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Abstract

Lower bounds on the rate of decrease in time of a uniform radius of spatial analyticity for solutions of the derivative Schrödinger equation are derived. The bounds depend algebraically on time. They are valid as long as the initial datum is of order one in L^2-norm and satisfies suitable spatial decay requirements on the real axis.

1. Introduction

Consideration is given to the initial-value problem for the derivative-nonlinear Schrödinger equation

$$\begin{cases} iu_t + u_{xx} = i(|u|^2 u)_x, \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

Attention will be focused on solutions $u(x,t)$ of (1.1) that may be continued analytically to a complex strip $S_\sigma = \{z = x + iy\colon\ |y| < \sigma\}$, at least for small values of σ. Assuming this to be the case at $t = 0$, the point arises as to whether or not this state of affairs is maintained for nonzero t.

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Besides being interesting in its own right, this question is connected to issues of singularity formation that have come to the fore in recent years in the study of dispersive equations (see, e.g., [1,2]).

The general issue in view has been studied by several authors. An early appreciation may be found in K. Kato and Masuda [14]. More recent work includes the results of Hayashi [8], de Bouard, Hayashi and K. Kato [4], Hayashi and Ozawa [12] and K. Kato and Ogawa [13] in the context of KdV and NLS equations. General theory for the initial-value problem for the derivative Schrödinger equation was developed by Tsutsumi and Fukuda [19,20], Hayashi [9], Hayashi and Ozawa [10] and Takaoka [18]. In [11], Hayashi and Ozawa proved for (1.1) that if u_0 decays exponentially, then the solution $u(\cdot,t)$ lies in the strip $S_{2|t|}$ at least for small values of $|t|$. Recently, this latter result was supplemented by one of Grujić and Kalisch [7] showing that if the strip of analyticity of u_0 is positive, it does not shrink, at least for a short time, even in the absence of exponential decay.

At the center of the analysis in the present article is the study of the width σ of the strip of analyticity for large t. It will be proved that in certain circumstances, the radius of analyticity remains positive for all time and can decrease at most algebraically as $|t| \to \infty$.

The function spaces considered in this article are known as analytic Gevrey spaces. They first appeared in nonlinear PDE theory in the work of Foias and Temam [5] on the Navier–Stokes equations, and are defined as follows. For $\sigma > 0$ and $s \in \mathbb{R}$, define $G_{\sigma,s}$ to be the subspace of $L^2(\mathbb{R})$ for which

$$\|u_0\|_{G_{\sigma,s}}^2 = \int_{-\infty}^{\infty} \bigl(1+|\xi|\bigr)^{2s} e^{2\sigma(1+|\xi|)} \bigl|\hat{u}_0(\xi)\bigr|^2 \, d\xi \tag{1.2}$$

is finite where $\hat{u}_0$ is the Fourier transform of u_0. Functions in $G_{\sigma,s}$ automatically possess an analytic extension to the complex strip S_σ. The main result of this article is the following theorem.

Theorem 1. *Suppose that $u_0 \in G_{\sigma_0,s}$ for some $s \geqslant \frac{7}{8}$ and $\sigma_0 > 0$. Assume that $\|u_0\|_{L^2} \leqslant \sqrt{2\pi}$, and let $T > 0$ be arbitrary. Then there exists a constant $C_0 > 0$ depending only on s, σ_0 and $\|u_0\|_{G_{\sigma_0,s}}$, such that the solution u of (1.1) corresponding to the initial data u_0 lies in the space $C([-T,T], G_{\sigma(T)/4,s})$ where*

$$\sigma(T) \geqslant \min\bigl\{\sigma_0, C_0 T^{-120}\bigr\}.$$

Remark 2. As is pointed out in Section 6, the presumption that the data is analytic can be replaced by exponential decay of the initial data and its first two derivatives.

It should be remarked also that the power 120 is simply an artifact of the proof. We do not expect it to be sharp; indeed, we know it can be improved somewhat at the expense of more complex estimates.

It is also worth pointing out that the restriction on the size of the initial data in $L^2(\mathbb{R})$ is consistent with the small amplitude assumptions that came to the fore in the derivation of (1.1) as a model of physical phenomena.

Since the nonlinear term $i\partial_x(|u|^2u)$ presents difficulties owing to the loss of a derivative, it is helpful to first apply a gauge transformation, as has been done in previous studies (cf. [10]). If $u(x,t)$ is a solution of (1.1), define a function $w(x,t)$ by

$$w(x,t) = \exp\left(-i\int_{-\infty}^{x} |u(y,t)|^2\,dy\right) u(x,t). \tag{1.3}$$

Then w formally satisfies the initial-value problem

$$\begin{cases} iw_t + w_{xx} = -iw^2\bar{w}_x - \frac{1}{2}|w|^4 w, \\ w(x,0) = w_0(x), \end{cases} \tag{1.4}$$

with initial datum $w_0(x) = e^{-i\int_{-\infty}^{x}|u_0(y)|^2\,dy} u_0(x)$. To see that this transform is continuous in the $G_{\sigma,s}$-norm, first consider the situation when $s = 0$. It is convenient to use the equivalent Hardy-space norm (the $\mathcal{H}^2$-norm) on a strip, namely,

$$\|u_0\|_{\mathcal{H}^2(S_\sigma)}^2 = \int_{-\infty}^{\infty} \bigl|u_0(x+i\sigma)\bigr|^2\,dx + \int_{-\infty}^{\infty} \bigl|u_0(x-i\sigma)\bigr|^2\,dx, \tag{1.5}$$

instead of the form (1.2). By complexifying the path integral in the definition of w above, and observing that the gauge transformation preserves the L^2-norm, it is easily ascertained that the gauge transformation is continuous with respect to the $G_{\sigma,0}$ norm. When s is an integer, one may differentiate w with respect to x s times and use the same analysis as for $s = 0$ to see that the gauge transformation is continuous. Finally, the general case can be obtained by interpolation between integer values. The gauge transformation results in an equation which still has a derivative nonlinearity. However, the derivative nonlinearity appearing in (1.4) can be controlled using Bourgain-type function spaces. The price to be paid is the quintic nonlinearity appearing in (1.4), but it turns out this does not pose any special challenge, at least for moderate sized initial data.

The paper proceeds as follows. Appropriate notation and function spaces are discussed in the next section, while Section 3 contains some auxiliary linear estimates. Multilinear estimates are proved in Section 4, and the proof of the main theorem is given in Section 5. The main result of Section 5 is combined in Section 6 with a local smoothing effect due to Hayashi and Ozawa to yield global analyticity result for exponentially decaying initial data.

2. Function spaces

The Fourier transform of a function v_0 defined on $\mathbb{R}$ belonging to the Schwartz class, say, is defined to be

$$\hat{v}_0(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v_0(x) e^{-ix\xi}\,dx.$$

For a function $v(x,t)$ of two variables, the spatial Fourier transform is denoted by

$$\mathcal{F}_x v(\xi,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(x,t) e^{-ix\xi}\,dx,$$

whereas the notation $\hat{v}(\xi,\tau)$ designates the space–time Fourier transform

$$\hat{v}(\xi,\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v(x,t) e^{-ix\xi} e^{-it\tau}\,dx\,dt.$$

Define Fourier multiplier operators A and Λ by

$$\widehat{Av}(\xi,\tau) = \big(1+|\xi|\big)\hat{v}(\xi,\tau) \quad \text{and} \quad \widehat{\Lambda v}(\xi,\tau) = \big(1+|\tau|\big)\hat{v}(\xi,\tau).$$

The following notation is used to signify the L^p–L^q space–time norms;

$$\|v\|_{L^pL^q} = \|v\|_{L_x^pL_t^q} = \left\{\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\big|v(x,t)\big|^q\,dt\right|^{\frac{p}{q}}dx\right\}^{\frac{1}{p}}.$$

The Sobolev space $H^s(\mathbb{R})$ is defined as the domain of A^s in $L^2(\mathbb{R})$, with the norm

$$\|v_0\|_{H^s}^2 = \int_{-\infty}^{\infty}\big(1+|\xi|\big)^{2s}\big|\hat{v}_0(\xi)\big|^2\,d\xi.$$

As mentioned already, a class of analytic functions suitable for our analysis is the analytic Gevrey class $G_{\sigma,s}$ used in [5]. This class is the domain of the operator $A^se^{\sigma A}$ in $L^2(\mathbb{R})$, with norm (1.2) as described in the introduction. It is straightforward to check that a function in $G_{\sigma,s}$ is the restriction to the real axis of a function analytic on the strip $S_\sigma = \{z = x+iy\colon\ |y| < \sigma\}$ symmetric about the real axis and of width 2σ.

To efficiently exploit the dispersive effects inherent in (1.1), the following norm is used. For $\sigma > 0$, $s \in \mathbb{R}$, and $b \in [-1,1]$, define $X_{\sigma,s,b}$ to be the subspace of $L^2(\mathbb{R}^2)$ for which the norm

$$\|v\|_{\sigma,s,b}^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big(1+\big|\tau+\xi^2\big|\big)^{2b}\big(1+|\xi|\big)^{2s}e^{2\sigma(1+|\xi|)}\big|\hat{v}(\xi,\tau)\big|^2\,d\xi\,d\tau$$

is finite. For $\sigma = 0$, the Banach space $X_{\sigma,s,b}$ coincides with the space $X_{s,b}$ introduced by Bourgain, and Kenig, Ponce and Vega. The norm on $X_{s,b}$ is denoted by $\|\cdot\|_{s,b}$ and is defined by the integral

$$\|v\|_{s,b}^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big(1+\big|\tau+\xi^2\big|\big)^{2b}\big(1+|\xi|\big)^{2s}\big|\hat{v}(\xi,\tau)\big|^2\,d\xi\,d\tau.$$

The space $X_{\sigma,s,b}$ was introduced in [7], where it was useful in obtaining local-in-time well-posedness of (1.1) in $G_{\sigma,s}$ for an appropriate range of s and with fixed σ. Here, interest is focused on the global behavior of solutions in $X_{\sigma,s,b}$, where σ will be allowed to vary in time.

3. Auxiliary estimates

The definition of the $X_{\sigma,s,b}$-norm is well coordinated with the linear part of the equation. In the following, some identities and linear estimates are listed that elucidate this relation. For the linear initial-value problem

$$\begin{cases} i\varphi_t + \varphi_{xx} = 0, \\ \varphi(x,0) = \varphi_0(x), \end{cases} \tag{3.1}$$

an explicit solution is given in terms of the NLS-group $S(t)$ defined by

$$\varphi(x,t) = S(t)\varphi_0 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ix\xi}e^{-it\xi^2}\hat{\varphi}_0(\xi)\,d\xi. \tag{3.2}$$

A relation between $S(t)$ and the space $X_{\sigma,s,b}$ is provided by the identity

$$\big\|S(t)\varphi\big\|_{\sigma,s,b} = \big\|A^se^{\sigma A}\Lambda^b\varphi\big\|_{L^2L^2}. \tag{3.3}$$

As is well known, the space $\mathcal{C}_{T,\sigma,s} = C([0,T],G_{\sigma,s})$ is a Banach space when equipped with the norm

$$|v|_{\mathcal{C}_{T,\sigma,s}} = \sup_{0\leqslant t\leqslant T}\big\|v(\cdot,t)\big\|_{G_{\sigma,s}}.$$

For $b > \frac{1}{2}$, the space $X_{\sigma,s,b}$ is embedded in $C([0,T],G_{\sigma,s})$ as is evident from the inequality

$$|v|_{\mathcal{C}_{T,\sigma,s}} \leqslant c\|v\|_{\sigma,s,b}, \tag{3.4}$$

which follows directly from (3.3) and the Sobolev embedding theorem.

To obtain estimates that are local in time, it is helpful to introduce a temporal cut-off function. Let ψ be an infinitely differentiable function on $\mathbb{R}$ such that

$$\psi(t) = \begin{cases} 0, & |t| \geqslant 2, \\ 1, & |t| \leqslant 1, \end{cases}$$

and let $\psi_T(t) = \psi(t/T)$.

Lemma 1. *Let $\sigma \geqslant 0$, $b > \frac{1}{2}$, $b-1 < b' < 0$, and $T > 0$. Let $v_0 \in G_{\sigma,s}$ and let v be the solution of* (3.1) *with initial data v_0 defined by* (3.2). *Then, there is a constant c such that*

$$\big\|\psi_T(t)S(t)v_0(x)\big\|_{\sigma,s,b} \leqslant c\big(T^{\frac{1}{2}} + T^{\frac{1-2b}{2}}\big)\|v_0\|_{G_{\sigma,s}}, \tag{3.5}$$

$$\big\|\psi_T(t)v(x,t)\big\|_{\sigma,s,b} \leqslant cT^{\frac{1-2b}{2}}\|v\|_{\sigma,s,b} \quad \text{and} \tag{3.6}$$

$$\left\|\psi_T(t)\int_0^t S(t-s)v(\cdot,s)\,ds\right\|_{\sigma,s,b} \leqslant c\big(T+T^{1-b+b'}\big)\|v\|_{\sigma,s,b'}. \tag{3.7}$$

Proof. The proof of (3.5) is immediate from the definition of $X_{\sigma,s,b}$, the linearity of the operator $e^{\sigma A}$ and [16, Lemma 3.1]. In the same way, (3.6) follows from [16, Lemma 3.2] (see also [17]). For the proof of (3.7), see [6, Lemma 2.1]. □

To control the nonlinear part of the equation, multilinear estimates are needed. Some of these rest on the following Kato-type smoothing estimates and maximal function-type inequalities. For a suitable function f, define F_ρ via its Fourier transform $\widehat{F}_\rho$ to be

$$\widehat{F}_\rho(\xi,\tau) = \frac{|f(\xi,\tau)|}{(1+|\tau+\xi^2|)^\rho}. \tag{3.8}$$

Lemma 2. (Bourgain [3]) *Let $\rho > \frac{1}{4}$ be given. Then there is a constant c, depending on ρ, such that*

$$\left\| A^{\frac{1}{4}} F_\rho \right\|_{L^4L^2} \leqslant c\|f\|_{L^2L^2}. \tag{3.9}$$

Lemma 3. (Kenig, Ponce and Vega [15]) *Let s and ρ be given real numbers. There is a constant c, depending on s and ρ, such that*

(i) *if $\rho > \frac{1}{2}$, then*

$$\left\| A^{\frac{1}{2}} F_\rho \right\|_{L^\infty L^2} \leqslant c\|f\|_{L^2L^2}; \tag{3.10}$$

(ii) *if $\rho > \frac{1}{2}$ and $s > \frac{1}{4}$, then*

$$\left\| A^{-s} F_\rho \right\|_{L^4L^\infty} \leqslant c\|f\|_{L^2L^2}; \tag{3.11}$$

(iii) *if $\rho > \frac{1}{2}$ and $s > \frac{1}{2}$, then*

$$\left\| A^{-s} F_\rho \right\|_{L^\infty L^\infty} \leqslant c\|f\|_{L^2L^2}. \tag{3.12}$$

Inequality (3.10) was proved in [15], and estimates (3.11) and (3.12) were proved in [7]. The next inequality is the basis for the nonlinear estimate in the next section.

Lemma 4. *Let $s \geqslant \frac{7}{8}$, $\frac{1}{2} < b \leqslant \frac{9}{16}$, and let b' be such that $-b \leqslant b' \leqslant -\frac{3}{8}$. Let $d\mu = d\xi_2\, d\tau_2\, d\xi_1\, d\tau_1\, d\xi\, d\tau$, and suppose f, g, and h are in $L^2(\mathbb{R}^2)$. Then, there exists a constant c depending only on s, b and b', such that*

$$\int_{\mathbb{R}^6} \frac{|\xi|^{\frac{1}{8}}(1+|\xi|)^s h(\xi,\tau)}{(1+|\tau+\xi^2|)^{-b'}} \times \frac{f(\xi_1,\tau_1)(1+|\xi_1|)^{-s}}{(1+|\tau_1+\xi_1^2)^b} \times \frac{f(\xi_2,\tau_2)(1+|\xi_2|)^{-s}}{(1+|\tau_2+\xi_2^2|)^b} \times \frac{g(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)(1+|\xi-\xi_1-\xi_2|)^{-s}}{(1+|\tau-\tau_1-\tau_2+(\xi-\xi_1-\xi_2)^2|)^b}\, d\mu$$
$$\leqslant c\|h\|_{L^2_\xi L^2_\tau}\|f\|^2_{L^2_\xi L^2_\tau}\|g\|_{L^2_\xi L^2_\tau}.$$

This lemma is similar to a lemma proved in [18]. The proof of these inequalities is standard fare, and is therefore omitted. The crucial step in the proof relies on estimating the algebraic expression

$$\frac{(1+|\xi|)^{\frac{1}{8}}(1+|\xi|)^s(1+|\xi-\xi_1-\xi_2|)^{1-s}}{(1+|\xi|_1)^s(1+|\xi|_2)^s(1+|\xi-\xi_1|)^{1-b}(1+|\xi-\xi_2|)^{1-b}},$$

and it turns out that this quantity can be dominated by one of the expressions

$$\frac{\min\{(1+|\xi|)^{\frac{1}{2}},(1+|\xi-\xi_1-\xi_2|)^{\frac{1}{2}}\}}{(1+|\xi_1|)^{\frac{1}{4}}(1+|\xi_2|)^{\frac{1}{4}}} \quad \text{or}$$
$$\frac{\max\{(1+|\xi|)^{\frac{1}{2}},(1+|\xi-\xi_1-\xi_2|)^{\frac{1}{2}}\}}{\max\{(1+|\xi_1|)^{\frac{1}{4}}(1+|\xi_2|)^{\frac{1}{4}}\}\min\{(1+|\xi|)^{\frac{1}{4}},(1+|\xi-\xi_1-\xi_2|)^{\frac{1}{4}}\}}.$$

The proof of these latter inequalities is straightforward.

4. Multilinear estimates in Bourgain–Gevrey spaces

The goal of this section is to prove some multilinear estimates in analytic Bourgain–Gevrey spaces which feature explicit dependence on the radius of spatial analyticity σ. These inequalities will play a key role in obtaining the algebraically decreasing temporal asymptotics for σ.

Theorem 3. *Let $\sigma > 0$, $s \geqslant \frac{7}{8}$, $\frac{1}{2} < b \leqslant \frac{9}{16}$, and $-\frac{1}{2} < b' < -\frac{3}{8}$. Then there exists a constant $c > 0$ depending only on s, b and b', such that for any $v \in X_{\sigma,s,b}$,*

$$\left\| v^2 v_x \right\|_{\sigma,s,b'} \leqslant c\|v\|^3_{s,b} + c\sigma^{\frac{1}{8}}\|v\|^3_{\sigma,s,b}. \tag{4.1}$$

Proof. First note that (4.1) can be written more explicitly as

$$\left\| \left(1+|\tau+\xi^2|\right)^{b'}(1+|\xi|)^s e^{\sigma(1+|\xi|)}\widehat{v^2 v_x}(\xi,\tau) \right\|_{L^2_\xi L^2_\tau} \leqslant c\|v\|^3_{s,b} + c\sigma^{\frac{1}{8}}\|v\|^3_{\sigma,s,b}. \tag{4.2}$$

Observe that if one defines

$$f(\xi,\tau) = \left(1+|\tau+\xi^2|\right)^b(1+|\xi|)^s e^{\sigma(1+|\xi|)}\hat{v}(\xi,\tau) \quad \text{and}$$
$$g(\xi,\tau) = \left(1+|\tau-\xi^2|\right)^b(1+|\xi|)^s e^{\sigma(1+|\xi|)}\hat{v}(-\xi,-\tau),$$

then proving inequality (4.1) is the same as establishing the estimate

$$\left\| \frac{(1+|\xi|)^s e^{\sigma(1+|\xi|)}}{(1+|\tau+\xi^2|)^{-b'}} \int_{\mathbb{R}^4} \frac{f(\xi_1,\tau_1)e^{-\sigma(1+|\xi_1|)}(1+|\xi_1|)^{-s}}{(1+|\tau_1+\xi_1^2|)^b} \times \frac{f(\xi_2-\xi_1,\tau_2-\tau_1)e^{-\sigma(1+|\xi_2-\xi_1|)}(1+|\xi_2-\xi_1|)^{-s}}{(1+|\tau_2-\tau_1+(\xi_2-\xi_1)^2|)^b} \right.$$

$$\times \frac{g(\xi-\xi_2,\tau-\tau_2)e^{-\sigma(1+|\xi-\xi_2|)}(1+|\xi-\xi_2|)^{1-s}}{(1+|\tau-\tau_2-(\xi-\xi_2)^2|)^b}\,d\xi_1\,d\tau_1\,d\xi_2\,d\tau_2\Bigg\|_{L^2_\xi L^2_\tau}$$

$$\leqslant c\big\|e^{-\sigma(1+|\xi|)}f\big\|^2_{L^2_\xi L^2_\tau}\big\|e^{-\sigma(1+|\xi|)}g\big\|_{L^2_\xi L^2_\tau}+c\sigma^{\frac{1}{8}}\|f\|^2_{L^2_\xi L^2_\tau}\|g\|_{L^2_\xi L^2_\tau}.$$

Using duality in the standard way, it thus suffices to estimate a 6-fold integral of the form

$$\int_{\mathbb{R}^6}\frac{h(\xi,\tau)(1+|\xi|)^s e^{\sigma(1+|\xi|)}}{(1+|\tau+\xi^2|)^{-b'}}\,\frac{f(\xi_1,\tau_1)e^{-\sigma(1+|\xi_1|)}(1+|\xi_1|)^{-s}}{(1+|\tau_1+\xi_1^2|)^b}$$

$$\times\frac{f(\xi_2-\xi_1,\tau_2-\tau_1)e^{-\sigma(1+|\xi_2-\xi_1|)}(1+|\xi_2-\xi_1|)^{-s}}{(1+|\tau_2-\tau_1+(\xi_2-\xi_1)^2|)^b}$$

$$\times\frac{g(\xi-\xi_2,\tau-\tau_2)e^{-\sigma(1+|\xi-\xi_2|)}(1+|\xi-\xi_2|)^{1-s}}{(1+|\tau-\tau_2-(\xi-\xi_2)^2|)^b}\,d\mu,$$

where h is an arbitrary element of the unit ball B in $L^2(\mathbb{R}^2)$ and $d\mu=d\xi_2\,d\tau_2\,d\xi_1\,d\tau_1\,d\xi\,d\tau$ as before. Using the simple inequality

$$e^{\sigma(1+|\xi|)}\leqslant e+\sigma^{\frac{1}{8}}\big(1+|\xi|\big)^{\frac{1}{8}}e^{\sigma(1+|\xi|)},\tag{4.3}$$

it is plain that the latter integral is bounded by I_1+I_2, where

$$I_1=e\sup_{h\in B}\int_{\mathbb{R}^6}\frac{|h(\xi,\tau)|(1+|\xi|)^s}{(1+|\tau+\xi^2|)^{-b'}}\,\frac{|f(\xi_1,\tau_1)|e^{-\sigma(1+|\xi_1|)}(1+|\xi_1|)^{-s}}{(1+|\tau_1+\xi_1^2|)^b}$$

$$\times\frac{|f(\xi_2-\xi_1,\tau_2-\tau_1)|e^{-\sigma(1+|\xi_2-\xi_1|)}(1+|\xi_2-\xi_1|)^{-s}}{(1+|\tau_2-\tau_1+(\xi_2-\xi_1)^2|)^b}$$

$$\times\frac{|g(\xi-\xi_2,\tau-\tau_2)|e^{-\sigma(1+|\xi-\xi_2|)}(1+|\xi-\xi_2|)^{1-s}}{(1+|\tau-\tau_2-(\xi-\xi_2)^2|)^b}\,d\mu\quad\text{and}$$

$$I_2=\sigma^{\frac{1}{8}}\sup_{h\in B}\int_{\mathbb{R}^6}\frac{|h(\xi,\tau)|(1+|\xi|)^s(1+|\xi|)^{\frac{1}{8}}e^{\sigma(1+|\xi|)}}{(1+|\tau+\xi^2|)^{-b'}}\,\frac{|f(\xi_1,\tau_1)|e^{-\sigma(1+|\xi_1|)}(1+|\xi_1|)^{-s}}{(1+|\tau_1+\xi_1^2|)^b}$$

$$\times\frac{|f(\xi_2-\xi_1,\tau_2-\tau_1)|e^{-\sigma(1+|\xi_2-\xi_1|)}(1+|\xi_2-\xi_1|)^{-s}}{(1+|\tau_2-\tau_1+(\xi_2-\xi_1)^2|)^b}$$

$$\times\frac{|g(\xi-\xi_2,\tau-\tau_2)|e^{-\sigma(1+|\xi-\xi_2|)}(1+|\xi-\xi_2|)^{1-s}}{(1+|\tau-\tau_2-(\xi-\xi_2)^2|)^b}\,d\mu.$$

After the change of variables $\xi_2\mapsto\xi_2+\xi_1$, the inequality

$$I_1\leqslant c\big\|e^{-\sigma(1+|\xi|)}f\big\|^2_{L^2_\xi L^2_\tau}\big\|e^{-\sigma(1+|\xi|)}g\big\|_{L^2_\xi L^2_\tau}\tag{4.4}$$

follows immediately from Lemma 4 with $f(\xi,\tau)$ replaced by $e^{-\sigma(1+|\xi|)}f(\xi,\tau)$ and $g(\xi,\tau)$ replaced by $e^{-\sigma(1+|\xi|)}g(\xi,\tau)$. To establish the estimate

$$I_2\leqslant c\sigma^{\frac{1}{8}}\|f\|^2_{L^2_\xi L^2_\tau}\|g\|_{L^2_\xi L^2_\tau},\tag{4.5}$$

observe that

$$e^{\sigma(1+|\xi|)}\leqslant e^{\sigma(1+|\xi_1|)}e^{\sigma(1+|\xi-\xi_2|)}e^{\sigma(1+|\xi_2-\xi_1|)},$$

and then use Lemma 4 again. Theorem 3 is thereby established. □

The next result provides control of the quintic term in Eq. (1.4).

Theorem 4. *Suppose $v\in X_{\sigma,s,b}$, where $\sigma>0$, $s>\frac{1}{2}$, $b>\frac{1}{2}$, and $b'<-\frac{1}{4}$. Then there exists a constant c depending only on s, b and b', such that*

$$\big\|v^3\bar v^2\big\|_{\sigma,s,b'}\leqslant c\|v\|^5_{s,b}+c\sigma^{\frac{1}{8}}\|v\|^5_{\sigma,s,b}.\tag{4.6}$$

Proof. As before, let

$$f(\xi,\tau)=\big(1+|\tau+\xi^2|\big)^b\big(1+|\xi|\big)^s e^{\sigma(1+|\xi|)}\hat v(\xi,\tau)\quad\text{and}$$

$$g(\xi,\tau)=\big(1+|\tau-\xi^2|\big)^b\big(1+|\xi|\big)^s e^{\sigma(1+|\xi|)}\hat v(-\xi,-\tau).$$

Then, as in the proof of Theorem 2, the inequality is concluded as soon as one has appropriate estimates of the integrals

$$I_1=c\sup_{h\in B}\int_{\mathbb{R}^{10}}\frac{h(\xi,\tau)(1+|\xi|)^s e^{\sigma(1+|\xi|)}}{(1+|\tau+\xi^2|)^{-b'}}\,\frac{f(\xi_1,\tau_1)e^{-\sigma(1+|\xi_1|)}(1+|\xi_1|)^{-s}}{(1+|\tau_1+\xi_1^2|)^b}$$

$$\times\frac{g(\xi-\xi_4,\tau-\tau_4)e^{-\sigma(1+|\xi-\xi_4|)}(1+|\xi-\xi_4|)^{-s}}{(1+|\tau-\tau_4-(\xi-\xi_4)^2|)^b}$$

$$\times\frac{f(\xi_4-\xi_3,\tau_4-\tau_3)e^{-\sigma(1+|\xi_4-\xi_3|)}(1+|\xi_4-\xi_3|)^{-s}}{(1+|\tau_4-\tau_3+(\xi_4-\xi_3)^2|)^b}$$

$$\times\frac{g(\xi_3-\xi_2,\tau_3-\tau_2)e^{-\sigma(1+|\xi_3-\xi_2|)}(1+|\xi_3-\xi_2|)^{-s}}{(1+|\tau_3-\tau_2-(\xi_3-\xi_2)^2|)^b}$$

$$\times\frac{f(\xi_2-\xi_1,\tau_2-\tau_1)e^{-\sigma(1+|\xi_2-\xi_1|)}(1+|\xi_2-\xi_1|)^{-s}}{(1+|\tau_2-\tau_1+(\xi_2-\xi_1)^2|)^b}\,d\mu\quad\text{and}$$

$$I_2=\sigma^{\frac{1}{8}}\sup_{h\in B}\int_{\mathbb{R}^{10}}\frac{h(\xi,\tau)(1+|\xi|)^s(1+|\xi|)^{\frac{1}{8}}e^{\sigma(1+|\xi|)}}{(1+|\tau+\xi^2|)^{-b'}}\,\frac{f(\xi_1,\tau_1)e^{-\sigma(1+|\xi_1|)}(1+|\xi_1|)^{-s}}{(1+|\tau_1+\xi_1^2|)^b}$$

$$\times\frac{g(\xi-\xi_4,\tau-\tau_4)e^{-\sigma(1+|\xi-\xi_4|)}(1+|\xi-\xi_4|)^{-s}}{(1+|\tau-\tau_4-(\xi-\xi_4)^2|)^b}$$

$$\times \frac{f(\xi_4-\xi_3,\tau_4-\tau_3)e^{-\sigma(1+|\xi_4-\xi_3|)}(1+|\xi_4-\xi_3|)^{-s}}{(1+|\tau_4-\tau_3+(\xi_4-\xi_3)^2|)^b}$$
$$\times \frac{g(\xi_3-\xi_2,\tau_3-\tau_2)e^{-\sigma(1+|\xi_3-\xi_2|)}(1+|\xi_3-\xi_2|)^{-s}}{(1+|\tau_3-\tau_2-(\xi_3-\xi_2)^2|)^b}$$
$$\times \frac{f(\xi_2-\xi_1,\tau_2-\tau_1)e^{-\sigma(1+|\xi_2-\xi_1|)}(1+|\xi_2-\xi_1|)^{-s}}{(1+|\tau_2-\tau_1+(\xi_2-\xi_1)^2|)^b}\,d\mu,$$

where, now, $d\mu = d\xi_4\,d\tau_4\,d\xi_3\,d\tau_3\,d\xi_2\,d\tau_2\,d\xi_1\,d\tau_1\,d\xi\,d\tau$ and h is again drawn from the unit ball in $L^2(\mathbb{R}^2)$. Using the inequality $|\xi| \leqslant |\xi_1| + |\xi-\xi_4| + |\xi_4-\xi_3| + |\xi_3-\xi_2| + |\xi_2-\xi_1|$ and the monotonicity of the exponential, I_1 is bounded as follows:

$$I_1 \leqslant \sup_{h\in B}\int_{\mathbb{R}^{10}} \frac{|h(\xi,\tau)|\,|\xi|^s}{(1+|\tau+\xi^2|)^{-b'}} \frac{|f(\xi_1,\tau_1)|(1+|\xi_1|)^{-s}}{(1+|\tau_1+\xi_1^2)^b} \cdot \frac{|g(\xi-\xi_4,\tau-\tau_4)|(1+|\xi-\xi_4|)^{-s}}{(1+|\tau-\tau_4-(\xi-\xi_4)^2|)^b}$$
$$\times \frac{|f(\xi_4-\xi_3,\tau_4-\tau_3)|(1+|\xi_4-\xi_3|)^{-s}}{(1+|\tau_4-\tau_3+(\xi_4-\xi_3)^2|)^b} \cdot \frac{|g(\xi_3-\xi_2,\tau_3-\tau_2)|(1+|\xi_3-\xi_2|)^{-s}}{(1+|\tau_3-\tau_2-(\xi_3-\xi_2)^2|)^b}$$
$$\times \frac{|f(\xi_2-\xi_1,\tau_2-\tau_1)|(1+|\xi_2-\xi_1|)^{-s}}{(1+|\tau_2-\tau_1+(\xi_2-\xi_1)^2|)^b}\,d\mu.$$

To analyze this integral, the domain of integration with respect to ξ, ξ_1, ξ_2, ξ_3 and ξ_4 is divided into 24 regions corresponding to combinations of inequalities such as $|\xi-\xi_4| \leqslant |\xi_4-\xi_3| \leqslant |\xi_3-\xi_2| \leqslant |\xi_2-\xi_1| \leqslant |\xi_1|$. The integral is then estimated on each region separately. The portion of I_1 corresponding to the particular region just delineated can be dominated by the supremum over all h in B of the term

$$\langle H_{-b'}, F_b A^{-s} G_b A^{-s} F_b A^{-s} G_b A^{-s} F_b\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(\mathbb{R}^2)$, and $H_{-b'}$, F_b and G_b are defined as in (3.8). The estimate continues by noting that

$$\begin{aligned}
&\langle H_{-b'}, F_b A^{-s} G_b A^{-s} F_b A^{-s} G_b A^{-s} F_b\rangle\\
&\quad\leqslant c\|H_{-b'}\|_{L_x^4 L_t^2}\|F_b\|_{L_x^4 L_t^2}\|A^{-s}G_b\|_{L_x^4 L_t^\infty}\|A^{-s}F_b\|_{L_x^4 L_t^\infty}\|A^{-s}G_b\|_{L_x^\infty L_t^\infty}\|A^{-s}F_b\|_{L_x^\infty L_t^\infty}\\
&\quad\leqslant c\|h\|_{L_\xi^2 L_\tau^2}\|f\|^3_{L_\xi^2 L_\tau^2}\|g\|^2_{L_\xi^2 L_\tau^2}.
\end{aligned}$$

The integral I_2 can be bounded above in an analogous way, resulting in the inequality

$$\begin{aligned}
I_2 &\leqslant \langle A^{\frac{1}{16}} H_{-b'}, A^{\frac{1}{16}} F_b A^{-s} G_b A^{-s} F_b A^{-s} G_b A^{-s} F_b\rangle\\
&\leqslant c\|A^{\frac{1}{16}} H_{-b'}\|_{L_x^4 L_t^2}\|A^{\frac{1}{16}} F_b\|_{L_x^4 L_t^2}\|A^{-s}G_b\|_{L_x^4 L_t^\infty}\\
&\quad\times \|A^{-s}F_b\|_{L_x^4 L_t^\infty}\|A^{-s}G_b\|_{L_x^\infty L_t^\infty}\|A^{-s}F_b\|_{L_x^\infty L_t^\infty}\\
&\leqslant c\|h\|_{L_\xi^2 L_\tau^2}\|f\|^3_{L_\xi^2 L_\tau^2}\|g\|^2_{L_\xi^2 L_\tau^2}.
\end{aligned}$$

The desired result now follows. □

5. Algebraic lower bounds on σ

In this section, it is proved that the radius σ of analyticity of a solution of (1.4) decreases at most algebraically when viewed as a function of time t. The principal step in achieving this goal is to obtain an a priori bound in $G_{\sigma(T),s}$ on the solutions of (1.4) for a fixed but arbitrary $T > 0$. To derive such a bound, a sequence of approximations to (1.4) is defined and proved to be bounded in $G_{\sigma(T),s}$ for an appropriate value of $\sigma(T)$. This bound, combined with the local existence theory in [7] and the global theory in [11] will enable us to prove the desired result.

The local and global results to be used in the proof of the main theorem are the following.

Theorem 5. (Grujić and Kalisch [7]) *Let $s > \frac{1}{2}$ and $\sigma_0 > 0$. For given initial data $w_0 \in G_{\sigma_0,s}$, there exists a positive time $t_0 = t_0(\|w_0\|_{G_{\sigma_0,s}})$ such that the initial-value problem (1.4) has a unique solution w in $C([-t_0,t_0], G_{\sigma_0,s})$. Moreover, there exist $b > \frac{1}{2}$ and b' with $b' < -\frac{3}{8}$ and $1-b+b' > 0$, a function $v \in X_{\sigma_0,s,b}$, and a constant c such that*

$$\|v\|_{\sigma_0,s,b} \leqslant cr + 2c^2\, t_0^{1-b+b'}\big(2c(r+1)\big)^5, \tag{5.1}$$

where $r = \|w_0\|_{G_{\sigma_0,s}}$, and for $t \in [-t_0,t_0]$, v agrees with the solution w of (1.4) corresponding to w_0.

Note that this theorem shows that a local solution of (1.4) exists in a fixed strip for sufficiently small values of t. The following theorem asserts the existence of global-in-time solutions in the usual Sobolev spaces provided the initial data are of order one in the L^2-norm.

Theorem 6. (Hayashi and Ozawa [11]) *Assume that $w_0 \in H^m$ for some integer $m \geqslant 1$, and $\|w_0\|_{L^2} \leqslant \sqrt{2\pi}$. Then, there exists a unique solution w of (1.4) in $C((-\infty,\infty), H^m)$ and a constant M_m, such that*

$$\sup_{t\in(-\infty,\infty)} \|w(\cdot,t)\|_{H^m} \leqslant M_m\|w_0\|_{H^m}.$$

Next, consider a lemma relating the boundedness of a Sobolev norm to the boundedness of a Bourgain-type norm.

Lemma 5. *Let $s > \frac{1}{2}$, $-1 < b < 1$, $t_0 > 0$, and let w be a solution of problem (1.4) in $C([-2T,2T], H^{s+1})$. There exists a constant c depending only on s, b and t_0 such that if $T > t_0$, then*

$$\|\psi_T(t)w(\cdot,t)\|_{s,b} \leqslant cT^{\frac{1}{2}}\big(1+\alpha_T(w)\big), \tag{5.2}$$

where

$$\alpha_T(w) \equiv \sup_{t\in[-2T,2T]}\big\{\|w(\cdot,t)\|_s^2\|w(\cdot,t)\|_{s+1} + \|w(\cdot,t)\|_s^5\big\}. \tag{5.3}$$

Proof. Changing variables in the definition of the norm, it follows immediately that

$$\begin{aligned}\|\psi_T(t)w(x,t)\|_{s,b}^2 &= \int_{-\infty}^{\infty}(1+|\xi|)^{2s}\int_{-\infty}^{\infty}\left|\Lambda^b\left(\psi_T(t)e^{i\xi^2 t}\mathcal{F}_x w(\xi,t)\right)\right|^2 dt\,d\xi\\ &\leqslant c\int_{-\infty}^{\infty}(1+|\xi|)^{2s}\int_{-\infty}^{\infty}\left|\psi_T(t)e^{i\xi^2 t}\mathcal{F}_x w(\xi,t)\right|^2 dt\,d\xi\\ &\quad + c\int_{-\infty}^{\infty}(1+|\xi|)^{2s}\int_{-\infty}^{\infty}\left|\partial_t\left(\psi_T(t)e^{i\xi^2 t}\mathcal{F}_x w(\xi,t)\right)\right|^2 dt\,d\xi.\end{aligned}$$

Applying Leibniz's rule, the last integrand is seen to be

$$\frac{1}{T}\psi_T'(t)e^{i\xi^2 t}\mathcal{F}_x w(\xi,t)+\psi_T(t)\left(i\xi^2\right)e^{i\xi^2 t}\mathcal{F}_x w(\xi,t)+\psi_T(t)e^{i\xi^2 t}\mathcal{F}_x w_t(\xi,t).$$

Using the equation $iw_t=-iw^2\bar{w}_x-\frac{1}{2}|w|^4w-w_{xx}$ satisfied by w, the last term above may be replaced by

$$\psi_T(t)e^{i\xi^2 t}i\mathcal{F}_x\left(w^2w_x\right)(\xi,t)+\frac{1}{2}\psi_T(t)e^{i\xi^2 t}i\mathcal{F}_x\left(|w|^4w\right)(\xi,t)-\psi_T(t)\left(i\xi^2\right)e^{i\xi^2 t}\mathcal{F}_x w(\xi,t).$$

Notice that the terms containing the second derivative cancel. Thus, there appears the inequality

$$\begin{aligned}\|\psi_T(t)w(x,t)\|_{s,b}^2 &\leqslant c\int_{-\infty}^{\infty}(1+|\xi|)^{2s}\int_{-\infty}^{\infty}\left|\psi_T(t)e^{i\xi^2 t}\mathcal{F}_x w(\xi,t)\right|^2 dt\,d\xi\\ &\quad + c\int_{-\infty}^{\infty}(1+|\xi|)^{2s}\int_{-\infty}^{\infty}\left|\frac{1}{T}\psi_T'(t)e^{i\xi^2 t}\mathcal{F}_x w(\xi,t)\right|^2 dt\,d\xi\\ &\quad + c\int_{-\infty}^{\infty}(1+|\xi|)^{2s}\int_{-\infty}^{\infty}\left|\psi_T(t)e^{i\xi^2 t}i\mathcal{F}_x\left(w^2\bar{w}_x\right)(\xi,t)\right|^2 dt\,d\xi\\ &\quad + \frac{c}{2}\int_{-\infty}^{\infty}(1+|\xi|)^{2s}\int_{-\infty}^{\infty}\left|\psi_T(t)e^{i\xi^2 t}i\mathcal{F}_x\left(|w|^4w\right)(\xi,t)\right|^2 dt\,d\xi\\ &\leqslant\left(1+\frac{1}{t_0}\right)c\int_{-\infty}^{\infty}(1+|\xi|)^{2s}\int_{-2T}^{2T}\left|\mathcal{F}_x w(\xi,t)\right|^2 dt\,d\xi\\ &\quad + c\int_{-\infty}^{\infty}(1+|\xi|)^{2s}\int_{-2T}^{2T}\left|\mathcal{F}_x\left(w^2\bar{w}_x\right)(\xi,t)\right|^2 dt\,d\xi\end{aligned}$$

$$\begin{aligned}&\quad + \frac{c}{2}\int_{-\infty}^{\infty}(1+|\xi|)^{2s}\int_{-2T}^{2T}\left|\mathcal{F}_x\left(|w|^4w\right)(\xi,t)\right|^2 dt\,d\xi\\ &\leqslant 4c\left(1+\frac{1}{t_0}\right)T\sup_{t\in|-2T,2T|}\left\|w(\cdot,t)\right\|_{H^s}^2+4cT\sup_{t\in|-2T,2T|}\left\|w^2\overline{w}_x(\cdot,t)\right\|_{H^s}^2\\ &\quad + 2cT\sup_{t\in|-2T,2T|}\left\||w|^4w(\cdot,t)\right\|_{H^s}^2.\end{aligned}$$

It is now clear that inequality (5.2) holds. □

With these preliminary results in place, we are ready to tackle the proof of Theorem 1. First, define a sequence of approximations to (1.4) as follows. Consider the initial-value problems

$$\begin{cases} iw_t^n+w_{xx}^n=-i\left(\eta_n*\psi_T w^n\right)^2\left(\overline{\eta_n*\psi_T w^n}\right)_x-\left|\eta_n*\psi_T w^n\right|^4\left(\eta_n*\psi_T w^n\right),\\ w^n(x,0)=w_0(x),\end{cases}\tag{5.4}$$

for n in $\mathbb{N}$ and $T>0$, where the convolution is taken only in the spatial variable, η_n is defined via its Fourier transform to be

$$\hat{\eta}_n(\xi)=\begin{cases}0, & |\xi|\geqslant 2n,\\ 1, & |\xi|\leqslant n,\end{cases}$$

and $\hat{\eta}_n$ is smooth and monotone on $(-2n,-n)$ and $(n,2n)$.

Note that the proof of Theorem 5 holds, with only minor modifications, also for the problem (5.4). Hence,

$$\left\|w^n\right\|_{\sigma_0,s,b}\leqslant cr+2c^2t_0^{1-b+b'}\left(2c(r+1)\right)^5,\tag{5.5}$$

where r, t_0, c, b and b' are as in Theorem 5.

Standard $X_{r,b}$-estimates on w^n-w and a priori regularity of w show that, for n large enough, solutions of (5.4) exist on the same interval as the solutions w of (1.4), and that indeed $\{w^n\}_{n\in\mathbb{N}}$ converges to w in the space $C([-2T,2T],H^r(\mathbb{R}))$. These facts are summarized in the following lemma.

Lemma 6. *Let $r\geqslant 0$ and $w_0\in H^r(\mathbb{R})$, and suppose w is the solution of (1.4) in $C([-2T,2T],H^r(\mathbb{R}))$ for some $T>0$ corresponding to the initial data w_0.*

Then, for any n large enough, there exists a solution w^n of (5.4) in $C([-2T,2T],H^r(\mathbb{R}))$ originating at w_0. Moreover, the sequence $\{w^n\}_{n\in\mathbb{N}}$ converges to w in $C([-2T,2T],H^r(\mathbb{R}))$, and the equivalent of Lemma 5 holds for each w^n.

Henceforth, it is assumed that w is a solution of (1.4) in $C([-4T,4T],H^{s+1})$ with initial data w_0 in $G_{\sigma_0,s}$ for some $\sigma_0>0$ and $s\geqslant\frac{7}{8}$. Let w^n be the solution of (5.4) with initial data w_0, $n\in\mathbb{N}$. Note that

$$\psi_T(t)w^n = \psi_T(t)S(t)w_0 - \psi_T(t)\int_0^t S(t-s)\big(\eta_n * \psi_T w^n\big)^2\big(\overline{\eta_n * \psi_T w^n}\big)_x\,ds$$
$$-\psi_T(t)\int_0^t S(t-s)\big|\eta_n * \psi_T w^n\big|^4\big(\eta_n * \psi_T w^n\big)\,ds \tag{5.6}$$

holds for all t in $(-\infty,\infty)$. This representation reveals that, for n large enough, $\psi_T w^n$ belongs to $X_{\sigma,s,b}$ for any $\sigma>0$. To see this fact, utilize the linear estimates (3.5)–(3.7), the multilinear estimates obtained in [7, Theorems 2 and 3], and Lemma 6 to derive the inequality

$$\begin{aligned}
\big\|\psi_T(t)w^n\big\|_{\sigma,s,b} &\leqslant \big\|\psi_T(t)S(t)w_0\big\|_{\sigma,s,b}\\
&\quad+\Big\|\psi_T(t)\int_0^t S(t-s)\big(\eta_n * \psi_T w^n\big)^2\big(\overline{\eta_n * \psi_T w^n}\big)_x\,ds\Big\|_{\sigma,s,b}\\
&\quad+\Big\|\psi_T(t)\int_0^t S(t-s)\big|\eta_n * \psi_T w^n\big|^4\big(\eta_n * \psi_T w^n\big)\,ds\Big\|_{\sigma,s,b}\\
&\leqslant c\big(T^{\frac12}+t_0^{\frac{1-2b}{2}}\big)\|w_0\|_{G_{\sigma_0,s}}\\
&\quad+c\big(T+T^{1-b+b'}\big)\big(\big\|\eta_n * \psi_T w^n\big\|_{\sigma,s,b}^3+\big\|\eta_n * \psi_T w^n\big\|_{\sigma,s,b}^5\big)\\
&\leqslant c\big(T^{\frac12}+t_0^{\frac{1-2b}{2}}\big)\|w_0\|_{G_{\sigma_0,s}}\\
&\quad+cK_{n,\sigma}\big(T+T^{1-b+b'}\big)\big(\big\|\psi_T w^n\big\|_{s,b}^3+\big\|\psi_T w^n\big\|_{s,b}^5\big)\\
&\leqslant c\big(T^{\frac12}+t_0^{\frac{1-2b}{2}}\big)\|w_0\|_{G_{\sigma_0,s}}\\
&\quad+cK_{n,\sigma}\big(T+T^{1-b+b'}\big)\big(\big[T^{\frac12}\big(1+\alpha_T(w^n)\big)\big]^3+\big[T^{\frac12}\big(1+\alpha_T(w^n)\big)\big]^5\big)
\end{aligned}$$

which is finite.

Our goal now is to show that, for n large enough, there exists a $\sigma(T)$ and a suitable $R(T)$ such that the sequence $\{\psi_T w^n\}_{n\in\mathbb{N}}$ lies in the ball $B_{R(T)}\subset X_{\sigma(T),s,b}$ centered at zero and of radius $R(T)$.

Proposition 1. *Let $T>0$, $\sigma_0>0$, $s\geqslant\frac78$ and $\frac12<b\leqslant\frac{9}{16}$. Suppose w is a solution of* (1.4) *in $C([-4T,4T],H^{s+1})$ with initial data $w_0\in G_{\sigma_0,s}$. Then, there exists a constant $K>0$ depending on s, b, $\|w_0\|_{G_{\sigma_0,s}}$ and $\alpha_T(w)$ (see* (5.3)*) such that the sequence $\{\psi_T w^n\}_{n\in\mathbb{N}}$ is bounded in $X_{\sigma(T),s,b}$ as long as*

$$\sigma(T)\leqslant\min\big\{\sigma_0, KT^{-120}\big\}. \tag{5.7}$$

Remark 7. The exponent 120 in (5.7) can be improved, but we eschew this exercise here.

Proof of Proposition 1. Let $t_0=t_0(\|w_0\|_{G_{\sigma_0,s}})>0$ be the existence time in the local well-posedness of (1.4) in $X_{\sigma_0,s,b}$ obtained in Theorem 5, and suppose without loss of generality that $T\geqslant t_0$. From Eq. (5.6), the linear estimates (3.5)–(3.7), the multilinear estimates in Theorems 3 and 4, and Lemmas 5 and 6, it follows that

$$\begin{aligned}
\big\|\psi_T(t)w^n\big\|_{\sigma,s,b} &\leqslant \big\|\psi_T(t)S(t)w_0\big\|_{\sigma,s,b}\\
&\quad+\Big\|\psi_T(t)\int_0^t S(t-s)\big(\eta_n * \psi_T w^n\big)^2\big(\overline{\eta_n * \psi_T w^n}\big)_x\,ds\Big\|_{\sigma,s,b}\\
&\quad+\Big\|\psi_T(t)\int_0^t S(t-s)\big|\eta_n * \psi_T w^n\big|^4\big(\eta_n * \psi_T w^n\big)\,ds\Big\|_{\sigma,s,b}\\
&\leqslant c\big(T^{\frac12}+t_0^{\frac{1-2b}{2}}\big)\|w_0\|_{G_{\sigma_0,s}}+c\big(T+T^{1-b+b'}\big)\big(\big\|\psi_T w^n\big\|_{s,b}^3+\big\|\psi_T w^n\big\|_{s,b}^5\big)\\
&\quad+c\big(T+T^{1-b+b'}\big)\sigma^{\frac18}\big(\big\|\psi_T w^n\big\|_{\sigma,s,b}^3+\big\|\psi_T w^n\big\|_{\sigma,s,b}^5\big)\\
&\leqslant c\big(T^{\frac12}+t_0^{\frac{1-2b}{2}}\big)\|w_0\|_{G_{\sigma_0,s}}+c\big(T+T^{1-b+b'}\big)\big(\|\psi_T w\|_{s,b}^3+\|\psi_T w\|_{s,b}^5\big)\\
&\quad+c\big(T+T^{1-b+b'}\big)\sigma^{\frac18}\big(\big\|\psi_T w^n\big\|_{\sigma,s,b}^3+\big\|\psi_T w^n\big\|_{\sigma,s,b}^5\big)\\
&\leqslant c\big(T^{\frac12}+t_0^{\frac{1-2b}{2}}\big)\|w_0\|_{G_{\sigma_0,s}}\\
&\quad+c\big(T+T^{1-b+b'}\big)\big(\big[T^{\frac12}\big(1+\alpha_T(w)\big)\big]^3+\big[T^{\frac12}\big(1+\alpha_T(w)\big)\big]^5\big)\\
&\quad+c\big(T+T^{1-b+b'}\big)\sigma^{\frac18}\big(\big\|\psi_T w^n\big\|_{\sigma,s,b}^3+\big\|\psi_T w^n\big\|_{\sigma,s,b}^5\big)
\end{aligned}\tag{5.8}$$

holds for n large enough, and an absolute constant $c>0$. Here, b' lies in the interval $(-\frac{7}{16},-\frac38)$.

Next, notice that $\|\psi_{t_0}w^n\|_{\sigma_0,s,b}$ is bounded by a constant M_{t_0}, viz.

$$\big\|\psi_{t_0}w^n\big\|_{\sigma_0,s,b}\leqslant ct_0^{\frac{1-2b}{2}}\big\|w^n\big\|_{\sigma_0,s,b}\leqslant M_{t_0}, \tag{5.9}$$

where $\|w^n\|_{\sigma_0,s,b}$ is bounded as in (5.5).

Defining dependent variables z, a and d by

$$\begin{aligned}
z(T)&=\big\|\psi_T w^n\big\|_{\sigma(T),s,b},\\
a(T)&=M_{t_0}+c\big(T^{\frac12}+t_0^{\frac{1-2b}{2}}\big)\|w_0\|_{G_{\sigma_0,s}}\\
&\quad+c\big(T+T^{1-b+b'}\big)\big(\big[T^{\frac12}\big(1+\alpha_T(w)\big)\big]^3+\big[T^{\frac12}\big(1+\alpha_T(w)\big)\big]^5\big),\quad\text{and}\\
d(T)&=c\big(T+T^{1-b+b'}\big),
\end{aligned}$$

a somewhat weakened version of (5.8) reads

$$z\leqslant a+d\sigma^{\frac18}(T)\big(z^3+z^5\big). \tag{5.10}$$

Henceforth, consider this inequality for any T' with $t_0 \leqslant T' \leqslant T$. Let $\sigma(T')$ be defined by

$$\sigma(T') = \frac{\delta^8}{d^8(2^3a^3 + 2^5a^5)^8},$$

and let $y = y(T') = z(T')/2a(T')$. Then (5.10) becomes

$$y\bigl(1 - \delta y^2 - \delta y^4\bigr) \leqslant \frac{1}{2}.$$

It follows that by choosing δ small enough, there are constants m^* and M^* with $\frac{1}{2} < m^* < 1 < M^*$, such that either $y \leqslant m^*$ or $y \geqslant M^*$. Recalling (5.9) and the definition of a, it is seen that $z(t_0) = \|\psi_{t_0} w^n\|_{\sigma_0,s,b} \leqslant M_{t_0} < a(t_0)$, so that $y(t_0) < \frac{1}{2} < m^*$.

Because $\|\psi_{T'} w^n_{T'}\|_{\sigma(T'),s,b}$ is a continuous function of T' for $t_0 \leqslant T' \leqslant T$, it follows that $y \leqslant m^* < 1$ for all T', which means, in particular, that $z(T) \leqslant 2a(T)$. This yields the desired estimate with a constant K depending on s, b, b', $\|w_0\|_{G_{\sigma_0,s+1}}$ and $\alpha_T(w)$. The following continuity argument is provided for completeness of the exposition. Let $t_0 \leqslant U \leqslant V \leqslant T$. Then,

$$\bigl|\|\psi_U w^n_U\|_{\sigma(U),s,b} - \|\psi_V w^n_V\|_{\sigma(V),s,b}\bigr| \leqslant \|\psi_U w^n_U - \psi_V w^n_V\|_{\sigma(U),s,b} + \frac{\bigl|\|\psi_V w^n_V\|^2_{\sigma(U),s,b} - \|\psi_V w^n_V\|^2_{\sigma(V),s,b}\bigr|}{\|\psi_V w^n_V\|_{\sigma(U),s,b} + \|\psi_V w^n_V\|_{\sigma(V),s,b}}.$$

The second term goes to 0 as $U \to V$ by the Lebesgue dominated convergence theorem. To analyze the first term, define

$$N_{n,S} = -i\bigl(\eta_n * \psi_S w^n_S\bigr)^2\bigl(\overline{\eta_n * \psi_S w^n_S}\bigr)_x - \bigl(\eta_n * \psi_S w^n_S\bigr)^4\bigl(\eta_n * \psi_S w^n_S\bigr)$$

for $n \in \mathbb{N}$. Then, it transpires that

$$\begin{aligned}
\psi_U w^n_U(t) - \psi_V w^n_V(t) &= \psi_U S(t)w_0 - \psi_V S(t)w_0 \\
&\quad + \psi_U \int_0^t S(t-s)N_{n,U}(s)\,ds - \psi_V \int_0^t S(t-s)N_{n,V}(s)\,ds \\
&= (\psi_U - \psi_V)S(t)w_0 + (\psi_U - \psi_V)\int_0^t S(t-s)N_{n,U}(s)\,ds \\
&\quad + \psi_V \int_0^t S(t-s)[N_{n,U} - N_{n,V}](s)\,ds.
\end{aligned}$$

Each of the terms on the right-hand side of this relation goes to 0 as $U \to V$ in the $X_{\sigma(U),s,b}$-norm. □

The main theorem, already stated in the introduction, now follows easily from Proposition 1.

Theorem 8. *Suppose that $w_0 \in G_{\sigma_0,s}$ for some integer $s \geqslant \frac{7}{8}$ and for some $\sigma_0 > 0$. Assume that $\|w_0\|_{L^2} \leqslant \sqrt{2\pi}$, and let $T > 0$. Then, there exists a constant $K > 0$ depending only on s, σ_0 and $\|w_0\|_{G\sigma_0,s}$, such that the solution w to (1.4) corresponding to the initial value w_0 lies in $C([-T, T], G_{\sigma(T)/4,s})$, where $\sigma(T)$ is given by*

$$\sigma(T) = \min\bigl\{\sigma_0, KT^{-120}\bigr\}.$$

Proof. Because of the restriction on the L^2-norm of the initial data, Theorem 6 assures the global existence of solutions to (1.1) corresponding to the initial value w_0. The proof of the theorem follows from a compactness argument presented already in [2]. The bound on the sequence $\{\psi_T w^n\}_{n\in\mathbb{N}}$ in $X_{\sigma(T),s,b}$ follows from the a priori bounds in Proposition 1. Since $b > \frac{1}{2}$, the Sobolev embedding theorem yields boundedness of the sequence $\{w^n\}_{n\in\mathbb{N}}$ in $G_{\sigma(T),s}$, uniformly on $[-T, T]$. Recall again that the analytic Gevrey spaces $G_{\sigma(T),s}$ are equivalent to the corresponding Hardy spaces on the symmetric complex strip around the real axis of width $2\sigma(T)$ (cf. (1.5)). Consequently, all spatial derivatives of w^n can be uniformly dominated on a closed strip of total width $\sigma(T)/2$ via the Cauchy formula applied on the disks whose radius equals $\sigma(T)/4$. Temporal derivatives are likewise bounded since each of the members of the sequence satisfies the differential equation (5.4). Locally uniform convergence of the sequences $\{w^n\}_{n\in\mathbb{N}}$, $\{\partial_t w^n\}_{n\in\mathbb{N}}$, $\{\partial_x w^n\}_{n\in\mathbb{N}}$, and $\{\partial_{xx} w^n\}_{n\in\mathbb{N}}$ on $S_{\sigma(T)} \times (0, T)$ then follows from the Arzela–Ascoli theorem. In particular, locally uniform convergence on $\mathbb{R} \times (0, T)$ allows us to pass to the limit in (5.4). The limit w is a spatially analytic function inheriting the a priori bounds from the sequence, which is to say, $w \in L^\infty((-T, T), G_{\sigma(T)/4,s})$. Theorem 5 then yields continuity in the temporal variable.

To conclude, observe that for any r, the Sobolev-norm $\|w_0\|_{H^r}$ is bounded by $\|w_0\|_{G_{\sigma_0,s}}$. Hence, Theorem 6 implies uniform boundedness of $\alpha_T(w)$ in terms of $\|w_0\|_{G_{\sigma_0,s}}$, thereby finishing the proof. □

6. Global analyticity for exponentially decaying initial data

It is well known that the solution map for the transformed initial-value problem (1.4) trades decay in space of the initial data for smoothness in space of the solution for positive time (see, e.g., [4] in the context of KdV and NLS equations). The following theorem states that exponential decay of the initial data implies spatial analyticity of local solutions for the derivative Schrödinger equation.

Theorem 9. (Hayashi and Ozawa [11]) *Assume that $\cosh(x)w_0(x)$ is in $H^2(\mathbb{R})$. Then, there exists $T^* = T^*(\|\cosh(\cdot)w_0\|_{H^2}) > 0$, a constant $M > 0$, and a local solution w to (1.4) satisfying*

$$\sup_{t\in[-T^*,T^*]} \bigl\|w(\cdot,t)\bigr\|_{G_{2|t|,2}} \leqslant M\bigl(\bigl\|\cosh(\cdot)w_0\bigr\|_{H^2}\bigr).$$

This local smoothing effect, combined with Theorem 8, yields global analyticity for exponentially decaying initial data having $L^2(\mathbb{R})$-norm of order one.

Theorem 10. *Suppose that $\cosh(x)w_0(x)$ is in $H^2(\mathbb{R})$. Assume that $\|w_0\|_{L^2} \leqslant \sqrt{2\pi}$, and let T^* be as in Theorem 9 relative to w_0. Then, there exists a positive constant C_1, depending only on $\|\cosh(\cdot)w_0\|_{H^2}$, such that the solution w to (1.4) corresponding to the initial value w_0 exists globally in time and, for each $t > 0$, $w(\cdot, t) \in G_{\sigma(t),2}$ where*

$$\sigma(t) = \begin{cases} 2t, & \text{if } t \leqslant T^* \\ \min\{2T^*, C_1(t + 1 - T^*)^{-120}\} & \text{if } t > T^*. \end{cases}$$

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