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NONLINEAR DISPERSIVE WAVES ON TREES

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ABSTRACT. We investigate the well-posedness of a class of nonlinear dispersive waves on trees, in connection with the mathematical modeling of the human cardiovascular system. Specifically, we study the Benjamin-Bona-Mahony (BBM) equation, also known as the regularized long wave equation, posed on finite trees, together with standard junction and terminal boundary conditions. We prove that the Cauchy problem for the BBM equation is well-posed in an appropriate space on arbitrary finite trees.

1 Introduction In the theory of shallow-water waves, long wave approximations lead to classical nonlinear dispersive wave equations such as the Benjamin-Bona-Mahony (BBM) equation

$$\text{(BBM):} \qquad u_t + u_x + uu_x - u_{xxt} = 0.$$

where $u = u(x,t)$ represents the displacement at location x and time t. This is related to the well-known Korteweg-de Vries equation, which is a completely integrable system and supports soliton solutions. When modeling wave propagation over finite distances, the BBM equation seems more suitable, since the number of boundary conditions which need to be imposed for its well-posedness over finite intervals [**3**], is two, whereas KdV requires three boundary conditions [**8**]. Moreover, comparison between the two models presented in [**4**] indicates that solutions of the two models stay 'close' to each other over relatively long time intervals.

In this paper we show that the BBM equation is well suited for modeling wave propagation on trees, by proving it is well-posed subject to standard junction conditions, motivated by several 1D models of the arterial system. In Section 2 we introduce the notation of the trees and of certain differential operators on trees. Section 3 contains the treatment of the BBM equation on various tree structures, where we state and

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prove the main well-posedness result. Section 4 contains some observations based on numerical simulations. We conclude with some remarks and comments about future directions of research.

2 Differential operators on trees In what follows $\mathcal{T}$ will represent a (finite or infinite) metric tree. Denote the collection of vertices of $\mathcal{T}$ by $\mathcal{V} = \{v_i, i \in I\}$ and the collection of edges $\mathcal{E}$, which is identified with a subset of $\mathcal{V} \times \mathcal{V}$. In short, we may write $\mathcal{T} = \{\mathcal{V}, \mathcal{E}\}$. v_0 will denote the (unique) root vertex (or origin vertex) of the tree. A boundary (or terminal) vertex v_b is a vertex (different from the root vertex, which may have several outgoing edges) which is adjacent to only one edge in the tree (such an edge is also called a leaf edge). The collection of boundary vertices will be denoted $\mathcal{B}$ and the collection of boundary edges (leaves) by $\mathcal{E}_{\mathcal{B}}$. Each edge in $\mathcal{E}$, say $e = \{v_i, v_j\}$, has a length $l = l_{ij}$ and a positive orientation, from the vertex that is closest to the root, say v_i, to the one farthest from the root, say v_j. We write $e = [v_i, v_j]$ to indicate its orientation and, when necessary, identify it with the interval $[0, l_{ij}]$.

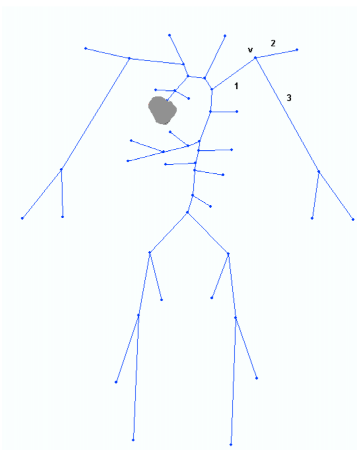


FIGURE 1: A schematic representation of the major (46) arteries in the human arterial system

Linear differential operators on trees have been considered in the literature in the context of the theory of quantum graphs [**10, 11, 12**] and, more recently, in connection with biological trees [**13**]. Throughout the paper, we will employ the 1D Laplacian on the tree $\mathcal{T}$, associated with the differential expression

$$D = -\frac{d^2}{dx^2}, \quad x \in \mathcal{T}. \tag{2.1}$$

We assume the standard junction conditions at each interior vertex $v \in \operatorname{int}(\mathcal{T})$:

$$\begin{cases} f(v-) = f(v+) & \text{(continuity at the vertex)} \\ f_x(v-) = \sum f_x(v+) & \text{(flux-balance condition).} \end{cases}$$

Here $v-$ represents the vertex v treated as the boundary of the incoming edge and $v+$ represents the same vertex v as a boundary of an outgoing edge. The summation in the flux-balance condition is over all outgoing edges adjacent to the vertex v. The positive direction of the edges in the tree is considered starting from the root vertex towards the terminal vertices.

3 BBM equation posed on a tree The Benjamin-Bona-Mahony (BBM) equation is considered here (rather than the KdV equation), as a 1D model for the unidirectional propagation of waves in shallow-water channels (see [**1**]). The initial-value problem of BBM posed on the real axis, on the half-line (quarter-plane) and on a finite interval have previously been discussed in [**1, 6, 9**]. Here we extend the treatment of the BBM equation given in [**3**] from intervals to trees of finite and infinite length.

In this paper we will restrict ourselves to well-posedness results in the Sobolev space $H^1 = \{f \in L^2, f' \in L^2\}$. We note that well-posedness for less regular initial data exists in the literature (see [**5**]). Our choice for the H^1-regularity stems from an interesting result for the BBM on the entire axis [**21**], where the existence and stability of N-solitary waves in the H^1-norm have been announced.

3.1 BBM on half-line We recall [**7, 9**] that the initial-boundary value problem for the BBM equation on the half-line is well-posed in $H^1(\mathbb{R}_+)$:

$$(\text{BBM})_{HL} \qquad \begin{aligned} & u_t - \alpha^2 u_{xxt} + \beta u_x + \gamma u u_x = 0, && x \in \mathbb{R}_+,\ t \geq 0, \\ & u(0,t) = h(t), \quad u(\infty,t) = 0, && t \geq 0 \\ & u(x,0) = \varphi(x), && x \geq 0. \end{aligned}$$

We remark that the solution u is in distributional sense, but for $\varphi \in H^1(\mathbb{R}_+) \cup C_b^2(\mathbb{R}_+)$ and $h \in C^1(0,T)$ for some $T > 0$, the solution $u = u(x,t)$ is classical, that is, $u \in C^1(0,T; C_b^2(\mathbb{R}_+))$. This can be seen by rewriting BBM in the (formally equivalent) form

$$u_t = h'(t)\phi_\alpha - (I - \alpha^2\partial_x^2)^{-1}(\beta u_x + \gamma u u_x), \tag{3.1}$$

where $\phi_\alpha(x) = e^{-x/\alpha}$ and $(I - \alpha^2\partial_x^2)^{-1}$ is the inverse of the operator $I - \alpha^2\partial_x^2$ acting on the space $H^2(\mathbb{R}_+)$ with domain

$$H_0^2(\mathbb{R}_+) = \{w \in H^2(\mathbb{R}_+), w(0) = 0\}.$$

For notational simplicity, we will rescale the spatial variable x to $\tilde{x} = x/\alpha$, so that $\partial_x = \partial_{\tilde{x}}/\alpha$. We also introduce $\tilde{u}(\tilde{x}, t) = u(x,t)$, $\tilde{\beta} = \beta/\alpha$ and $\tilde{\gamma} = \gamma/\alpha$. In what follows we will use these rescaled variables and omit the $\tilde{}$ notation.

Using the normalization $\alpha = 1$, the initial-boundary value problem $(\mathrm{BBM})_{HL}$ can be written in the integral form (see, e.g., [**2**])

$$u_t(x,t) = h'(t)e^{-x} + \int_0^\infty P(x,y)[\beta u_y + \gamma u u_y](y,t)\, dy \tag{3.2}$$

where $P(x,y) = \frac{1}{2}(e^{-(x+y)} - e^{|x-y|})$. Integrating by parts, we further obtain

$$u_t(x,t) = h'(t)e^{-x} + \int_0^\infty K(x,y)[\beta u + \gamma u^2/2](y,t)\, dy \tag{3.3}$$

where $K(x,y) = \frac{1}{2}(e^{-(x+y)} + sgn(x-y)e^{-|x-y|})$, and, after integrating in time,

$$\begin{aligned} u(x,t) = {} & \varphi(x) + (h(t) - h(0))e^{-x} \\ & + \int_0^t \int_0^\infty K(x,y)[\beta u + \gamma u^2/2](y,s)\, dy\, ds. \end{aligned} \tag{3.4}$$

For convenience, we present here a well-posedness result for the initial-boundary value problem on the half-line.

Theorem 3.1. ([**7**]) *Let $\varphi \in H^1(\mathbb{R}_+) \cap C_b^2(\mathbb{R}_+)$ and $h \in C^1(0,T)$ with $h(0) = \varphi(0)$. Then, the IBVP on the half-line has a unique (classical) solution $u \in C(0,T; H^1(\mathbb{R}_+))$ with $\partial_x^2 \partial_t u \in C(0,T; C_b(\mathbb{R}_+))$. Moreover, if $h \in C^1(\mathbb{R}_+) \cap H^1(\mathbb{R}_+)$, then $u \in C_b(\mathbb{R}_+; H^1(\mathbb{R}_+))$.*

Note that the regularity of the solution $u(\cdot, t)$ for any $t > 0$ is the same as that of the initial condition, since $v(x,t) = u(x,t) - \varphi(x)$ is differentiable and

$$v_x(x,t) = -(h(t) - h(0))e^{-x} + \int_0^t [\beta u + \gamma u^2/2](x,s)\, ds - \frac{1}{2}\int_0^t \int_0^\infty M(x,y)[\beta u + \gamma u^2/2](y,s) dy\, ds \tag{3.5}$$

with $M(x,y) = \frac{1}{2}(e^{-(x+y)} + e^{-|x-y|})$.

3.2 BBM on a $\mathcal{Y}$-junction We first formulate the initial-value problem for BBM for an $\mathcal{Y}$-junction with infinite edges, i.e., the incoming (parent) edge and outgoing (children) edges are half-lines as in the figure below

$$\mathcal{Y} = e_1 \cup e_2 \cup e_3 = (-\infty_1, 0] \cup [0, \infty_2) \cup [0, \infty_3). \tag{3.6}$$

Here the notation ∞_i indicates the edge $(i = 1, 2, 3)$ where ∞ is considered.

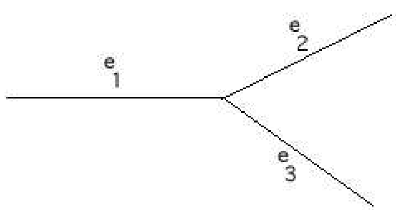


FIGURE 2: $\mathcal{Y}$-junction with infinite edges

On each edge e_i, $i = 1, 2, 3$, define $\overline{u} = \overline{u}_i$ to satisfy the BBM equation

$$(\text{BBM})_i : \quad \overline{u}_{i,t} - \overline{u}_{i,\overline{x}\overline{x}t} + \overline{u}_{i,\overline{x}} + \overline{u}_i \overline{u}_{i,\overline{x}} = 0, \quad \overline{x} \in e_i,\ t \geq 0 \tag{3.7}$$

and impose the following continuity and flux-balance junction conditions

$$\begin{aligned} \overline{u}_1(0,t) &= \overline{u}_2(0,t) = \overline{u}_3(0,t), \\ \overline{u}_{1,x}(0,t) &= \overline{u}_{2,x}(0,t) + \overline{u}_{3,x}(0,t) \end{aligned} \tag{3.8}$$

where $\overline{a}_i > 0,\ i = 1,2,3$. Making the change of variable $\overline{x} \to x = -\overline{x}$ for $\overline{u}_1$, we obtain the initial-value problem with three BBM equations for $u_i = u_i(x,t)[= \overline{u}_i(\overline{x},t)]$ on $\mathbb{R}_+$ $(i = 1,2,3)$

$$(\mathrm{BBM})_{\mathcal{Y}} \qquad \begin{aligned} &u_{i,t} - u_{i,xxt} + \sigma_i u_{i,x} + \sigma_i u_i u_{i,x} = 0, && x \in \mathbb{R}_+,\ t \geq 0,\\ &u_1 = u_2 = u_3, \quad u_{1,x} + u_{2,x} + u_{3,x} = 0, && x = 0,\ t \geq 0\\ &u_i(+\infty, t) = 0, && t \geq 0\\ &u_i(x,0) = \varphi_i(x), && x \in \mathbb{R}_+, \end{aligned}$$

where $\sigma_1 = -1,\ \sigma_2 = \sigma_3 = +1$. We will assume that the initial condition (at $t = 0$) satisfies the compatibility conditions

$$\varphi_1 = \varphi_2 = \varphi_3, \quad \varphi_{1,x} + \varphi_{2,x} + \varphi_{3,x} = 0 \quad \text{at } x = 0. \tag{3.9}$$

From (3.4) we obtain the integral formulation $(i = 1,2,3)$

$$\begin{aligned} u_i(x,t) = \varphi_i(x) &+ (h(t) - h(0))e^{-x} \\ &+ \sigma_i \int_0^t \int_0^\infty K(x,y)[u_i + u_i^2/2](y,s)\,dy\,ds, \end{aligned} \tag{3.10}$$

where $h(t) = u_1(0,t) = u_2(0,t) = u_3(0,t)$ is unknown and $h(0) = \varphi(0)$. Recall that the integral kernel has the form

$$K(x,y) = \frac{1}{2}(e^{-(x+y)} + \operatorname{sgn}(x-y)e^{-|x-y|}), \quad x, y \geq 0.$$

We now use the flux-balance condition in (3.9) to eliminate $h(t) - h(0)$. From (3.2) we obtain

$$u_{i,xt}(0,t) = -h'(t) - \sigma_i \int_0^\infty e^{-y}[u_{i,y} + u_i u_{i,y}](y,s)\,dy\,ds,$$

hence the flux-balance condition becomes

$$0 = \sum_{i=1}^3 u_{i,x}(0,t) = -3h'(t) - \int_0^\infty e^{-y} \sum_{i=1}^3 \sigma_i [u_{i,y} + u_i u_{i,y}](y,s)dy\,ds.$$

In other words, the unknown function h must satisfy

$$\begin{aligned} h(t) - h(0) = \frac{1}{3} \int_0^t \int_0^\infty e^{-y} \big[&(1+u_1)u_{1,y} \\ &- (1+u_2)u_{2,y} - (1+u_3)u_{3,y}\big](y,s)\,dy\,ds. \end{aligned} \tag{3.11}$$

From (3.10) and (3.11) we conclude that $\mathbf{u} = [u_1, u_2, u_3]^T$ must satisfy the integral equation below, which is (formally) equivalent to $(\mathrm{BBM})_{\mathcal{Y}}$,

$$\mathbf{u}(x,t) = \varphi(x) + \Phi(\mathbf{u})(t)e^{-x} + \sigma \int_0^t \int_0^\infty \mathbf{K}(x,y)\mathbf{P}[\mathbf{u}(y,s)]dy\,ds, \tag{3.12}$$

where $\Phi(\mathbf{u})$ is the right hand side in (3.11)

$$\begin{aligned}\Phi(\mathbf{u})(t) = \frac{1}{3}\int_0^t \int_0^\infty & e^{-y}\big[(1+u_1)u_{1,y} \\ & - (1+u_2)u_{2,y} - (1+u_3)u_{3,y}\big](y,s)\,dy\,ds,\end{aligned} \tag{3.13}$$

and

$$\mathbf{K}(x,y) = K(x,y)I_3, \quad \sigma = diag(-1,+1,+1),$$

$$\mathbf{P}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 + u_1^2/2 \\ u_2 + u_2^2/2 \\ u_3 + u_3^2/2 \end{bmatrix}.$$

Let us first introduce some notation. Denote

$$C(\mathcal{Y}) = \{\mathbf{v} = [v_1, v_2, v_3]^T \in C(\mathbb{R}_+)^3 \mid v_1(0) = v_2(0) = v_3(0)\}$$

and, for $k \geq 1$,

$$C_a^k(\mathcal{Y}) = \left\{\mathbf{v} = [v_1, v_2, v_3]^T \in C^k(\mathbb{R}_+)^3 \cap C(\mathcal{Y}) \,\middle|\, \sum_i v_i'(0) = 0\right\}.$$

We define the usual Sobolev spaces for functions defined on the junction $\mathcal{Y}$, $L^2(\mathcal{Y}) = L^2(\mathbb{R}_+)^3$, $H^1(\mathcal{Y}) = H^1(\mathbb{R}_+)^3 \cap C(\mathcal{Y})$ and, for $k \geq 2$, $H_a^k(\mathcal{Y}) = H^k(\mathbb{R}_+)^3 \cap C_a^{k-1}(\mathcal{Y})$.

Theorem 3.2. *The initial value problem for BBM on the $\mathcal{Y}$-junction is well posed in $H^1(\mathcal{Y})$, for initial conditions $\varphi \in H^1(\mathcal{Y}) \cap C(0,T; C_a^2(\mathcal{Y}))$.*

Proof. Define the map $\mathbb{B}$ by

$$\begin{aligned}\mathbb{B}[\mathbf{v}](x,t) = \varphi(x) &+ \Phi(\mathbf{u})(t)e^{-x} \\ &+ \sigma \int_0^t \int_0^\infty \mathbf{K}(x,y)\mathbf{P}[\mathbf{v}(y,s)]dy\,ds.\end{aligned} \tag{3.14}$$

One proves that $\mathbb{B}$ is a contraction on a ball of radius R centered at the origin in the space $C(0,T; H^1(\mathcal{Y}))$, for sufficiently small T. ☐

To prove global well posedness of $(\mathrm{BBM})_{\mathcal{Y}}$ we now proceed in the standard fashion: we rely on the local existence of solutions and use *a priori* bounds to show that solutions can be extended for all times. We note that the functionals which are invariant under the BBM equation on the entire line, do not remain 'exact' invariants for BBM posed on half-line or trees, due to the presence of the junction conditions.

3.3 BBM on finite intervals We recall [**3**] that the initial-boundary value problem for BBM on the finite interval $[0,1]$, $(\alpha > 0)$

$$u_t - \alpha^2 u_{xxt} + \beta u_x + \gamma u u_x = 0, \quad x \in [0,1],\ t \geq 0, \tag{3.15}$$

$$u(0,t) = g_0(t), \quad u(1,t) = g_1(t), \quad t \geq 0, \tag{3.16}$$

$$u(x,0) = f(x), \qquad x \in [0,1]. \tag{3.17}$$

is well posed in $H^1(0,1)$. In the case BBM is defined on an interval of length $l \neq 1$, say on $[0,l]$, then we can reduce it to the interval $[0,1]$, by making the change of spatial variables $\tilde{x} = x/l$, and consequently $\tilde{\alpha} = \alpha/l$, $\tilde{\beta} = \beta/l$ and $\tilde{\gamma} = \gamma/l$. Hence, the considerations below easily extend to solutions of the BBM equation on arbitrary finite intervals.

Equation (3.15) can be (formally) rewritten as [**3**]

$$(I - \alpha^2 \partial_x^2) u_t = -(\beta u_x + \gamma u u_x), \tag{3.18}$$

or,

$$u_t = g_0'(t)\phi_0(x) + g_1'(t)\phi_1(x) - (I - \alpha^2\partial_x^2)^{-1}(\beta u_x + \gamma u u_x) \tag{3.19}$$

where

$$\phi_{0,\alpha}(x) = \frac{\sinh((1-x)/\alpha)}{\sinh(1/\alpha)} \quad \text{and} \quad \phi_{1,\alpha}(x) = \frac{\sinh(x/\alpha)}{\sinh(1/\alpha)} \tag{3.20}$$

satisfy $(I - \alpha^2\partial_x^2)\phi_{i,\alpha} = 0$ and $\phi_{i,\alpha}(j) = \delta_{ij}$, for $i,j = 0,1$. In (3.19), $(I - \alpha^2\partial_x^2)^{-1}$ is given by the resolvent $R(\lambda; D) = (D - \lambda I)^{-1}$, with $\lambda = -1/\alpha^2$, of the operator $D = -\partial_x^2$ on $[0,1]$ with Dirichlet boundary conditions, *viz.*

$$(I - \alpha^2\partial_x^2)^{-1} f = -1/\alpha^2 R(-1/\alpha^2; D) f. \tag{3.21}$$

Recall that

$$R(\lambda; D) f = \frac{1}{W(\alpha)} \int_0^1 P_\alpha(x,\xi) f(\xi)\, d\xi \tag{3.22}$$

with

$$P_\alpha(x,\xi) = \begin{cases} \phi_{0,\alpha}(x)\phi_{1,\alpha}(\xi), & \text{if } \xi \le x \\ \phi_{1,\alpha}(x)\phi_{0,\alpha}(\xi), & \text{if } \xi \ge x \end{cases} \tag{3.23}$$

and

$$\begin{aligned} W(\alpha) = W[\phi_{0,\alpha},\phi_{1,\alpha}] &= \phi_{0,\alpha}(x)\phi'_{1,\alpha}(x) - \phi'_{0,\alpha}(x)\phi_{1,\alpha}(x) \\ &\equiv (\alpha\sinh(1/\alpha))^{-1}. \end{aligned}$$

Substituting back into (3.19) then leads to

$$\begin{aligned} u_t(x,t) &= g'_0(t)\phi_{0,\alpha}(x) + g'_1(t)\phi_{1,\alpha}(x) \\ &\quad - k(\alpha)\int_0^1 P_\alpha(x,\xi)\big[\beta u_\xi(\xi,t) + \gamma u u_\xi(\xi,t)\big]\,d\xi \end{aligned} \tag{3.24}$$

with $k(\alpha) = (-1/\alpha^2)W(\alpha)^{-1} = -1/\alpha\sinh(1/\alpha)$. We now integrate by parts ($P_\alpha(x,0) = P_\alpha(x,l) \equiv 0$) to obtain

$$\begin{aligned} u_t(x,t) &= g'_0(t)\phi_{0,\alpha}(x) + g'_1(t)\phi_{1,\alpha}(x) \\ &\quad + k(\alpha)\int_0^1 P_{\alpha,\xi}(x,\xi)\big[\beta u(\xi,t) + \gamma u^2(\xi,t)/2\big]\,d\xi \end{aligned} \tag{3.25}$$

and, by an integration with respect to t, we obtain the integral equation

$$u(x,t) = \tilde{g}_0(t)\phi_0(x) + \tilde{g}_1(t)\phi_1(x) + B_\alpha(u)(x,t) \tag{3.26}$$

where $\tilde{g}_i(t) = g_i(t) - g_i(0)$, $i = 0,1$, and $B_\alpha(u)$ is the nonlinear operator given by

$$B_\alpha(u)(x,t) = \int_0^t \int_0^1 K_\alpha(x,\xi)\big[\beta u(\xi,t) + \gamma u^2(\xi,t)/2\big]\,d\xi. \tag{3.27}$$

Here

$$K_\alpha(x,\xi) = k(\alpha)P_{\alpha,\xi}(x,\xi) = k(\alpha)\begin{cases} \phi_{0,\alpha}(x)\phi'_{1,\alpha}(\xi), & \text{if } \xi \le x \\ \phi_{1,\alpha}(x)\phi'_{0,\alpha}(\xi), & \text{if } \xi \ge x \end{cases} \tag{3.28}$$

and $k(\alpha) = -1/\alpha\sinh(1/\alpha)$.

The results in [**3**] show well-posedness of the BBM on finite interval $[0,1]$, due to the fact that the right-hand side of (3.26) defines a contraction mapping on a suitable space of continuous functions. We now extend these ideas to the case of a finite tree.

3.4 BBM on finite trees Let $\mathcal{T}$ be a tree with finite number of vertices (hence edges). On each edge e_i, $i = I$, define the BBM equation for $u = u_i$

$$(\text{BBM})_i : \quad u_t - \alpha_i^2 u_{xxt} + \beta_i u_x + \gamma_i u u_x = 0, \quad x \in e_i,\ t \geq 0. \tag{3.29}$$

We say that a function $u = u(x,t)$, $x \in \mathcal{T}, t \geq 0$ solves the initial-boundary value problem for BBM on the tree $\mathcal{T}$ if the restriction of u to each edge e_i satisfies the corresponding $(\text{BBM})_i$ equation and u satisfies the interior boundary conditions

$$u(v_i-,t) = u(v_i+,t) \tag{3.30}$$

$$u_x(v_i-,t) = \sum u_x(v_i+,t) \quad \text{for } v_i \in \operatorname{int}(\mathcal{T}), \tag{3.31}$$

the boundary conditons

$$u(v_e,t) = g(v_e,t) \quad \text{for } v_e \in \partial\mathcal{T} \tag{3.32}$$

and the initial condition

$$u(x,0) = f(x), \quad x \in \mathcal{T}. \tag{3.33}$$

By a remark made above, we can rescale all lengths of the edges to unit length, this way modifying only the coefficients c_i, γ_i and α_i in the formulas above. Since in this section we only consider a finite number of edges, we can simply rename these coefficients and hence, without loss of generality, we can assume all $l_i = 1$.

For the sake of clarity, we start with a simple tree, a $\mathcal{Y}$-junction, which consists of 4 vertices $\mathcal{V} = \{v_0, v_1, v_2, v_3\}$ and three edges $\mathcal{E} = \{e_1, e_2, e_3\}$, where $e_1 = [v_0, v_1]$ is the root edge and $e_2 = [v_1, v_2]$ and $e_3 = [v_1, v_3]$ are its children edges.

Then $u_i = u_i(x,t)$, the restrictions of u to e_i, must satisfy ($x \in [0,1],\ t \geq 0$)

$$u_{1t} - \alpha_1^2 u_{1xxt} + \beta_1 u_{1x} + \gamma_1 u_1 u_{1x} = 0,$$

$$u_{2t} - \alpha_2^2 u_{1xxt} + \beta_2 u_{2x} + \gamma_2 u_2 u_{2x} = 0,$$

$$u_{3t} - \alpha_3^2 u_{1xxt} + \beta_3 u_{3x} + \gamma_3 u_3 u_{3x} = 0,$$

the junction conditions at v_1

$$u_1(1,t) = u_2(0,t) = u_3(0,t), \qquad u_{1x}(1,t) = u_{2x}(0,t) + u_{3x}(0,t)$$

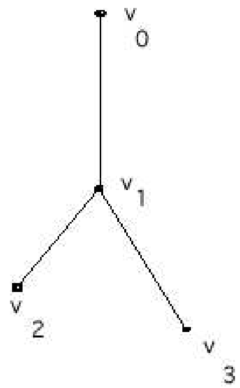


FIGURE 3: $\mathcal{Y}$-junction with finite edges

and the boundary conditions (at v_0, v_2 and v_3)

$$u_1(0,t) = g_0(t), u_2(1,t) = g_2(t) \quad \text{and} \quad u_3(1,t) = g_3(t).$$

In addition, we impose the initial conditions

$$u_i(x,0) = f_i(x), \quad x \in e_i, \quad (i = 1,2,3).$$

We apply formula (3.26) obtained in the finite interval case to conclude that u_i, $i = 1,2,3$, must satisfy the system

$$\begin{cases} u_1(x,t) = g_0(t)\phi_{0,\alpha_1}(x) + \mathbf{g}_1(t)\phi_{1,\alpha_1}(x) + B_{\alpha_1}(u_1)(x,t) \\ u_2(x,t) = \mathbf{g}_1(t)\phi_{0,\alpha_2}(x) + g_2(t)\phi_{1,\alpha_2}(x) + B_{\alpha_2}(u_2)(x,t) \\ u_3(x,t) = \mathbf{g}_1(t)\phi_{0,\alpha_3}(x) + g_3(t)\phi_{1,\alpha_3}(x) + B_{\alpha_3}(u_3)(x,t) \end{cases}$$

of nonlinear integral equations, where

$$\phi_{0,\alpha}(x) = \frac{\sinh((1-x)/\alpha)}{\sinh(1/\alpha)} \quad \text{and} \quad \phi_{1,\alpha}(x) = \frac{\sinh(x/\alpha)}{\sinh(1/\alpha)}$$

($\alpha = \alpha_i$, $i = 1,2,3$) are as in (3.20) and

$$B_{\alpha_i}(u)(x,t) = \int_0^t \int_0^1 K_{\alpha_i}(x,\xi)\big[c_i u(\xi,t) + \gamma_i u^2(\xi,t)/2\big]\, d\xi \tag{3.34}$$

with K_α given in (3.28), $\alpha = \alpha_i,\ i = 1, 2, 3..$

Since $\mathbf{g}_1 = \mathbf{g}_1(t)$ is not determined *a priori* as a vertex or boundary condition, it must be considered as an additional unknown in the system above. Using the flux-balance condition at the junction vertex v_1, we seek an expression for the unknown $\mathbf{g}_1(t)$ in terms of $u = (u_1, u_2, u_3)$

$$\begin{aligned} 0 &= u_{1,xt}(1,t) - u_{2,xt}(0,t) - u_{3,xt}(0,t) \\ &= g_0'(t)\phi_{0,\alpha_1}'(1) + \mathbf{g}_1'(t)\phi_{1,\alpha}'(1) + B_{\alpha_1}(u_1)_x(1,t) \\ &\quad - \mathbf{g}_1'(t)[\phi_{0,\alpha_2}'(0) + \phi_0'(0,\alpha_3)] \\ &\quad + g_2'(t)\phi_{1,\alpha_2}'(0) + g_3'(t)\phi_{1,\alpha_3}'(0) + B_{\alpha_2}(u_2)_x(0,t) + B_{\alpha_3}(u_3)x(0,t). \end{aligned}$$

Hence,

$$\begin{aligned} &\mathbf{g}_1'(t)[\phi_{1,\alpha_1}'(1) - \phi_{0,\alpha_2}'(0) - \phi_{0,\alpha_3}'(0)] \\ &\qquad = g_2(t)\phi_{1,\alpha_2}'(0) + g_3(t)\phi_{1,\alpha_3}'(0) - g_0(t)\phi_{0,\alpha_1}'(1) \\ &\qquad\qquad + B_{\alpha_2}(u_2)_x(0,t) + B_{\alpha_3}(u_3)_x(0,t) - B_{\alpha_1}(u_1)_x(1,t). \end{aligned}$$

Note that, for any $\alpha > 0$, $\phi_{1,\alpha}'(1) = -\phi_{0,\alpha}'(0) = 1/\alpha \coth(1/\alpha) > 0$ and $\phi_{1,\alpha}'(0) = -\phi_{0,\alpha}'(1) = 1/(\alpha \sinh(1/\alpha)) > 0$. Also,

$$\begin{aligned} &B_\alpha(u)_x(0,t) \\ &\qquad = \int_0^t \int_0^1 K_{\alpha,x}(0,\xi)[\beta u(\xi,t) + \gamma u^2(\xi,t)/2]\ d\xi \end{aligned} \tag{3.35}$$

$$= k(\alpha) \int_0^t \int_0^1 \phi_{1,\alpha}'(0)\phi_{0,\alpha}(\xi)[\beta u(\xi,t) + \gamma u^2(\xi,t)/2]\ d\xi \tag{3.36}$$

and

$$\begin{aligned} &B_\alpha(u)_x(1,t) \\ &\qquad = \int_0^t \int_0^1 K_{\alpha,x}(1,\xi)[\beta u(\xi,t) + \gamma u^2(\xi,t)/2]\ d\xi \end{aligned} \tag{3.37}$$

$$= k(\alpha) \int_0^t \int_0^1 \phi_{0,\alpha}'(1)\phi_{1,\alpha}(\xi)[\beta u(\xi,t) + \gamma u^2(\xi,t)/2]\ d\xi. \tag{3.38}$$

Combining these expressions, it follows that there exist constants $c_j,\ j = 0, \ldots, 3$ (depending only on α_1, α_2 and α_3) and a 1×3 matrix-valued kernel $\widetilde{\mathbf{K}}(\xi) = [\widetilde{K}_1(\xi), \widetilde{K}_2(\xi), \widetilde{K}_3(\xi)]$ such that $\mathbf{g}_1(t)$ can be written in terms of $\mathbf{u} = [u_1, u_2, u_3]^T$ in the form

$$c_0 g_0(t) + c_1 \mathbf{g}_1(t) + c_2 g_2(t) + c_3 g_3(t) = \int_0^t \int_0^1 \widetilde{\mathbf{K}}(\xi) \mathbf{F}(\mathbf{u})(\xi, t)\, d\xi. \tag{3.39}$$

where $\mathbf{F}(\mathbf{u}) = [c_1 u_1 + \gamma_1 u_1^2/2, c_2 u_2 + \gamma_2 u_2^2/2, c_3 u_3 + \gamma_3 u_3^2/2]^T$. Substituting back $\mathbf{g}_1(t)$ into the system of BBM equations, we obtain an integral system for $\mathbf{u} = [u_1, u_2, u_3]^T$, which is equivalent to the initial-boundary value problem posed on $\mathcal{T}$.

$$\begin{pmatrix} u_1(x,t) \\ u_2(x,t) \\ u_3(x,t) \end{pmatrix} = \mathbf{C} \begin{pmatrix} g_0(t) \\ g_2(t) \\ g_3(t) \end{pmatrix} + \int_0^t \int_0^1 \mathbf{K}(x, \xi) \begin{pmatrix} \beta_1 u_1 + \gamma_1 u_1^2/2 \\ \beta_2 u_2 + \gamma_2 u_2^2/2 \\ \beta_3 u_3 + \gamma_3 u_3^2/2 \end{pmatrix} d\xi \tag{3.40}$$

where $\mathbf{C}$ is a 3×3 constant matrix determined by α_1, α_2 and α_3 only and $K(x, \xi)$ is a 3×3 matrix valued kernel whose entries are rational expressions of $\phi_{0,\alpha}$, $\phi_{1,\alpha}$, $\phi'_{0,\alpha}$ and $\phi'_{1,\alpha}$.

We are now ready to state a result for a general finite tree. As before, let $\mathcal{T}$ be a finite tree with the set of vertices $\mathcal{V} = \{v_j \,|\, j = 0 \dots n\}$ and set of edges $\mathcal{E} = \{e_j \,|\, j = 1 \dots n\}$ such that e_j is the edge whose terminal vertex is v_j. Each edge $e \in \mathcal{E}$ is identified with the interval $[0, l_e]$, where l_e is the length of the interval. The edges are then rescaled to length 1 as before. Without loss of generality, we can assume that $\mathcal{J} = \{1, \dots k\}$ is the set of indices corresponding to the boundary vertices (hence terminal edges) $\mathcal{B} = \{v_j\}_{j \in \mathcal{J}}$ and $\mathcal{E}_{\mathcal{B}} = \{e_j\}_{j \in \mathcal{J}}$.

Theorem 3.3. *Let $u = u(x,t) \in C(\mathcal{T} \times [0,T])$ and denote u_i the restriction of u to the edge e_i. Then $u = u(x,t)$ solves the initial-boundary value problem if and only if $\mathbf{u} = [u_1, u_2, \dots, u_n]$ satisfies the integral equation*

$$\mathbf{u}(x,t) = c_0 g_0(t) + \sum_{v_b \in \mathcal{B}} c_b g_b(t) + \mathbf{B}(\mathbf{u})(x,t) \tag{3.41}$$

$$= \sum_{b \in \mathcal{B} \cup \{0\}} c_b g_b(t) + \int_0^t \int_0^1 \mathbf{K}(x, \xi) \mathbf{F}(\mathbf{u})(\xi, t)\, dt \tag{3.42}$$

where c_b *are constants (determined by* α_i*'s), the integral kernel* $\mathbf{K}(x,\xi)$ *is an* $n \times n$ *matrix-valued operator whose entries are rational functions of* ϕ_α *and* ϕ'_α *and* $\mathbf{F}(\mathbf{u}) = [\beta_i u_i + \gamma_i u^2/2]_{i=1,n}$.

Proof. The proof is by induction by n, the total number of vertices in the finite tree. For $n = 1$ the tree reduces to an interval, and the statement immediate. For $n = 3$ the statement was proved above in the case of an Y-junction (the other configurations of a tree with 3 edges are treated similarly.

Assume the statement is true for any finite tree with at most $n - 1$ edges. Let $\mathcal{T} = \{\mathcal{V}, \mathcal{E}\}$ be a tree with n edges. We consider the subtree $\mathcal{T}'$ obtained by trimming the leaf edges, say corresponding to the vertices $v \in \mathcal{B}$ (boundary vertices). Thus, $\mathcal{T}' = \{\mathcal{V}', \mathcal{E}'\}$, where

$$\mathcal{V}' = \mathcal{V} \setminus \mathcal{B}, \qquad \mathcal{E}' = \mathcal{E} \setminus \mathcal{E}_{\mathcal{B}}.$$

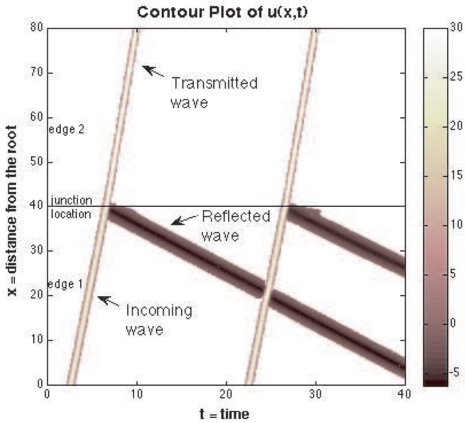
The system of BBM equations posed on each edge $e \in \mathcal{E}$

$$u_{i,t} - \alpha^2 u_{i,xxt} + \beta u_{i,x} + \gamma u_i u_{i,x} = 0, \quad x \in [0,1],\ t \geq 0, \tag{3.43}$$

can be rewritten using the known boundary values and unknown junction values. These unknowns can be found from imposing the junction conditions. □

4 Numerical Simulations To study the dynamics of solutions to BBM on a tree, we discretized the BBM equation using the following numerical scheme: The spatial discretization is based on the Chebyshev differentiation matrix [**33**], while the temporal discretization is based on 3rd-order Milne's predictor-corrector scheme [**23**]. We discretize a $\mathcal{Y}$-junction using the prescribed junction conditions. The root (inlet) boundary conditions is Dirichlet, while the outlet (leaf) conditions are Neumann. When applying an incoming solitary wave moving to the right along the root edge, the resulting scattering at the bifurcation creates a transmitted solitary wave of same shape (different velocities) and a reflected negative wave, as indicated in the figure below (top).

Based on these numerical findings, it appears that there is a linear dependence of the transmitted and reflected wave amplitudes to the incoming waves. While a theoretical understanding of this phenomenon is out of reach at this stage, it seems reasonable to expect this also in the Korteweg-de Vries context, for instance in the presence of a shelf (variable topography). While in the derivation of the models the angle of the bifurcation did not play a role, it is expected that it will, once the angle is factored into formulation of the junction conditions.



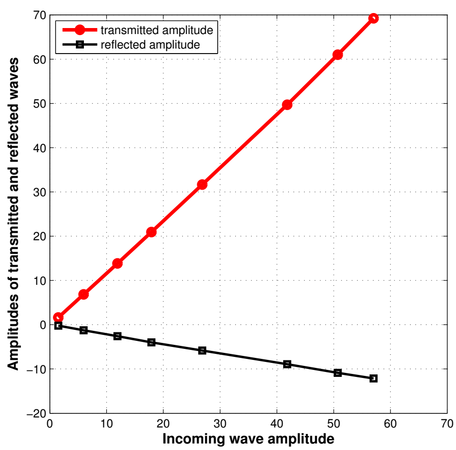


FIGURE 4: Scattering of solitary waves at a junction. An incoming wave modifies its speed past the junction and creates a reflected wave (top). Dependence of reflected and transmitted wave amplitudes to the incoming amplitude (bottom).

5 Conclusions In this paper we showed that the BBM equation is well-posed on finite trees. This nonlinear dispersive system is well suited to capture wave propagation phenomena in the cardiovascular system, under several simplifying assumptions on the physical domain, while preserving the nonlinear characteristics of the dynamics. Transmission and reflection waves are detected in numerical simulations for this simplified model and a theoretical underpinning seems natural to pursue. Moreover, deciding which specific assumptions work best to capture the 1D features of the pulse wave propagation in the human arterial system remains crucial in developing realistic models for the human cardiovascular system. Numerical results of such models must be compared with experimental data collected during various physiological regimes (exercise, disease). A natural extension to the theoretical results in this paper would be to consider different boundary conditions for the BBM equation on the finite tree, which would better model the peripheral circulation. From a numerical point of view, it even makes sense to consider absorbing boundary conditions [**22**] which are well suited for a truncated domain. In particular, an optimal choice of boundary conditions for the finite trees would come from modeling the peripheral circulation itself using nonlinear dispersive waves. More specifically, one could consider the BBM equation on infinite trees, with a main finite sub-tree (the root tree) and fractal trees attached to the leaves of the root tree (see [**16**]). It is expected that similar well-posedness results hold true in the infinite tree context and that dynamics of BBM equation on infinite trees is well approximated by the dynamics of truncated, finite trees, with appropriate boundary conditions.

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