'''$$

with domain $\mathcal{D}(A_R) = H^3(R)$ and $v(x,t) = W_R(t)\phi^*(x)$.

Proposition 2.13. *For any $\phi \in L_2(0,1)$, if ϕ^* is its zero-extension to R, then $W_0(t)\phi$ may be written in the form*

$$W_0(t)\phi = W_R(t)\phi^* - W_b(t)\vec{g}$$

for any $x \in (0,1)$, $t > 0$, where $\vec{g} = (g_1, g_2, g_3)$,

$$g_1(t) = v(0,t), \quad g_2(t) = v(1,t), \quad g_3(t) = v_x(1,t),$$

with $v = W_R(t)\phi^*$.

Remark 2.14. Of course, $\vec{g}$ depends upon the particular extension ϕ^* of ϕ chosen here. However, the intrusion of ϕ^* is simply as an intermediary for obtaining the trace estimate in Proposition 2.16 below. It plays no other role in the theory.

To have appropriate estimates of $W_0(t)\phi$, the following trace result related to the semi-group $W_R(t)$ is needed.

Lemma 2.15. *There exists a constant* C *such that for any* $\psi \in L_2(R)$, $v(x,t) = W_R(t)\psi(x)$ *satisfies*

$$\sup_{x\in R} \|v(x,\cdot)\|_{H^{1/3}(R)} \leq C\|\psi\|_{L_2(R)}.$$

Moreover, $v_x \in C_b(R; L_2(R))$ *and*

$$\sup_{x\in R} \|v_x(x,\cdot)\|_{L_2(R)} \leq C\|\psi\|_{L_2(R)}.$$

Proof. This lemma follows as a special case of Lemma 2.1 in Kenig et al., (1991b) except the continuity of $v_x(x,\cdot)$ from R to the space $L_2(R)$, which can be verified using Fatou's lemma and the argument that appears in the proof of Proposition 2.7. □

The following estimate for $W_0(t)$ follows from Proposition 2.13, Lemma 2.15, and the estimates of $W_b(t)$ established earlier in Theorem 2.10.

Proposition 2.16. *For any* $\phi \in L_2(0,1)$, $u = W_0(t)\phi$ *has the property*

$$u_x \in C_b([0,1]; L_{2,t}(R^+))$$

and there exists a constant C *such that*

$$\sup_{x\in(0,1)} \|u_x(x,\cdot)\|_{L_2(R^+)} \leq C\|\phi\|_{L_2(0,1)}.$$

We conclude this section with the following proposition which, like the foregoing results, will be needed later (see Proposition 3.2).

Proposition 2.17. *Let* $T > 0$ *be given and*

$$u(x,t) = \int_0^t W_0(t-\tau)f(\cdot,\tau)\,d\tau.$$

Then

$$\sup_{x\in(0,1)} \|u_x(x,\cdot)\|_{L^2(0,T)} \leq C\int_0^T \|f(\cdot,\tau)\|_{L^2(0,1)}\,d\tau.$$

Proof. Observe that

$$u_x(x,t) = \int_0^t \partial_x(W_0(t-\tau)f(\cdot,\tau))\,d\tau = \int_0^T \xi_{(0,t)}(\tau)\partial_x(W_0(t-\tau)f(\cdot,\tau))\,d\tau$$

where

$$\xi_{(0,t)}(\tau) = \begin{cases} 1 & \text{if } \tau \in (0, t); \\ 0 & \text{if } \tau > t. \end{cases}$$

Using the Minkowski's integral inequality gives us

$$\begin{aligned} \|u_x(x,\cdot)\|_{L^2(0,T)} &\le \int_0^T \left(\int_0^T \left| \xi_{(0,t)}(\tau)\partial_x(W_0(t-\tau)f(\cdot,\tau)) \right|^2 dt \right)^{1/2} d\tau \\ &= \int_0^T \left(\int_\tau^T \left| \partial_x(W_0(t-\tau)f(\cdot,\tau)) \right|^2 dt \right)^{1/2} d\tau. \end{aligned}$$

Thus, invoking Proposition 2.16 (with the initial time τ) gives us

$$\begin{aligned} \sup_{x\in(0,1)} \|u_x(x,\cdot)\|_{L^2(0,T)} &\le \int_0^T \sup_{x\in(0,1)} \left(\int_\tau^T \left| \partial_x(W_0(t-\tau)f(\cdot,\tau)) \right|^2 dt \right)^{1/2} d\tau \\ &\le C \int_0^T \|f(\cdot,\tau)\|_{L^2(0,1)}\, d\tau. \end{aligned}$$

The proof is complete. □

Remark 2.18. It is worth highlighting the crucial role played by the formulas (2.14) and following, which provided an explicit representation of solutions directly in terms of the boundary data. Our theory devolves in large part on the efficacy of these formulas.

3. LOCAL WELL-POSEDNESS

In this section, attention will be given to the full nonlinear IBVP

$$\left.\begin{aligned} &u_t + u_x + uu_x + u_{xxx} = 0, \qquad u(x,0) = \phi(x), \\ &u(0,t) = h_1(t), \quad u(1,t) = h_2(t), \quad u_x(1,t) = h_3(t) \end{aligned}\right\} \tag{3.1}$$

introduced at the outset of our discussion.

For any $T > 0$ and $s \ge 0$, let $X_{s,T}$ be as defined in (1.11) with its usual product topology and let $Y_{s,T}$ be the collection of

$$v \in C([0,T]; H^s(0,1)) \cap L_2([0,T]; H^{s+1}(0,1))$$

with $v_x \in C([0,1]; L_2(0,T))$. A norm $\|\cdot\|_{Y_{s,T}}$ on the space $Y_{s,T}$ is defined by

$$\|v\|_{Y_{s,T}} := \left(\|v\|^2_{C([0,T];H^s(0,1))} + \|v\|^2_{L_2([0,T];H^{s+1}(0,1))} + \|v_x\|^2_{C([0,1];L_2(0,T))} \right)^{1/2}$$

for $v \in Y_{s,T}$.[a] The space $Y_{s,T}$ possesses the following helpful property.

[a]The reader may notice that the space $Y_{s,T}$ need not include the finiteness of $\|v_x\|_{C([0,1];L_2(0,T))}$ for the arguments that follow to be valid, and hence for proving well-posedness of the IBVP Eq. (3.1) in the space $H^s(0,1)$. However, by keeping this term, we are able to determine at a stroke that solutions of Eq. (3.1) possess the sharp Kato smoothing effect.

Lemma 3.1. *Let $s \geq 0$ be given. There exists a constant C such that for any $T > 0$ and $u, v \in Y_{s,T}$,*

$$\int_0^T \|(u(\cdot,t)v(\cdot,t))_x\|_{H^s(0,1)}\,dt \leq C(T^{1/2}+T^{1/3})\|u\|_{Y_{s,T}}\|v\|_{Y_{s,T}}. \tag{3.2}$$

Proof. The proof is given for $0 \leq s \leq 1$. The proof for other values of s is similar. Notice first that

$$\int_0^T \|(u(\cdot,t)v(\cdot,t))_x\|_{L_2(0,1)}\,dt \leq \int_0^T \|u_x(\cdot,t)v(\cdot,t)\|_{L_2(0,1)}\,dt + \int_0^T \|u(\cdot,t)v_x(\cdot,t)\|_{L_2(0,1)}\,dt.$$

Using the Poincare inequality, there obtains

$$\begin{aligned}\|u(\cdot,t)v_x(\cdot,t)\|_{L_2(0,1)} &\leq \|u(\cdot,t)\|_{L_\infty(0,1)}\|v_x(\cdot,t)\|_{L_2(0,1)}\\ &\leq C(\|u(\cdot,t)\|_{L_2(0,1)} + \|u(\cdot,t)\|_{L_2(0,1)}^{1/2}\|u_x(\cdot,t)\|_{L_2(0,1)}^{1/2})\|v_x(\cdot,t)\|_{L_2(0,1)}.\end{aligned}$$

These two terms, when integrated with respect to t, are bounded thusly:

$$\begin{aligned}\int_0^T \|u(\cdot,t)\|_{L_2(0,1)}\|v_x(\cdot,t)\|_{L_2(0,1)}\,dt &\leq \sup_{0\leq t\leq T}\|u(\cdot,t)\|_{L_2(0,1)}\int_0^T \|v_x(\cdot,t)\|_{L_2(0,1)}\,dt\\ &\leq T^{1/2}\sup_{0\leq t\leq T}\|u(\cdot,t)\|_{L_2(0,1)}\left(\int_0^T \|v_x(\cdot,t)\|_{L_2(0,1)}^2\,dt\right)^{1/2}\\ &\leq CT^{1/2}\|u\|_{Y_{0,T}}\|v\|_{Y_{0,T}}\end{aligned}$$

and

$$\begin{aligned}&\int_0^T \|u(\cdot,t)\|_{L_2(0,1)}^{1/2}\|u_x(\cdot,t)\|_{L_2(0,1)}^{1/2}\|v_x(\cdot,t)\|_{L_2(0,1)}\,dt\\ &\leq \sup_{0\leq t\leq T}\|u(\cdot,t)\|_{L_2(0,1)}^{1/2}\left(\int_0^T \|u_x(\cdot,t)\|_{L_2(0,1)}^2\,dt\right)^{1/4}\left(\int_0^T \|v_x(\cdot,t)\|_{L_2(0,1)}^{4/3}\,dt\right)^{3/4}\\ &\leq CT^{1/3}\|u\|_{Y_{0,T}}\|v\|_{Y_{0,T}}.\end{aligned}$$

The last three inequalities combine to establish that

$$\int_0^T \|u(\cdot,t)v_x(\cdot,t)\|_{L_2(0,1)}\,dt \leq C(T^{1/2}+T^{1/3})\|u\|_{Y_{0,T}}\|v\|_{Y_{0,T}}.$$

Similarly, one sees that

$$\int_0^T \|u_x(\cdot,t)v(\cdot,t)\|_{L_2(0,1)}\,dt \leq C(T^{1/2}+T^{1/3})\|u\|_{Y_{0,T}}\|v\|_{Y_{0,T}}.$$

In consequence, estimate (3.2) holds with $s = 0$. To see that (3.2) is true for $s = 1$, argue as follows. Observe that

$$\|(u(\cdot,t)v(\cdot,t))_x)\|_{H^1(0,1)} \leq \|(u(\cdot,t)v(\cdot,t))_x\|_{L_2(0,1)} + \|(u(\cdot,t)v(\cdot,t))_{xx}\|_{L_2(0,1)}$$

and that

$$\|(u(\cdot,t)v(\cdot,t))_{xx}\|_{L_2(0,1)} \le \|(u_x(\cdot,t)v(\cdot,t))_x\|_{L_2(0,1)} + \|(u(\cdot,t)v_x(\cdot,t))_x\|_{L_2(0,1)}.$$

The inequality (3.2) with $s=0$, just established, gives

$$\begin{aligned}\int_0^T \|(u(\cdot,t)v(\cdot,t))_{xx}\|_{L_2(0,1)}\,dt &\le C(T^{1/2}+T^{1/3})(\|u_x\|_{Y_{0,T}}\|v\|_{Y_{0,T}} + \|u\|_{Y_{0,T}}\|v_x\|_{Y_{0,T}})\\ &\le C(T^{1/2}+T^{1/3})\|u\|_{Y_{1,T}}\|v\|_{Y_{1,T}}\end{aligned}$$

which together with (3.2) (again with $s=0$) yields

$$\int_0^T \|u(\cdot,t)v(\cdot,t)\|_{H^1(0,1)}\,dt \le C(T^{1/2}+T^{1/3})\|u\|_{Y_{1,T}}\|v\|_{Y_{1,T}}.$$

The estimate (3.2) with $0<s<1$ now follows from the nonlinear interpolation theory developed in Bona and Scott (1976). The proof is complete. □

The next step is to show that the IBVP (3.1) is locally well-posed in the space $X_{0,T}$.

Proposition 3.2. *Let $T>0$ be given. For any $(\phi,\vec{h}) \in X_{0,T}$ with $\vec{h}=(h_1,h_2,h_3)$, there is a $T^* \in (0,T]$ depending on $\|(\phi,\vec{h})\|_{X_{0,T}}$ such that the IBVP Eq.* (3.1) *admits a unique solution $u \in Y_{0,T^*}$. Moreover, for any $T' < T^*$, there is a neighborhood U of $(\phi,\vec{h})$ such that the IBVP Eqs.* (1.1)–(1.3) *admits a unique solution in the space $Y_{0,T'}$ for any $(\psi,\vec{h}_1) \in U$ and the corresponding solution map from U to $Y_{0,T'}$ is Lipschitz continuous.*

Proof. Write the IBVP (3.1) in its integral equation form

$$u(t) = W_0(t)\phi + W_b(t)\vec{h} - \int_0^t W_0(t-\tau)(uu_x)(\tau)\,d\tau \tag{3.3}$$

where the operator $W_b(t)$ is as defined in formulas (2.14) and (2.22) in Sec. 2 and the spatial variable is suppressed throughout. For given $(\phi,\vec{h}) \in X_{0,T}$, let $r>0$ and $\theta>0$ be constants to be determined. Let

$$S_{\theta,r} = \{v \in Y_{0,\theta}, \|v\|_{Y_{0,\theta}} \le r\}.$$

The set $S_{\theta,r}$ is a closed, convex, and bounded subset of the space $Y_{0,\theta}$ and therefore is a complete metric space in the topology induced from $Y_{0,\theta}$. Define a map Γ on $S_{\theta,r}$ by

$$\Gamma(v) = W_0(t)\phi + W_b(t)\vec{h} - \int_0^t W_0(t-\tau)(vv_x)(\tau)\,d\tau$$

for $v \in S_{\theta,r}$. The crux of the matter is the following inequality. For any $v \in S_{\theta,r}$,

$$\begin{aligned}\|\Gamma(v)\|_{Y_{0,\theta}} &\le C_0\|(\phi,\vec{h})\|_{X_{0,T}} + C_1\int_0^\theta \|vv_x(\cdot,\tau)\|_{L_2(0,1)}\,d\tau\\ &\le C_0\|(\phi,\vec{h})\|_{X_{0,T}} + C_1(\theta^{1/2}+\theta^{1/3})\|v\|^2_{Y_{0,\theta}}\end{aligned}$$

where C_0 and C_1 are constants. As the norm on $Y_{0,\theta}$ has three parts, this amounts to three inequalities, all of which follow immmediately from the linear estimates in Sec. 2 and in Lemma 3.1. Choosing $r > 0$ and $\theta > 0$ so that

$$\begin{cases} r = 2C_0\|(\phi, \vec{h})\|_{X_{0,T}}, \\ C_1(\theta^{1/2} + \theta^{1/3})r \leq \frac{1}{2}, \end{cases} \tag{3.4}$$

then

$$\|\Gamma(v)\|_{Y_{0,\theta}} \leq r$$

for any $v \in S_{\theta,r}$. Thus, with such a choice of r and θ, Γ maps $S_{\theta,r}$ into $S_{\theta,r}$. The same inequalities allow one to deduce that for r and θ chosen as in (3.4),

$$\|\Gamma(v_1) - \Gamma(v_2)\|_{Y_{0,\theta}} \leq \frac{1}{2}\|v_1 - v_2\|_{Y_{0,\theta}}$$

for any $v_1, v_2 \in S_{\theta,r}$. In other words, the map Γ is a contraction mapping of $S_{r,\theta}$. Its fixed point $u = \Gamma(u)$ is the unique solution of the IBVP (1.1)–(1.3) in $S_{\theta,r}$. □

Consider the forced linear problem

$$\left.\begin{aligned} &u_t + u_x + u_{xxx} = f, \quad u(x,0) = \phi(x), \\ &u(0,t) = h_1(t), \quad u(1,t) = h_2(t), \quad u_x(1,t) = h_3(t). \end{aligned}\right\} \tag{3.5}$$

Applying the linear estimates derived in Sec. 2, for $(\phi, \vec{h}) \in X_{0,T}$ and $f \in L_1(0, T; L_2(0,1))$, the corresponding solution u of (3.5) belongs to the space $Y_{0,T}$ and satisfies

$$\|u\|_{Y_{0,T}} \leq C(\|(\phi, \vec{h})\|_{X_{0,T}} + \|f\|_{L_1(0,T;L_2(0,1))}) \tag{3.6}$$

for some constant C independent of ϕ, h_j, $j = 1,2,3$ and f. The next lemma gives an estimate for solutions of (3.5) in the space $Y_{s,T}$ with s in the range of $0 \leq s \leq 3$.

Lemma 3.3. *For given $T > 0$ and s in the range* $[0,3]$, *let there be given $f \in W^{s/3,1}([0,T]; L_2(0,1))$ and $(\phi, \vec{h}) \in X_{s,T}$ satisfying the compatibility conditions*

$$\begin{cases} \phi(0) = h_1(0), \quad \phi(1) = h_2(0) & \text{if } 1/2 < s \leq 3/2, \text{ or} \\ \phi(0) = h_1(0), \quad \phi(1) = h_2(0), \quad \phi'(1) = h_3(0) & \text{if } 3/2 < s \leq 3. \end{cases} \tag{3.7}$$

Then Eq. (3.5) *admits a unique solution $u \in Y_{s,T}$ and*

$$\|u\|_{Y_{s,T}} \leq C(\|(\phi, \vec{h})\|_{X_{s,T}} + \|f\|_{W^{s/3,1}(0,T;L_2(0,1))}) \tag{3.8}$$

for some constant $C > 0$ independent of ϕ, $\vec{h}$, and f. Moreover, if $s = 3$, $u_t \in Y_{0,T}$ and

$$\|u_t\|_{Y_{0,T}} \leq C(\|(\phi, \vec{h})\|_{X_{3,T}} + \|f\|_{W^{s/3,1}(0,T;L_2(0,1))}).$$

Proof. The proof is provided for $s = 3$ since the result for other values of s can be established by interpolation and (3.6). For the solution u of (3.5), let $v = u_t$. Then the function v is a solution of

$$\left.\begin{aligned} &v_t + v_x + v_{xxx} = f_t, \quad v(x,0) = f(x,0) - \phi'''(x) - \phi'(x), \\ &v(0,t) = h_1'(t), \quad v(1,t) = h_2'(t), \quad v_x(1,t) = h_3'(t). \end{aligned}\right\} \tag{3.9}$$

Applying (3.6) to v in (3.9) yields that

$$\|v\|_{Y_{0,T}} \leq C(\|f_t\|_{L_1(0,T;(0,1))} + \|(f(\cdot,0) - \phi'''(\cdot) - \phi'(\cdot), \vec{h}')\|_{X_{0,T}}).$$

Define

$$u(x,t) = \int_0^t v(x,\tau)\,d\tau + \phi(x).$$

Then $u(x,0) = \phi(x)$ and

$$\begin{aligned} u(0,t) &= \int_0^t v(0,\tau)\,d\tau + \phi(0) \\ &= \int_0^t h_1'(\tau)\,d\tau + \phi(0) \\ &= h_1(t) - h_1(0) + \phi(0) = h_1(t). \end{aligned}$$

Similarly, $u(1,t) = h_2(t)$ and $u_x(1,t) = h_3(t)$. Furthermore, it is easily verified that

$$\begin{aligned} &u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) \\ &\quad = v(x,t) + \int_0^t (v_x(x,\tau) + v_{xxx}(x,\tau))\,d\tau + \phi'(x) + \phi'''(x) \\ &\quad = v(x,0) + \int_0^t (f_t(x,\tau) - v_x(x,\tau) - v_{xxx}(x,\tau))\,d\tau \\ &\qquad + \int_0^t (v_x(x,\tau) + v_{xxx}(x,\tau))\,d\tau + \phi'(x) + \phi'''(x) = 0. \end{aligned}$$

Thus u solves the IBVP (3.5). Since

$$u_{xxx} = f - u_t - u_x = f - v - u_x,$$

it follows that $u \in Y_{3,T}$ and satisfies (3.8) with $s = 3$. The proof is complete. □

Here is the promised local well-posedness result for the IBVP (3.1) in $X_{s,T}$.

Theorem 3.4. *Let $T > 0$ and $s \geq 0$ be given. Suppose that $(\phi, \vec{h}) \in X_{s,T}$ satisfies the s-compatibility conditions. Then there exists a $T^* \in (0,T]$ depending only on $\|(\phi, \vec{h})\|_{X_{s,T}}$ such that Eq. (3.1) admits a unique solution $u \in Y_{s,T^*}$ with*

$$\partial_t^j u \in Y_{s-3j,T^*}$$

for $j = 0, 1, 2, \ldots, [s/3] - 1, [s/3]$. *Moreover, for any* $T' < T^*$, *there is a neighborhood* U *of* $(\phi, \vec{h})$ *such that the IBVP Eq.* (3.1) *admits a unique solution in the space* $Y_{s,T'}$ *for any* $(\psi, \vec{h}_1) \in U$ *and the corresponding solution map is Lipschitz continuous.*

Proof. For given s-compatible $(\phi, \vec{h}) \in X_{s,T}$, let $r > 0$ and $\theta > 0$ be given and $S_{\theta,r}$ be the collection of functions v in the space $C([0,\theta]; L_2(0,1)) \cap L_2(0,\theta; H^1(0,1))$ satisfying

$$\partial_t^j v \in Y_{3,\theta}, \quad \text{for } j = 0, 1, 2, \ldots, [s/3] - 1 \text{ and } \partial_t^{[s/3]} v \in Y_{s-3[s/3],\theta},$$

and

$$\|\partial_t^{[s/3]} v\|_{Y_{s-3[s/3]}} + \sum_{j=0}^{[s/3]-1} \|\partial_t^j v\|_{Y_{3,\theta}} \le r.$$

Let

$$\mathcal{Y}_{s,\theta} = Y_{s-3[s/3],\theta} \times \prod_{j=0}^{[s/3]-1} Y_{3,\theta}$$

with the usual product topology. Then the set $S_{\theta,r}$ may be viewed as a closed subset of $\mathcal{Y}_{s,\theta}$ via the mapping $v \to (v, \partial_t v, \ldots, \partial_x^{[s/3]} v) \equiv \vec{v}$, and therefore is a complete metric space. For any $v \in S_{\theta,r}$, consider the system of equations

$$\left.\begin{array}{ll} u_t^{(k)} + u_x^{(k)} + u_{xxx}^{(k)} = -\frac{1}{2}\partial_x\left(\sum_{j=0}^{k} \frac{k!}{j!(k-j)!} v^{(j)} v^{(k-j)}\right), & u^{(k)}(x,0) = \phi_k(x), \\ u(0,t) = h_1^{(k)}(t), \quad u(1,t) = h_2^{(k)}(t), \quad u_x(1,t) = h_3^{(k)}(t), & \end{array}\right\} \tag{3.10}$$

for $k = 0, 1, 2, \ldots, [s/3]$, where $u^{(k)} \equiv \partial_t^k u$, $v^{(k)} \equiv \partial_t^k v$ and ϕ_k, $h_1^{(k)}$, $h_2^{(k)}$, $h_3^{(k)}$ are defined in (1.7) and (1.8). By Lemma 3.3, the IBVP (3.10) defines a map Γ from $S_{\theta,r}$ to the space $\mathcal{Y}_{s,\theta}$. Moreover,

$$\|\Gamma(\vec{v})\|_{\mathcal{Y}_{s,\theta}} \le C\|(\phi, \vec{h})\|_{X_{s,T}} + C(\theta^{1/2} + \theta^{1/3})\|\vec{v}\|_{\mathcal{Y}_{s,\theta}}$$

for some constant C independent of $\vec{h}, \phi$, and θ. Thus, the argument presented in the proof of Proposition 3.2 shows that Γ is a contraction map from $S_{\theta,r}$ to $S_{\theta,r}$ if r and θ are appropriately chosen. As a result, its fixed point $\vec{u} \in S_{\theta,r}$ is the unique solution of (3.5). Thus the proof is complete when $s \le 3$. In case $s > 3$, the result just established shows that

$$u^{(j)} \in C([0,\theta]; H^3(0,1)) \cap L_2([0,\theta]; H^4(0,1))$$

for $j = 0, 1, \ldots, [s/3] - 1$ and

$$u^{[s/3]} \in Y_{s-3[s/3],\theta} = C([0,\theta]; H^{s-3[s/3]}(0,1)) \cap L_2([0,\theta]; H^{s+1-3[s/3]}(0,1)).$$

In case $k = [s/3] - 1$, (3.10) implies that

$$u_{xxx}^{([s/3]-1)} = -u_t^{([s/3]-1)} - u_x^{([s/3]-1)} - \frac{1}{2}\partial_x\left(\sum_{j=0}^{([s/3]-1)} C_{[s/3]-1}^j u^{(j)} u^{([s/3]-1-j)}\right).$$

We thereby arrive at the conclusion

$$u^{([s/3]-1)} \in C([0,\theta]; H^{s-3[s/3]+2}(0,1)).$$

It is further implied that the left-hand side of the last equation belongs to

$$C([0,\theta]; H^{s-3([s/3]}(0,1)) \cap L_2([0,\theta]; H^{s+1-3[s/3]}(0,1)).$$

Consequently, it must be the case that

$$u^{([s/3]-1)} \in C([0,\theta]; H^{s+3-3([s/3]}(0,1)) \cap L_2([0,\theta]; H^{s+4-3[s/3]}(0,1)).$$

Repeating this argument if necessary yields that $u \in C([0,\theta]; H^s(0,1)) \cap L_2([0,\theta]; H^{s+1}(0,1))$ with

$$\partial_t^j u \in C([0,\theta]; H^{s-3j}(0,1)) \cap L_2([0,\theta]; H^{s+1-3j}(0,1))$$

for $j = 1, 2, \ldots, [s/3]-1$ and

$$\partial_t^{[s/3]} u \in C([0,\theta]; H^{s-3[s/3]}(0,1)) \cap L_2([0,\theta]; H^{s+1-3[s/3]}(0,1)).$$

The proof is complete. □

4. GLOBAL WELL-POSEDNESS

The results presented in Theorem 3.4 are local in the sense that the time interval $(0, T^*)$ on which the solution exists depends on $\|(\phi, \vec{h})\|_{X_{s,T}}$. In general, the larger $\|(\phi, \vec{h})\|_{X_{s,T}}$, the smaller will be T^*. However, if $T^* = T$ no matter what the size of $\|(\phi, \vec{h})\|_{X_{s,T}}$, the IBVP (3.1) is said to be globally well-posed. In this section we study global well-posedness of the problem (3.1). First we introduce a helpful Banach space. For given $s \geq 0$ and $T > 0$, let

$$Z_{s,T} \equiv H^s(0,1) \times H^{\epsilon+(5s+9)/18}(0,T) \times H^{\epsilon+(5s+9)/18}(0,T) \times H^{\epsilon+(5s+3)/18}(0,T)$$

if $0 \leq s \leq 3$ and

$$Z_{s,T} \equiv X_{s,T}$$

if $s > 3$, where ϵ is any positive constant. Of course, for $s \leq 3$, $Z_{s,T}$ depends on ϵ, but this dependence is suppressed. The Sobolev indices when s lies in $[0, 3]$ may look a little odd. We feel it likely that they are an artifact of our proof. The strange indices derive from slightly inadequate smoothing results and are the best we can do with what is in hand. Note this inelegance ceases as soon as $s \geq 3$, and hence for the case of classical solutions when $s > 7/2$. The same issue arose in Bona et al., (2001) for the quarter-plane problem, so the issue does not necessarily devolve upon the third boundary condition $u_x(1,t) = h_3(t)$.

Theorem 4.1. *Let $T > 0$ and $s \geq 0$. For any s-compatible $(\phi, \vec{h}) \in Z_{s,T}$, the IBVP Eq.* (3.1) *admits a unique solution $u \in Y_{s,T}$ with $\partial_t^j u \in Y_{s-3j,T}$ for $j = 0, 1, 2, \ldots, [s/3]$. Moreover, the corresponding solution map of the IBVP Eq.* (3.1) *is Lipschitz continuous.*

Proof of Theorem 4.1. In the context of an established local well-posedness result, it suffices to prove the following global a priori H^s-estimate for smooth solutions of the IBVP (3.1).

Proposition 4.2. *For given $T > 0$ and $s \geq 0$, there exists a continuous and non-decreasing function $\gamma_s : R^+ \to R^+$ such that for any smooth solution u of Eq.* (3.1),

$$\sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{H^s(0,1)} \leq \gamma_s(\|(\phi, \vec{h})\|_{Z_{s,T}}). \tag{4.1}$$

The proof of Proposition 4.3 consists of four parts. In part (i), estimate (4.1) is shown to be true for $s = 0$. In part (ii), estimate (4.1) is shown to be true for $s = 3$. Then, in part (iii), Tartar's nonlinear interpolation theory is used to show that (4.1) holds for $0 < s < 3$. The validity of (4.1) for other values of s is established in part (iv).

Part (i). For a smooth solution u of the IBVP (3.1), write $u = w + v$, where v solves

$$\left.\begin{aligned} &v_t + v_x + v_{xxx} = 0, \qquad v(x,0) = \psi(x), \\ &v(0,t) = h_1(t), \quad v(1,t) = h_2(t), \quad v_x(1,t) = h_3(t), \end{aligned}\right\} \tag{4.2}$$

with

$$\psi(x) = (1-x)h_1(0) + xh_2(0) + x(1-x)(h_3(0) - h_2(0) + h_1(0))$$

and w solves

$$\left.\begin{aligned} &w_t + w_x + ww_x + w_{xxx} = -(wv)_x - vv_x, \qquad w(x,0) = \phi(x) - \psi(x), \\ &w(0,t) = 0, \quad w(1,t) = 0, \quad w_x(1,t) = 0. \end{aligned}\right\} \tag{4.3}$$

By Lemma 3.3

$$\|v\|_{Y_{s,T}} \leq C\|(\psi, \vec{h})\|_{X_{s,T}} \tag{4.4}$$

for $0 \leq s \leq 3$. In particular, for $3/2 < s \leq 3$,

$$\|v\|_{Y_{s,T}} \leq C(\|h_1\|_{H^{s/3}(0,T)} + \|h_2\|_{H^{s/3}(0,T)} + \|h_3\|_{H^{(s-1)/3}(0,T)}) \tag{4.5}$$

since

$$\|\psi\|_{H^s(0,1)} \leq C(\|h_1\|_{H^{s/3}(0,T)} + \|h_2\|_{H^{s/3}(0,T)} + \|h_3\|_{H^{(s-1)/3}(0,T)}).$$

Multiply both sides of the equation in (4.3) by w and integrate over $(0, 1)$ with respect to x. Integration by parts leads to

$$\frac{d}{dt}\|w(\cdot,t)\|^2_{L_2(0,1)} \leq C\int_0^1 |v_x(\cdot,t)w^2(\cdot,t)|\,dx + C\int_0^1 |v_x(\cdot,t)v(\cdot,t)w(\cdot,t)|\,dx.$$

Observe that

$$\int_0^1 |v_x(\cdot,t)w^2(\cdot,t)|\,dx \le \sup_{x\in(0,1)} |v_x(x,t)|\,\|w(\cdot,t)\|^2_{L_2(0,1)}$$
$$\le C_\epsilon \|v(\cdot,t)\|_{H^{3/2+\epsilon}(0,1)}\|w(\cdot,t)\|^2_{L_2(0,1)}$$

and

$$\int_0^1 |v_x(\cdot,t)v(\cdot,t)w(\cdot,t)|\,dx \le \sup_{x\in(0,1)} |v_x(x,t)|\,\|v(\cdot,t)\|_{L_2(0,1)}\|w(\cdot,t)\|_{L_2(0,1)}$$
$$\le C_\epsilon \|v(\cdot,t)\|^2_{H^{3/2+\epsilon}(0,1)}\|w(\cdot,t)\|_{L_2(0,1)}$$

where ϵ is any fixed positive constant.

In consequence, one has that

$$\frac{d}{dt}\|w(\cdot,t)\|_{L_2(0,1)} \le C_\epsilon\|v(\cdot,t)\|_{H^{3/2+\epsilon}(0,1)}\|w(\cdot,t)\|_{L_2(0,1)} + C_\epsilon\|v(\cdot,t)\|^2_{H^{3/2+\epsilon}(0,1)}$$

for any $t \ge 0$ The estimate (4.1) with $s = 0$ then follows by using Gronwall's inequality and (4.5).

Part (ii). For a smooth solution u, $v = u_t$ solves

$$\left.\begin{aligned} &v_t + v_x + (uv)_x + v_{xxx} = 0, \quad v(x,0) = \phi^*(x) \\ &v(0,t) = h_1'(t), \quad v(1,t) = h_2'(t), \quad v_x(1,t) = h_3'(t) \end{aligned}\right\}$$

where $\phi^*(x) = -\phi'(x) - \phi'(x)\phi(x) - \phi'''(x)$. By Lemma 3.3, there exists a constant $C > 0$ such that for any $T' \le T$,

$$\|v\|_{Y_{0,T'}} \le C\|(\phi^*, \vec{h}')\|_{X_{0,T}} + C(T'^{1/2} + T'^{1/3})\|u\|_{Y_{0,T}}\|v\|_{Y_{0,T}}.$$

Choose $T' \le T$ such that $C(T'^{1/2} + T'^{1/3})\|u\|_{Y_{0,T}} = 1/2$; with such a choice, $\|v\|_{Y_{0,T'}} \le 2C\|(\phi^*, \vec{h}')\|_{X_{0,T}}$. Note that T' only depends on $\|u\|_{Y_{0,T}}$, and therefore depends only on $\|(\phi, \vec{h})\|_{Z_{0,T}}$ by the estimate proved in Part (i). By a standard density argument,

$$\|v\|_{Y_{0,T}} \le C_1\|(\phi, \vec{h})\|_{Z_{3,T}}$$

where C_1 depends only on T and $\|(\phi, \vec{h})\|_{Z_{0,T}}$. The estimate (4.1) with $s = 3$ then follows from

$$v = -(u_{xxx} + u_x + uu_x)$$

by a now familiar argument.

Part (iii). Here is a précis of the (real) interpolation theory as it will be used below. Let B_0 and B_1 be two Banach spaces such that $B_1 \subset B_0$ with the inclusion map continuous. Let $f \in B_0$ and, for $t \ge 0$, define

$$K(f,t) = \inf_{g\in B_1}\{\|f - g\|_{B_0} + t\|g\|_{B_1}\}.$$

For $0 < \theta < 1$ and $1 \le p \le +\infty$, define

$$[B_0, B_1]_{\theta,p} = B_{\theta,p} = \left\{ f \in B_0 : \|f\|_{\theta,p} = \left(\int_0^{+\infty} K(f,t)^p t^{-\theta p - 1} dt \right)^{1/p} < +\infty \right\}$$

with the usual modification for the case $p = +\infty$. Then $B_{\theta,p}$ is a Banach space with norm $\| \|_{\theta,p}$. Given two pairs of indices (θ_1, p_1) and (θ_2, p_2) as above, then $(\theta_1, p_1) \prec (\theta_2, p_2)$ means

$$\begin{cases} \theta_1 < \theta_2, & \text{or} \\ \theta_1 = \theta_2 & \text{and } p_1 > p_2. \end{cases}$$

If $(\theta_1, p_1) \prec (\theta_2, p_2)$ then $B_{\theta_2,p_2} \subset B_{\theta_1,p_1}$ with the inclusion map continuous.

Theorem 4.3. (Bona and Scott, 1976) *Let B_0^j and B_1^j be Banach spaces such that $B_1^j \subset B_0^j$ with continuous inclusion mappings, $j = 1, 2$. Let λ and q lie in the ranges $0 < \lambda < 1$ and $1 \leq q \leq +\infty$. Suppose A is a mapping such that*

(*i*) $A : B^1_{\lambda,q} \to B_0^2$ *and for* $f,\ g \in B^1_{\lambda,q}$,

$$\|Af - Ag\|_{B_0^2} \leq C_0(\|f\|_{B^1_{\lambda,q}} + \|g\|_{B^1_{\lambda,q}})\|f - g\|_{B_0^1}$$

and

(ii) $A :\ B_1^1 \to B_1^2$ *and for* $h \in B_1^1$

$$\|Ah\|_{B_1^2} \leq C_1(\|h\|_{B^1_{\lambda,q}})\|h\|_{B_1^1},$$

where $C_j : R^+ \to R^+$ *are continuous nondecreasing functions,* $j = 0,\ 1$.

Then if $(\theta, p) \succeq (\lambda, q)$, *A maps* $B^1_{\theta,p}$ *into* $B^2_{\theta,p}$ *and for* $f \in B^1_{\theta,p}$

$$\|Af\|_{B^2_{\theta,p}} \leq C(\|f\|_{B^1_{\lambda,q}})\|f\|_{B^1_{\theta,p}},$$

where for $r > 0$, $C(r) = 4C_0(4r)^{1-\theta}C_1(3r)^{\theta}$.

Remark 4.4. This theorem is identical with Theorem 2 of Tartar (1972) except that Tartar makes the more restrictive assumption that the constants C_0 and C_1 depend only on the B_0^1 norms of the functions in question. Theorem 4.3 was used by Bona and Scott to give the first proof of global well-posedness of the pure initial-value problem for the KdV-equation on the whole line in fractional order Sobolev spaces $H^s(R)$.

To prove that estimate (4.1) holds for $T > 0$ and $0 \leq s \leq 3$, let

$$\mathcal{Z}_{s,T} = \{(\phi, \vec{h}) \in Z_{s,T} \text{ satisfying } s\text{-compatibility condition}\}$$

with the inherited norm from the space $Z_{s,T}$. Choose

$$B_0^1 = \mathcal{Z}_{0,T}, \quad B_1^1 = \mathcal{Z}_{3,T}, \quad B_0^2 = C([0,T]; L_2(0,1)), \quad B_1^2 = C([0,T]; H^3(0,1)).$$

Let A be the solution map of the IBVP (3.1): $u = A(\phi, \vec{h})$. For given s with $0 < s < 3$, choose $p = 2$ and $\theta = s/3$. Then

$$B^2_{\theta,p} = C([0,T]; H^s(0,1)), \qquad B^1_{\theta,p} = \mathcal{Z}_{s,T}.$$

In this case, assumption (ii) of Theorem 4.3 is (4.1) with $s = 3$, which we have already proved. It remains to verify assumption (i) of Theorem 4.3.

To this end, let $u_1 = A(\phi_1, \vec{h}_1)$, $u_2 = A(\phi_2, \vec{h}_2)$, and $w = u_1 - u_2$. It is seen that w solves the variable coefficient problem

$$\left.\begin{array}{l} w_t + w_x + (zw)_x + w_{xxx} = 0, \quad v(x,0) = \phi_1(x) - \phi_2(x) \\ w(0,t) = h_{1,1}(t) - h_{2,1}(t), \quad w_x(1,t) = h_{1,2}(t) - h_{2,2}(t), w(1,t) = h_{1,3}(t) - h_{2,3}(t) \end{array}\right\}$$

with $z = (1/2)(u_1 - u_2)$. Applying Lemma 3.3 with $s = 0$ yields that, for any $0 \le T' \le T$,

$$\begin{aligned} \|w\|_{Y_{0,T'}} &\le C(\|(\phi_1, \vec{h}_1) - (\phi_2, \vec{h}_2)\|_{X_{0,T}} + \|zw\|_{L_1(0,T';L_2(0,1))}) \\ &\le C\|(\phi_1, \vec{h}_1) - (\phi_2, \vec{h}_2)\|_{X_{0,T}} + C\big(T'^{1/2} + T'^{1/3}\big)\|z\|_{Y_{0,T}}\|w\|_{Y_{0,T'}}. \end{aligned}$$

Because of Part (ii), the estimate

$$\|z\|_{Y_{0,T}} \le \gamma(\|(\phi_1, \vec{h}_1\|_{Z_{0,T}} + \|(\phi_2, \vec{h}_2)\|_{Z_{0,T}})$$

is obtained for z. If T' is chosen such that

$$C\big(T'^{1/2} + T'^{1/3}\big)\|z\|_{Y_{0,T}} = 1/2$$

then it follows that

$$\|w\|_{Y_{0,T'}} \le 2C\|(\phi_1, \vec{h}_1) - (\phi_2, \vec{h}_2)\|_{X_{0,T}}. \tag{4.6}$$

Since T' only depends on $\|z\|_{Y_{0,T}}$ which in turn only depends on $\|(\phi_1, \vec{h}_1)\|_{Z_{0,T}} + \|(\phi_2, \vec{h}_2)\|_{Z_{0,T}}$, by a standard extension argument, one arrives at

$$\|w\|_{Y_{0,T}} \le \gamma(\|(\phi_1, \vec{h}_1)\|_{Z_{0,T}} + \|(\phi_2, \vec{h}_2)\|_{Z_{0,T}})\|(\phi_1, \vec{h}_1) - (\phi_2, \vec{h}_2)\|_{X_{0,T}}.$$

Thus assumption (i) of Theorem 4.3 is satisfied. Estimate (4.1) is established for $0 < s < 3$ by invoking Theorem 4.3.

Part (iv). We prove that (4.1) holds for $3 < s < 6$. The same argument can be invoked for $s \ge 6$. For a smooth solution u of the IBVP (3.1), $v = u_t$ solves

$$\left.\begin{array}{l} v_t + v_x + (uv)_x + v_{xxx} = 0, \quad v(x,0) = \phi^*(x) \\ v(0,t) = h_1'(t), \quad v(1,t) = h_2'(t), \quad v_x(1,t) = h_3'(t). \end{array}\right\}$$

Applying Lemma 3.3 for any $0 < T' \le T$ gives the inequality

$$\|v\|_{Y_{s-3,T'}} \le C\|(\phi, \vec{h})\|_{Z_{s,T}} + C\big(T'^{1/2} + T'^{1/3}\big)\|v\|_{Y_{s-3,T'}}\|u\|_{Y_{s-3,T}}$$

for some constant $C > 0$ independent of T' and $(\phi, \vec{h})$. Thus, if one chooses T' such that

$$C(T'^{1/2} + T'^{1/3})\|u\|_{Y_{s-3,T}} = 1/2,$$

then

$$\|v\|_{Y_{s-3,T'}} \le 2C\|(\phi, \vec{h})\|_{Z_{s,T}}.$$

Because T' only depends on $\|u\|_{Y_{s-3,T}}$, which by the estimate (4.1) proved in Part (iii) only depends on $\|(\phi, \vec{h})\|_{Z_{s-3,T}}$, one obtains

$$\|v\|_{Y_{s-3,T}} \le \gamma_{s-3}(\|(\phi, \vec{h})\|_{Z_{s-3,T}})\|(\phi, \vec{h})\|_{Z_{s,T}}.$$

Consequently,

$$\|u\|_{Y_{s,T}} \le C\gamma_{s-3}(\|(\phi, \vec{h})\|_{Z_{s-3,T}})\|(\phi, \vec{h})\|_{Z_{s,T}}$$

and the proof is complete. □

5. ANALYTICITY

For given $T > 0$ and $s \ge 0$, let $\mathcal{X}_{s,T}$ be the collection of all s-compatible functions $(\phi, \vec{h}) \in X_{s,T}$. By the definition of s-compatibility, $\mathcal{X}_{s,T}$ is a linear subspace of the Banach space $X_{s,T}$ if and only if $0 \le s \le 7/2$. When $0 \le s \le 7/2$, we consider $\mathcal{X}_{s,T}$ as a Banach space with its norm inherited from $X_{s,T}$. By the results established in Secs. 3 and 4, the IBVP (3.1) defines a nonlinear map K_I from $\mathcal{X}_{s,T}$ to the space $Y_{s,T}$ for any $s \ge 0$. From the proofs of the results given in Sec. 3, the map K_I is known to be locally Lipschitz continuous from $\mathcal{D}(K_I)$, the domain of K_I, to $Y_{s,T}$. In this section it is shown that this nonlinear map K_I is *analytic*. More precisely, when $0 \le s \le 7/2$, for any $g \in \mathcal{D}(K_I)$, there exists an $\eta > 0$ such that for any $w \in \mathcal{X}_{s,T}$ with $\|w\|_{\mathcal{X}_{s,T}} \le \eta$, we have $g + w \in \mathcal{X}_{s,T}$ and $K_I(g + w)$ has the following Taylor series expansion:

$$K_I(g + w) = K_I(g) + \sum_{n=1}^{\infty} \frac{K_I^{(n)}(g)[w^n]}{n!}$$

where $K_I^{(n)}(g)$ is the nth order Fréchet derivative of K_I evaluated at g and the series converges in the space Y_T^s. In case $s > 7/2$, the Taylor series expansion does not hold as just written since the space $\mathcal{X}_{s,T}$ is no longer a linear vector space. In this case, we consider the initial-boundary value problem for a general m-nonlinear system, which includes the IBVP (3.1) as a special case, and show that the corresponding nonlinear solution map $\mathcal{K}_I$ is analytic in this context.

In pursuit of this program, we present a well-posedness result for the linearized KdV-equation with variable coefficients, viz.

$$\left.\begin{aligned} &u_t + u_x + (au)_x + u_{xxx} = f(x,t), \qquad u(x,0) = \phi(x), \\ &u(0,t) = h_1(t), \quad u(1,t) = h_2(t), \quad u_x(1,t) = h_3(t). \end{aligned}\right\} \tag{5.1}$$

Proposition 5.1. *Let $0 \leq s \leq 3$ and $T > 0$ be given. Assume that $a \in Y_{s,T}$. Then for any $f \in H^{s/3}([0,T]; H^{(3-s)/3}(0,1))$ and $(\phi, \vec{h}) \in \mathcal{X}_{s,T}$, Eq. (5.1) admits a unique solution $u \in Y_{s,T}$ satisfying*

$$\|u\|_{Y_{s,T}} \leq C(\|(\phi, \vec{h})\|_{X_{s,T}} + \|f\|_{H^{s/3}([0,T];H^{s/3}(0,1))})$$

where $C > 0$ only depends on $\|a\|_{Y_{s,T}}$.

Proof. The proof is similar to that of Theorem 3.4, and so we only provide a sketch. For given $0 < \beta \leq T$ and $r > 0$, let

$$S_{\beta,r} = \{w \in Y_{s,\beta} : \|w\|_{Y_{s,\beta}} \leq r\}.$$

For specified $a \in Y_{s,T}, f \in H^{s/3}([0,T]; H^{s/3}(0,1))$ and $(\phi, \vec{h}, f) \in X_{s,T} \times H^{s/3}([0,T]; H^{s/3}(0,1))$ with $(\phi, \vec{h}) \in \mathcal{X}_{s,T}$, consider a map $\Gamma: \ S_{\beta,r} \to Y_{s,\beta}$ defined by

$$u = \Gamma(v)$$

where u is the unique solution of

$$\left.\begin{array}{l} u_t + u_x + u_{xxx} = f(x,t) - (av)_x, \qquad u(x,0) = \phi(x), \\ u(0,t) = h_1(t), \quad u(1,t) = h_2(t), \quad u_x(1,t) = h_3(t), \end{array}\right\}$$

for $v \in S_{\beta,r}$. Applying Lemmas 3.3 and 3.1 yields

$$\|\Gamma(v))\|_{Y_{s,\beta}} \leq C\|(\phi, \vec{h}, f)\|_{X_{s,T}\times H^{s/3}(0,T;H^{s/3}(0,1))} + C(\beta^{1/3} + \beta^{1/2})\|a\|_{Y_{s,T}}\|v\|_{Y_{s,\beta}}.$$

Choose $0 < \beta \leq T$ and r such that

$$r = 2C\|(\phi, \vec{h}, f)\|_{X_{s,T}\times H^{s/3}(0,T;H^{s/3}(0,1))} \tag{5.2}$$

and

$$C(\beta^{1/3} + \beta^{1/2})\|a\|_{Y_{s,T}} r \leq 1/2. \tag{5.3}$$

It follows that

$$\|\Gamma(v)\|_{Y_{s,\beta}} \leq r$$

for any $v \in S_{\beta,r}$ and that for any $v_1, v_2 \in S_{\beta,r}$,

$$\|(\Gamma(v_1) - \Gamma(v_2))\|_{Y_{s,\beta}} \leq \frac{1}{2}\|v_1 - v_2\|_{Y_{s,\beta}}.$$

Thus Γ is a contraction from $S_{\beta,r}$ to $S_{\beta,r}$. Its unique fixed point is the desired solution of (5.1) for $0 \leq t \leq \beta$. However, since β is chosen according to (5.2) and (5.3) which only depends on $\|a\|_{Y_{s,T}}$, this local argument can be iterated to extend the solution to the entire temporal interval $0 \leq t \leq T$. The proof is complete. □

Formally, if K_I is an analytic mapping from $\mathcal{X}_{s,T}$ to $Y_{s,T}$, then, for $n = 0, 1, 2, \ldots$, its n-th order Fréchet derivative $K_I^{(n)}(g)$ at $g \in \mathcal{X}_{s,T}$ exists and is the symmetric, n-linear map from $\mathcal{X}_{s,T}$ to $Y_{s,T}$ given by

$$K_I^{(n)}(g)[w_1, \ldots, w_n] = \left\{ \frac{\partial^n}{\partial \xi_1 \cdots \partial \xi_n} K_I \left(g + \sum_{k=1}^{n} \xi_k w_k \right) \right\}_{0,\ldots,0}$$

for any $w_1, w_2, \ldots, w_n \in \mathcal{X}_{s,T}$. The homogeneous polynomial $K_I^{(n)}(g)[w^n]$ of degree n induced by $K_I^{(n)}(g)$, where $w^n = (w, w, \ldots, w)$ (n components), is

$$K_I^{(n)}(g)[w^n] = \left\{ \frac{d^n}{d\xi^n} K_I(g + \xi w) \right\}_{\xi=0}$$

for $w = (w_\phi, w_h) \in \mathcal{X}_{s,T}$. If we define y_n by

$$y_n = K_I^{(n)}(g)[w^n],$$

then it is formally ascertained that for $0 < t < T$, $(y_1, y_2, \ldots, y_n)$ solves the system of the equations

$$\left. \begin{aligned} &\partial_t y_1 + \partial_x y_1 + \partial_x(u y_1) + \partial_x^3 y_1 = 0, \qquad y_1(x, 0) = w_\phi(x), \\ &y_1(0, t) = w_{h_1}(t), \quad y_1(1, t) = w_{h_2}(t), \quad \partial_x y_1(1, t) = w_{h_3}(t) \end{aligned} \right\} \tag{5.4}$$

and

$$\left. \begin{aligned} &\partial_t y_k + \partial_x y_k + \partial_x(u y_k) + \partial_x^3 y_k = -\frac{1}{2} \sum_{j=0}^{k-1} \binom{k}{j} \partial_x(y_j y_{k-j}), \\ &y_k(x, 0) = 0, \quad y_k(0, t) = 0, \quad y_k(1, t) = 0, \quad \partial y_k(1, t) = 0 \end{aligned} \right\} \tag{5.5}$$

for $2 \leq k \leq n$, where $u = K_I(g)$ and $w = (w_\phi, w_{h_1}, w_{h_2}, w_{h_3}) \in \mathcal{X}_{s,T}$.

On the other hand, for any $g = (\phi, \vec{h}) \in \mathcal{D}(K_I)$, let $u = K_I(g)$ and consider solving the linear systems (5.4)–(5.5). It follows from Proposition 5.1 that (5.4)–(5.5) define a homogeneous polynomial of degree n from $\mathcal{X}_{s,T}$ to $Y_{s,T}$ as described by the following proposition.

Proposition 5.2. *Let $T > 0$, $0 \leq s \leq 3$, and $g \in \mathcal{X}_{s,T}$ be given and let $u = K_I(g)$. Then Eqs.* (5.4)–(5.5) *define a homogeneous polynomial $K_I^{(n)}(g)[w^n]$ of degree n from $\mathcal{X}_{s,T}$ to $Y_{s,T}$. Moreover, there exists a constant c_3 such that*

$$\|y_n\|_{Y_{s,T}} \leq c_3^n n! \|w\|_{\mathcal{X}_{s,T}}^n \tag{5.6}$$

for any $n \geq 2$, where $c_3 = c_3(T, \|u\|_{Y_{s,T}})$, and it may be that $c_3 \to +\infty$ as $T \to +\infty$ or $\|u\|_{Y_{s,T}} \to +\infty$, but in any case $c_3 \to 0$ if $T \to 0$.

Proof. The proof is a straightforward consequence of the linear estimates in Sec. 2 and Proposition 5.1 (cf. Zhang, 1995b), Proposition 3.3 for a detailed argument in related circumstances). □

Define a Taylor polynomial $P_n(w)$ of degree n for $w \in \mathcal{X}_{s,T}$ by

$$P_n(w) = \sum_{k=0}^{n} \frac{K_I^{(k)}(g)[w^k]}{k!} = K_I(g) + \sum_{k=1}^{n} \frac{y_k}{k!} \tag{5.7}$$

and a Taylor series by

$$P(w) = \sum_{k=0}^{\infty} \frac{K_I^{(k)}(g)[w^k]}{k!}. \tag{5.8}$$

Proposition 5.3. *Let $T > 0$ and $0 \leq s \leq 3$ be given. For any $g = (\phi, \vec{h}) \in \mathcal{D}(K_I)$, there exists an $\eta > 0$ depending only on $\|K_I(g)\|_{Y_{s,T}}$ such that the formal Taylor series (5.8) is uniformly convergent in the space $Y_{s,T}$ with respect to $w \in \mathcal{X}_{s,T}$ with $\|w\|_{\mathcal{X}_{s,T}} \leq \eta$. Moreover, if $v = P(w)$, then $v \in Y_{s,T}$ solves the problem*

$$\left.\begin{array}{l} v_t + v_x + vv_x + v_{xxx} = 0, \qquad v(x,0) = \phi(x) + w_\phi(x) \\ v(0,t) = h_1 + w_{h_1}, \quad v(1,t) = h_2 + w_{h_2}, \quad v_x(1,t) = h_3 + w_{h_3} \end{array}\right\} \tag{5.9}$$

for $0 \leq t \leq T$.

Proof. It is readily seen that the sequence $\{P_n(w)\}_{n=0}^{\infty}$ of Taylor polynomials is Cauchy in $Y_{s,T}$ uniformly for w in the ball of radius η in $\mathcal{X}_{s,T}$ for suitable η. Indeed, because of Proposition 5.2, for $m \geq n \geq 0$,

$$\|P_n(w) - P_m(w)\|_{Y_{s,T}} = \left\| \sum_{k=n}^{m} \frac{y_k}{k!} \right\|_{Y_{s,T}} \leq \sum_{k=n}^{m} \frac{\|y_k\|_{Y_{s,T}}}{k!} \leq \sum_{k=n}^{m} c_3^k \|h\|_{\mathcal{X}_{s,T}}^k.$$

If η is chosen so that

$$\eta \leq 1/(2c_3), \tag{5.10}$$

then for $w \in \mathcal{X}_{s,T}$ with $\|w\|_{\mathcal{X}_{s,T}} \leq \eta$,

$$\|P_n(w) - P_m(w)\|_{Y_{s,T}} \leq \sum_{k=n}^{m} \frac{1}{2^k}$$

which goes to zero uniformly as $n, m \to \infty$.

Since $\{P_n(w)\}_{n=0}^{\infty}$ is a Cauchy sequence in the space $Y_{s,T}$, it makes sense to define $v = P(w) \in Y_{s,T}$ as its limit as $n \to \infty$. It is then readily verified that v solves the IBVP (3.1). The proof is complete. □

The following theorem is now adduced.

Theorem 5.4. (Analyticity) *For any $T > 0$ and $0 \leq s \leq 3$, the IBVP Eq. (3.1) establishes an analytic map K_I from the space $\mathcal{X}_{s,T}$ to the space $Y_{s,T}$ in the sense that for any $g \in \mathcal{D}(K_I)$ there exists an $\eta > 0$ such that for any $w \in \mathcal{X}_{s,T}$ with $\|w\|_{\mathcal{X}_{s,T}} \leq \eta$, the Taylor series expansion*

$$K_I(g + w) = \sum_{n=0}^{\infty} \frac{K_I^{(n)}(g)[w^n]}{n!}$$

converges in the space $Y_{s,T}$. Moreover, the convergence is uniform with regard to w in the aforementioned ball in $\mathcal{X}_{s,T}$.

Remark 5.5. The above theorem holds also for $3 < s \leq 7/2$. Since the reasoning in this case is similar to that put forward for the system discussed below, we include analysis for this range of s in our next theorem.

Now, consider the case wherein $s > 3$. This situation is a little more involved than the previous case because the compatibility conditions are no longer linear restrictions. One could attempt to deal directly with the geometric situation implied by the nonlinear compatibility conditions, but another approach presents itself which is more transparent. That is to link the single equation faithfully with a class of systems to be discussed presently.

As in the Sec. 4, for any $s > 3$, write $s = 3m + s'$ where $m > 0$ is an integer and $0 < s' \leq 3$. For $T > 0$, define the space $\mathcal{Y}_T^s$ to be

$$\mathcal{Y}_T^s = Y_{3,T} \times Y_{3,T} \times \cdots \times Y_{3,T} \times Y_{s',T}$$

and the space $\mathcal{X}_T^s$ as

$$\mathcal{X}_T^s = X_{3,T} \times X_{3,T} \times \cdots \times X_{3,T} \times X_{s',T}.$$

Consider the system

$$\left.\begin{aligned} &\vec{u}_t + \vec{u}_x + (F(\vec{u})\vec{u})_x + \vec{u}_{xxx} = 0, \qquad \vec{u}(x,0) = \vec{\phi}(x) \\ &\vec{u}(0,t) = \vec{h}_1, \quad \vec{u}(1,t) = \vec{h}_2, \quad \vec{u}_x(1,t) = \vec{h}_3 \end{aligned}\right\} \tag{5.11}$$

where

$$\vec{u} = (u_0, u_1, \ldots, u_m)^T, \quad \vec{\phi} = (\phi_0, \phi_1, \ldots, \phi_m)^T,$$
$$\vec{h}_j = (h_{j,0}, h_{j,1}, \ldots, h_{j,m})^T$$

for $j = 1, 2, 3$ and

$$F(\vec{u}) = (-1/2)\left(u_0^2,\, 2u_0u_1, \ldots, \sum_{k=0}^{m}\binom{m}{k}u_k u_{m-k}\right)^T.$$

By Theorem 4.1, for any s-compatible $(\phi, h_1, h_2, h_3) \in X_{s,T}$, the IBVP (3.1) has a unique solution $u \in Y_{s,T}$. If one defines ϕ_0 by $\phi_0 = \phi$ and let ϕ_k be obtained from ϕ by (3.4) with $h_{j,k} = h_j^{(k)}$, $u_k = \partial_t^k u$ for $j = 1, 2, 3$ and $k = 0, 1, \ldots, m$, then $(\vec{\phi}, \vec{h}_1, \vec{h}_2, \vec{h}_3) \in \mathcal{X}_T^s$ and $\vec{u}$ is a solution of (5.11). In this sense, the IBVP (3.1) is a specialization of the system (5.11).

Theorem 5.6. *Let $T > 0$ and $s > 3$ be given with $s = 3m + s'$ and $0 \leq s' < 3$. Then for any $(\vec{\phi}, \vec{h}) \in \mathcal{X}_T^s$, the system (5.11) admits a unique solution $\vec{u} \in \mathcal{Y}_T^s$.*

Proof. Observe that the nonlinear system (5.11) consists of initial-boundary-value problems for $m+1$ scalar equations. Among them, the first one is the IBVP (3.1) which only involves u_0. The second one involves only u_0 and u_1. If u_0 is known, then the second IBVP is a linear problem. Similar remarks apply for the rest of the equations. Thus we can solve the nonlinear system by solving for u_0 from the first equation, plugging u_0 into the second equation and solving the corresponding linearized problem to obtain u_1, etc. Using Theorem 4.1 and Proposition 5.1, it is deduced inductively that $u_k \in Y_{3,T}$ for $k = 0, 1, \dots, m-1$. Now the equation related to u_m has the form

$$\left.\begin{array}{l} \partial_t u_m + \partial_x u_m + \partial_x(a u_m) + \partial_x^3 u_m = f, \qquad u_m(x,0) = \phi_m(x) \\ u_m(0,t) = h_{1,m}(t), \quad u_m(1,t) = h_{2,m}(t), \quad \partial_x u_m(1,t) = h_{3,m}(t) \end{array}\right\} \tag{5.12}$$

where $f \in C([0,T]; H^{s'}(0,1))$ and $a \in \mathcal{Y}_T^{3m}$ are known. Using Lemmas 4.1–4.6, the contraction principle and arguments similar to those appearing in the proof of Theorem 4.1, it can be shown that for any $(\phi_m, h_{1,m}, h_{2,m}, h_{3,m}) \in \mathcal{X}_{s',T}$, (5.12) admits a unique solution $u_m \in Y_{s',T}$. The proof is complete. □

The last result implies the nonlinear system (5.11) defines a nonlinear map $\mathcal{K}_I$ from the space $\mathcal{X}_T^s$ to $\mathcal{Y}_T^s$ for given $T > 0$ and $s = 3m + s'$ with $0 \le s' < 3$. We claim this map $\mathcal{K}_I$ is analytic from $\mathcal{X}_T^s$ to $\mathcal{Y}_T^s$. For the purpose of establishing this contention, consider the linearized system corresponding to the nonlinear system (5.11), namely

$$\left.\begin{array}{l} \partial_t \vec{w} + \partial_x \vec{w} + \partial_x(J(\vec{a})\vec{w}) + \partial_x^3 \vec{w} = \vec{f}, \qquad \vec{w}(x,0) = \vec{\phi}(x), \\ \vec{w}(0,t) = \vec{h}_1(t), \quad \vec{w}(1,t) = \vec{h}_2(t), \quad \partial_x \vec{w}(1,t) = \vec{h}_3(t) \end{array}\right\} \tag{5.13}$$

where J is the Jacobian matrix of F at $\vec{u} = \vec{a}$, viz.

$$\begin{aligned} J(\vec{a}) &= \left.\frac{\partial F(\vec{u})}{\partial \vec{u}}\right|_{\vec{u}=\vec{a}} \\ &= \left(\sum_{j=0}^{k} \binom{k}{j} (\delta(i,j) a_{k-j} + a_j \delta(i,k-j)) \right)_{0 \le k,\ i \le m} \end{aligned}$$

and

$$\delta(i,j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \ne j. \end{cases}$$

Proposition 5.7. *Let $T > 0$ and $s > 3$ be given. Suppose $\vec{a} \in \mathcal{Y}_T^s$ and*

$$\vec{f} \in F_T^s = L_1(0,T; H^3(0,1)) \times \cdots \times L_1(0,T; H^3(0,1)) \times L_1(0,T; H^{s'}(0,1)).$$

Then for any $(\vec{\phi}, \vec{h}_1, \vec{h}_2, \vec{h}_3) \in \mathcal{X}_T^s$, Eq. (5.13) admits a unique solution $\vec{w} \in \mathcal{Y}_T^s$. Moreover,

$$\|\vec{w}\|_{\mathcal{Y}_T^s} \le \gamma(\|\vec{a}\|_{\mathcal{Y}_T^s})(\|(\vec{\phi}, \vec{h}_1, \vec{h}_2, \vec{h}_3)\|_{\mathcal{X}_T^s} + \|\vec{f}\|_{F_T^s})$$

where $\gamma : R^+ \to R^+$ is a continuous nondecreasing function.

Proof. The proof is similar to that of Proposition 5.1 and therefore omitted. □

For given $\vec{u} = \mathcal{K}_I((\vec{\phi}, \vec{h}))$ with $(\vec{\phi}, \vec{h}) \in \mathcal{X}_T^s$, consider the linear systems

$$\left.\begin{aligned} &\partial_t \vec{y}_1 + \partial_x \vec{y}_1 + \partial_x (J(\vec{u})\vec{y}_1) + \partial_x^3 \vec{y}_1 = 0, \qquad \vec{y}_1(x,0) = \vec{w}_{\vec{\phi}}, \\ &\vec{y}_1(0,t) = \vec{w}_{\vec{h}_1}(t), \quad \vec{y}_1(1,t) = \vec{w}_{\vec{h}_2}(t), \quad \partial_x \vec{y}_1(1,t) = \vec{w}_{\vec{h}_3}(t) \end{aligned}\right\} \tag{5.14}$$

and

$$\left.\begin{aligned} &\partial_t \vec{y}_n + \partial_x \vec{y}_n + \partial_x (J(\vec{u})\vec{y}_n) + \partial_x^3 \vec{y}_n = F_n(\vec{y}_1, \dots, \vec{y}_{n-1}), \qquad \vec{y}_n(x,0) = 0, \\ &\vec{y}_n(0,t) = 0, \quad \vec{y}_n(1,t) = 0, \quad \partial_x \vec{y}_n(1,t) = 0 \end{aligned}\right\} \tag{5.15}$$

for $2 \le n \le N$, where

$$F_n = (f_{n,0}, f_{n,1}, \dots, f_{n,m})^T$$

with

$$f_{n,k} = -\frac{1}{2}\partial_x \left(\sum_{j=0}^{k} \sum_{i=1}^{n-1} \binom{k}{j} \binom{n}{i} y_{i,j} y_{n-i,k-j} \right)$$

for $k = 0, 1, \dots, m$.

Proposition 5.8. *Given $T > 0$, $s > 3$, and $\vec{g} = (\vec{\phi}, \vec{h}) \in \mathcal{Y}_T^s$, let $\vec{u} = \mathcal{K}_I((\vec{\phi}, \vec{h}))$. Then the systems* (5.14)–(5.15) *defines a homogeneous polynomial $\mathcal{K}_I^{(n)}(\vec{g})[\vec{w}^n]$ of degree n from $\mathcal{X}_T^s$ to $\mathcal{Y}_T^s$. Moreover, there exists a constant $C > 0$ such that*

$$\|\vec{y}_n\|_{\mathcal{Y}_T^s} \le C^n n! \|\vec{w}\|_{\mathcal{X}_T^s}^n$$

for any $n \ge 2$, where $C = C(T, \|\vec{u}\|_{\mathcal{Y}_T^s})$. Here C may go to $+\infty$ when $T \to \infty$ or $\|\vec{u}\|_{\mathcal{Y}_T^s} \to \infty$, but must go to 0 if $T \to 0$,

Proof. This follows from Proposition 5.4 by direct computation. □

Define a Taylor polynomial $P_n(\vec{w})$ of degree n for $\vec{w} \in \mathcal{X}_T^s$ by

$$P_n(\vec{w}) = \sum_{k=0}^{N} \frac{\mathcal{K}_I^{(k)}(g)[\vec{w}^k]}{k!} = \mathcal{K}_I(\vec{g}) + \sum_{k=1}^{N} \frac{\vec{y}_k}{k!}, \tag{5.16}$$

and a Taylor series by

$$P(\vec{w}) = \sum_{k=0}^{\infty} \frac{\mathcal{K}_I^{(k)}(\vec{g})[\vec{w}^k]}{k!}. \tag{5.17}$$

A proof similar to that given for Proposition 5.3 yields the following proposition.

Proposition 5.9. *Let $T > 0$ and $s > 0$ be given. For any $\vec{g} = (\vec{\phi}, \vec{h}_1, \vec{h}_2, \vec{h}_3)) \in \mathcal{X}^s_T$, there exists an $\eta > 0$ depending only on $\|\mathcal{K}_I(\vec{g})\|_{\mathcal{Y}^s_T}$ such that the formal Taylor series (5.17) is uniformly convergent in the space $\mathcal{Y}^s_T$ with respect to $\vec{w} \in \mathcal{X}^s_T$ with $\|\vec{w}\|_{\mathcal{X}^s_T} \leq \eta$. Moreover, if $\vec{v} = P(\vec{w})$, then $\vec{v} \in \mathcal{Y}^s_T$ solves the problem*

$$\left.\begin{aligned} &\partial_t \vec{v} + \partial_x \vec{v} + \partial_x (F(\vec{v})_x \vec{v}) + \partial_x^3 \vec{v} = 0, \qquad \vec{v}(x,0) = \vec{\phi}(x) + \vec{w}_{\vec{\phi}}(x) \\ &\vec{v}(0,t) = \vec{h}_1(t) + \vec{w}_{\vec{h}_1}(t), \quad \vec{v}(1,t) = \vec{h}_2(t) + \vec{w}_{\vec{h}_2}(t), \quad \partial_x \vec{v}(1,t) = \vec{h}_3(t) + \vec{w}_{\vec{h}_3}(t) \end{aligned}\right\} \tag{5.18}$$

for $0 \leq t \leq T$.

Consequently, we have the following theorem.

Theorem 5.10. (Analyticity) *For any $T > 0$ and $s > 3$, the nonlinear problem (5.11) establishes a map $\mathcal{K}_I$ from the space $\mathcal{X}^s_T$ to the space $\mathcal{Y}^s_T$. The map $\mathcal{K}_I$ is analytic from $\mathcal{X}^s_T$ to $\mathcal{Y}^s_T$ in the sense that for any $\vec{g} \in \mathcal{X}^s_T$, there exists an $\eta > 0$ such that for any $\vec{w} \in \mathcal{X}^s_T$ with $\|\vec{w}\|_{\mathcal{X}^s_T} \leq \eta$, the Taylor series expansion*

$$\mathcal{K}_I(\vec{g} + \vec{w}) = \sum_{n0}^{\infty} \frac{\mathcal{K}_I^{(n)}(\vec{g})[\vec{w}^n]}{n!}$$

converges in the space $\mathcal{Y}^s_T$. Moreover, the convergence is uniform with regard to $\vec{h}$ in the aforementioned ball in $\mathcal{X}^s_T$.

ACKNOWLEDGMENTS

JLB was partially supported by the National Science Foundation and by the W. M. Keck Foundation. SMS was partially supported by the National Science Foundation. BYZ was partially supported by the Charles H. Taft Memorial Fund. The manuscript benefitted very considerably from detailed commentary by an anonymous referee. The authors are grateful for this advice.

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Received May 2002
Revised March 2003