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STABILITY OF CNOIDAL WAVES

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Abstract. This paper is concerned with the stability of periodic travelling-wave solutions of the Korteweg-de Vries equation

$$u_t + uu_x + u_{xxx} = 0.$$

Here, u is a real-valued function of the two variables $x, t \in \mathbb{R}$ and subscripts connote partial differentiation. These special solutions were termed cnoidal waves by Korteweg and de Vries. They also appear in earlier work of Boussinesq. It is shown that these solutions are stable to small, periodic perturbations in the context of the initial-value problem. The approach is that of the modern theory of stability of solitary waves, but adapted to the periodic context. The theory has prospects for the study of periodic travelling-wave solutions of other partial differential equations.

1. Introduction

The Korteweg– de Vries equation

$$u_t + uu_x + u_{xxx} = 0 \tag{1.1}$$

for the function $u = u(x,t)$ was first derived by Boussinesq in 1877, and later by Korteweg and de Vries in 1895, as an approximate description of

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surface water waves propagating in a canal. Here, u, x and t are real variables and subscripts connote partial differentiation. This equation has since found application to a range of problems in solid and fluid mechanics as well as plasma physics and astrophysics. Some indication of the range of its applicability can be ascertained by consulting [24], [23]-[35], [48], [36], [45] and especially the early reviews [6], [29] and [44].

One of the centrally important properties of this evolution equation is its travelling-wave solutions, Scott Russell's solitary waves, later termed solitons, and Boussinesq's and Korteweg de Vries' cnoidal waves. The solitons are the single crested, symmetric, localized travelling waves whose $sech^2$-profiles have become so well known. The cnoidal waves are also travelling waves, but their spatial structure is periodic.

It is our purpose here to consider the stability of these latter waveforms. For solitary waves, stability theory was begun by Benjamin [5] and has since been refined and improved in several ways (see, for example, [4], [8], [11], [22], [25], [26], [37], [38], [39], [40], [43], [49], [50]). It is otherwise with the spatially periodic cnoidal waves, whose orbital stability has received comparatively little attention.

We intend to cast light on this issue. As our general experience with nonlinear, dispersive evolution equations indicates that travelling waves, when they exist, are of fundamental importance in the development of a broad range of disturbances, we expect the issue of stability of cnoidal waves to be of interest.

We approach the question of cnoidal wave stability by way of the general methods that have proven successful for deriving stability theory for solitary waves. In particular, we make no use of the KdV-equation's complete integrability as did McKean [42] in his study of the periodic initial-value problem. The periodic problem presents new points not encountered when considering stability issues related to the solitary waves. The outcome of our analysis appears in Sections 5 and 6. Roughly speaking, we show that, indeed, cnoidal waves are orbitally stable to disturbances of the same period. Moreover, we show that the perturbed solution propagates at about the same speed as does the unperturbed cnoidal wave. It is worth pointing out that theory has recently been developed by H. Chen [18] which includes stability results for periodic traveling-waves. Her theory applies to a general class of model equations, but the information gleaned is not quite as specific as that obtained here in the context of the KdV-equation.

The scheme of the paper is as follows. Section 2 contains some classical preliminaries about cnoidal waves and an appreciation and critique of earlier work of Benjamin reported in [6] on the issue in view here. We also indicate why the general theory of Grillakis, Shatah and Strauss [25], [26] cannot be applied directly to the problem at hand. Various characterizations of the cnoidal waves are made in Sections 3 and 4, and these are used to infer the stability theory in Section 5. The theory in Section 5 takes advantage of the ideas in Bona and Soyeur [12]. The theory in Section 5 is extended to higher-order Sobolev spaces in Section 6 using the idea appearing in Bona, Liu and Nguyen [10] in their study of stability of solitary waves. An Appendix A collecting for the reader's convenience some facts about the Jacobi elliptic functions is followed by Appendix B where an alternative, variational argument for stability is sketched. While straightforward, the latter method fails to provide the detailed aspects about stability that emerge from the analysis in the body of the paper.

2. Notation and preliminaries

2.1. **Function Classes.** The following, mostly standard notational conventions will be in force throughout. If Ω is an open set in $\mathbb{R}$ and $1 \leq p \leq \infty$, then $L^p(\Omega)$ is the usual Banach space of (equivalence classes of) real or complex-valued Lebesgue measurable functions defined on Ω with the norm

$$|f|^p_{L^p(\Omega)} = \int_\Omega |f|^p dx, \tag{2.1}$$

and with the usual modification when $p = \infty$. When Ω is understood, we write simply L^p for $L^p(\Omega)$. The inner product in $L^2(\Omega)$ of two functions f and g is written as

$$(f, g) = \int_\Omega f\overline{g}dx. \tag{2.2}$$

The L^2-based Sobolev spaces of periodic functions are defined as follows (for further details see [II]). Let $\mathcal{P} = C^\infty_{per}$ denote the collection of all the functions $f : \mathbb{R} \to \mathbb{C}$ which are C^∞ and periodic with period $2\ell > 0$. The collection $\mathcal{P}'$ of all continuous linear functionals from $\mathcal{P}$ into $\mathbb{C}$ is the set of *periodic distributions.* If $\Psi \in \mathcal{P}'$ we denote the value of Ψ at φ by

$$\Psi(\varphi) =< \Psi, \varphi > . \tag{2.3}$$

For $k \in \mathbb{Z}$, let $\Theta_k(x) = \exp(ik\pi x/\ell)$ for $x \in \mathbb{R}$. The Fourier transform of $\Psi \in \mathcal{P}'$ is the function $\widehat{\Psi} : \mathbb{Z} \to \mathbb{C}$ defined by the formula

$$\widehat{\Psi}(k) = \frac{1}{2\ell} < \Psi,\ \Theta_{-k} >, \qquad k \in \mathbb{Z}.$$

As usual, a function Ψ in $L^1(-\ell,\ell)$ is realized as an element of $\mathcal{P}'$ by defining

$$\langle \Psi, \varphi\rangle = \frac{1}{2\ell}\int_{-\ell}^{\ell} \Psi(x)\varphi(x)dx$$

for $\varphi \in \mathcal{P}$. If $\Psi \in L^p(-\ell,\ell)$ for some $p \geq 1$, then, for $k \in \mathbb{Z}$,

$$\widehat{\Psi}(k) = \frac{1}{2\ell}\int_{-\ell}^{\ell} \Psi(x)e^{-ik\pi x/\ell}dx.$$

The space $\mathcal{P}'$ carries the usual weak-star topology, but it will not be needed here. For $s \in \mathbb{R}$, the Sobolev space $H^s_{2\ell} = H^s_{per}([-\ell,\ell])$ is the set of all $f \in \mathcal{P}'$ such that

$$\|f\|^2_{H^s_{2\ell}} = 2\ell \sum_{k=-\infty}^{\infty} (1+|k|^2)^s|\widehat{f}(k)|^2 < \infty.$$

The collection $H^s_{2\ell}(\mathbb{R})$ is a Hilbert space with respect to the inner product

$$(f|g)_s = 2\ell \sum_{k=-\infty}^{\infty} (1+|k|^2)^s \widehat{f}(k)\overline{\widehat{g}(k)}.$$

In case $s = 0$, $H^0_{2\ell}$ is a Hilbert space that is isometrically isomorphic to $L^2(-\ell,\ell)$, and

$$(f|g)_0 = (f,g) = \int_{-\ell}^{\ell} f\overline{g}dx. \tag{2.4}$$

The space $H^0_{2\ell}$ will be denoted by $L^2_{2\ell}$ and its norm by $|\cdot|_{L^2(-\ell,\ell)}$. Of course $H^s_{2\ell} \subset L^2_{2\ell}$, for any $s \geq 0$, and, for every $n \in \mathbb{N}$, the norm $\|f\|^2_{H^n_{2\ell}}$ of a function f is equivalent to the norm

$$\Big(\sum_{j=0}^{n} \|f^{(j)}\|^2_{L^2_{2\ell}}\Big)^{1/2} = \Big(\sum_{j=0}^{n} \int_{-\ell}^{\ell} |f^{(j)}(x)|^2 dx\Big)^{1/2},$$

where $f^{(j)}$ is the j^{th} derivative of f taken in the sense of $\mathcal{P}'$. Moreover, $(H^s_{2\ell})'$, the topological dual of $H^s_{2\ell}$, is isometrically isomorphic to $H^{-s}_{2\ell}$ for all $s \in \mathbb{R}$. The duality is implemented concretely by the pairing

$$\langle f,g\rangle_s = 2\ell \sum_{k=-\infty}^{\infty} \widehat{f}(k)\overline{\widehat{g}(k)}, \qquad \text{for} \qquad f \in H^{-s}_{2\ell}, \quad g \in H^s_{2\ell}.$$

Thus, if $f \in L^2_{2\ell}$ and $g \in H^s_{2\ell}$, it follows that $\langle f,g\rangle_s = (f,g)$. One of Sobolev's lemmas in this context states that if $s > \frac{1}{2}$ and

$$C_{2\ell} = \{f : \mathbb{R} \to \mathbb{C} : f \text{ is continuous and periodic with period } 2\ell\},$$

then $H^s_{2\ell} \hookrightarrow C_{2\ell}$. If X is a Banach space like $H^s_{2\ell}$, and $T > 0$, then $C(0,T;X)$ is the space of continuous mappings from $[0,T]$ to X and, for $k \geq 0$, $C^k(0,T;X)$ is the subspace of mappings $t \mapsto u(t)$ such that $\partial_t^j u \in C(0,T;X)$ for $0 \leq j \leq k$, where the derivative is taken in the sense of vector-valued distributions. This space carries the standard norm

$$\|u\|_{C^k(0,T;X)} = \sum_{j=0}^{k} \max_{0\leq t\leq T} \|\partial_t^j u(t)\|_X.$$

2.2. **The Initial-Value Problem.** Logically prior to questions of stability to perturbations of the initial data is the issue of well posedness for the initial-value problem for (1.1). The initial-value problem for (1.1) with periodic data has been extensively studied, and very satisfactory theory is available (see e.g. [13], [14], [30], [31], [32], [15], [20], [46] and [33]). The more subtle aspects of recent theory do not find use here. All that is needed is the following result.

Theorem 2.1. *Let $s \geq 1$ be given. For each $u_0 \in H^s_{2\ell}$ there is a unique solution u of (1.1) that, for each $T > 0$, lies in $C(0,T;H^s_{2\ell})$. Moreover, the correspondence $u_0 \longmapsto u$ is an analytic mapping of the relevant function spaces.*

Remark. In what follows, we will occasionally use more regularity than is implied by membership in H^1_{2l}. This extra regularity is only needed at intermediate stages; the final results only feature elements requiring H^1_{2l} regularity. To justify the intermediate steps, we follow the standard procedure of regularizing the initial data, making the calculations which are easily justified for the more regular solutions and then passing to the limit in the final result. In this latter step, the full power of the well posedness is needed to insure that if a sequence of smooth initial data ψ_n converges to a rougher initial value ψ, then the associated solutions u_n converge strongly to the solution u emanating from ψ.

3. Classical results about cnoidal waves and stability theory

3.1. **Facts about Cnoidal Waves.** Travelling-wave solutions of the KdV-equation are obtained by searching for solutions u of (1.1) of the form

$$u(x,t) = \varphi_c(x - ct), \tag{3.1}$$

where c is the speed of propagation. Physically relevant solutions of the KdV-equation written in travelling coordinates as in (1.1) and with mean zero, require $0 < c << 1$ in the travelling frame of reference that underlies the tidy form (1.1). (In the original physical variables, this corresponds to speeds of propagation $\tilde{c}$ just in excess of the kinematic wave velocity $\tilde{c}_0 = \sqrt{gh_0}$, where h_0 is the undisturbed depth and g the gravity constant.)

Such waveforms are certainly covered here, but our theory is not restricted to this range. Indeed, the KdV-equation is invariant under the Galilean transformation

$$v(x,t) = u(x+\gamma t, t) - \gamma, \tag{3.2}$$

where γ is any real number. That is, if u solves (1.1), then so does v. For a travelling wave φ_c as in (3.1), this means that if $\gamma \in \mathbb{R}$, the function

$$\phi_e(z) = \varphi_c(z) - \gamma$$

is a travelling-wave solution of (1.1) where $e = c - \gamma$, which is to say

$$\phi_e(x - et) = \varphi_c(x - (c-\gamma)t) - \gamma \tag{3.3}$$

is also a solution of the KdV-equation. This familiar fact (see e.g. Miura's historical perspective in [M]) allows one to normalize considerations. One has a choice of either restricting the range of speeds or else fixing the mean-value of the wave. In case φ_c is periodic of period L, this means specifying the value of

$$\mathcal{H}(\varphi_c) = \frac{1}{L}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} \varphi_c(x)dx$$

whereas if $\varphi_c = \phi_c$ is of solitary-wave type, the limit

$$\phi_\infty = \lim_{L\mapsto\infty} \frac{1}{L}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} \phi_c(x)dx$$

may be specified. Note that the spacial period $[0, L]$ of a periodic traveling wave solution φ_c of the KdV-equation does not change under the Galilean transformation (3.2). For the moment, fix the speed c to be positive.

Substitution of the form (3.1) into (1.1) yields an ordinary differential equation for $\varphi_c(z)$, say, where $z = x - ct$, which may be integrated once to reach the second-order equation

$$\varphi_c'' + \frac{1}{2}\varphi_c^2 - c\varphi_c = A_{\varphi_c}. \tag{3.4}$$

Here, the constant A_{φ_c} of integration need not vanish. Upon multiplying (3.4) by the integrating factor φ_c', a second exact integration is possible, yielding the first-order equation

$$3(\varphi_c')^2 = -\varphi_c^3 + 3c\varphi_c^2 + 6A_{\varphi_c}\varphi_c + 6B_{\varphi_c} \tag{3.5}$$

where B_{φ_c} is another constant of integration. This latter equation may be solved implicitly, *viz.*

$$\int_{\varphi_c(0)}^{\varphi_c(z)} \frac{\sqrt{3}dy}{\sqrt{-y^3+3cy^2+6A_{\varphi_c}y+6B_{\varphi_c}}} + c_0 = z, \tag{3.6}$$

where c_0 is a final constant of integration. A class of solutions to (3.6), found already in the 19^{th}-century work of Boussinesg [5] and Korteweg and de Vries [34], may be written in terms of the Jacobi elliptic function as

$$\varphi_c(z) = \beta_2 + (\beta_3 - \beta_2)cn^2\Big(\sqrt{\frac{\beta_3-\beta_1}{12}}z; k\Big), \tag{3.7}$$

where

$$\beta_1 < \beta_2 < \beta_3,\ \beta_1+\beta_2+\beta_3 = 3c \quad \text{and} \quad k^2 = \frac{\beta_3-\beta_2}{\beta_3-\beta_1}.$$

Here is a classical argument leading exactly to these formulas. Fix a speed of propagation $c > 0$ and suppose φ to be a non-constant, smooth, periodic solution of (3.5)-(3.6). The formula (3.5) may be written

$$\varphi'(z)^2 = \frac{1}{3}F_\varphi(\varphi(z)) \tag{3.8}$$

with $F_\varphi(t) = -t^3 + 3ct^2 + 6A_\varphi t + 6B_\varphi$ a cubic polynomial. If F_φ has only one real root β, say, then $\varphi'(z)$ can vanish only when $\varphi(z) = \beta$. This means the maximum value φ takes on in its period domain $[-\ell, \ell]$ is the same as its minimum value there, and so φ is constant, contrary to presumption. Therefore F_φ must have three real roots, say $\beta_1 < \beta_2 < \beta_3$ (the degenerate cases will be considered presently), so F_φ has the form

$$F_\varphi(t) = (t-\beta_1)(t-\beta_2)(\beta_3 - t) \tag{3.9}$$

where we have incorporated the minus sign into the third factor. Of course, we must have

$$\beta_1+\beta_2+\beta_3 = 3c, \quad -\frac{1}{6}(\beta_1\beta_2+\beta_1\beta_3+\beta_2\beta_3) = A_\varphi, \quad \frac{1}{6}\beta_1\beta_2\beta_3 = B_\varphi.$$

It follows immediately from (3.8)-(3.9) that φ, if it is to exist, must take values in the range $\beta_2 \le \varphi \le \beta_3$. Since $c > 0$, $\beta_3 > 0$ and we can normalize φ by letting $\rho = \varphi/\beta_3$ so that (3.8)-(3.9) becomes

$$(\rho')^2 = \frac{\beta_3}{3}(\rho-\eta_1)(\rho-\eta_2)(1-\rho) \tag{3.10}$$

where $\eta_i = \beta_i/\beta_3,\ i = 1, 2$. The variable ρ lies in the interval $[\eta_2, 1]$. By translation of the spatial coordinates, we may locate a maximum value of ρ

at $z = 0$. As the only critical points of ρ for values of ρ in $[\eta_2, 1]$ are when $\rho = \eta_2 < 1$ and when $\rho = 1$, it must be the case that $\rho(0) = 1$. One checks that $\rho'' > 0$ when $\rho = \eta_2$ and $\rho'' < 0$ when $\rho = 1$. Thus, it is clear that our putative periodic solution must oscillate monotonically between the values $\rho = \eta_2$ and $\rho = 1$. A simple analysis would now allow us to conclude such periodic solutions exist, but we are pursuing the formula (3.10), not just existence.

Change variables again by letting

$$\rho = 1 + (\eta_2 - 1)\sin^2\psi$$

with $\psi(0) = 0$ and ψ continuous. Substituting into (3.10) yields the equation

$$(\psi')^2 = \frac{\beta_3}{12}(1-\eta_1)\Big[1 - \frac{1-\eta_2}{1-\eta_1}\sin^2\psi\Big]$$

with $\psi(0) = 0$. To put this in standard form, define

$$k^2 = \frac{1-\eta_2}{1-\eta_1} \quad \text{and} \quad \lambda = \frac{\beta_3}{12}(1-\eta_2).$$

Of course $0 \le k^2 \le 1$ and $\lambda > 0$. We may solve for ψ implicitly to obtain

$$F(\psi; k) = \int_0^{\psi(z)} \frac{dt}{\sqrt{1-k^2\sin^2 t}} = \sqrt{\lambda}\, z. \tag{3.11}$$

The left-hand side of (3.11) is just the standard elliptic integral of the first kind (see Appendix A). As mentioned in Appendix A, the elliptic function $sn(z;k)$ is, for fixed k, defined in terms of the inverse of the mapping $\psi \longmapsto F(\psi;k)$. Hence, (3.11) may be rewritten as

$$\sin\psi = \mathrm{sn}(\sqrt{\lambda}\, z; k),$$

and, therefore,

$$\rho = 1 + (\eta_2 - 1)sn^2(\sqrt{\lambda}\, z; k).$$

As $sn^2 + cn^2 = 1$, it transpires that $\rho = \eta_2 + (1-\eta_2)cn^2(\sqrt{\lambda}\, z; k)$, which, when properly unwrapped, is exactly (3.7).

A moment's reflection about the degenerate cases is worthwhile. First, fix the value of $c > 0$ and consider whether or not periodic solutions can persist if $\beta_1 = \beta_2$ or $\beta_2 = \beta_3$. As φ can only take values in the interval $[\beta_2, \beta_3]$, we conclude that the second case leads only to the constant solution $\varphi(z) \equiv \beta_2 = \beta_3$. Indeed, the limit of (3.7) as $\beta_2 \to \beta_3$ is uniform in z and is exactly this constant solution. If, on the other hand, c and β_1 are fixed, say, $\beta_2 \downarrow \beta_1$ and $\beta_3 = 3c - \beta_2 - \beta_1$, then $k_\varphi \to 1$, the elliptic function cn

converges, uniformly on compact sets, to the hyperbolic function *sech* and (3.7) becomes, in this limit,

$$\lim_{\beta_2\downarrow\beta_1} \varphi_c(z) = \phi_c(z) = \phi_\infty + a\, sech^2\Big(\sqrt{\frac{a}{12}}z\Big)$$

with $\phi_\infty = \beta_1$ and $a = \beta_3 - \beta_1$. If β_1 happens to be zero, the standard solitary-wave solution

$$\phi_c(z) = 3c\, sech^2\Big(\frac{\sqrt{c}}{2}z\Big)$$

of speed c of the KdV-equation is recovered. Note $\beta_2 = \beta_1 = 0$ exactly when $A_\varphi = B_\varphi = 0$, as one would expect.

3.2. **Generalities about stability and Benjamin's theory.** We present here a brief review of the general stability theory that will come to the fore in our analysis. In what follows, it is supposed that the travelling-wave solution φ_c is periodic of period $2\ell = L > 0$ and we let $\Omega = [0, L]$ be a minimal period. The two functionals

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega (u_x^2 - \frac{1}{3}u^3)dx \quad \text{and} \quad \mathcal{F}(u) = \frac{1}{2}\int_\Omega u^2 dx \tag{3.12}$$

play a central role in the stability results that follow. They are well-defined C^∞-mappings of $H_L^1 = H_{per}^1(\Omega) = H_{per}^1$ to $\mathbb{R}$ and each is independent of t when evaluated on H_{per}^1-solutions of the KdV-equation (1.1).

Starting with [5], the stability theory for solitary waves has always relied upon a local analysis in a neighborhood of the solitary wave whose stability is in question. This is also the case with the present theory of stability of cnoidal waves. One approach to demonstrating stability of cnoidal waves which is suggested by the associated stability theory for solitary waves is to show that if $u(\cdot, 0) = u_0$ is close enough to a cnoidal wave φ_c and $\mathcal{F}(u_0) = \mathcal{F}(\varphi_c)$, then for all $t \geq 0$,

$$\mathcal{E}(u(\cdot, t)) - \mathcal{E}(\varphi_c) \geq f(d_1(u, \varphi_c)) \tag{3.13}$$

where f is a continuous and strictly monotone increasing function at least near 0 and $f(0) = 0$. Here, for $s \geq 0$, and $g, h \in H_{per}^s = H_{per}^s([0, L])$,

$$d_s(g, h) = \inf_{y\in\mathbb{R}} \|g(\cdot) - h(\cdot + y)\|_{H_{per}^s} \tag{3.14}$$

is the usual pseudo-metric arising in this context. The inequality (3.13) implies that for any given $\epsilon > 0$, if $\|u_0 - \varphi_c\|_{H_{per}^1}$ is small enough, then

$$d_1(u(\cdot, t), \varphi_c) \leq \epsilon \tag{3.15}$$

for all $t \geq 0$, and this is simply an expression of orbital stability since $\{\varphi_c(\cdot + y)\}_{y\in\mathbb{R}}$ is precisely the orbit of the solitary wave φ_c. To see that (3.15) is a consequence of (3.13), let $\epsilon > 0$ be given and argue as follows. If u_0 is near enough to φ_c in the H^1_{per}–norm, then the quantity

$$\mathcal{E}(u(\cdot,t)) - \mathcal{E}(\varphi_c) = \mathcal{E}(u_0) - \mathcal{E}(\varphi_c)$$

is less than any prescribed value $\gamma > 0$ for all $t \geq 0$. This means that $f(d_1(u(\cdot,t),\varphi_c)) < \gamma$ for all t. Let Θ be the orbit of φ_c, so $\Theta = \{\varphi_c(\cdot + y) : y \in \mathbb{R}\}$ is a closed set. The triangle inequality implies that for any t, $t' \geq 0$,

$$\begin{aligned} |d_1(u(\cdot,t),\varphi_c) - d_1(u(\cdot,t'),\varphi_c)| &= |dist(u(\cdot,t),\Theta) - dist(u(\cdot,t'),\Theta)| \\ &\leq \|u(\cdot,t) - u(\cdot,t')\|_{H^1_{per}}, \end{aligned}$$

where the distance is referred to the H^1_{per}-norm. Since $u(\cdot,t)$ is a continuous function of t with values in H^1_{per}, it thus follows that $d_1(u(\cdot,t),\varphi_c)$ is a continuous function of t. Because f is monotone, we may choose $\gamma_0 > 0$ so that $r \leq \epsilon$ whenever $f(r) \leq \gamma_0$. Thus, if $\|u_0 - \varphi_c\|_{H^1_{per}} \leq \delta$ is small enough that $\mathcal{E}(u_0) - \mathcal{E}(\varphi_c) \leq \gamma_0$, then (3.15) must hold.

The general case where $\mathcal{F}(u_0) \neq \mathcal{F}(\varphi_c)$ now follows from the triangle inequality. In a little more detail, fix c and let φ_c be a cnoidal wave whose stability is in question. We will see in Section 4 that there is a C^1-branch of cnoidal waves $\{\varphi_d\}_{|d-c|\leq\eta}$ passing through φ_c, all of the same spatial period L, and that the mapping $d \to \mathcal{F}(\varphi_d)$ is continuous and strictly monotone increasing. Let $\epsilon > 0$ be given and let u_0 be initial data for (1.1) for which $\|u_0 - \varphi_c\|_{H^1_{per}} \leq \delta$ where δ will be determined presently. For δ small enough, there is a value e near c such that $\mathcal{F}(u_0) = \mathcal{F}(\varphi_e)$. Moreover,

$$\|u_0 - \varphi_e\|_{H^1_{per}} \leq \|u_0 - \varphi_c\|_{H^1_{per}} + \|\varphi_c - \varphi_e\|_{H^1_{per}} \leq \delta + o(1)$$

as $\delta \to 0$, since the branch of cnoidal waves is continuous. The assumption (3.13), the just derived conclusion that if δ is small enough, then $d_1(u,\varphi_e) \leq \frac{1}{2}\epsilon$ for all t, and the triangle inequality assure that

$$d_1(u,\varphi_c) \leq d_1(u,\varphi_e) + d_1(\varphi_e,\varphi_c) \leq \frac{1}{2}\epsilon + o(1)$$

as $\delta \to 0$. Moreover, this inequality is valid for all t, since $d_1(\varphi_e,\varphi_c)$ is obviously independent of t. The desired result follows.

Thus the issue of orbital stability of cnoidal waves would be reduced to establishing (3.13) and the existence of the aforementioned branches of cnoidal waves. However as will be apparent in Section 5, provision of a suitable version of (3.13) is not quite as straightforward in the periodic context as it is for solitary waves. One reason for this is easily appreciated. The function

$$F(r,t) = \int_{-\frac{1}{2}L}^{\frac{1}{2}L} \Big(u(x,t) - \varphi_c(x+r) \Big)^2 dx$$

figures prominently in the stability analysis. Here, u is a solution corresponding to initial data ψ, say, that is periodic of period L and close to the cnoidal wave φ_c at $t = 0$. What is important about F is that, for each t, there are translations r such that

$$\frac{\partial F}{\partial r}\bigg|_{(r,t)} = 0.$$

In the case of the solitary wave where $L = +\infty$, the extended stability theory (see [BS]) implies that there is a unique $r = r(t)$ for which the latter holds, whereas in the periodic context there are always at least two such points, one of which is, say at time t, the point r_0 where

$$F(r_0, t) = \max_{0 \le r \le L} F(r,t).$$

Thus, points where $\frac{\partial F}{\partial r} = 0$ need not be points where F is small. This potential obstacle is overcome in Section 5 by the derivation of a slightly more subtle version of (3.13) and a dynamical determination of $r(t)$. The latter approach has the salutary effect of yielding sharper results that those obtaining from the analysis just outlined, were it to be correct.

Except for the caution just raised, the preceding argument is the same as one made for solitary-wave solutions of (1.1) in [11] or [4], and indeed for a general class of KdV-type equations of the form

$$u_t + f(u)_x - Mu_x = 0 \tag{3.16}$$

where $f : \mathbb{R} \to \mathbb{R}$ is typically a polynomial and M is a Fourier multiplier operator defined via its Fourier transform by the formula $\widehat{Mv} = m\widehat{v}$ where the circumflex connotes the Fourier transform in the spatial variable x and m is the symbol of M. The existence of solitary-wave or periodic travelling-wave solutions of (3.16) has been dealt with in a number of works, e.g. [2], [3], [7], [9], [17] and [49].

Fix a speed c and let ϕ_c be a solitary-wave solution of (3.16). There are two principal hypotheses leading to inequalities like (3.13), and hence to the conclusion of stability.

(I) For a suitably chosen value of s depending upon m, the self-adjoint, unbounded linear operator $\mathcal{L}$ defined on the subspace H^s of L^2 by $\mathcal{L}h = Mh + (c - f'(\phi_c))h$ has exactly one negative eigenvalue which is simple, the zero eigenvalue is simple with eigenfunction ϕ_c', and the rest of the spectrum is positive and bounded away from zero. The Sobolev-space index s depends upon the large-ξ asymptotics of the symbol m of M; provided such a value exists, s is chosen so that the norm

$$||f||_m^2 = \int_{-\infty}^{\infty} [1 + m(\xi)] \, |\hat{f}(\xi)|^2 d\xi$$

is equivalent to the H^s-norm, where $\hat{f}$ is the Fourier transform of f.

(II) The correspondence $c \in (0, \infty) \longmapsto \phi_c \in H^s(\mathbb{R})$ is a C^1-mapping and, for $c > 0$, the function $d : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$d(c) = \mathcal{E}(\phi_c) + c\mathcal{F}(\phi_c)$$

is convex, which is to say $d''(c) > 0$.

The conclusion to be drawn from these hypotheses is that ϕ_c is orbitally stable in the sense mentioned in (3.15), except relative to d_s. It deserves remark that theory exists giving checkable conditions under which (I) holds (see e.g. [1]).

The approach taken here to the question of stability of cnoidal waves follows the same general lines as those that were successful in the just described theory for solitary waves. Benjamin [6] in his wide ranging lectures on nonlinear waves also put forward an approach to the stability of cnoidal waves. He claimed that the cnoidal wave, "with appropriate choices of β_2, β_3 and c, to make L the fundamental period", realized the absolute minimum of $\mathcal{E}(u)$ subject to $\mathcal{F}(u) = \lambda$ where λ was restricted by $\lambda > 32\pi^4/L^3$ so as to make $c > 4\pi^2/L^2$. Benjamin went on to assert that, for such values of λ, the quantity $\mathcal{E}(u) - \mathcal{E}(\varphi_c)$ is bounded below in terms of $d_1(u, \varphi_c)$ and the invariance of $\mathcal{E}$ then implies stability as previously indicated.

Benjamin did not provide a detailed justification of his assertions, and several aspects seem problematic. The most important gap is the identification of the minimizer of the constrained variational problem with the cnoidal wave φ_c one starts with. Certainly the variational problem has a minimum ψ as we show in Appendix B. However, the identification of ψ with φ_c requires a uniqueness theorem for the variational problem that seems troublesome. Thus, the conclusion available by this analysis is just that the set of absolute minimizers is stable. The fact that the cnoidal wave is in the set is not certain without further argument. A related point concerns the spectral analysis of the operator $\mathcal{L}$ in the periodic case. This type of analysis was the core of Benjamin's original proof of stability of the solitary wave, but he makes no attempt to provide it in the periodic case.

Here, a complete theory is provided of the stability of cnoidal waves. While inspired by Benjamin's commentary, our analysis takes a different path than he envisioned.

3.3. **Stability of the constant solutions.** Unlike the situation that arises for solitary waves where the natural physically relevant assumption is that $\phi_c(z) \to 0$ as $z \to \pm\infty$ (or what is the same, $\mathcal{H}(\phi_c) = 0$), for which the only trivial solution is $\phi(z) \equiv 0$, the cnoidal-wave problem throws up two trivial solutions. This is easily appreciated from (3.4), for example. If a constant solution taking the value λ is sought, then one sees that λ must satisfy a quadratic equation. If we let $\psi = \varphi_c + r$ where $r + c = \sqrt{c^2 + 2A_{\varphi_c}}$, then ψ satisfies (3.4) with $A_{\varphi_c} = 0$. Note that according to the formulas below (3.9) for c and A_{φ_c} in terms of $\beta_1, \beta_2, \beta_3$,

$$c^2 + 2A_{\varphi_c} = \frac{1}{18}\Big[(\beta_2 - \beta_1)^2 + (\beta_3 - \beta_1)^2 + (\beta_3 - \beta_2)^2\Big] > 0,$$

so this is a real-valued transformation. Thus, for ψ, the two constant solutions are $\psi = \varphi_0 \equiv 0$ and $\psi = \psi_0 \equiv 2(c + r)$.

The state φ_0 is clearly stable in H^1_{per}. This follows immediately from the time independence of $\mathcal{E}$ and $\mathcal{F}$. For if $\|\varphi_0 - u_0\|_{H^1_{per}} = \|u_0\|_{H^1_{per}} \le \delta$, then

$$\int_0^L (u - \varphi_0)^2 dx = \int_0^L u^2 dx = \int_0^L u_0^2 \, dx \le \delta^2$$

and

$$\begin{aligned}
&\int_0^L (u_x - \varphi_{0_x})^2 \, dx = \int_0^L u_x^2 \, dx = \frac{1}{3}\int_0^L u^3 \, dx + \int_0^L \Big(u_{0_x}^2 - \frac{1}{3}u_0^3\Big) \, dx \\
&\le \frac{1}{3}\|u\|_{L^\infty_{per}}\delta^2 + \delta^2 + C\delta^3 \le \frac{1}{3}\delta^2\Big[\frac{1}{L}\|u\|^2_{L^2_{per}} + \|u\|_{L^2_{per}}\|u_x\|_{L^2_{per}}\Big]^{\frac{1}{2}} + \delta^2 + C\delta^3 \\
&\le \frac{1}{3}\delta^2\Big[\frac{1}{L}\delta^2 + \delta\|u_x\|_{L^2_{per}}\Big]^{\frac{1}{2}} + \delta^2 + C\delta^3.
\end{aligned}$$

These two inequalities imply the advertised stability result. Note that in the second inequality, use has been made of the elementary result

$$\sup_{0 \le x \le L} |h(x)|^2 \le \frac{1}{L}\int_0^L h^2 \, dx + \Big(\int_0^L h^2 \, dx \int_0^L h_x^2 \, dx\Big)^{\frac{1}{2}}, \tag{3.17}$$

which will be needed again later.

To study stability of the constant state ψ_0, we rephrase Benjamin's argument as follows. Let $u = \psi_0 + h$ where $h \in H^1_{per}$ and suppose $\mathcal{F}(\psi_0) = \mathcal{F}(u_0)$ for the moment. Then, by invariance of $\mathcal{F}$,

$$0 = \mathcal{F}(u) - \mathcal{F}(\psi_0) = \frac{1}{2}\int_0^L (2\psi_0 h + h^2)\ dx. \tag{3.18}$$

In consequence, it transpires that

$$\begin{aligned} \Delta\mathcal{E} &= \mathcal{E}(u) - \mathcal{E}(\psi_0) = \mathcal{E}(\psi_0 + h) - \mathcal{E}(\psi_0) \\ &= \frac{1}{2}\int_0^L [(h_x)^2 + (c - \psi_0)h^2]\ dx - \frac{1}{6}\int_0^L h^3\ dx \qquad (3.19) \\ &= \frac{1}{2}\int_0^L [(h_x)^2 - ch^2]\ dx - \frac{1}{6}\int_0^L h^3\ dx. \end{aligned}$$

A lower bound for the conditional second variation $\delta^2\mathcal{E}$ of $\mathcal{E}$, the quadratic portion of the right-hand side of (3.19), is obtained as follows. Define

$$h^\perp = h - \frac{1}{L}\int_0^L h\ \ dx = h - \mathcal{H}(h). \tag{3.20}$$

Clearly, $\int_0^L h^\perp dx = 0$, so it follows from Poincaré's inequality that

$$\int_0^L (h^\perp_x)^2\ dx \geq \Big(\frac{2\pi}{L}\Big)^2 \int_0^L (h^\perp)^2\ dx. \tag{3.21}$$

Formula (3.18) implies that

$$\mathcal{H}(h) = -\frac{1}{4cL}\int_0^L h^2 dx = -\frac{1}{4cL}\|h\|^2_{L^2_{per}}. \tag{3.22}$$

It thus transpires that

$$\int_0^L h^2 dx = \int_0^L (h^\perp)^2 dx + L\mathcal{H}(h)^2 = \int_0^L (h^\perp)^2 dx + \frac{1}{16c^2L}\|h\|^4_{L^2_{per}}. \tag{3.23}$$

As a consequence of (3.17) and Young's inequality, for any $\lambda < 1$, we have that

$$\begin{aligned} &\Big|\int_0^L h^3 dx\Big| \leq \sup_{0\leq x\leq L}|h(x)|\int_0^L h^2 dx \\ &\leq \frac{1}{L^{1/2}}\Big(\int_0^L h^2 dx\Big)^{3/2} + \Big(\int_0^L h^2 dx\Big)^{5/4}\Big(\int_0^L h_x^2\ dx\Big)^{1/4} \qquad (3.24) \\ &\leq \frac{1}{L^{1/2}}\Big(\int_0^L h^2 dx\Big)^{3/2} + \frac{1-\lambda}{4}\int_0^L h_x^2\ dx + \frac{1}{(1-\lambda)^{1/3}}\Big(\int_0^L h^2\ dx\Big)^{5/3}. \end{aligned}$$

We are now in a position to obtain the aforementioned lower bound. Fix $c < (2\pi/L)^2$ and choose $\lambda \in (0,1)$ such that

$$\lambda\Big(\frac{2\pi}{L}\Big)^2 - c \geq 1 - \lambda. \tag{3.25}$$

Starting with (3.19), argue as follows using (3.21), (3.22), (3.24) and the choice of λ in (3.25):

$$\begin{aligned}
\Delta\mathcal{E} &= \frac{1}{2}\int_0^L (h_x^2 - ch^2)\,dx - \frac{1}{6}\int_0^L h^3\,dx \\
&\geq \frac{1}{2}(1-\lambda)\int_0^L h_x^2\,dx + \frac{1}{2}\int_0^L [\lambda(h^\perp)_x^2 - c(h^\perp)^2]\,dx - \frac{1}{32cL}\|h\|_{L^2_{per}}^4 \\
&\quad -\frac{1}{24}(1-\lambda)\int_0^L h_x^2\,dx - \frac{1}{6L^{\frac{1}{2}}}\|h\|_{L^2_{per}}^3 - \frac{1}{6(1-\lambda)^{\frac{1}{3}}}\|h\|_{L^2_{per}}^{\frac{10}{3}} \\
&\geq \frac{1}{4}(1-\lambda)\int_0^L h_x^2\,dx + \frac{1}{2}\Big[\lambda\Big(\frac{2\pi}{L}\Big) - c\Big]\int_0^L (h^\perp)^2\,dx \\
&\quad -\frac{1}{6L^{\frac{1}{2}}}\|h\|_{L^2_{per}}^3 - \frac{1}{6(1-\lambda)^{\frac{1}{3}}}\|h\|_{L^2_{per}}^{\frac{10}{3}} - \frac{1}{32cL}\|h\|_{L^2_{per}}^4 \\
&\geq \frac{1}{4}(1-\lambda)\int_0^L (h_x^2 + h^2)\,dx - \frac{1}{6L^{\frac{1}{2}}}\|h\|_{L^2_{per}}^3 \\
&\quad -\frac{1}{6(1-\lambda)^{\frac{1}{3}}}\|h\|_{L^2_{per}}^{\frac{10}{3}} - \frac{1}{32cL}\|h\|_{L^2_{per}}^4.
\end{aligned}$$

Proceeding just as in the argument following (3.13) allows the conclusion of stability in $H^1_{per}([0,L])$. The side condition $F(u) = F(\psi_0)$ is then easily removed by a simple use of the triangle inequality. Reverting to the original variables $\varphi_c = \psi - r$ with $r = \sqrt{c^2 + 2A_{\varphi_c}} - c$ yields the following formal conclusion.

Proposition 3.1. *Let $L > 0$ and $c > 0$ be given. Let φ_0 and ψ_0 be the two constant solutions of the cnoidal-wave equation* (3.4). *Then φ_0 is stable in H^1_{per}, and ψ_0 is stable in H^1_{per} provided $c < (2\pi/L)^2$.*

Remarks. Notice that the translation group acts trivially on φ_0 and ψ_0, so stability up to translation is simply stability in this case.

If $c > (2\pi/L)^2$, the second variation of $\mathcal{E}$ at ψ_0 can be negative. Indeed, in this case, if $h_0 = \epsilon\cos(2\pi x/L)$, then h_0 has mean zero on $[0,L]$ and a straightforward calculation reveals that

$$\delta^2\mathcal{E}(h_0) = \frac{1}{2}(\mathcal{L}_0 h_0, h_0) = \frac{1}{2}\Big[\Big(\frac{2\pi}{L}\Big)^2 - c\Big]\int_0^L h_0^2\,dx < 0,$$

where $\mathcal{L}_0 f = -f_{xx} - cf$ is the operator appearing in the general stability theory outlined previously. Thus the approach to stability just indicated fails in this case. Even when $c = (2\pi/L)^2$,

$$\ker\mathcal{L}_0 = \operatorname{span}\Big\{\sin(2\pi x/L), \cos(2\pi x/L)\Big\}$$

is two dimensional. Thus, in this case, Condition I is not met. Of course, neither of these implies instability, but the general theory of Grillakis, *et al.* [25] does not apply because the spectrum of $\mathcal{L}_0$ has at least two negative eigenvalues when $c > (2\pi/L)^2$. We note also that the theory in [26] can not be applied. Indeed, it seems likely that the constant solution $\psi_0 \equiv 2c$ loses stability to a branch of cnoidal waves as c crosses $(2\pi/L)^2$.

4. Existence of smooth curves of cnoidal waves

In Subsection 3.3 the issue of stability of cnoidal waves was connected to checking hypotheses I and II in a periodic context. It will be shown in Section 5 that if Condition II and something like Condition I hold in the periodic situation, then stability obtains. Our purpose in this section is to deal with smooth branches of cnoidal waves, passing through a given wave, all having the same fundamental period.

Branches of cnoidal waves having mean zero are constructed here. The condition of zero mean, namely that

$$\int_0^L \varphi_c(\xi)\,d\xi = 0,$$

physically amounts to demanding that the wavetrain has the same mean depth as does the undisturbed free surface (this is a very good presumption for waves generated by an oscillating wavemaker in a channel, for example, as no mass is added in such a configuration). Wavetrains with non-zero mean are readily derived from this special case as will be remarked presently.

Let a phase speed c be given and consider three constants $\beta_1, \beta_2, \beta_3$ and k as in (3.7). The complete elliptic integral of the first kind (see Appendix A) is the function $K(k)$ defined by the formula

$$K \equiv K(k) \equiv \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{4.1}$$

The fundamental period of the cnoidal wave φ in (3.7) is

$$T_\varphi \equiv T_\varphi(\beta_1,\beta_2,\beta_3) = \frac{4\sqrt{3}}{\sqrt{\beta_3-\beta_1}}K(k), \tag{4.2}$$

with K as in (4.1). The period of cn is $4K(k)$ and cn is antisymmetric about its half period, from which (4.2) follows.

The mean value of φ over a period $[0,T_\varphi]$ is easily determined to be

$$\overline{\varphi} = \beta_2 + (\beta_3-\beta_2)\frac{1}{2K}\int_0^{2K} cn^2(\xi;k)\,d\xi. \tag{4.3}$$

Demanding that φ have mean zero asks exactly that the expression in (4.3) vanishes. The integral in (4.3) may be evaluated because

$$\int_0^{2K} cn^2(\xi;k)\,d\xi = 2\int_0^{K} cn^2(u;k)\,du = \frac{2}{k^2}[E(k)-k'^2K(k)]$$

where $k' = (1-k^2)^{1/2}$ and $E(k)$ is the complete elliptic integral of the second kind (see again Appendix A). This formula may be found in [16], page 193. Thus the zero mean value condition is exactly

$$\beta_2 + (\beta_3-\beta_2)\frac{E(k)-k'^2K(k)}{k^2K(k)} = 0. \tag{4.4}$$

Because $(\beta_3-\beta_2)k'^2 = (\beta_2-\beta_1)k^2$, the relation (4.4) has the equivalent form

$$\beta_1K(k) + (\beta_3-\beta_1)E(k) = 0. \tag{4.5}$$

And since

$$\frac{dK(k)}{dk} = \frac{E(k)-k'^2K(k)}{kk'^2}$$

(see Appendix A), having mean zero also implies that

$$\frac{dK}{dk} = -\frac{\beta_2}{\beta_3-\beta_2}\,\frac{k}{k'^2}\,K, \tag{4.6}$$

another formula that will find use presently.

From (4.5), the following result may be deduced.

Lemma 4.1. (**Cnoidal Waves with Mean Zero**) *For every $\beta_3 > 3$, there are unique constants β_1, β_2 satisfying $\beta_1 < \beta_2 < 0 < \beta_3$ and $\beta_1+\beta_2+\beta_3 = 3$, such that the cnoidal wave $\varphi_1 = \varphi_1(\cdot\,;\beta_i)$ in (3.7) has mean zero, speed $c = 1$ and spatial period T_{φ_1}. Moreover, the map $\beta_3 \in (3,\infty) \to \beta_2 \equiv \beta_2(\beta_3)$ is continuous.*

Proof. Let $\beta_3 > 3$, $\alpha = \frac{\beta_3-3}{2}$, $\beta = \frac{1}{\alpha}$ and $I = (-\alpha, 0)$. For $x \in I$, define the continuous function J on the interval I by

$$J(x) = \frac{K(k(x))}{E(k(x))} + \frac{\beta_3}{3-\beta_3-x} - 1,$$

where $k(x) = \sqrt{\frac{\beta_3-x}{2\beta_3+x-3}}$. It will be proved that for every $\beta_3 > 3$, there is a unique $\beta_2 = \beta_2(\beta_3) \in I$ such that $J(\beta_2) = 0$. Let A be the function defined for $x \in I$ by the integral

$$A(x) = \int_0^1 \frac{1}{\sqrt{1-t^2}} \frac{-\beta_3 + (\beta_3 - x)t^2}{\sqrt{2\beta_3 - 3 + x - (\beta_3 - x)t^2}}\, dt. \tag{4.7}$$

A calculation shows that $J(x) = 0$ if and only if $A(x) = 0$. If it is demonstrated that A has at least one zero on I and is strictly monotone there, it will follow that J has a unique zero, say x_0, in I. The choices $k = k(x_0)$, β_3 as given, β_2 as determined and then $\beta_1 = 3 - \beta_3 - \beta_2$ will then satisfy (4.5).

Toward establishing this fact, note that there is at least one point $x_0 \in I$ such that $J(x_0) = 0$. In fact, using (4.1) and referring to the integral defining E in Appendix A, the existence of such a point is a consequence of the relations

$$\lim_{x\to-\alpha} A(x) = \lim_{x\to-\alpha} \Big[\frac{\beta_3 - 3 + x}{\sqrt{2\beta_3 - 3 + x}} K(k(x)) - \sqrt{2\beta_3 - 3 + x}\; E(k(x))\Big] = +\infty,$$

$$A(0) = \int_0^1 \frac{1}{\sqrt{1-t^2}} \frac{\beta_3(t^2-1)}{\sqrt{2\beta_3 - 3 - \beta_3 t^2}}\, dt < 0.$$

To prove uniqueness, make the change of variables $s = (\sqrt{\beta_3 - x})\, t$ in (4.7) to obtain

$$\begin{aligned} A(x) &= \int_0^{\sqrt{\beta_3}} \frac{-\beta_3 + s^2}{\sqrt{\beta_3 - x - s^2}\sqrt{2\beta_3 - 3 + x - s^2}}\, ds \\ &+ \int_{\sqrt{\beta_3}}^{\sqrt{\beta_3 - x}} \frac{-\beta_3 + s^2}{\sqrt{\beta_3 - x - s^2}\sqrt{2\beta_3 - 3 + x - s^2}}\, ds. \end{aligned} \tag{4.8}$$

Denote by $N(x)$ and $P(x)$ the first and second integrals, respectively, on the right-hand side of (4.8). It will be ascertained that both are decreasing functions of x on I.

(i) Let f be the parabola

$$f(x; s) = (\beta_3 - x - s^2)(2\beta_3 - 3 + x - s^2).$$

For $x \in (-\alpha, 0)$,

$$N'(x) = -\frac{1}{2}\int_0^{\sqrt{\beta_3}} \frac{(-\beta_3 + s^2)\partial_x f(x;s)}{[f(x;s)]^{3/2}} ds.$$

It is easily seen that for $x \in [-\alpha, 0]$ and $s \in [0, \sqrt{\beta_3}]$, $f(x;s) > 0$, $\partial_x f(x;s) \leqq 0$ and $-\beta_3 + s^2 \leqq 0$. In consequence, $N'(x) < 0$ in $(-\alpha, 0)$ and thus N is decreasing.

(ii) Letting $t^2 = -\beta_3 + s^2$ in $P(x)$, it follows that

$$P(x) = \int_0^{\sqrt{-x}} \frac{t^3}{\sqrt{-x-t^2}\sqrt{\beta_3 - 3 + x - t^2}\sqrt{\beta_3 + t^2}}\, dt.$$

If we make the change of variables $s = \sqrt{\frac{\alpha}{-x}}t$, the integral defining P is changed into an integral over the fixed domain $[0, \sqrt{\alpha}]$. The integrand, $h(x,s)$ that appears in this representation of P can be differentiated with respect to x and the result shown to be negative for x in $(-\alpha, 0)$, thereby demonstrating that P is also decreasing there.

Thus, A is a strictly decreasing function on I, and it is concluded that A and, therefore, J, has a unique zero on I. Hence, if $\beta_2 \in (\frac{3-\beta_3}{2}, 0)$ is such that $J(\beta_2) = 0$ then by defining $\beta_1 \equiv 3 - \beta_2 - \beta_3$, we obtain (4.5) and so establish the existence of a cnoidal wave φ_1 of mean zero on $[0, T_{\varphi_1}]$.

The continuity of the map $\beta_3 \in (3, \infty) \mapsto \beta_2 = \beta_2(\beta_3)$ is a consequence of noticing that $\partial_x A(\beta_2(\beta_3), \beta_3) < 0$, where $A(x, \beta_3)$ is as in (4.7), and the implicit function theorem. □

Remark. It follows from Lemma 4.1 that if the fundamental period T_{φ_1} of $\varphi_1 = \varphi_1(\cdot; \beta_i)$ is regarded as a function of the parameter β_3, namely,

$$T_{\varphi_1}(\beta_3) = \frac{4\sqrt{3}}{\sqrt{2\beta_3 + \beta_2(\beta_3) - 3}}\, K\Big(\sqrt{\frac{\beta_3 - \beta_2(\beta_3)}{2\beta_3 + \beta_2(\beta_3) - 3}}\Big),$$

then $T_{\varphi_1}(\beta_3) \to +\infty$ as $\beta_3 \to 3$, since $K(k) \to +\infty$ as $k \to 1$. On the other hand, since $J(\beta_2) = 0$ and $\frac{3-\beta_3}{2} < \beta_2 < 0$, it also follows that

$$\begin{aligned} 0 \le T_{\varphi_1}(\beta_3) &= 4\sqrt{3}\, \frac{\sqrt{2\beta_3 + \beta_2 - 3}}{\beta_3 + \beta_2 - 3}\, E(k(\beta_2)) \le 2\pi\sqrt{3}\, \frac{\sqrt{2\beta_3 + \beta_2 - 3}}{\beta_3 + \beta_2 - 3} \\ &\le 4\pi\sqrt{3}\, \frac{\sqrt{2\beta_3 + \beta_2 - 3}}{\beta_3 - 3} < 4\pi\sqrt{3}\, \frac{\sqrt{2\beta_3 - 3}}{\beta_3 - 3}, \end{aligned} \tag{4.9}$$

and so $T_{\varphi_1}(\beta_3) \to 0$ as $\beta_3 \to +\infty$.

It is now demonstrated that for an arbitrary positive number L, there is a smooth branch of cnoidal waves, $c \in \mathcal{I} \mapsto \varphi_1(\cdot;\beta_i(c))$, dependent on the wave speed c in an appropriate interval, to mean zero cnoidal waves with spatial period $[0, L\sqrt{c}]$.

Lemma 4.2. *Let $L > 0$ be fixed and consider β_i, $i = 1, 2, 3$, an arbitrary but fixed trio as adduced in Lemma* 4.1. *Define $c_0 = (T_{\varphi_1}/L)^2$, where $T_{\varphi_1} = T_{\varphi_1}(\beta_3)$ as in Lemma* 4.1. *Then the following conclusions hold.*

(1) *There exists an open interval $\mathcal{I}(c_0)$ about c_0, an open neighborhood $\mathcal{B}(\overrightarrow{\beta})$ in $\mathbb{R}^3$ of $\overrightarrow{\beta} = (\beta_1, \beta_2, \beta_3)$ and a unique smooth function*

$$\Pi : \mathcal{I}(c_0) \to \mathcal{B}(\overrightarrow{\beta}), \qquad d \mapsto \big(\alpha_1(d), \alpha_2(d), \alpha_3(d)\big)$$

such that $\Pi(c_0) = (\beta_1, \beta_2, \beta_3)$ and $\alpha_i \equiv \alpha_i(d)$ satisfy the relations

$$\begin{cases} \dfrac{4\sqrt{3}}{\sqrt{\alpha_3 - \alpha_1}} K(k) = L\sqrt{d} \\ \alpha_1 + \alpha_2 + \alpha_3 = 3, \quad \alpha_2 + (\alpha_3 - \alpha_2)\dfrac{E(k) - k'^2 K(k)}{k^2 K(k)} = 0, \end{cases} \tag{4.10}$$

where $k^2 = (\alpha_3 - \alpha_2)/(\alpha_3 - \alpha_1)$.

(2) *The cnoidal wave $\varphi_1 = \varphi_1(\cdot;\alpha_i)$ given by* (3.7), *with α_i instead of β_i, has fundamental period $L\sqrt{d}$, mean zero over $[0, L\sqrt{d}]$ and satisfies* (3.4) *with $c = 1$. Moreover, for all $d \in \mathcal{I}(c_0)$*

$$A_{\varphi_1}(d) = \frac{1}{2L\sqrt{d}} \int_0^{L\sqrt{d}} \varphi_1^2(x)dx = -\frac{1}{6}\sum_{i<j} \alpha_i(d)\alpha_j(d).$$

(3) *The interval $\mathcal{I}(c_0)$ can be chosen to be $(0, +\infty)$, independently of c_0.*

Proof. The proof begins with an application of the implicit function theorem. Define

$$\Omega = \{(\alpha_1, \alpha_2, \alpha_3, d) : \alpha_1 < \alpha_2 < 0 < \alpha_3,\ \alpha_3 > 3 \text{ and } d > 0\ \} \subset \mathbb{R}^4$$

and $\Phi : \Omega \to \mathbb{R}^3$ by

$$\begin{aligned} &\Phi(\alpha_1, \alpha_2, \alpha_3, c) \\ &= \Big(\frac{4\sqrt{3}\, K(k)}{\sqrt{\alpha_3 - \alpha_1}} - L\sqrt{c}\,, \alpha_1 + \alpha_2 + \alpha_3 - 3\,, \alpha_2 + (\alpha_3 - \alpha_2)\frac{E(k) - k'^2 K(k)}{k^2 K(k)}\Big) \\ &\equiv (\Phi_1, \Phi_2, \Phi_3), \end{aligned} \tag{4.11}$$

with $k^2 \equiv (\alpha_3 - \alpha_2)/(\alpha_3 - \alpha_1)$. Then, from Lemma 4.1, we see that

$$\Phi(\beta_1, \beta_2, \beta_3, c_0) = (0, 0, 0).$$

To compute $\frac{\partial}{\partial \alpha_j}\Phi_i(\beta_1,\beta_2,\beta_3,c_0)$, use (4.6)

$$\frac{dK}{dk} = \frac{E(k) - k'^2K(k)}{kk'^2}\Big|_{(\alpha_1,\alpha_2,\alpha_3)=(\beta_1,\beta_2,\beta_3)} = -\frac{\beta_2}{\beta_3-\beta_2}\frac{k_1}{k_1'^2}K(k_1)$$

and the relation

$$\frac{4\sqrt{3}}{\sqrt{\beta_3-\beta_1}}K(k_1) = L\sqrt{c_0}$$

with $k_1^2 = (\beta_3-\beta_2)/(\beta_3-\beta_1)$ to obtain

$$\begin{aligned}
\frac{\partial}{\partial \alpha_1}\Phi_1(\beta_1,\beta_2,\beta_3,c_0) &= -\frac{4\sqrt{3}\,\beta_1}{2(\beta_3-\beta_1)^{3/2}(\beta_2-\beta_1)}\,K(k_1)\\
&= \frac{-\beta_1 L\sqrt{c_0}}{2(\beta_3-\beta_1)(\beta_2-\beta_1)},\\
\frac{\partial}{\partial \alpha_2}\Phi_1(\beta_1,\beta_2,\beta_3,c_0) &= \frac{4\sqrt{3}\,\beta_2}{2\sqrt{\beta_3-\beta_1}\,(\beta_2-\beta_1)(\beta_3-\beta_2)}\,K(k_1)\\
&= \frac{\beta_2 L\sqrt{c_0}}{2(\beta_2-\beta_1)(\beta_3-\beta_2)}, \qquad (4.12)\\
\frac{\partial}{\partial \alpha_3}\Phi_1(\beta_1,\beta_2,\beta_3,c_0) &= \frac{-4\sqrt{3}\,\beta_3}{2(\beta_3-\beta_1)^{3/2}(\beta_3-\beta_2)}\,K(k_1)\\
&= \frac{-\beta_3 L\sqrt{c_0}}{2(\beta_3-\beta_1)(\beta_3-\beta_2)}.
\end{aligned}$$

From the definition of Φ_2, it follows immediately that

$$\nabla_{(\alpha_1,\alpha_2,\alpha_3)}\Phi_2(\beta_1,\beta_2,\beta_3,c_0) = (1,1,1). \qquad (4.13)$$

Regarding Φ_3, the formula

$$\Phi_3(\alpha_1,\alpha_2,\alpha_3,c_0) = \alpha_2 + (\alpha_3-\alpha_2)\frac{k'^2}{kK(k)}\frac{dK}{dk}$$

obtains from the fact that

$$\frac{E(k)-k'^2K(k)}{k^2K(k)} = \frac{k'^2}{kK(k)}\frac{dK}{dk},$$

for all k. Since K satisfies the *hypergeometric differential equation* (see Appendix A)

$$kk'^2\frac{d^2K}{dk^2} + (1-3k^2)\frac{dK}{dk} - kK(k) = 0, \qquad (4.14)$$

and

$$\frac{k'^2}{kK(k)}\frac{dK}{dk}\Big|_{(\alpha_1,\alpha_2,\alpha_3)=(\beta_1,\beta_2,\beta_3)} = -\frac{\beta_2}{\beta_3-\beta_2}$$

(see again (4.6)), it is concluded that

$$\begin{aligned}&\nabla_{(\alpha_1,\alpha_3,\alpha_3)}\Phi_3(\beta_1,\beta_2,\beta_3,c_0) \qquad (4.15)\\ &= \left(\frac{\beta_2\beta_3-\beta_1\beta_2-\beta_1\beta_3}{2(\beta_2-\beta_1)(\beta_3-\beta_1)}, \frac{\beta_2\beta_3-\beta_1\beta_3+\beta_1\beta_2}{2(\beta_3-\beta_2)(\beta_2-\beta_1)}, \frac{\beta_1\beta_2-\beta_1\beta_3-\beta_2\beta_3}{2(\beta_3-\beta_2)(\beta_3-\beta_1)}\right).\end{aligned}$$

The Jacobian determinant of $\Phi(\cdot,\cdot,\cdot,d)$ at $(\beta_1,\beta_2,\beta_3,c_0)$ is therefore

$$\frac{\partial(\Phi_1,\Phi_2,\Phi_3)}{\partial(\alpha_1,\alpha_3,\alpha_3)}\Big|_{(\beta_1,\beta_2,\beta_3,c_0)} = L\sqrt{c_0}\,\frac{\beta_1\beta_2+\beta_1\beta_3+\beta_2\beta_3}{4(\beta_2-\beta_1)(\beta_3-\beta_2)(\beta_3-\beta_1)}, \qquad (4.16)$$

as a consequence of (4.12), (4.13) and (4.15). The properties of the cnoidal wave φ_1 imply that

$$\frac{1}{2}\int_0^{T_{\varphi_1}} \varphi_1^2(\xi)\,d\xi = A_{\varphi_1}T_{\varphi_1} = A_{\varphi_1}L\sqrt{c_0},$$

and so (4.16) yields

$$\frac{\partial(\Phi_1,\Phi_2,\Phi_3)}{\partial(\alpha_1,\alpha_3,\alpha_3)}\Big|_{(\beta_1,\beta_2,\beta_3,c_0)} = \frac{-3}{4(\beta_2-\beta_1)(\beta_3-\beta_2)(\beta_3-\beta_1)}\int_0^{T_{\varphi_1}}\varphi_1^2(\xi)\,d\xi \neq 0. \qquad (4.17)$$

Therefore, the implicit function theorem implies the existence of neighborhoods $\mathcal{I}(c_0)$ of c_0 in $\mathbb{R}$ and $\mathcal{B}(\beta_1,\beta_2,\beta_3)$ of $(\beta_1,\beta_2,\beta_3)$ in Ω, and a unique smooth function $\Pi:\mathcal{I}(c_0)\to\mathcal{B}(\beta_1,\beta_2,\beta_3)$ such that

$$\Phi(\Pi(d),d) = (0,0,0) \qquad \text{for all} \quad d\in\mathcal{I}(c_0), \qquad (4.18)$$

and this in turn implies (4.10).

Finally, from the remark following Lemma 4.1, for each $\beta_3>3$, $T_{\varphi_1}(\beta_3)$ can take any value in $(0,\infty)$ as β_3 ranges over $(3,\infty)$. So, we may choose $c_0=(T_{\varphi_1}/L)^2$ arbitrarily in $(0,\infty)$. By the local uniqueness guaranteed by the implicit function theorem, the mapping Π can thus be defined on $(0,+\infty)$. This proves the lemma. □

Our next goal is to obtain formulas for the derivatives $\frac{d}{dc}\alpha_i$, $1\le i\le 3$.

Lemma 4.3. *Let $\Pi:\mathcal{I}(c_0)\to\mathcal{B}(\beta_1,\beta_2,\beta_3)$ be the smooth function defined in Lemma 4.2. If we write $\Pi(c)=(\alpha_1(c),\alpha_2(c),\alpha_3(c))$, then*

$$\frac{d}{dc}\alpha_1(c) = \frac{1}{c(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)}\Big(\alpha_3^2(\alpha_1-\alpha_2)+\alpha_2^2(\alpha_1-\alpha_3)\Big),$$

$$\frac{d}{dc}\alpha_2(c) = \frac{1}{c(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)}\Big(\alpha_1^2(\alpha_2 - \alpha_3) + \alpha_3^2(\alpha_2 - \alpha_1)\Big),$$
$$\frac{d}{dc}\alpha_3(c) = \frac{1}{c(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)}\Big(\alpha_1^2(\alpha_3 - \alpha_2) + \alpha_2^2(\alpha_3 - \alpha_1)\Big). \quad (4.19)$$

Proof. The formulas in (4.19) are obtained by differentiating (4.18) with respect to c and using (4.16), (4.17) (changing β_i to α_i, $i = 1, 2, 3$, and d to c), *viz.*

$$\frac{d}{dc}\Pi(c) = \begin{pmatrix} \nabla_\alpha \Phi_1 \\ \nabla_\alpha \Phi_2 \\ \nabla_\alpha \Phi_3 \end{pmatrix}^{-1} \begin{pmatrix} \frac{L}{2\sqrt{c}} \\ 0 \\ 0 \end{pmatrix},$$

with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. □

The main theorem of this section is established next. In what follows, the notation used will be that appearing in Lemma 4.2 except that we use c to connote the wave speed.

Theorem 4.4. *Let $L > 0$ be a fixed number and n any positive integer. Then there exists a smooth branch $c \in (0, +\infty) \mapsto \phi_c \in H^n_{per}([0, L])$ of cnoidal waves such that $\int_0^L \phi_c(\xi)\, d\xi = 0$ for every c, and*

$$\phi_c''(\xi) + \frac{1}{2}\phi_c^2(\xi) - c\,\phi_c(\xi) = A_{\phi_c}, \qquad \text{for all} \quad \xi \in \mathbb{R}, \quad (4.20)$$

where

$$A_{\phi_c} = \frac{1}{2L}\int_0^L \phi_c^2(\xi)\, d\xi = -\frac{c^2}{6}\sum_{i<j}\alpha_i(c)\alpha_j(c). \quad (4.21)$$

Proof. Let $\varphi_1 \equiv \varphi_1(\cdot; \alpha_i)$ be the cnoidal wave determined in Lemma 4.2. Define

$$\phi_c(\xi) = c\,\varphi_1(\sqrt{c}\,\xi), \qquad \text{for all} \ \ \xi \in \mathbb{R}. \quad (4.22)$$

Then, ϕ_c has the form (3.7) with $\beta_i = c\alpha_i$ and $\sum \beta_i = 3c$. Moreover, ϕ_c has period L, mean value zero on $[0, L]$ and satisfies the equation

$$\phi_c''(\xi) + \frac{1}{2}\phi_c^2(\xi) - c\phi_c(\xi) = c^2 A_{\varphi_1}(c).$$

Lemma 4.2 implies the relation

$$A_{\varphi_1}(c) = \frac{1}{2L\sqrt{c}}\int_0^{L\sqrt{c}} \varphi_1^2(\xi)d\xi = \frac{1}{2Lc^2}\int_0^L \phi_c^2(\xi)d\xi \equiv \frac{1}{c^2}A_{\phi_c}(c), \quad (4.23)$$

and therefore ϕ_c satisfies (4.20).

Finally, since the functions α_1, α_2 and α_3 are smooth, the same property is deduced for the curve $c \mapsto \phi_c$. This proves the theorem. □

Corollary 4.5. *Let $L > 0$ and $\mu \in \mathbb{R}$ be fixed and n a positive integer. Then there is a smooth branch $e \in (\mu, \infty) \mapsto \psi_e \in H^n_{per}([0, L])$ such that $\frac{1}{L}\int_0^L \psi_e(\xi)d\xi = \mu$ for all e and*

$$\psi_e''(\xi) + \frac{1}{2}\psi_e^2(\xi) - e\psi_e(\xi) = A_{\psi_e} \tag{4.24}$$

where

$$A_{\psi_e} = A_{\varphi_c} - c\mu - \frac{1}{2}\mu^2, \tag{4.25}$$

$c = e - \mu$, and A_{φ_c} is as in Theorem 4.4.

Proof. This follows directly from the Galilean transformation (3.2)-(3.3). More precisely, define

$$\psi_e(z) = \varphi_c(z) + \mu,$$

where $c = e - \mu > 0$ and $\varphi_c(z)$ is as in Theorem 4.4. Obviously, ψ_e has mean value μ for all values of e. Then, a calculation reveals that ψ_e satisfies (4.24) where A_{ψ_e} is exactly as in (4.25). □

We close this section with a result about the monotonicity of the modulus k as a function of the speed c.

Proposition 4.6. *Consider $c \in (0, \infty)$ and define the modulus-function*

$$k(c) = \frac{\sqrt{\alpha_3(c) - \alpha_2(c)}}{\sqrt{\alpha_3(c) - \alpha_1(c)}}.$$

Then $\frac{d}{dc}k(c) > 0$.

Proof. Denoting $A(c) = c(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)$ and $B(c) = 2(\alpha_3 - \alpha_1)^{3/2}\sqrt{\alpha_3 - \alpha_2}$, it is seen that

$$\begin{aligned} \frac{d}{dc}k(c) &= \frac{1}{B(c)}\Big[(\alpha_3 - \alpha_2)\frac{d\alpha_1(c)}{dc} - (\alpha_3 - \alpha_1)\frac{d\alpha_2(c)}{dc} + (\alpha_2 - \alpha_1)\frac{d\alpha_3(c)}{dc}\Big] \\ &= -\frac{2}{A(c)B(c)}(\alpha_3 - \alpha_2)(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)(\alpha_1 + \alpha_2 + \alpha_3) \\ &= -\frac{3(\alpha_2 - \alpha_1)}{c(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)}k(c) > 0. \end{aligned}$$

The above computation makes use of the results of Lemmas 4.2 and 4.3. □

5. Stability of cnoidal waves

In this section, attention is turned to stability results for the cnoidal-wave solutions φ_c determined by Theorem 4.4 or Corollary 4.5. The first objective is to obtain the basic inequality (3.13), predicated upon the constraints $\mathcal{F}(u) = \mathcal{F}(\varphi_c)$ and $\mathcal{H}(u) = 0$. For this aspect, a modification of the general theory is used. Stability of the orbit $\{\varphi_c(\cdot + y)\}_{y\in\mathbb{R}}$ in the closed subspace $\{f \in H^1_{per}([0,L]) : \int f dx = 0\}$ is thereby obtained. Stability in $H^1_{per}([0,L])$ without this restriction will then follow readily by making a change of variables in equation (1.1) and using Poincaré's inequality

$$\int_0^L |h(x)|^2\, dx \leqq \frac{1}{L}\Big|\int_0^L h(x)\, dx\Big|^2 + C(L)\int_0^L |h'(x)|^2\, dx. \tag{5.1}$$

From Section 3, we know that the two basic hypotheses I and II are sufficient conditions to apply the theory in [25] or [4] to solitary waves. The next two subsections deal with these conditions in the periodic context.

5.1. **Spectral Analysis of the Operator $\mathcal{L}_{cn} = -\frac{d^2}{dx^2} + c - \varphi_c$.** As already mentioned, the study of the periodic eigenvalue problem considered on $[0,L]$ will allow us to mount a stability theory much as that outlined earlier for solitary waves. The spectral problem in question is

$$\mathcal{L}_{cn}\chi \equiv (-\frac{d^2}{dx^2} + c - \varphi_c)\chi = \lambda\chi, \qquad \chi(0) = \chi(L),\;\; \chi'(0) = \chi'(L), \tag{5.2}$$

where $c > 0$ is fixed and φ_c is the cnoidal wave solution given in Theorem 4.4. The following result obtains in this context.

Theorem 5.1. *For $c \in (0,\infty)$, let φ_c be the cnoidal wave given in Theorem* 4.4 *for some $c \in (0,\infty)$. Let*

$$\lambda_0 \leqq \lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \lambda_4 \leqq \cdots,$$

connote the eigenvalues of the problem (5.2). *Then $\lambda_0 < \lambda_1 = 0 < \lambda_2$ are all simple whilst, for $j \geq 3$, the λ_j are double eigenvalues. The λ_j accumulate only at $+\infty$.*

Theorem 5.1 is a consequence of Floquet theory (Magnus and Winkler [41]) together with some particular facts about the Lamé equation. For the reader's convenience, we quickly outline the basic results that are needed in the proof of Theorem 5.1.

Introduce the so called semi-periodic eigenvalue problem

$$\mathcal{L}_{cn}\xi = \mu\xi, \qquad \xi(0) = -\xi(L),\;\; \xi'(0) = -\xi'(L). \tag{5.3}$$

It follows directly from the spectral theory of compact symmetric operators that both (5.2) and (5.3) have a countable infinity of real eigenvalues,

$$\lambda_0 \leqq \lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \cdots, \tag{5.4}$$

and

$$\mu_0 \leqq \mu_1 \leqq \mu_2 \leqq \mu_3 \leqq \cdots,$$

that accumulate only at $+\infty$. As the operators are of second order, the eigenvalues are either simple or double, and double eigenvalues are displayed twice in the listings in (5.4). The eigenfunctions associated with λ_n and μ_n are denoted χ_n and ξ_n, respectively, $n = 0, 1, 2, \cdots$. The boundary conditions in (5.2) on χ_n imply that χ_n can be extended to all of $\mathbb{R}$ as a continuously differentiable function which is periodic of period L. Similarly, the boundary conditions on ξ_n allow it to be extended as a periodic solution of period $2L$ of

$$\mathcal{L}_{cn} f = \gamma f \tag{5.5}$$

with $\gamma = \mu_n$ (define $\xi_n(L + x) = \xi_n(L - x)$ for $0 \leq x \leq L$). Indeed, the theory goes on to assert that the only periodic solutions of (5.5) of period L correspond to $\gamma = \lambda_j$ for some j whilst the only periodic solutions of period $2L$ are either those associated with $\gamma = \lambda_j$, but viewed on $[0, 2L]$, or those corresponding to $\gamma = \mu_j$, but extended as just indicated, for some $j = 0, 1, \cdots$. We remind the reader that

$$\begin{array}{lll} (i) & \chi_0 \ \text{has no zeros in} \ [0,L], & \\ (ii) & \chi_{2n+1} \ \text{and} \ \chi_{2n+2} \ \text{have exactly} \ 2n+2 \ \text{zeros in} \ [0,L), & \\ (iii) & \xi_{2n} \ \text{and} \ \xi_{2n+1} \ \text{have exactly} \ 2n+1 \ \text{zeros in} \ [0,L). & \end{array} \tag{5.6}$$

Sturm's oscillation theory then implies that the sequences in (5.4) are intertwined, *viz.*

$$\lambda_0 < \mu_0 \leqq \mu_1 < \lambda_1 \leqq \lambda_2 < \mu_2 \leqq \mu_3 < \lambda_3 \leqq \cdots.$$

For a given value γ, if all solutions of (5.5) are bounded, then γ is called a *stable* value, whereas if there is an unbounded solution, γ is called *unstable.* The open intervals $(\lambda_0, \mu_0), (\mu_1, \lambda_1), (\lambda_2, \mu_2), (\mu_3, \lambda_3), \cdots$, are called *intervals of stability.* The endpoints of these intervals are generally unstable. This is always so for $\gamma = \lambda_0$ as λ_0 is always simple. The intervals $(-\infty, \lambda_0), (\mu_0, \mu_1)$, $(\lambda_1, \lambda_2), (\mu_2, \mu_3), \cdots$, and so on are called *intervals of instability.* Of course, at a double eigenvalue, the interval is empty and omitted from the discussion. Absence of an instability interval means there is a value of γ for which all solution of (5.5) are either periodic of basic period L or periodic with basic period $2L$.

Proof. (**Theorem 5.1**) We certainly know that $\lambda_0 < \lambda_1 \leqq \lambda_2$. Since $\mathcal{L}_{cn}\varphi_c' = 0$ and φ_c' has 2 zeros in $[0, L)$, it follows that 0 is either λ_1 or λ_2. We will show that $0 = \lambda_1 < \lambda_2$. First, define the transformation

$$T_\eta \chi(x) \equiv \chi(\eta x), \qquad \text{for} \quad \eta^2 = \frac{12}{\beta_3 - \beta_1},$$

where the β_i are as in (3.6) in the initial discussion of φ_c and they also reappear in Theorem 4.4 (N.B. $\beta_i = c\alpha_i$ where the α_i are as in Lemma 4.2). Then, using the explicit form (3.4) for φ_c, the problem (5.2) is equivalent to the eigenvalue problem

$$\frac{d^2}{dx^2}\Lambda + \left[\rho - 12k^2 sn^2(x)\right]\Lambda = 0, \quad \Lambda(0) = \Lambda(2K), \quad \Lambda'(0) = \Lambda'(2K), \tag{5.7}$$

for $\Lambda \equiv T_\eta \chi$, where

$$\rho = -\frac{12(c - \beta_3 - \lambda)}{\beta_3 - \beta_1}. \tag{5.8}$$

The second-order differential equation (5.7) is the *Jacobian form of Lamé's equation.* Floquet theory ([41], Theorem 7.8) informs us that this equation has exactly 4 intervals of instability, namely

$$(-\infty, \rho_0), \quad (\mu_0', \mu_1'), \quad (\rho_1, \rho_2), \quad (\mu_2', \mu_3'),$$

where the μ_i', are the eigenvalues associated to the semi-periodic problem determined by (5.7), (5.8) (see (5.3)). Therefore, the first three eigenvalues ρ_0, ρ_1, ρ_2 are simple and the remainder of the eigenvalues for (5.7) are double, so, $\rho_3 = \rho_4$, $\rho_5 = \rho_6, \cdots$, etc.

The first three eigenvalues ρ_0, ρ_1, ρ_2 and their corresponding eigenfunctions $\Lambda_0, \Lambda_1, \Lambda_2$ are known explicitly. Since $\rho_1 = 4 + 4k^2$ is a simple eigenvalue of (5.7) with eigenfunction

$$\Lambda_1(x) \equiv cn(x)sn(x)dn(x) = CT_\eta\varphi_c'(x),$$

it follows from (5.8) that $\lambda = 0$ is a simple eigenvalue of problem (5.2) with eigenfunction φ_c'. The functions Λ_0, Λ_2 (sometimes called *Lamé polynomials*) defined by

$$\begin{aligned} \Lambda_0(x) &= dn(x)\left[1 - (1 + 2k^2 - \sqrt{1 - k^2 + 4k^4}\,)sn^2(x)\right], \\ \Lambda_2(x) &= dn(x)\left[1 - (1 + 2k^2 + \sqrt{1 - k^2 + 4k^4}\,)sn^2(x)\right], \end{aligned}$$

are the eigenfunctions associated to the other two eigenvalues, ρ_0, ρ_2. These eigenvalues must satisfy the equation

$$\rho = k^2 + \frac{5k^2}{1 + \frac{9}{4}k^2 - \frac{1}{4}\rho}$$

(see Ince [I]). This latter equation is quadratic in the variable ρ and so has the two roots, $\rho_0 = 2+5k^2-2\sqrt{1-k^2+4k^4}$ and $\rho_2 = 2+5k^2+2\sqrt{1-k^2+4k^4}$. Since Λ_0 has no zeros in $[0, 2K]$ and Λ_2 has exactly 2 zeros in $[0, 2K)$, it must be the case that Λ_0 is the eigenfunction associated to ρ_0, the first eigenvalue of (5.7). On the other hand, since $\rho_0 < \rho_1$ for every $k \in (0, 1)$, there obtains from (5.8) and the relation $-\beta_1(1 + k^2) = (2 - k^2)\alpha_3 - 3c$, the inequality

$$12\lambda_0 = 3\frac{\beta_3 - c}{k^2+1}\rho_0 + 12(c - \beta_3) < 0.$$

As a consequence, the first eigenvalue λ_0 of $\mathcal{L}_{cn}$ is negative and has the eigenfunction $\chi_0(x) = \Lambda_0(\frac{1}{\eta}x)$. It is also true that $\rho_1 < \rho_2$ for every $k \in (0, 1)$, so it follows from (5.8) that

$$12\lambda_2 = 3\frac{\beta_3 - c}{k^2+1}\rho_2 + 12(c - \beta_3) > 0,$$

and so λ_2 is the third eigenvalue to $\mathcal{L}_{cn}$ with eigenfunction $\chi_2(x) = \Lambda_2(\frac{1}{\eta}x)$.

It is straightforward to ascertain that the first two eigenvalues of Lamé's equation in the semi-periodic case are

$$\mu_0' = 5 + 2k^2 - 2\sqrt{4-k^2+k^4} \quad \text{and} \quad \mu_1' = 5 + 5k^2 - 2\sqrt{4-7k^2+4k^4}.$$

The associated eigenfunctions are

$$\xi_0(x) = \ cn(x)\Big[1 - (2 + k^2 - \sqrt{4-k^2+k^4}\ )sn^2(x)\Big],$$

$$\xi_1(x) = 3sn(x) - \Big(2 + 2k^2 - \sqrt{4-7k^2+4k^4}\ \Big)sn^3(x),$$

respectively, both of which have exactly one zero in $[0, 2K)$. Since $\mu_0' < \mu_1' < 4k^2 + 4$, it is concluded from the relation

$$\mu_i' = -\frac{12(c - \beta_3 - \mu_i)}{\beta_3 - \beta_1}, \qquad \text{for} \quad i \geqq 0, \tag{5.9}$$

that the first three instability intervals associated to $\mathcal{L}_{cn}$ are

$$(-\infty, \lambda_0),\ (\mu_0, \mu_1),\ (\lambda_1, \lambda_2).$$

The third and fourth eigenvalues are

$$\mu_2' = 5 + 2k^2 + 2\sqrt{4-k^2+k^4} \quad \text{and} \quad \mu_3' = 5 + 5k^2 + 2\sqrt{4-7k^2+4k^4}$$

with associated eigenfunctions

$$\xi_2(x) = cn(x)\Big[1 - (2 + k^2 + \sqrt{4 - k^2 + k^4}\ )sn^2(x)\Big],$$

$$\xi_3(x) = 3sn(x) - \Big(2 + 2k^2 + \sqrt{4 - 7k^2 + 4k^4}\ \Big)sn^3(x),$$

respectively, both of which have exactly three zeros in $[0, 2K)$. Finally, it follows from (5.9) that the last instability interval of $\mathcal{L}_{cn}$ is (μ_2, μ_3). This concludes the proof. □

5.2. **Convexity of the Function d(c).** Attention is turned to the convexity property of the map $d(c) = \mathcal{E}(\varphi_c) + c\mathcal{F}(\varphi_c)$, where the smooth branch of cnoidal waves $c \in (0, \infty) \mapsto \varphi_c \in H^n_{per}([0, L])$ is that adduced in Theorem 4.4. Since φ_c satisfies $(\mathcal{E}' + c\mathcal{F}')\varphi_c = -A_{\varphi_c}$, there obtains immediately that

$$d'(c) = (-A_{\varphi_c}, \frac{d}{dc}\varphi_c) + \mathcal{F}(\varphi_c) = \frac{1}{2}\int_0^L \varphi_c^2(x)\ dx = -\frac{c^2 L}{6}\sum_{i<j}\alpha_i(c)\alpha_j(c), \tag{5.10}$$

where, in the last equality, use has been made of (4.21).

Theorem 5.2. *For every $c \in (0, \infty)$ the function $c \mapsto d(c)$ is strictly convex. More precisely,*

$$d''(c) = \frac{2304}{L^3}\Big(K(k) - E(k)\Big)E(k)K(k)\frac{dK}{dk}\frac{dk}{dc} > 0. \tag{5.11}$$

The equality in (5.11) is obtained via the explicit formulas for the functions $\alpha_i = \alpha_i(c)$ in (5.10). With the notation set out in Lemma 4.2, for every $c \in (0, +\infty)$,

$$\frac{4\sqrt{3}}{\sqrt{\alpha_3 - \alpha_1}}K(k) = L\sqrt{c}$$

where $k \equiv k(c) = \frac{\sqrt{\alpha_3 - \alpha_2}}{\sqrt{\alpha_3 - \alpha_1}}$. An immediate consequence is that $\alpha_3 - \alpha_2 = 48k^2K(k)^2/cL^2$ and, because of (4.10), there is derived the relation

$$\alpha_2 = -(\alpha_3 - \alpha_2)\frac{E(k) - k'^2K(k)}{k^2K(k)} = -\frac{48K(k)}{cL^2}\Big[E(k) - k'^2K(k)\Big]. \tag{5.12}$$

Using $k^2 + k'^2 = 1$ and solving for α_3 in (5.12) gives

$$\alpha_3 = \frac{48k^2K(k)^2}{cL^2} + \alpha_2 = \frac{48K(k)}{cL^2}\Big[K(k) - E(k)\Big]. \tag{5.13}$$

The third equality in (4.10) implies $\alpha_2(E(k)-K(k)) = \alpha_3(E(k)-(1-k^2)K(k))$ and so $(\alpha_2-\alpha_3)(E(k)-K(k)) = \alpha_3 k^2 K(k)$. Using the definition of k^2 yields

$$K(k)-E(k) = \frac{\alpha_3}{\alpha_3-\alpha_1}\,K(k), \quad \text{which implies} \quad -\frac{\alpha_1}{\alpha_3-\alpha_1} = \frac{E(k)}{K(k)}. \tag{5.14}$$

The values obtained in (5.12) and (5.13) imply the further relation

$$\begin{aligned}\alpha_1 = \frac{E(k)}{E(k)-K(k)}\,\alpha_3 &= \frac{48K(k)}{cL^2}\frac{E(k)}{E(k)-K(k)}\Big[K(k)-E(k)\Big]\\ &= -\frac{48K(k)}{cL^2}E(k).\end{aligned} \tag{5.15}$$

Note that formulas (5.12), (5.13) and (5.15) together with the condition $\alpha_1+\alpha_2+\alpha_3 = 3$ imply that the speed c associated to the cnoidal wave φ_c satisfies

$$c = -\frac{16}{L^2}K(k)\Big[3E(k)+K(k)(k^2-2)\Big]. \tag{5.16}$$

A calculation using the MAPLE software shows that the unique root of $3E(k)+K(k)(k^2-2)$ is approximately $k = k_0 \sim 0.9804$. Since $c \mapsto k(c)$ is a strictly increasing mapping, the cnoidal wave solutions found in Theorem 4.4 are determined by a modulus $k > k_0$.

We are in position to give a proof of Theorem 5.2.

Proof. (**Theorem 5.2**) The relations

$$\frac{dK}{dk} = \frac{E(k)-k'^2K(k)}{kk'^2}, \qquad \frac{dE}{dk} = \frac{E(k)-K(k)}{k}, \tag{5.17}$$

and (5.10), (5.12), (5.13), lead to the formula

$$-\frac{6L^3}{48^2}\,d'(c) = K(k)^2\Big[kk'^2E(k)\frac{dK}{dk}+k^2k'^2\frac{dK}{dk}\frac{dE}{dk}+kE(k)\frac{dE}{dk}\Big] \equiv K(k)^2D(k). \tag{5.18}$$

It remains to determine the quantity

$$\frac{d}{dc}D(k) = \frac{d}{dk}D(k)\frac{dk}{dc}.$$

We will show that

$$\frac{d}{dk}D(k) = \frac{dK}{dk}\Big[2k'^2(E(k)-K(k))-2k^2E(k)\Big]. \tag{5.19}$$

Indeed, as $K(k)$ satisfies (4.14) and $E(k)$ satisfies the hypergeometric differential equation (see again Appendix A)

$$kk'^2\frac{d^2E}{dk^2}+k'^2\frac{dE}{dk}+kE(k)=0, \tag{5.20}$$

one comes to the formulas

$$\begin{cases} \Big[(1-3k^2)\dfrac{dK}{dk}+kk'^2\dfrac{d^2K}{dk^2}\Big]E(k)=kK(k)E(k),\\[2ex] \Big[2k(1-2k^2)\dfrac{dK}{dk}+k^2k'^2\dfrac{d^2K}{dk^2}\Big]\dfrac{dE}{dk}=\Big[kk'^2\dfrac{dK}{dk}+k^2K(k)\Big]\dfrac{dE}{dk},\\[2ex] k^2k'^2\dfrac{dK}{dk}\dfrac{d^2E}{dk^2}=-k^2E(k)\dfrac{dK}{dk}-kk'^2\dfrac{dK}{dk}\dfrac{dE}{dk},\\[2ex] kE(k)\dfrac{d^2E}{dk^2}=-E(k)\dfrac{dK}{dk}, \end{cases} \tag{5.21}$$

where, in the last equation in (5.21), we used $\frac{d^2E}{dk^2}=-\frac{1}{k}\frac{dK}{dk}$ (see again Appendix A). As a consequence of (5.21), it follows that

$$\begin{aligned} \frac{d}{dk}D(k) \quad &= kK(k)E(k)+kk'^2\frac{dK}{dk}\frac{dE}{dk}+k^2K(k)\frac{dE}{dk}\\ &-k^2E(k)\frac{dK}{dk}+E(k)\frac{dE}{dk}+k\Big(\frac{dE}{dk}\Big)^2-E(k)\frac{dK}{dk}. \end{aligned} \tag{5.22}$$

Another collection of relations are obtained from (5.17), *viz.*

$$\begin{cases} kk'^2\dfrac{dK}{dk}\dfrac{dE}{dk}=(1-k^2)\dfrac{dK}{dk}E(k)+(k^2-1)K(k)\dfrac{dK}{dk},\\[2ex] (k^2-1)K(k)\dfrac{dK}{dk}=\dfrac{(1-k^2)K(k)^2-K(k)E(k)}{k},\\[2ex] k^2K(k)\dfrac{dE}{dk}=kK(k)E(k)-kK(k)^2,\\[2ex] E(k)\dfrac{dE}{dk}+k\Big(\dfrac{dE}{dk}\Big)^2=\dfrac{2E(k)^2-3E(k)K(k)+K(k)^2}{k}. \end{cases} \tag{5.23}$$

Collecting the results from (5.23), (5.17) and (5.22), it is deduced that

$$\frac{d}{dk}D(k)=2kE(k)K(k)-kK(k)^2-2k^2E(k)\frac{dK}{dk}+(k^2-1)K(k)\frac{dK}{dk}$$

$$
\begin{aligned}
&+\frac{2E(k)^2-3E(k)K(k)+K(k)^2}{k}\\
&=2kK(k)E(k)-2kK(k)^2-2k^2E(k)\frac{dK}{dk}+\frac{K(k)^2-E(k)K(k)}{k}\\
&+\frac{2E(k)^2-3E(k)K(k)+K(k)^2}{k}\\
&=2[E(k)-K(k)]\Big[\frac{E(k)-k'^2K(k)}{k}\Big]-2k^2E(k)\frac{dK}{dk}\\
&=\frac{dK}{dk}\left[2k'^2(E(k)-K(k))-2k^2E(k)\right],
\end{aligned}
\tag{5.24}
$$

as claimed in (5.19).

The following expression for $D(k)$ comes next:

$$
D(k)=[E(k)-K(k)][3E(k)-K(k)+k^2K(k)]+k^2E(k)K(k). \tag{5.25}
$$

Using (5.17), (5.20) and the fact that $\dfrac{d^2E}{dk^2}=-\dfrac{1}{k}\dfrac{dK}{dk}$, there obtains

$$
\begin{cases}
kk'^2E(k)\dfrac{dK}{dk} & =kE(k)\left(k'^2\dfrac{dE}{dk}+kE(k)\right)=kE(k)\left(\dfrac{dE}{dk}+kK(k)\right),\\[2ex]
k^2k'^2\dfrac{dK}{dk}\dfrac{dE}{dk} & =k^2\dfrac{dE}{dk}\left((1-k^2)\dfrac{dE}{dk}+kE(k)\right)\\[2ex]
& =\dfrac{dE}{dk}\left[k(E(k)-K(k))+k^3K(k)\right],
\end{cases}
\tag{5.26}
$$

which leads to

$$
D(k)=\frac{dE}{dk}\Big[2kE(k)+k(E(k)-K(k))+k^3K(k)\Big]+k^2E(k)K(k), \tag{5.27}
$$

and this in turn gives (5.25) because of (5.17). Thus, (5.19) and (5.27) imply the equality

$$
\begin{aligned}
&-\frac{6L^3}{48^2}\,d''(c)\\
&=2K(k)\frac{dK}{dk}\frac{dk}{dc}\left[D(k)+k'^2K(k)\Big(E(k)-K(k)\Big)-k^2E(k)K(k)\right]\\
&=2K(k)\frac{dK}{dk}\frac{dk}{dc}\left[\Big(E(k)-K(k)\Big)\Big(3E(k)-K(k)+k^2K(k)+k'^2K(k)\Big)\right]\\
&=6\Big(E(k)-K(k)\Big)E(k)K(k)\frac{dK}{dk}\frac{dk}{dc}.
\end{aligned}
\tag{5.28}
$$

Since $K(k)$ is a strictly increasing function, $\frac{dk}{dc} > 0$ (Theorem 4.6) and since $K(k) > E(k)$ for $k \in (0,1)$, the formula (5.28) implies the convexity property of $d(c)$. □

5.3. **Stability of Cnoidal Waves.** The stability theory for the branches of cnoidal waves determined by Theorem 4.4 is developed in this subsection. The periodic problem throws up points not arising in the considerations concerning stability of solitary waves. We combine ideas present in [4] and [25] with perspectives in [8] and [12] to obtain the following result.

Theorem 5.3. *Let $c \in (0,\infty)$ and let $\{\varphi_c\}$ be the cnoidal wave branch of period L given in Theorem 4.4. Then, for each c, the orbit $\mathcal{O}_c = \{\varphi_c(\cdot + s)\}_{s\in\mathbb{R}}$ is stable in $H^1_{per}([0,L])$ with regard to L-periodic perturbations and the KdV-flow. More precisely, given any $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ and a C^2-function $r : \mathbb{R} \to \mathbb{R}$ such that if $\psi \in H^1_{per}([0,L])$ and $d_1(\psi, \mathcal{O}_c) < \delta$, then*

$$d_1(u(\cdot,t), \mathcal{O}_c) \leq \|u(\cdot,t) - \varphi_c(\cdot + r(t))\|_{H^1_{per}([0,L])} \leq \epsilon$$

for all $t \in \mathbb{R}$, where $u(x,t)$ is the solution of the KdV-equation with initial value ψ and $r'(t) = -c + O(\epsilon)$ as $\epsilon \downarrow 0$, uniformly for $t \in \mathbb{R}$.

Several preliminary results are needed to prepare for the proof of Theorem 5.3. These will occupy attention for the moment.

Focus on a mean-zero branch $\{\varphi_c\}$ of cnoidal waves as guaranteed by Theorem 4.4. As the correspondence $c \in (0,\infty) \mapsto \varphi_c \in H^1_{per}([0,L])$ is a C^1-mapping, it must be the case that

$$0 = \frac{d}{dc}\int_0^L \varphi_c(x)dx = \int_0^L \frac{d}{dc}\varphi_c(x)dx$$

and, in addition, differentiating (4.20) with regard to c, it also transpires that

$$-\mathcal{L}_{cn}\Big(\frac{d}{dc}\varphi_c\Big) = \varphi_c + \frac{d}{dc}A_{\varphi_c}.$$

Thus, (4.20) yields the basic relations

$$\begin{aligned} d'(c) &= \langle \mathcal{E}'(\varphi_c) + c\,\mathcal{F}'(\varphi_c), \frac{d}{dc}\varphi_c\rangle_1 + \mathcal{F}(\varphi_c) \\ &= -A_{\varphi_c}\int_0^L \frac{d}{dc}\varphi_c\,dx + \mathcal{F}(\varphi_c) = \frac{1}{2}\int_0^L \varphi_c^2\,dx, \qquad (5.29) \\ d''(c) &= (\varphi_c, \frac{d}{dc}\varphi_c) = -(\mathcal{L}_{cn}\frac{d}{dc}\varphi_c, \frac{d}{dc}\varphi_c). \end{aligned}$$

Lemma 5.4. *Let $c \in (0,\infty)$ and suppose $d''(c) > 0$ in the above context. Define*

$$\mathcal{A} = \Big\{ \psi \in H^1_{per}([0,L]) : \mathcal{H}(\psi) = \frac{1}{L}\int_0^L \psi dx = 0,\ (\psi,\varphi_c) = (\psi,\varphi_c') = 0, \\ \text{and } \|\psi\|_{L^2_{per}([0,L])} = 1 \Big\}.$$

Then, $\zeta = \inf\{\langle \mathcal{L}_{cn}\psi,\psi\rangle_1 : \psi \in \mathcal{A}\} > 0$, and consequently $\langle \mathcal{L}_{cn}\psi,\psi\rangle_1 \geqq \zeta\|\psi\|^2_{H^1_{per}}$ for all ψ with $\mathcal{H}(\psi) = (\psi,\varphi_c) = (\psi,\varphi_c^{'}) = 0$.

Proof. The proof is standard. It is first shown that $\zeta \geqq 0$. The second formula in (5.29) then insures that $0 < d''(c) = -\langle \mathcal{L}_{cn}\frac{d}{dc}\varphi_c, \frac{d}{dc}\varphi_c\rangle_1$. In more detail, using Theorem 5.1, we may write $\frac{d}{dc}\varphi_c = a_0\chi_0 + b_0\varphi_c' + p_0$, where $\mathcal{L}_{cn}\chi_0 = \lambda_0\chi_0$ with $\|\chi_0\| = 1$, $\lambda_0 < 0$, and $\langle \mathcal{L}_{cn}p_0, p_0\rangle_1 > 0$. Hence, $\langle \mathcal{L}_{cn}p_0,p_0\rangle_1 < -a_0^2\lambda_0$. Let $\psi \in \mathcal{A}$ and write $\psi = a\chi_0 + p$ with p in the positive subspace of $\mathcal{L}_{cn}$. Since

$$0 = -(\varphi_c,\psi) = \langle \mathcal{L}_{cn}\frac{d}{dc}\varphi_c + \frac{d}{dc}A_{\varphi_c},\psi\rangle_1 = a_0 a\lambda_0 + \langle \mathcal{L}_{cn}p_0,p\rangle_1,$$

it follows that

$$\langle \mathcal{L}_{cn}\psi,\psi\rangle_1 = a^2\lambda_0 + \langle \mathcal{L}_{cn}p,p\rangle_1 \geqq a^2\lambda_0 + \frac{\langle \mathcal{L}_{cn}p,p_0\rangle_1^2}{\langle \mathcal{L}_{cn}p_0,p_0\rangle_1} > a^2\lambda_0 - \frac{(a_0 a\lambda_0)^2}{a_0^2\lambda_0} = 0.$$

Therefore, $\zeta \geqq 0$. Now suppose $\zeta = 0$, then following the analysis in Albert and Bona [4] for example, we obtain the existence of a $\psi^* \in \mathcal{A}$ such that $\langle \mathcal{L}_{cn}\psi^*,\psi^*\rangle_1 = 0$, which is a contradiction. So, we conclude that $\zeta > 0$.

Finally, it follows easily from a homogeneity argument that $\langle \mathcal{L}_{cn}\psi,\psi\rangle_1 \geqq \zeta\|\psi\|^2_{H^1_{per}}$ for all ψ with $\mathcal{H}(\psi) = (\psi,\varphi_c) = (\psi,\varphi_c^{'}) = 0$. □

The crucial inequality, an improvement on (3.13), is addressed next. Fix an $\epsilon > 0$, $c > 0$ and an element φ_c from the branch of cnoidal waves. Without loss of generality, we may suppose ϵ is small; the precise restriction will appear later and will only depend on c, L and φ_c. Let $\psi \in H^1_{per}([0,L])$ be an initial datum for the KdV-equation which is a small perturbation of φ_c, and let $u \in C(\mathbb{R}; H^1_{per}([0,L]))$ be the solution corresponding to ψ by imposing $u = \psi$ at $t = 0$. By translating ψ if necessary, say by considering $\psi(x + r_0)$, it is assumed that

$$\|\psi - \varphi_c\|^2_{L^2_{per}([0,L])} = d_0(\psi,\varphi_c) \le d_1(\psi,\varphi_c) \le \delta_0 \tag{5.30}$$

where $\delta_0 > 0$ will be determined later. Define the function $F : \mathbb{R}^2 \to \mathbb{R}$ by

$$F(r,t) = \int_0^L \Big(u(x,t) - \varphi_c(x+r)\Big)^2 dx. \tag{5.31}$$

Since $\varphi_c \in H^\infty_{per}([0,L])$, F is a C^∞-function of r and, as will appear momentarily, a C^1-function of t. Moreover, because of the first equality in (5.30), the quantity

$$-2\int_0^L u(x,t)\varphi_c'(x+r)dx = \frac{\partial F}{\partial r} = G(r,t) \tag{5.32}$$

must vanish when $(r,t) = (0,0)$. That is to say, $(\psi, \varphi_c') = 0$. The following corollary to Lemma 5.4 thus applies to ψ.

Corollary 5.5. *Let $\rho \in H^1_{per}([0,L])$ be such that $\mathcal{H}(\rho) = \frac{1}{L}\int_0^L \rho(x)dx = 0$, $\mathcal{F}(\rho) = \mathcal{F}(\varphi_c)$ and $\int_0^L \rho\varphi_c' dx = 0$. Then there are positive constants A_0, B_0, A_1 and B_1 depending only on c and φ_c such that*

$$\begin{aligned} A_0\, d_1(\rho,\varphi_c)^2 + B_0\, d_1(\rho,\varphi_c)^3 &\geq \mathcal{E}(\rho) - \mathcal{E}(\varphi_c) \\ &\geq A_1\|\rho-\varphi_c\|^2_{H^1_{per}([0,L])} - B_1\|\rho-\varphi_c\|^4_{L^2_{per}([0,L])}, \end{aligned} \tag{5.33}$$

where $\mathcal{F}$ and $\mathcal{E}$ are the KdV-invariants defined in (3.9).

Proof. Write ρ in the form

$$\rho(x) = (1+a)\varphi_c(x) + \omega(x),$$

where $(\omega,\varphi_c) = 0$ and a is a scalar. Note that $\int_0^L \omega(x)dx = 0$. Since $\mathcal{F}(\rho) = \mathcal{F}(\varphi_c)$, it follows immediately that if $h = \rho - \varphi_c = a\varphi_c + \omega$, then

$$\int_0^L \varphi_c^2 dx = \int_0^L \rho^2 dx = \int_0^L (\varphi_c + h)^2 dx = \int_0^L (\varphi_c^2 + 2\varphi_c h + h^2)dx,$$

whence

$$a\int_0^L \varphi_c^2 dx = -\frac{1}{2}\int_0^L h^2 dx. \tag{5.34}$$

Another calculation reveals that

$$\begin{aligned} \mathcal{E}(\rho) - \mathcal{E}(\varphi_c) &= \mathcal{E}(\rho) - c\mathcal{F}(\rho) - (\mathcal{E}(\varphi_c) - c\mathcal{F}(\varphi_c)) \\ &= \langle \mathcal{L}_{c_n} h, h\rangle - \frac{1}{6}\int_0^L h^3 dx. \end{aligned} \tag{5.35}$$

Both $\mathcal{E}$ and $\mathcal{F}$ are invariant under translation of the independent variables x, which is to say, $\mathcal{E}(\tau_s\rho) = \mathcal{E}(\rho)$ and $\mathcal{F}(\tau_s\rho) = \mathcal{F}(\rho)$, where $\tau_s\rho(x) = \rho(x+s)$. In consequence, (5.35) implies

$$\mathcal{E}(\rho) - \mathcal{E}(\varphi_c) \leq A_0 d_1(\rho,\varphi_c)^2 + B_0 d_1(\rho,\varphi_c)^3 \tag{5.36}$$

where A_0 and B_0 are constants depending only on φ_c, c and the period L, all of which are fixed in the present considerations.

If ω and h are as above, then because of Lemma 5.4,

$$\langle \mathcal{L}_{c_n}\omega,\omega\rangle_1 \geq \zeta\|\omega\|^2_{H^1_{per}([0,L])} \tag{5.37}$$

where $\zeta > 0$ depends only on c and φ_c. Because of the elementary inequalities

$$\langle \mathcal{L}_{c_n}\omega,\omega\rangle_1 = \langle \mathcal{L}_{c_n}(h-a\varphi_c), h-a\varphi_c\rangle_1 \leq \langle \mathcal{L}_{c_n}h,h\rangle + 2\Gamma a^2\|\varphi_c\|^2_{H^1_{per}([0,L])}$$

where Γ depends only on c and φ_c, and

$$\|\omega\|^2_{H^1_{per}([0,L])} \geq \frac{1}{2}\|h\|^2_{H^1_{per}([0,L])} - \frac{3}{2}a^2\|\varphi_c\|^2_{H^1_{per}([0,L])},$$

(5.37) can be expressed in terms of h, *viz.*

$$\begin{aligned}\langle \mathcal{L}_{c_n}h,h\rangle_1 &\geq \langle \mathcal{L}_{c_n}\omega,\omega\rangle - \Gamma a^2\|\varphi_c\|^2_{H^1_{per}([0,L])} \geq \frac{1}{2}\zeta\|\omega\|^2_{H^1_{per}([0,L])} - \Gamma_1 a^2 \\ &\geq \frac{1}{4}\zeta\|h\|^2_{H^1_{per}([0,L])} - \Gamma_2 a^2 \geq \frac{1}{4}\zeta\|h\|^2_{H^1_{per}([0,L])} - \Gamma_3\|h\|^4_{L^2_{per}([0,L])}\end{aligned} \tag{5.38}$$

where, in the last step, (5.34) is used and Γ_3 is a constant depending only on c and φ_c.

Finally, the cubic term in (5.35) needs to be considered. Clearly, we have

$$\begin{aligned}\int_0^L h^3dx &\leq \|h\|_{L^\infty_{per}([0,L])}\|h\|^2_{L^2_{per}([0,L])} \leq \|h\|^{\frac{5}{2}}_{L^2_{per}([0,L])}\|h_x\|^{\frac{1}{2}}_{L^2_{per}([0,L])} \\ &\leq \frac{1}{8}\|h_x\|^2_{L^2_{per}([0,L])} + b\|h\|^{\frac{10}{3}}_{L^2_{per}([0,L])} \\ &\leq \frac{1}{8}\|h_x\|^2_{L^2_{per}([0,L])} + \frac{1}{8}\zeta\|h\|^2_{L^2_{per}([0,L])} + D\|h\|^4_{L^2_{per}([0,L])}\end{aligned}$$

where b and D depend only on ζ. In summary,

$$\begin{aligned}d_1(\rho,\varphi_c)^2 + d_1(\rho,\varphi_c)^3 &\geq \mathcal{E}(\rho) - \mathcal{E}(\varphi_c) \\ &\geq \tfrac{1}{8}\zeta\|h\|^2_{H^1_{per}([0,L])} - B_1\|h\|^4_{L^2_{per}([0,L])}\end{aligned}$$

as advertised in (5.33) with $A_1 = \frac{1}{8}\zeta$. The corollary is established. □

Consider the polynomial

$$P(y) = A_1 y^2 - B_1 y^4. \tag{5.39}$$

The maximum positive excursion of P is $\frac{1}{4}\frac{A_1^2}{B_1}$, taken on when $y^2 = \frac{1}{2}\frac{A_1}{B_1}$. If $0 < \gamma < \frac{1}{4}\frac{A_1^2}{B_1}$ and y is such that $P(y) \le \gamma$, then either

$$0 \le y^2 \le x_-(\gamma) \tag{5.40}$$

or

$$x_+(\gamma) \le y^2 \tag{5.41}$$

where

$$x_\pm(\gamma) = \frac{A_1 \pm \sqrt{A_1^2 - 4\gamma B_1}}{2B_1}.$$

Notice that as $\gamma \downarrow 0$, $x_-(\gamma) \downarrow 0$ and $x_+(\gamma) \uparrow \frac{A_1}{B_1}$. For $0 < \alpha < \frac{1}{2}\frac{A_1}{B_1}$, let $\gamma(\alpha) > 0$ be such that $x_-(\gamma(\alpha)) = \alpha$. A calculation reveals that if α is in the range just mentioned, then

$$\gamma(\alpha) = A_1\alpha - B_1\alpha^2. \tag{5.42}$$

Thus, for such values of α, $P(y) \le \gamma(\alpha)$ and $y \ge 0$ implies that $y \le \alpha$ or

$$y \ge x_+(\gamma(\alpha)) = \frac{A_1}{B_1} - \alpha > \alpha = x_-(\gamma(\alpha)). \tag{5.43}$$

Returning to (5.32), note that

$$\begin{aligned}\frac{\partial G}{\partial r} = \frac{\partial^2 F}{\partial r^2} &= -2\int_0^L u(x,t)\varphi_c''(x+r)dx \\ &= 2\int_0^L \Big[(\varphi_c'(x+r))^2 - h(x,t)\varphi_c''(x+r)\Big]\,dx,\end{aligned} \tag{5.44}$$

where $h(x,t) = u(x,t) - \varphi_c(x + r(t))$ and the C^1-function $r(t)$ is soon to be specified. In particular, if

$$\|h(\cdot,t)\|_{L^2_{per}([0,L])} < \frac{\|\varphi_c'\|^2_{L^2_{per}([0,L])}}{\|\varphi_c''\|_{L^2_{per}([0,L])}} = M_0, \tag{5.45}$$

then

$$\left.\frac{\partial G}{\partial r}\right|_{(r(t),t)} > 0.$$

Evaluating at $(r,t)=(0,0)$, we see that

$$\begin{aligned}\left.\frac{\partial G}{\partial r}\right|_{(0,0)} &\geq 2\|\varphi_c'\|^2_{L^2_{per}([0,L])} - 2\|\psi-\varphi_c\|_{L^2_{per}([0,L])}\|\varphi_c''\|_{L^2_{per}([0,L])} \\ &\geq 2\left(\|\varphi_c'\|^2_{L^2_{per}([0,L])} - \delta_0\|\varphi_c''\|_{L^2_{per}([0,L])}\right). \end{aligned} \tag{5.46}$$

The following lemma concerning the regularity of the function G will be helpful.

Lemma 5.6. *Let u be a spatially periodic solution of the KdV-equation* (1.1) *of period L lying in $C(\mathbb{R}; H^k_{per}([0,L]))$ where $k \geq 0$ is an integer. Let φ_c be any cnoidal-wave solution of* (1.1), *also of period L. Then the function*

$$G(r,t) = \int_0^L u(x,t)\varphi_c'(x,t)dx$$

is an L-periodic, C^∞-function of r and a C^{k+1}-function of t.

Proof. The fact that G is an infinitely differentiable function of r follows immediately since, for each t, $u(x,t) \in L^2([0,L])$ and $\varphi_c \in H^\infty_{per}([0,L])$. Since the map $t \mapsto u(\cdot,t)$ is continuous from $\mathbb{R}$ to $L^2([0,L])$, it follows at once that G is a continuous function of t.

The higher-order smoothness in t is established by straightforward calculations indicated now, using the fact that u satisfies (1.1). In these computations, at intermediate stages, extra regularity appears to be needed, but the final result does not reflect more regularity than is hypothesized. As mentioned before, one justifies these calculations by regularizing the initial data $u(\cdot,0)$, making the computations for the smoother solution emanating from the smoother data, and then passing to the limit in the final result as the regularization becomes weaker, making use of the well posedness of (1.1) in $H^s_{per}([0,L])$ for $s \geq 0$.

Consider first the case $k=0$ and note that for smooth solutions u of (1.1),

$$\begin{aligned}\frac{\partial G}{\partial t} &= \int_0^L u_t(x,t)\varphi_c'(x,t)dx \\ &= -\int_0^L (uu_x + u_{xxx})\varphi_c' dx = \frac{1}{2}\int_0^L u^2\varphi_c'' dx + \int_0^L u\varphi_c'''' dx. \end{aligned} \tag{5.47}$$

Integrating in time leads to

$$G(r,q) - G(r,s) = \int_s^q \int_0^L \left(\frac{1}{2}u^2\varphi_c'' + u\varphi_c''''\right) dxdt. \tag{5.48}$$

Formula (5.47) continues to hold as the regularization vanishes, and this formula together with the fact that $u \in C(\mathbb{R}; L^2_{per}([0,L]))$ implies G is C^1 and that (5.47) holds pointwise in $\mathbb{R}\times\mathbb{R}$. In terms of $h(x,t) = u(x,t) - \varphi_c(x+r)$, and using the equation $\varphi_c''' + \frac{1}{2}\varphi_c^2 = c\varphi_c$ satisfied by the cnoidal wave, this formula may be rewritten as

$$\frac{\partial G}{\partial t} = -c\int_0^L [(\varphi_c')^2 - h\varphi_c'']dx - \int_0^L h[(\varphi_c')^2 - h\varphi_c'']dx, \tag{5.49}$$

a formula that will be useful momentarily.

Again assuming temporarily that u is smooth, it is deduced after appropriate integrations by parts that

$$\frac{\partial^2 G}{\partial t^2} = \int_0^L \left(-\frac{3}{4}u_x^2\varphi_c''' - u_x\varphi_c^{(iv)} + \frac{1}{3}u^3\varphi_c''' + \frac{1}{2}u_x^2\varphi_c^{(v)}\right)dx. \tag{5.50}$$

Assuming $u \in C(\mathbb{R}, H^1_{per}([0,L]))$ and arguing as in the case $k=0$ shows that G is C^2 as a function of t and that its second partial derivative with respect to t is given by (5.50). A slightly tedious, but straightforward induction on k concludes the proof. □

Returning to (5.45), notice that if

$$\delta_0 \le \frac{1}{2}M_0, \tag{5.51}$$

then $\frac{\partial G}{\partial r}\big|_{(0,0)} > 0$ and so the implicit function theorem applied to G implies there is a $T>0$ and a C^2-function $r : [-T,T] \to \mathbb{R}$ with $r(0)=0$ such that

$$G(r(t),t) = 0 \tag{5.52}$$

for all $t \in [-T,T]$. (Because of Lemma 5.6, if u happens to be smoother, say $u \in C(\mathbb{R}; H^k_{per}([0,L]))$ for some $k>1$, then r will be a C^{k+1} function of t.) By choosing T possibly smaller, it may also be presumed that

$$\|h(\cdot,t)\|_{L^2_{per}([0,L])} = \|u(\cdot,t) - \varphi_c(\cdot + r(t))\|_{L^2_{per}([0,L])} < M_0 \tag{5.53}$$

for $t \in [-T,T]$. In consequence, for t in this range, implicit differentiation yields

$$r'(t) = -\frac{\frac{\partial G}{\partial t}(r(t),t)}{\frac{\partial G}{\partial r}(r(t),t)} = -c + \frac{\int_0^L h\left(\frac{1}{2}h\varphi_c'' - (\varphi_c')^2\right)dx}{\int_0^L ((\varphi_c')^2 - h\varphi_c'')\,dx} \tag{5.54}$$

according to (5.44) and (5.49). Naturally, (5.54) holds as long as the C^1-function r exists, satisfies $G(r(t),t)=0$ and (5.45)-(5.53) continues to hold.

We are now in a position to prove Theorem 5.3.

Proof. (**Theorem 5.3**) The proof commences with the special case wherein the initial datum ψ respects the additional constraints

$$\mathcal{H}(\psi) = \frac{1}{L}\int_0^L \psi dx = 0,$$
$$\mathcal{F}(\psi) = \frac{1}{2}\int_0^L \psi^2 dx = \frac{1}{2}\int_0^L \varphi_c^2 dx = \mathcal{F}(\varphi_c). \tag{5.55}$$

These restrictions are easily removed after the result is established in case (5.55) holds. First, restrict the δ_0 appearing in (5.30) as in (5.51), which is to say

$$\|\psi - \varphi_c\|_{L^2_{per}([0,L])} \le \delta_0 \le \frac{1}{2}M_0, \tag{5.56}$$

where M_0 is defined in (5.45). Let $r(t)$ be the C^2-function determined via the implicit function theorem as above. Initially, r is only defined on a finite interval $[-T, T]$ and is such that

$$0 = G(r(t), t) = -2\int_0^L u(x,t)\varphi_c^{'}(x + r(t))dx. \tag{5.57}$$

We use the notation $h(x,t) = u(x,t) - \varphi_c(x + r(t))$. Since both φ_c and ψ have mean zero, so does h. Moreover, since $\mathcal{F}(\psi) = \mathcal{F}(\varphi_c)$ and $\mathcal{F}$ is a KdV-invariant, it must be the case that $\mathcal{F}(u(\cdot,t)) = \mathcal{F}(\varphi_c)$. Coupling these two points with (5.57) allows us to apply Corollary 5.5 and conclude that

$$\begin{aligned} A_0\delta_0^2 + B_0\delta_0^3 &\ge A_1\|h(\cdot,t)\|^2_{H^1_{per}([0,L])} - B_1\|h(\cdot,t)\|^4_{L^2_{per}([0,L])} \\ &\ge P\left(\|h(\cdot,t)\|_{L^2_{per}([0,L])}\right) \end{aligned} \tag{5.58}$$

at least for $-T \le t \le T$ where $P(y) = A_1y^2 - B_1y^4$ as before. Restrict δ_0 further by requiring that

$$A_0\delta_0^2 + B_0\delta_0^3 \le \gamma(\mu) \tag{5.59}$$

where $\mu = \epsilon\sqrt{\frac{A_1}{2B_1}}$ and ϵ itself is restricted by

$$\epsilon < \sqrt{\frac{A_1}{2B_1}}. \tag{5.60}$$

Notice that, in consequence of the two conditions above, μ lies in the interval $(0, \frac{1}{2}\frac{A_1}{B_1})$ and hence (5.58) and (5.59) imply that either

$$0 \le \|h(\cdot,t)\|^2_{L^2_{per}([0,L])} \le \mu \tag{5.61}$$

or

$$x_+(\gamma(\mu)) \leq \|h(\cdot,t)\|^2_{L^2_{per}([0,L])}. \tag{5.62}$$

As the mapping $t \mapsto \|h(\cdot,t)\|^2_{L^2_{per}([0,L])}$ is continuous on $[-T,T]$ and $\mu < x_+(\gamma(\mu))$ (see (5.43)), one or the other of (5.61) or (5.62) must hold for all $t \in [-T,T]$. Hence, if

$$\delta_0^2 < x_+(\gamma(\mu)), \tag{5.63}$$

then we conclude (5.61) holds at $t = 0$ and hence for $t \in [-T,T]$. Returning to (5.58) armed with this information yields the inequality

$$A_0\delta_0^2 + B_0\delta_0^3 \geq A_1\|h(\cdot,t)\|^2_{H^1_{per}([0,L])} - B_1\mu^2, \tag{5.64}$$

whence,

$$\|h(\cdot,t)\|^2_{H^1_{per}([0,L])} \leq \frac{A_0\delta_0^2 + B_0\delta_0^3}{A_1} + \frac{1}{2}\,\epsilon^2 = \delta_1^2 \tag{5.65}$$

for $t \in [-T,T]$. Note that δ_1 just determined does not depend on t, but only on ϵ, δ_0 and constants depending only on c and φ_c.

To extend this argument to larger time intervals, it suffices to choose ϵ and δ_0 so that δ_1 satisfies the same restriction as did δ_0 and then reapply the implicit function theorem. The uniqueness aspect of the implicit function theorem assures that the function r thereby determined remains C^2 and, as remarked earlier, as long as (5.45) holds, the equation (5.54) for r continues to be valid.

The needed restrictions on δ_1 are that

$$\delta_1 \leq \epsilon, \quad \delta_1 \leq \frac{1}{2}M_0 \tag{5.66}$$

and

$$A_0\delta_1^2 + B_0\delta_1^3 \leq \gamma(\mu). \tag{5.67}$$

The restrictions (5.66) are easily managed, for example by insisting that, in addition to the restriction (5.56) and (5.59),

$$\epsilon < M_0 \quad \text{and} \quad \frac{A_0\delta_0^2 + A_1\delta_0^3}{A_1} < \frac{1}{2}\,\epsilon^2. \tag{5.68}$$

The inequality (5.67) is a little more complicated as ϵ appears on both sides. Because of (5.66)

$$A_0\delta_1^2 + B_0\delta_1^3 \leq A_0\epsilon^2 + B_0\epsilon^3. \tag{5.69}$$

On the other hand, because of (5.60), $\mu < \frac{1}{2}\frac{A_1}{B_1}$ and so (5.42) implies

$$\gamma(\mu) = A_1\mu - B_1\mu^2 = \epsilon A_1\sqrt{\frac{A_1}{2B_1}} - \frac{1}{2}\,\epsilon^2 A_1. \tag{5.70}$$

Thus, (5.67) is seen to hold as soon as ϵ is small enough that

$$\epsilon A_1\sqrt{\frac{A_1}{2B_1}} - \frac{1}{2}\,\epsilon^2 A_1 > A_0\epsilon^2 + B_0\epsilon^3.$$

The theorem is thus established in case the restrictions (5.55) on ψ are valid. Note the condition that $r^{'}(t) = -c+O(\epsilon)$ as $\epsilon \downarrow 0$ follows from the inequality

$$\|h(\cdot,t)\|_{H^1_{per}([0,L])} = \|u(\cdot,t) - \varphi_c(\cdot + r(t))\|_{H^1_{per}([0,L])} \le \epsilon$$

and the formula (5.54) for $r^{'}$.

The restrictions in (5.55) on ψ are now lifted. Let $\epsilon > 0$ and a cnoidal wave φ_c from the branch constructed in Theorem 4.4 be given and suppose

$$||\psi - \varphi_c||_{H^1_{per}} \le \delta$$

where $\delta > 0$ is to be determined. It follows immediately that

$$\mu = \frac{1}{L}\int_0^L (\psi(x) - \varphi_c(x))dx \le L^{-\frac{1}{2}}||\psi - \varphi_c||_{L^2([0,L])} \le \delta L^{-\frac{1}{2}}.$$

Let $\rho(x) = \psi(x) - \mu$ so that $\mathcal{H}(\rho) = 0$. Note that

$$||\rho - \varphi_c||_{H^1_{per}} \le 2\delta$$

by the triangle inequality. Notice also that

$$|\mathcal{F}(\rho) - \mathcal{F}(\varphi_c)| = \left|\int_0^L (\rho - \varphi_c)[\rho - \varphi_c + 2\varphi_c]dx\right|$$

$$\le 2\delta\left[2\delta + 2||\varphi_c||_{L^2([0,L])}\right] \le \delta\left[4 + 4||\varphi_c||_{L^2([0,L])}\right] = \delta M_1$$

provided $\delta \le 1$, which we now presume. Because $\mathcal{F}(\varphi_c)$ is a strictly monotone function of c (see (5.10), (5.11)), it follows that for δ small enough, there is an e near c for which $\mathcal{F}(\rho) = \mathcal{F}(\varphi_c)$. Of course e has to be near to c. Indeed,

$$M_1\delta \ge |\mathcal{F}(\rho) - \mathcal{F}(\varphi_c)| = |\mathcal{F}(\varphi_e) - \mathcal{F}(\varphi_c)| = d^{''}(\tilde{c})\;|e - c|,$$

where $\tilde{c}$ lies between e and c. Because $d^{''}(c)$ is bounded below on compact subsets of $(0,\infty)$, we conclude that $|e - c| \le M_2\delta$, where M_2 depends on M_1 and the formula (5.11) for $d^{''}$. Since the mapping $c \mapsto \varphi_c$ is C^2 from $\mathbb{R}^+$ to $H^1_{per}([0,L])$, it follows that

$$||\varphi_e - \varphi_c||_{H^1_{per}([0,L])} \le M_3\delta$$

where M_3 depends on M_2 and a local upper bound on the derivative of the curve of cnoidal waves.

The existing theory applied to ρ viewed as a perturbation of φ_e assures that if δ is small enough, then

$$||v(\cdot,t)-\varphi_e(\cdot+R(t))||_{H^1_{per}([0,L])} \leq \frac{1}{2}\epsilon$$

for all t, where v is the solution of (1.1) with initial data ρ. Here, the C^2-function $R(t)$ is the solution of the ordinary differential equation

$$R^{'}(t) = -e - \frac{\int_0^L \tilde{h}\left((\varphi_e^{'})^2 - \frac{1}{2}\tilde{h}\varphi_e^{''}\right)dx}{\int_0^L \left((\varphi_e^{'})^2 - \tilde{h}\varphi_e^{''}\right)dx}$$

where $\tilde{h}(x,t) = v(x,t) - \varphi_c(x+R(t))$.

By uniqueness, if u is the solution of (1.1) emanating from ψ, then $u(x,t) = v(x-\mu t,t)+\mu$. If we define $S(t) = R(t)-\mu t$, then it is straightforward to ascertain that

$$\begin{aligned}
&||u(x,t)-\varphi_c(x+S(t))||_{H^1_{per}([0,L])}\\
&= ||u(x+\mu t,t)-\mu+\mu-\varphi_c(x+R(t))||_{H^1_{per}([0,L])}\\
&\leq ||v(x,t)-\varphi_e(x+R(t))||_{H^1_{per}([0,L])} + L|\mu|\\
&+ ||\varphi_e(x+R(t))-\varphi_c(x+R(t))||_{H^1_{per}([0,L])} \leq \frac{1}{2}\epsilon + L^{\frac{1}{2}}\delta + M_3\delta \leq \epsilon
\end{aligned}$$

provided that δ is chosen small enough.

As for R, since $||\tilde{h}||_{H^1_{per}([0,L])} \leq \frac{1}{2}\epsilon$ for all t, it follows that

$$R^{'}(t) = -c + (c-e) + \mathcal{O}(\epsilon) = -c + \mathcal{O}(\delta) + \mathcal{O}(\epsilon) = -c + \mathcal{O}(\epsilon)$$

as $\epsilon \downarrow 0$, as required. □

Corollary 5.7. *Let $L > 0$ and let $\{\psi_c\}$ be the branch of cnoidal waves determined in Corollary 4.5. Then each ψ_c is stable to small $H^1_{per}([0,L])$ perturbations. That is, given ψ_c and $\epsilon > 0$, there is a $\delta = \delta(\epsilon,\psi_c) > 0$ such that if*

$$\|\psi - \psi_c\|_{H^1_{per}([0,L])} \leq \delta,$$

then, for all $t \in \mathbb{R}$

$$d_1(u(\cdot,t),\psi_c) \leq ||u(\cdot,t)-\psi_c(\cdot+r(t))||_{H^1_{per}([0,L])} \leq \epsilon,$$

where u is the solution of (1.1) with initial value ψ and $r:\mathbb{R}\to\mathbb{R}$ is a C^2-function such that

$$r^{'}(t) = -c + \mathcal{O}(\epsilon)$$

as $\epsilon \downarrow 0$, uniformly for $t \in \mathbb{R}$.

Proof. This follows from Theorem 5.3 and the Galilean transformation (3.2). □

Remarks. i) If the constant of integration A_{φ_c} in (3.4) is set equal to zero, we can show the existence of a branch of positive cnoidal-wave solutions, $c \in (\frac{4\pi^2}{L^2}, \infty) \mapsto \varphi_c$, of the form established in (3.7) with $0 < \beta_2 < 2c < \beta_3 < 3c$ and $\beta_2 \leqq \varphi_c \leqq \beta_3$. A proof of the existence and the stability of these positive solutions is analogous to the theory of stability of cnoidal-wave solutions to the Hirota-Satsuma system

$$\begin{cases} u_t - au_{xxx} + 6u_x u = 2bvv_x \\ v_t + v_{xxx} + 3uv_x = 0, \end{cases}$$

established by Angulo in [3].

ii) The ideas in this paper have been applied to other nonlinear evolution equations. For example, Angulo in [2] obtained a stability theory for periodic travelling-wave solutions of the form $u(x,t) = e^{i\omega t}\varphi_\omega(x)$ to the cubic nonlinear Schrödinger equation

$$iu_t + u_{xx} + |u|^2 u = 0,$$

where, for $\omega > 2\pi^2/L^2$, φ_ω is a real-valued function with fundamental period L and with a profile defined via the Jacobian elliptic function dn.

iii) Our stability result for the orbit $\{\varphi_c(\cdot + s)\}_{s\in\mathbb{R}}$ in $H^1_{per}([0,L])$ (Theorem 5.3) under the flow of the periodic KdV-equation is obtained for initial disturbances of φ_c having the same period L. It is a conjecture going back to Benjamin[1] that cnoidal waves of period L are unstable to perturbations of period $2L$, for example. Some evidence in favor of this scenario is available only in the case of the nonlinear Schrödinger equation at the moment (see Angulo [2]), where, for a profile φ_ω depending of the dnoidal function dn, the existence of three simple negative eigenvalues for the linear operator

$$\mathcal{L}_{dn} = -\frac{d^2}{dx^2} + \omega - 3\varphi_\omega^2$$

and the positivity of the function $d''(c) = \frac{d}{d\omega}\int_0^{2L} \varphi_\omega^2\, dx$ imply that the orbit $\{e^{iy}\varphi_\omega : y \in \mathbb{R}\}$ is $H^1_{per}([0,2L])$-unstable. We note that these same background points obtain in the case of the KdV-equation, since the proof of Theorem 5.1 and Theorem 5.2 imply that the linear operator $\mathcal{L}_{cn}$ defined on $H^2_{per}([0,2L])$ will have exactly three negative eigenvalues which are simple and that $d''(c) > 0$.

[1]Personal Communication to the second author.

6. Stability in higher order sobolev classes

The preceding theory was developed in the space $H^1_{per}([0,L])$ that arises naturally in the context of the two invariants $\mathcal{F}$ and $\mathcal{E}$ of the KdV-flow. Better control of high frequency components than would be afforded by being small in H^1_{per} can be obtained if one works in the higher order classes H^k_{per}, $k = 2, 3, \cdots$. As pointed out long ago by Saut and Temam [47], the mapping $\psi \mapsto u(\cdot, t)$ of initial data to the solution at time $t \neq 0$ of (1.1) is a one-to-one mapping of $H^k_{per}([0,L])$ onto itself, for any $k = 1, 2, 3, \cdots$. As a consequence, to have stability in $H^k_{per}([0,L])$, one must start with an $H^k_{per}([0,L])$ perturbation of a cnoidal wave. As it turns out, this is the only additional restriction needed to infer stability in these smaller spaces. The analysis presented in favor of this assertion follows closely the idea in the recent paper [10].

Theorem 6.1. *The cnoidal waves in Theorem* 5.3 *or Corollary* 5.7 *are stable in $H^k_{per}([0,L])$ for any integer $k \geq 1$. More precisely, let $L > 0$ be fixed and let $\{\varphi_c\}$ be any branch of cnoidal waves that are stable in $H^1_{per}([0,L])$ as asserted in Theorem* 5.3 *or Corollary* 5.7. *Then, for each speed c and $\epsilon > 0$, there is a $\delta = \delta(\epsilon, c) > 0$ such that if $\psi \in H^k_{per}([0,L])$ and*

$$d_k(\psi, \varphi_c) < \delta,$$

then there is a C^{k+1}-function $r : \mathbb{R} \to \mathbb{R}$ such that if u is the solution of the KdV-equation (1.1) *starting at ψ, then for all $t \in \mathbb{R}$,*

$$d_k(u(\cdot, t), \varphi_c) \leq ||u(\cdot, t) - \varphi_c(\cdot - r(t))||_{H^k_{per}([0,L])} \leq \epsilon.$$

Moreover,

$$r'(t) = -c + \mathcal{O}(\epsilon)$$

as $\epsilon \downarrow 0$, uniformly for $t \in \mathbb{R}$

Proof. In fact, it will transpire that the same function $r(t)$ determined in Theorem 5.3 or Corollary 5.7 suffices. In a little more detail, suppose $\epsilon > 0$ to be given and, for a given $\psi \in H^1_{per}([0,L])$, let $\delta_1 = \delta_1(\epsilon, \psi) > 0$ be such that

$$||u(\cdot, t) - \varphi_c(\cdot + r(t))||_{H^1_{per}([0,L])} \leq \epsilon_1 \tag{6.1}$$

if $d_1(\psi, \varphi_c) \leq \delta_1$, where r is determined by 5.57 and ϵ_1 will be specified below.

Suppose now that $\psi \in H^n_{per}([0,L])$ and that

$$d_n(\psi, \varphi_c) \leq \delta_n$$

where δ_n will be determined presently, but in any event $\delta_n \le \delta_1(\epsilon_n, \varphi_c)$. In particular, it follows from Lemma 5.6 that the function r lies in C^{n+1} and that (6.1) holds.

We show by induction on n that

$$d_k(u(\cdot,t),\varphi_c) \le ||u(\cdot,t) - \varphi_c(\cdot + r(t))||_{H^k_{per}([0,L])} \le \epsilon_k$$

for all $t \in \mathbb{R}$ and $k \le n$, where the ϵ_k are chosen appropriately for $k < n$. The case $n = 1$ is in hand so we consider next $n = 2$. For $f \in H^2_{per}([0,L])$, let

$$I_4(F) = \int_0^L \Big(\frac{9}{5} f_{xx}^2 - 3 f f_x^2 + \frac{1}{4} f^4\Big) dx.$$

The functional I_4 is invariant on H^2-flows of the KdV-equation, which is to say,

$$I_4(u(\cdot,t)) = I_4(\psi)$$

for all t if u is the solution of (1.1) with initial value ψ. Let $\epsilon_2 > 0$ be given. If $||\psi - \varphi_c||_{H^2_{per}([0,L])} = \delta_2$, it is straightforward to see that

$$I_4(\psi) - I_4(\varphi_c) \le C_2\delta_2 + C_2^{'}\delta_2^4,$$

where C_2 and $C_2^{'}$ are constants depending only on c and φ_c. Assume at the outset that $\delta_2 \le \delta_1(\epsilon_1)$ where ϵ_1 will be quantified shortly. As before, let

$$h(x,t) = u(x,t) - \varphi_c(x + r(t))$$

and, without loss of generality, assume ψ has been translated so that $r(0) = 0$. Since $\delta_2 \le \delta_1$, it follows that (6.1) holds. At time $t \ne 0$, note that

$$\begin{aligned} C_0\delta_2 + C_1\delta_2^4 &\ge I_4(\psi) - I_4(\varphi_c) = I_4(u(\cdot,t)) - I_4(\varphi_c) \\ &= I_4(h(\cdot,t) + \varphi_c(\cdot + r(t))) - I_4(\varphi_c(\cdot + r(t))) \\ &= \int_0^L \Big\{ \frac{9}{5} h_{xx}^2 + \frac{18}{5} h \varphi_{cxxxx} - 6\varphi_c\varphi_{cx} h_x - 3\varphi_c h_x^2 - 3h\varphi_{cx}^2 \\ &\quad - 6hh_x\varphi_{cx} - 3hh_x^2 + \frac{1}{4}h^4 + h^3\varphi_c + \frac{3}{2}h^2\varphi_c^2 + h\varphi_c^3 \Big\} dx. \end{aligned} \tag{6.2}$$

Because of (6.1), elementary considerations reveal that the quantity $I_4(u(x,t)) - I_4(\varphi_c)$ in (6.2) is bounded below by

$$\frac{9}{5}||h_{xx}||^2_{L^2([0,L])} - D_2\epsilon_1 - D_2^{'}\epsilon_1^4, \tag{6.3}$$

where D_2 and $D_2^{'}$ also only depend upon c and φ_c. It follows that for all t,

$$||h_{xx}||^2_{L^2([0,L])} \le \epsilon_1 M_2 + \delta_2 M_2^{'}$$

if ϵ_1 and δ_2 are small. As M_2 and M_2' are bounded above for $0 < \epsilon_1,\ \delta_2 \le 1$, say, it simply remains to choose $\epsilon_1 \le \frac{1}{2}\epsilon_2$ so that $\epsilon_1 M_2 \le \frac{1}{4}\epsilon_2$. The quantity δ_1 is then determined and we choose $\delta_2 \le \delta_1$ so that $\delta_2 M_2' \le \frac{1}{4}\epsilon_2$. It then follows that

$$\begin{aligned} ||u(x,t) - \varphi_c(x + r(t))||^2_{H^2_{per}([0,L])} &= ||h||^2_{H^2_{per}([0,L])} \\ &\le ||h||^2_{H^1_{per}([0,L])} + ||h_{xx}||^2_{L^2([0,L])} \le \frac{1}{4}\epsilon_1^2 + \frac{1}{4}\epsilon_2^2 \le \epsilon_2^2, \end{aligned}$$

thereby establishing the result for $n = 2$.

For $n = 3$ the argument relies upon the functional

$$I_5(f) = \int_0^L \Big\{\frac{108}{35} f_{xxx}^2 - \frac{36}{5} f f_{xx} + 6 f^2 f_x^2 - \frac{1}{5} f^5\Big\} dx, \tag{6.4}$$

which is invariant under H^3_{per}-flows of the KdV-equation and the result just established for H^2_{per}-perturbations. One considers the constant

$$I_5(u(\cdot,t)) - I_5(\varphi_c) \tag{6.5}$$

and bounds this above at $t = 0$ by $C_3\delta_3 - C_3'\delta_3^5$ with $\delta_3 \ge ||\psi - \varphi_c||_{H^2_{per}([0,L])}$. After writing $u = \varphi_c + h$ and expanding the integrals in (6.4) defining I_5, a lower bound on (6.5) is obtained of the form

$$\frac{108}{35}\int_0^L h_{xxx}^2 dx - D_3\epsilon_2 - D_3'\epsilon_2^5.$$

The desired time-independent bound on $||h||_{H^3_{per}([0,L])}$ now follows from the case $n = 2$ and appropriate choices for ϵ_2 and δ_3.

In general, one uses the functional

$$I_{n+2}(f) = \int_0^L \Big\{(\partial_x^n f)^2 + \cdots + a_n f^{n+2}\Big\} dx$$

(see [10] for example), which is invariant under $H^n_{per}([0,L])$-flows of the KdV-equation. The difference $I_{n+2}(u(\cdot,t)) - I_{n+2}(\varphi_c)$ is bounded above at $t = 0$ by $C_n\delta_n + C_n'\delta_n^{n+2}$ and below at any time by

$$\int_0^L (\partial_x^n h(x,t))^2 dx - D_n\epsilon_{n-1} - D_n'\epsilon_{n-1}^{n+2}.$$

The desired inequality follows after choosing ϵ_{n-1} and then δ_n appropriately. $\square$

APPENDIX A

In this Appendix, some basic properties of Jacobian elliptic integrals (see [16]) are collected for the reader's convenience. As before, the *normal elliptic integral of the first kind* is

$$\int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^\varphi \frac{d\theta}{\sqrt{1-k^2\ \sin^2\theta}} \equiv F(\varphi,k),$$

where $y = sin\varphi$, whereas, the *normal elliptic integral of the second kind* is

$$\int_0^y \sqrt{\frac{1-k^2t^2}{1-t^2}}\, dt = \int_0^\varphi \sqrt{1-k^2\ \sin^2\theta}\, d\theta \equiv E(\varphi,k).$$

The number k is called the *modulus* and belongs to the interval $(0,1)$. The number $k' = \sqrt{1-k^2}$ is called the *complementary modulus*. The parameter φ is called the *argument* of the normal elliptic integrals. It is usually understood that $0 \leqq y \leqq 1$ or, what is the same, $0 \leqq \varphi \leqq \pi/2$.

For $y = 1$, the integrals above are said to be *complete*. In this case, one writes

$$\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\ \sin^2\theta}} = F(\pi/2,k) \equiv K(k)$$

and

$$\int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\, dt = \int_0^{\pi/2} \sqrt{1-k^2\ \sin^2\theta}\, d\theta = E(\pi/2,k) \equiv E(k).$$

Clearly, we have $K(0) = E(0) = \pi/2$, whilst $E(1) = 1$ and $K(1) = +\infty$. For $k \in (0,1)$, $K'(k) > 0$, $K''(k) > 0$, $E'(k) < 0$, $E''(k) < 0$ and $E(k) < K(k)$. Moreover, $E(k) + K(k)$ and $E(k)K(k)$ are strictly increasing functions for every $k \in (0,1)$.

An important property of the complete elliptic integrals K and E is that they satisfy the equations

$$\begin{cases} kk'^2 \frac{d^2K}{dk^2} + (1-3k^2)\frac{dK}{dk} - kK = 0, \\ kk'^2 \frac{d^2E}{dk^2} + k'^2\frac{dE}{dk} + kE = 0, \end{cases}$$

respectively, which are special cases of the *hypergeometric equation*. In fact, if we consider the binomial expansion of $(1-k^2sin^2\theta)^{-1/2}$, namely,

$$(1-k^2sin^2\theta)^{-1/2} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!} k^{2n} sin^{2n}\theta,$$

where $(a)_n$ denotes the shifted factorial defined by $(a)_n = a(a+1)\cdots(a+n-1)$ for $n > 0$, $(a)_0 = 1$ (note that since $|k| < 1$ this series converges absolutely), and use the integral formula $\int_0^{\frac{\pi}{2}} sin^{2n}\theta \, d\theta = (\frac{1}{2})_n \frac{\pi}{2}/(1)_n$, we obtain that

$$K(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{2})_n}{(1)_n} \frac{k^{2n}}{n!} = \frac{\pi}{2} F(1/2, 1/2; 1; k^2).$$

Since the hypergeometric function $F(1/2, 1/2; 1; x)$ satisfies Euler's hypergeometric differential equation

$$x(1-x)\frac{d^2y}{dx^2} + (1-2x)\frac{dy}{dx} - \frac{1}{4}y = 0,$$

the first differential equation above obtains immediately. It follows similarly that $E(k) = \frac{\pi}{2} F(-1/2, 1/2; 1; k^2)$, from which follows the second differential equation above.

Next we record some derivatives of the complete elliptical integrals K and E used in this work, namely

$$\begin{cases} \dfrac{dK}{dk} &= \dfrac{E - k'^2 K}{kk'^2}, \\ \dfrac{dE}{dk} &= \dfrac{E-K}{k}, \\ \dfrac{d^2E}{dk^2} &= -\dfrac{1}{k}\dfrac{dK}{dk} = -\dfrac{E - k'^2K}{k^2k'^2}. \end{cases}$$

The *Jacobian elliptic functions* are usually defined as follows. Consider the elliptic integral

$$u(y_1; k) \equiv u = \int_0^{y_1} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\varphi} \frac{d\theta}{\sqrt{1-k^2 \, \sin^2\theta}} = F(\varphi, k),$$

which is a strictly increasing function of the variable y_1. Its inverse function is written $y_1 = \sin\varphi \equiv sn(u; k)$, or briefly $y_1 = sn(u)$ when it is not necessary to emphasize the modulus k. So, sn is an odd function. The other two basic elliptic functions, the **cnoidal** and **dnoidal** functions, are defined in terms of sn by

$$\begin{cases} cn(u;k) &= \sqrt{1-y_1^2} = \sqrt{1-sn^2(u;k)}, \\ dn(u;k) &= \sqrt{1-k^2y_1^2} = \sqrt{1-k^2sn^2(u;k)}. \end{cases}$$

Note that these functions are normalized by the requirement $sn(0,k) = 0$, $cn(0,k) = 1$ and $dn(0,k) = 1$. The functions $cn(\cdot;k)$ and $dn(\cdot;k)$ are therefore even functions. These functions are all periodic with

$$\begin{aligned} &sn(u+4K(k);k) = sn(u;k), \quad cn(u+4K(k);k) = cn(u;k), \\ &dn(u+2K(k);k) = dn(u;k). \end{aligned}$$

Moreover, the relations

$$\begin{cases} sn^2u + cn^2u = 1, \quad k^2sn^2u + dn^2u = 1, \quad k'^2sn^2u + cn^2u = dn^2u, \\ -1 \leqq sn(u;k) \leqq 1, \quad -1 \leqq cn(u;k) \leqq 1, \quad k'^2 \leqq dn(u;k) \leqq 1, \\ sn(u+2K;k) = -sn(u;k), \quad cn(u+2K;k) = -cn(u;k), \end{cases}$$

hold for all $k \in (0,1)$ and $u \in \mathbb{R}$. Moreover, these functions take on the following specific values:

$$sn\ (0) = 0, \quad cn\ (0) = 1, \quad sn\ (K) = 1, \quad cn\ (K) = 0.$$

Also, we have the limiting forms

$$sn(u;0) = \sin(u),\ cn(u;0) = cos(u),\ sn(u;1) = \tanh(u),\ cn(u;1) = sech(u).$$

Finally, the formulas

$$\frac{\partial}{\partial u} snu = cnu\ dn\ u, \quad \frac{\partial}{\partial u} cnu = -sn\ u\ dn\ u, \quad \frac{\partial}{\partial u} dnu = -k^2 sn\ u\ cn\ u$$

are straightforwardly deduced from the foregoing material.

Appendix B

In this Appendix, we sketch a theory of existence and stability of periodic travelling waves solutions with mean zero to the KdV-equation via a variational argument.

Consider the minimization problem

$$B(\lambda) = \inf\{\mathcal{E}(f) : f \in H^1_{per}([0,L]),\ \mathcal{F}(f) = \lambda \text{ and } \ \mathcal{H}(f) \equiv \frac{1}{L}\int f dx = 0\}$$

with $\lambda > 0$, and denote by G_λ the set of minimizers associated to $B(\lambda)$, namely,

$$G_\lambda = \{\psi \in H^1_{per}([0,L]) : \mathcal{E}(\psi) = B(\lambda),\ \mathcal{F}(\psi) = \lambda \text{ and } \ \mathcal{H}(\psi) = 0\ \}.$$

We claim that G_λ is a stable set of periodic travelling waves solutions to the KdV-equation in the following sense: for every $\epsilon > 0$ there exists $\delta > 0$ such that if

$$d_1(u_0, G_\lambda) \equiv \inf_{\psi \in G_\lambda} \|u_0 - \psi\|_{H^1_{per}} < \delta,$$

then the solution $u(x,t)$ of (1.1) with $u(x,0) = u_0$ satisfies

$$\inf_{\psi \in G_\lambda} \|u(t) - \psi\|_{H^1_{per}} < \epsilon, \text{ for all } t \in \mathbb{R}. \tag{$*$}$$

As a first step, it is ascertained that $G_\lambda \neq \emptyset$. Indeed, since Sobolev's embedding theorem implies that $H^1([0,L]) \hookrightarrow C([0,L])$ is compact and the functional $\mathcal{E}$ is weakly lower semi-continuous, so every minimizing sequence $\{f_n\}_{n\in\mathbb{N}} \subset H^1_{per}([0,L])$ associated to $B(\lambda)$ has a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ that converges uniformly to a function $\varphi \in H^1_{per}([0,L])$ which is a minimizer to the constrained problem in question. Of course, $\varphi(0) = \varphi(L)$. In consequence, there are Lagrange multipliers $c(\lambda)$ and $a(\lambda)$ such that

$$-\varphi'' - \frac{1}{2}\varphi^2 = c(\lambda)\varphi + a(\lambda).$$

It follows from standard arguments and the fact that φ is a minimizer that the minimizing sequence that converges weakly to φ in $H^1_{per}([0,L])$ and strongly in $C([0,L])$ must also converge strongly in $H^1_{per}([0,L])$. Moreover, since $B(\lambda) = \mathcal{E}(\varphi) < 0$ and $\int_0^L \varphi^3 dx > 0$ it follows that $c(\lambda) < 0$. Also, since $\mathcal{H}(\varphi) = 0$, it follows that $a(\lambda) = -\frac{1}{L}\mathcal{F}(\varphi)$.

Now we prove that G_λ is a stable set. The proof follows classical arguments (see Cazenave-Lions [19]). Initially we have that if $\{f_n\}_{n\in\mathbb{N}}$ is a minimizing sequence for $B(\lambda)$, then $\lim_{n\to\infty} d_1(f_n, G_\lambda) = 0$. In fact, supposing that this statement does not hold, there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ and a number $\epsilon > 0$ such that

$$d_1(f_{n_k}, G_\lambda) \geqq \epsilon \quad \text{for } k \in \mathbb{N}.$$

But, since $\{f_{n_k}\}_{k\in\mathbb{N}}$ itself is a minimizing sequence for $B(\lambda)$, there exists a $\psi_0 \in G_\lambda$ such that $f_{n_k} \to \psi_0$ in $H^1_{per}([0,L])$-norm, a contradiction that implies the stated result.

Next, it is shown that $(*)$ is true in the closed subspace $\chi = \{f \in H^1_{per}([0,L]) : \mathcal{H}(f) = 0\}$ of $H^1_{per}([0,L])$. Supposing this is false, there is an $\epsilon > 0$, a sequence $\{g_n\}_{n\in\mathbb{N}} \subset \chi$ and a sequence of times $\{t_n\}_{n\in\mathbb{N}}$ such that

$$d_1(g_n, G_\lambda) < \frac{1}{n} \ \text{ and } \ d_1(u_n(t_n), G_\lambda) \geqq \epsilon$$

for all n, where u_n solves (1.1) with $u_n(0) = g_n$. Then, since

$$\lim_{n\to\infty} d_1(g_n, G_\lambda) = 0$$

and G_λ is a bounded set in $H^1_{per}([0,L])$, it follows that $\{g_n\}$ is a bounded set in $H^1_{per}([0,L])$. Moreover, since $\mathcal{E}(g) = B(\lambda)$ and $\mathcal{F}(g) = \lambda$ for $g \in G_\lambda$, it follows that $\mathcal{E}(g_n) \to B(\lambda)$ and $\mathcal{F}(g_n) \to \lambda$. Choosing $\{\alpha_n\}_{n\in\mathbb{N}}$ such that $\mathcal{F}(\alpha_n g_n) = \lambda$ and defining $\psi_n = \alpha_n u(t_n)$ it follows that $\{\psi_n\}_{n\in\mathbb{N}}$ is a minimizing sequence for $B(\lambda)$ and so there is a sequence $\{\varphi_n\} \in G_\lambda$ such that $\|\psi_n - \varphi_n\|_{H^1_{per}} < \epsilon/2$ for large n. Because $\{g_n\}$ is bounded and $\mathcal{E}$ and $\mathcal{F}$ are invariant, it follows that $\{u_n(t_n)\}$ is bounded, say, $\|u_n(t_n)\|_{H^1_{per}} \leq M$ for all n. Hence, we have

$$\epsilon \leqq \|u(t_n) - \varphi_n\|_{H^1_{per}} \leqq |1 - \alpha_n| M + \frac{\epsilon}{2},$$

which is a contradiction since $\alpha_n \to 1$.

Finally, by using Poincaré's inequality (5.1) and the validity of $(*)$ in χ, it is inferred that $(*)$ holds in all of $H^1_{per}[0, L]$ and thus G_λ is seen to be a stable set.

We note that this variational approach to stability gives only information on the set of minimizers G_λ without providing information on the structure of this set, nor distinguishing among its possibly different orbits. In particular, without further information, an individual travelling wave might not be stable in a sense that would be recognized in the laboratory.

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