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Initial-boundary-value problems for linear and integrable nonlinear dispersive partial differential equations

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Abstract
It is suggested here that an interesting and important line of inquiry is the elaboration of methods of inverse scattering transform (IST) type in contexts where non-homogeneous boundary conditions intercede. The issue, which has practical relevance we indicate by example, appears ripe for development, thanks to recent new ideas interjected into the panoply of IST methodologies. A sketch of the principal steps envisaged in carrying out analysis of boundary-value problems using inverse scattering ideas is provided.

Mathematics Subject Classification: 30E25, 35Q15, 35Q35, 35Q51, 35Q53, 35Q58, 78B15, 76M40

1. Introduction

A central development in the theory of nonlinear partial differential equations (PDEs) in the second half of the 20th century, and continuing to the present, is the introduction of the inverse scattering transform (IST henceforth). The technique was put forward in the famous 1967 paper of Gardner *et al* [GGKM] in connection with the Korteweg–deVries (KdV) equation and the range of its applicability began to unfold with the publication of Zakharov and Shabat in 1972 [ZS] on the nonlinear Schrödinger (NLS) equation. Since both the KdV equation and the NLS equation arise as models in a number of different contexts, the interest in these new ideas was immediate. Following these seminal works, the hunt was on by the scientific community, broadly construed, for other equations having an IST theory, the paper of Ablowitz *et al* [AKNS] being especially important in showing the way.

By now, we have a well-developed theory of *integrable* evolution equations, by which we mean equations which can be analysed via IST machinery. The impact of this formalism is hard to overestimate. Firstly, for integrable equations we have learned detailed aspects of

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solution behaviour. This includes the long-time asymptotics of solutions and the central role played by the solitons. Secondly, it has became apparent that some of the lessons taught by integrable equations have applicability even in non-integrable situations. Indeed, many investigations in the last decade or so have their genesis in results coming from IST theory, albeit in non-integrable settings.

Heretofore, the IST methodology was pursued almost entirely for pure initial-value problems, which is to say, an integrable evolution equation

$$u_t = A(u), \qquad u(\cdot, 0) = \varphi, \tag{1.1}$$

where φ is drawn from suitable function classes defined on all of space $x \in \mathbb{R}$ and which might evanesce as $|x| \to \infty$ or which might be periodic in space. However, in many laboratory and field situations, the wave motion is initiated by what corresponds to the imposition of boundary conditions rather than initial conditions. In any case, it is a daunting task to determine a wavefield everywhere in space at a given instant of time, which is what is required in (1.1)! Much more tractable is to determine the wavefield as a function of time at given spatial points. Hence, the study of initial-boundary value problems for integrable equations and their relatives presents itself naturally.

Theory for initial-boundary-value problems for nonlinear dispersive wave equations made by the classical techniques of PDEs has lagged behind that for the pure initial-value problems, although the issue was already apparent in the 1970s (see [BB, BD]). For example, the elegant well-posedness results for the pure initial-value problems for KdV and NLS pioneered by Bourgain and others in the 1990s have only very recently seen analogues for initial-boundary value problems (see, e.g., [BL, BSZ1, BSZ3, CK, H]). Similarly, IST theory that is adapted to lateral boundary conditions had been noticeably lacking until recently, despite a need for the delicate conclusions such techniques might supply.

It is our purpose here to bring this forward as a potentially very fruitful line of development, and to indicate by a couple of simple examples the sort of payoff that would be in the offing.

The plan is to outline techniques in section 2 and to use them in section 3 in a linear circumstance to cast light on a recalcitrant issue arising in water wave theory. A brief commentary on nonlinear problems is provided in section 4.

2. IST for boundary-value problems

A unified approach for analysing initial-boundary-value problems for linear and integrable nonlinear PDEs, based on ideas of IST methodology, was introduced in [F1] in 1997 (see also [F2, FIS]). The forthcoming monograph [F3] chronicles further progress and also provides a guide to the existing literature.

As a paradigm problem, which will find use in section 3, attention is focused on the initial-boundary-value problem

$$\begin{aligned} q_t(x,t) + q_{xxx}(x,t) &= f(x,t), && x, t > 0, \\ q(x,0) &= q_0(x), && x > 0, \\ q(0,t) &= g_0(t), && t > 0, \end{aligned} \tag{2.1}$$

with the obvious compatibility condition $q_0(0) = g_0(0)$. General well-posedness results for this problem are readily available (see, e.g. [BSZ3]) corresponding to restricting q_0 and g_0 for minimal smoothness and requiring decent behaviour of $q_0(x)$ as $x \to +\infty$. The concern here is to develop IST type methods capable of dealing with quantitative issues. We foresee four steps to this process.

Step 1. Write the PDE as a *one-parameter* 'family' of equations, each of which is in *divergence form*.

For problem (2.1), a clear choice of an integrating factor to achieve this is $\tilde{q}(x,t) = \mathrm{e}^{-\mathrm{i}kx-\mathrm{i}k^3t}$, leading to the form

$$\left[\mathrm{e}^{-\mathrm{i}kx-\mathrm{i}k^3t}q\right]_t + \left[\mathrm{e}^{-\mathrm{i}kx-\mathrm{i}k^3t}\left(q_{xx}+\mathrm{i}kq_x-k^2q\right)\right]_x = \mathrm{e}^{-\mathrm{i}kx-\mathrm{i}k^3t}f. \tag{2.2}$$

One natural and more systematic way to get to (2.2) is to consider the formal adjoint, namely,

$$\tilde{q}_t + \tilde{q}_{xxx} = 0$$

and to note that

$$(\tilde{q}q)_t + [(\tilde{q}q)_{xx} - 3\tilde{q}_x q_x]_x = \tilde{q}f.$$

Upon choosing $\tilde{q}$ as above, (2.2) falls out.

Step 2. Use the divergence form to determine a *global relation* coupling the various boundary values.

In the case of (2.1), this takes the form

$$\mathrm{e}^{-\mathrm{i}k^3t}\hat{q}(k,t) = \hat{q}_0(k) + \tilde{g}(k,t) + F(k,t), \qquad t>0, \quad \operatorname{Im} k \leqslant 0, \tag{2.3}$$

where $\hat{q}$, $\hat{q}_0$, $\tilde{g}$ and F are defined as follows:

$$\hat{q}(k,t) = \int_0^\infty \mathrm{e}^{-\mathrm{i}kx}q(x,t)\,\mathrm{d}x, \quad \hat{q}_0(k) = \int_0^\infty \mathrm{e}^{-\mathrm{i}kx}q_0(x)\,\mathrm{d}x, \qquad \operatorname{Im} k \leqslant 0,$$

$$F(k,t) = \int_0^t\int_0^\infty \mathrm{e}^{-\mathrm{i}k\xi-\mathrm{i}k^3\tau}f(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau, \qquad t \geqslant 0, \quad \operatorname{Im} k \leqslant 0,$$

and for $t \geqslant 0$, $k \in \mathbb{C}$, $\omega(k) = -\mathrm{i}k^3$,

$$\tilde{g}(k,t) = \tilde{g}_2(\omega(k),t) + \mathrm{i}k\tilde{g}_1(\omega(k),t) - k^2\tilde{g}_0(\omega(k),t), \tag{2.4}$$

with

$$\tilde{g}_j(k,t) = \int_0^t \mathrm{e}^{k\tau}\partial_x^j q(0,\tau)\,\mathrm{d}\tau, \qquad j=0,1,2.$$

The global relation provides a link between the known boundary value g_0 and the two unknown boundary values $q_x(0,t)$ and $q_{xx}(0,t)$. The global relation follows from (2.2) by integrating in x over the half-line $\mathbb{R}^+ = \{x : x > 0\}$ to derive the formula

$$\left(\mathrm{e}^{-\mathrm{i}k^3t}\tilde{q}\right)_t = \int_0^\infty \mathrm{e}^{-\mathrm{i}kx-\mathrm{i}k^3t}f(x,t)\,\mathrm{d}x + \mathrm{e}^{-\mathrm{i}k^3t}\left[q_{xx}(0,t)+\mathrm{i}kq_x(0,t)-k^2q(0,t)\right]. \tag{2.5}$$

Integrating formula (2.5) over $(0,t)$ leads directly to (2.3) with the notational conventions just defined.

Step 3. Obtain an *integral representation* for the solution by inverting the global relation.

For the example (2.1), this takes the form

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx+\mathrm{i}k^3t}\left[\hat{q}_0(k)+F(k,t)\right]\mathrm{d}k + \frac{1}{2\pi}\int_{\partial D^+} \mathrm{e}^{\mathrm{i}kx+\mathrm{i}k^3t}\tilde{g}(k,t)\,\mathrm{d}k, \qquad x,t \geqslant 0, \tag{2.6}$$

where

$$D = \{k \in \mathbb{C} : \operatorname{Re}\omega(k) < 0\} = D^+ \cup D_1^- \cup D_2^-,$$

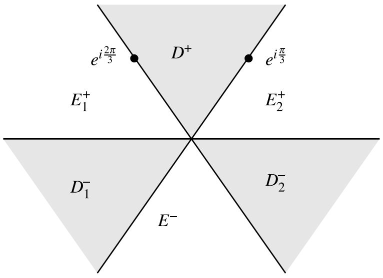

Figure 1. The Domains D and E.

with

$$D^+ = \{k \in \mathbb{C} : \mathrm{Re}\, \omega(k) < 0,\ \mathrm{Im}\, k \geqslant 0\}$$

and D_1^-, D_2^- similarly defined (see figure 1).

Indeed, taking the inverse Fourier transform of the left-hand side of the global relation (2.3) yields q expressed as the first integral on the right-hand side of (2.6) plus the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx+\mathrm{i}k^3t}\tilde{g}(k,t)\,\mathrm{d}k.$$

The contour of integration in the last integral can be deformed to lying along ∂D^+ by Cauchy's theorem (actually Jordan's lemma) together with the fact that for $x, t \geqslant 0$, $\mathrm{e}^{\mathrm{i}kx+\mathrm{i}k^3t}$ decays exponentially in the domains E_1^+ and E_2^+ between the real axis and D^+ and is bounded in the closure of this set.

Step 4. Use the global relation and certain transformations to eliminate the unknown boundary conditions, thereby obtaining an *effective integral representation* for the solution.

In the special case (2.1) under consideration as an example, this step proceeds as follows. Let $\alpha = \mathrm{e}^{\mathrm{i}(2\pi/3)}$ so that α, α^2 and α^3 are the three cube roots of unity. Note that the change of dependent variables $k \mapsto \alpha k$ and $k \mapsto \alpha^2 k$ leaves $\omega(k)$ invariant. Furthermore, if $k \in D^+$, then $\alpha k \in D_1^-$ and $\alpha^2 k \in D_2^-$.

The global relation (2.3) holds in $D_1^- \cup D_2^-$. Hence for $k \in D^+$, we see that

$$\mathrm{e}^{-\mathrm{i}k^3t}\hat{q}(\alpha k,t) = \hat{q}_0(\alpha k) + F(\alpha k,t) + \tilde{g}_2 + \mathrm{i}\alpha k\tilde{g}_1 - \alpha^2k^2\tilde{g}_0,$$
$$\mathrm{e}^{-\mathrm{i}k^3t}\hat{q}(\alpha^2 k,t) = \hat{q}_0(\alpha k) + F(\alpha^2 k,t) + \tilde{g}_2 + \mathrm{i}\alpha^2 k\tilde{g}_1 - \alpha k^2\tilde{g}_0.$$

View this as a pair of equations for $\tilde{g}_2$ and $\tilde{g}_1$, solve and substitute the resulting expressions into the definition of $\tilde{g}$ in (2.4) to reach the formula

$$\tilde{g}(k,t) = \alpha\left[\hat{q}_0(\alpha k) + F(\alpha k,t) - \mathrm{e}^{-\mathrm{i}k^3t}\hat{q}(\alpha k,t)\right]$$
$$+\,\alpha^2\left[\hat{q}_0(\alpha^2 k) + F(\alpha^2 k,t) - \mathrm{e}^{-\mathrm{i}k^3t}\hat{q}(\alpha^2 k,t)\right] - 3k^2\tilde{g}_0(k,t).$$

Replace $\tilde{g}$ in (2.6) by the above expression and use the fortunate fact that the terms involving $\hat{q}(\alpha,t)$ and $\hat{q}(\alpha^2 k,t)$ make a zero contribution, by Cauchy's theorem again, to obtain

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx+\mathrm{i}k^3t}[\hat{q}_0(k) + F(k,t)]\,\mathrm{d}k + \frac{1}{2\pi}\int_{\partial D^+}\mathrm{e}^{\mathrm{i}kx+\mathrm{i}k^3t}\left[\alpha\left(\hat{q}_0(\alpha k) + F(\alpha k,t)\right)\right.$$
$$\left. +\,\alpha^2\left(\hat{q}_0(\alpha^2 k) + F(\alpha^2 k,t)\right) - 3k^2\tilde{g}_0(\omega(k),t)\right]\mathrm{d}k. \tag{2.7}$$

Equation (2.7) provides an explicit formula for the solution of (2.1) depending only on the data g_0 and q_0 and the forcing f. Applications of this formula are discussed in section 3.

3. Periodic boundary conditions

In laboratory experiments performed in a flume and aimed at understanding how well KdV-type equations work in practice, a paddle-type wavemaker was oscillated periodically and the resulting wave motion was monitored at several points down the channel (see [BPS]). If $q(x,t)$ denotes the deviation of the surface from its rest position at the point x in the channel at the time t, then q is governed approximately by the KdV equation, of course with damping taken into account. (On laboratory scales, damping is as important as nonlinearity and dispersion, whereas it is often much less crucial on geophysical scales.) The measurements correspond to determining the functions

$$\tilde{q}(0,t) \quad \text{and} \quad \tilde{q}(x_j,t), \qquad j = 1,2,\ldots,$$

where the zero of the horizontal coordinate x along the channel has been located at the point where the first measurement is taken. To check the predictive power of the model, $g_0(t) = \tilde{q}(0,t)$ is used as boundary data (with $q(x,0) \equiv 0$ corresponding to everything being at rest at the beginning of the experiment). The evolution equation then predicts values for $q(x_j,t)$, $j = 1, 2, \ldots$, and a direct comparison can be made between these and the measured values $\tilde{q}(x_j,t)$. The reader may consult [BPS] to see the outcome; our attention is drawn to two interesting mathematical issues arising from this set of laboratory experiments.

It transpires that the boundary data $g_0(t)$ in the just mentioned experiments rapidly become periodic of period $2\pi/\Omega$ which is the period of the wavemaker. Moreover, at successively later times, $q(x_j,t)$ also becomes periodic of period $2\pi/\Omega$.

The first question is whether or not there is a corresponding, mathematically rigorous result about the initial-boundary-value problem for the KdV equation. Secondly, in the wavemaker experiment, there is obviously no mass added, on average. Is this a fact about the initial-boundary-value problem? These problems have received some attention in the literature (see, e.g. [BSZ2, AABCW, BWu]).

In the case of the linear KdV equation, we will now show that the ideas developed in section 2 yield answers to both these issues via elementary considerations.

The case we have in mind corresponds to (2.1) with $q_0 = f \equiv 0$, and so the solution given in equation (2.7) simplifies to

$$q(x,t) = -\frac{3}{2\pi}\int_{\partial D^+} k^2 \mathrm{e}^{\mathrm{i}kx}\left(\int_0^t \mathrm{e}^{\mathrm{i}k^3(t-\tau)} g_0(\tau)\,\mathrm{d}\tau\right)\mathrm{d}k. \tag{3.1}$$

Suppose now that $g_0(t)$ is a periodic function of frequency Ω (period $2\pi/\Omega$). Consider the difference between $q(\cdot,t)$ and q one period later, namely,

$$q\left(x,t+\frac{2\pi}{\Omega}\right)-q(x,t) = -\frac{3}{2\pi}\int_{\partial D^+} k^2\mathrm{e}^{\mathrm{i}kx}\left[\int_0^{t+\frac{2\pi}{\Omega}} \mathrm{e}^{\mathrm{i}k^3(t+\frac{2\pi}{\Omega}-\tau)} g_0(\tau)\,\mathrm{d}\tau - \int_0^t \mathrm{e}^{\mathrm{i}k^3(t-\tau)} g_0(\tau)\,\mathrm{d}\tau\right]. \tag{3.2}$$

The change of variables $s = \tau - (2\pi/\Omega)$ in the first temporal integral together with the presumption that g_0 is periodic of period $2\pi/\Omega$ allows the right-hand side of (3.2) to be simplified to

$$q\left(x,t+\frac{2\pi}{\Omega}\right) - q(x,t) = \frac{1}{2\pi}\int_{\partial D^+} 3k^2\mathrm{e}^{\mathrm{i}kx}\left(\int_0^{\frac{2\pi}{\Omega}} \mathrm{e}^{\mathrm{i}k^3(t-s)} g_0(s)\,\mathrm{d}s\right)\mathrm{d}k. \tag{3.3}$$

The right-hand side of (3.3) disperses away. More precisely the integral on the right-hand side of (3.3) tends to 0 as $t \to \infty$, uniformly on bounded subsets $\{x : 0 \leqslant x \leqslant L\}$. In fact, using the method of stationary phase, the decay is easily determined to be $O(t^{-2/3})$ for $x/t = O(1)$, as $t \to \infty$. Thus, q does indeed becomes asymptotically periodic.

Concerning the mass, suppose again that g_0 is periodic with frequency Ω and that g_0 does not add mass, which is to say $g_0 = h_0'$ where h_0 is also periodic with frequency Ω. If equation (2.1) with $f = 0$ is integrated over the half-line $\mathbb{R}^+$, then one obtains formally that

$$\partial_t \int_0^\infty q(x,t)\,\mathrm{d}x = q_{xx}(0,t).$$

Integrating this equation over $[0,t]$ and using the fact that $q(x,0) \equiv 0$, it is found that

$$M(t) = \int_0^\infty q(x,t)\,\mathrm{d}x = \int_0^t q_{xx}(0,s)\,\mathrm{d}s. \tag{3.4}$$

This formal calculation that ignores a possible boundary contribution at $+\infty$ is easily justified using for example the qualitative theory in [BWu]. The quantity $M(t)$ is the added mass in the channel at the time t, relative to the undisturbed state $q = 0$, since $q(x,t)$ represents the deviation of the free surface from the rest position. Of course, $M(0) = 0$. But M does not remain identically zero. Indeed, the mass in a real channel oscillates as the wavemaker is displaced. However, guided by intuition derived from the physical situation, $M(t)$ is expected to settle down to oscillations around zero. To see the validity of our intuition, fix T and consider the average mass in the channel over a wavemaker period starting at T, namely,

$$\bar{M}(T) = \frac{\Omega}{2\pi}\int_T^{T+\frac{2\pi}{\Omega}} M(t)\,\mathrm{d}t = \frac{\Omega}{2\pi}\int_T^{T+\frac{2\pi}{\Omega}} \left(\int_0^t q_{xx}(0,s)\,\mathrm{d}s\right)\mathrm{d}t. \tag{3.5}$$

To develop an appreciation of the double integral on the right-hand side of (3.5), return to formula (3.1) for q, differentiate twice with respect to x and integrate by parts using the fact that $g_0(0) = 0$ to obtain

$$q_{xx}(x,t) = -\frac{3\mathrm{i}}{2\pi}\int_{\partial D^+} k\mathrm{e}^{\mathrm{i}kx}\left(\int_0^t \mathrm{e}^{\mathrm{i}k^3(t-\tau)} g_0'(\tau)\,\mathrm{d}\tau\right)\mathrm{d}k. \tag{3.6}$$

Evaluate the latter relation at $x = 0$ and integrate over $[0,t]$ to reach the helpful formula

$$\int_0^t q_{xx}(0,s)\,\mathrm{d}s = -\frac{3\mathrm{i}}{2\pi}\int_{\partial D^+} k\left(\int_0^t \mathrm{e}^{\mathrm{i}k^3(t-\tau)} g_0(\tau)\,\mathrm{d}\tau\right)\mathrm{d}k. \tag{3.7}$$

As $t \geqslant \tau$, the change of variables $m = k(t-\tau)^{\frac{1}{3}}$ can be implemented in (3.7) to reach the alternative formula

$$\int_0^t q_{xx}(0,s)\,\mathrm{d}s = -\frac{3\mathrm{i}}{2\pi}\int_{\partial D^+} m\mathrm{e}^{\mathrm{i}m^3}\left(\int_0^t \frac{h_0'(\tau)}{(t-\tau)^{2/3}}\,\mathrm{d}\tau\right)\mathrm{d}m = \frac{3C\mathrm{i}}{2\pi}\int_0^t \frac{h_0'(\tau)}{(t-\tau)^{2/3}}\,\mathrm{d}\tau, \tag{3.8}$$

where

$$C = -\int_{\partial D^+} m\mathrm{e}^{\mathrm{i}m^3}\,\mathrm{d}m.$$

Thus, we are reduced to determining

$$\bar{M}(T) = \frac{3C\Omega\mathrm{i}}{4\pi^2}\int_T^{T+\frac{2\pi}{\Omega}}\left(\int_0^t \frac{h_0'(\tau)}{(t-\tau)^{2/3}}\,\mathrm{d}\tau\right)\mathrm{d}t. \tag{3.9}$$

Inverting the order of integration and performing the t-integration explicitly yields

$$\begin{aligned}\bar{M}(T) &= \frac{9C\Omega\mathrm{i}}{4\pi^2}\left[\int_0^T h_0'(\tau)(t-\tau)^{\frac{1}{3}}\Big|_{t=T}^{t=T+\frac{2\pi}{\Omega}}\mathrm{d}\tau + \int_T^{T+\frac{2\pi}{\Omega}} h_0'(\tau)(t-\tau)^{\frac{1}{3}}\Big|_{t=\tau}^{t=T+\frac{2\pi}{\Omega}}\mathrm{d}\tau\right]\\ &= \frac{9C\Omega\mathrm{i}}{4\pi^2}\left[\int_0^{T+\frac{2\pi}{\Omega}} h_0'(\tau)\left(T+\frac{2\pi}{\Omega}-\tau\right)^{\frac{1}{3}}\mathrm{d}\tau - \int_0^T h_0'(\tau)(T-\tau)^{\frac{1}{3}}\,\mathrm{d}\tau\right].\end{aligned}$$

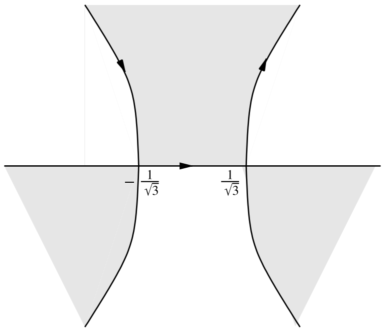

Figure 2. The contour ∂D^{+} for the linearized KdV.

Letting $\tau = s - 2\pi/\Omega$ in the second integral and using the fact that $g_0 = h_0'$ is periodic with period $2\pi/\Omega$, this may be simplified to

$$\bar{M}(T) = \frac{9C\Omega\mathrm{i}}{4\pi^2}\int_0^{\frac{2\pi}{\Omega}} h_0'(\tau)\left(T + \frac{2\pi}{\Omega} - \tau\right)^{\frac{1}{3}}\,\mathrm{d}\tau.$$

Integrating by parts and using the fact that $h(0) = 0$, and by periodicity $h(2\pi/\Omega) = 0$, there follows that

$$\bar{M}(T) = \frac{3C\mathrm{i}}{4\pi^2}\int_0^{\frac{2\pi}{\Omega}} \frac{h_0(\tau)}{(T + (2\pi/\Omega) - \tau)^{\frac{2}{3}}}\,\mathrm{d}\tau = \frac{3C\mathrm{i}}{4\pi^2 T^{\frac{2}{3}}}\int_0^{\frac{2\pi}{\Omega}} h_0(\tau)\,\mathrm{d}\tau$$
$$- \frac{C\mathrm{i}}{2\pi^2 T^{\frac{5}{3}}}\int_0^{\frac{2\pi}{\Omega}} h_0(\tau)\left(\frac{2\pi}{\Omega} - \tau\right)\mathrm{d}\tau + O\left(T^{-\frac{8}{3}}\right), \quad T \to \infty.$$

This shows clearly that asymptotically, the average mass in the channel tends to 0.

Similar results are valid for the case that the linear PDE also contains the term q_x. For example, equation (3.3) is modified as follows: k^3 in the exponent of the exponential is replaced with $k^3 - k$, the factor $3k^2$ is replaced by $3k^2 - 1$ and the contour ∂D^{+} is now depicted in figure 2, where the relevant curves are defined by $k_{\mathrm{I}}^2 - 3k_{\mathrm{R}}^2 + 1 = 0$.

4. Nonlinear problems

We very briefly indicate the way these ideas ramify in the presence of nonlinearity. This will be developed in more detail in a separate publication. Let us consider the following initial-boundary-value problems for the full KdV or NLS equations

$$\begin{aligned} q_t + q_x + qq_x + q_{xxx} &= 0, && x, t > 0,\\ q(x,0) &= q_0(x), && x > 0,\\ q(0,t) &= g_0(t), && t > 0 \end{aligned}$$

and

$$\begin{aligned} iq_t + q_{xx} + |q|^2 q &= 0, && x, t > 0,\\ q(x,0) &= q_0(x), && x > 0,\\ q(0,t) &= g_0(t), && t > 0. \end{aligned}$$

Steps 1, 2 and 3 may still be carried out. The nonlinearity is reflected in two ways. First, the analogues of $\hat{q}_0(k)$ and $\tilde{g}_j(k,t)$ apparently cannot be written in closed form but instead are obtained in terms of the eigenfunctions of the x and t parts of the associated Lax pair, evaluated on the appropriate boundary values of the solutions. For the KdV equation, these values are $q_0(x)$ and $\{\partial_x^j q(0,t)\}_{j=0}^2$, respectively; whilst for the cubic NLS equation they are $q_0(x)$ and $\{\partial_x^j q(0,t)\}_{j=0}^1$. The integral representation that results from carrying out steps 1, 2 and 3 involves components of a 2×2 matrix $\mu(x,t)$ which is the *simultaneous* solution of both parts of the Lax pair. This matrix may be obtained as the solution of a matrix Riemann–Hilbert problem defined in terms of the analogues of $\hat{q}_0(k)$ and $\tilde{g}(k,t)$. The global relation can then be solved *explicitly* for the unknown boundary values (in the KdV case, $\{\partial_x^j q(0,t)\}_{j=1}^2$) [BFS, F4, TF]. However, this 'solution' involves the eigenfunctions of the t-part of the Lax pair which itself depends on all the boundary values. Thus, one is left with a nonlinear integral equation to solve. Of course, well posedness of the initial-boundary-value problem for the KdV equation is known by other means, as already indicated. If the auxiliary data q_0 and g_0 are sufficiently smooth, (see [BW1,BW2] for theory in smooth spaces), the boundary values $\{\partial_x^j q(0,t)\}_{j=1}^2$ all exist. Hence, while the inverse scattering formalism developed here does not yield solutions in closed form, the associated integral equation does contain information not available by other means. For example, by applying the Deift–Zhou [DZ] theory to the matrix Riemann–Hilbert problem, it can be shown that for boundary data of essentially finite duration, the solution is dominated by solitons in the asymptotic range where x/t is of order 1 and $t\to\infty$.

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