DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES S
Volume **4**, Number **1**, February **2011**

doi:10.3934/dcdss.2011.4.15

pp. 15–50

LARGE-TIME ASYMPTOTICS OF THE GENERALIZED BENJAMIN-ONO-BURGERS EQUATION

Jerry L. Bona

Department of Mathematics, Statistics, and Computer Science
The University of Illinois at Chicago, Chicago, IL 60607, USA

Laihan Luo

Department of Mathematics, New York Institute of Technology
1855 Broadway, New York, NY 10023, USA

ABSTRACT. In this paper, attention is given to pure initial-value problems for the generalized Benjamin-Ono-Burgers (BOB) equation

$$u_t + u_x + (P(u))_x - \nu u_{xx} - \mathbb{H}u_{xx} = 0,$$

where $\mathbb{H}$ is the Hilbert transform, $\nu > 0$ and $P : \mathbb{R} \to \mathbb{R}$ is a smooth function. We study questions of global existence and of the large-time asymptotics of solutions of the initial-value problem. If $\Lambda(s)$ is defined by $\Lambda'(s) = P(s)$, $\Lambda(0) = 0$, then solutions of the initial-value problem corresponding to reasonable initial data maintain their integrity for all $t \geq 0$ provided that Λ and P' satisfy certain growth restrictions. In case a solution corresponding to initial data that is square integrable is global, it is straightforward to conclude it must decay to zero when t becomes unboundedly large. We investigate the detailed asymptotics of this decay. For generic initial data and weak nonlinearity, it is demonstrated that the final decay is that of the linearized equation in which $P \equiv 0$. However, if the initial data is drawn from more restricted classes that involve something akin to a condition of zero mean, then enhanced decay rates are established. These results extend the earlier work of Dix who considered the case where P is a quadratic polynomial.

1. Introduction. This paper is concerned with solutions of damped wave equations of the form

$$u_t + u_x + (P(u))_x - \nu u_{xx} - \mathbb{H}u_{xx} = 0, \quad (x \in \mathbb{R},\ t > 0) \tag{1.1}$$

posed with a specified initial condition

$$u(x, 0) = f(x), \quad (x \in \mathbb{R}). \tag{1.2}$$

In the above models, $u = u(x, t)$ is a real-valued function of the two real variables x and t, subscripts adorning u connote partial differentiation, ν is a positive number and $\mathbb{H}$ is the Hilbert transform defined in the first instance by the principle-value integral

$$\mathbb{H}u(x) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{u(y)}{x - y} dy.$$

2000 Mathematics Subject Classification. Primary: 35B40, 35C20, 35Q35; Secondary: 76D03, 76D09.

Key words and phrases. Generalized Benjamin-Ono-Burgers equation; dissipation; dispersion; nonlinearity; global well-posedness; decay rates.

This research was partially supported by the National Science Foundation, USA

The nonlinearity $P\colon \mathbb{R} \to \mathbb{R}$ belongs to a broad class to be spelled out presently.

When $\nu = 0$ and $P = \frac{1}{2}u^2$, (1.1) is the well-known Benjamin-Ono equation (BO-equation henceforth)

$$u_t + u_x + uu_x - \mathbb{H}u_{xx} = 0, \tag{1.3}$$

originally derived as a model for the propagation of internal waves in deep, stratified fluids [4, 38]. The Benjamin-Ono equation has attracted a lot of attention, both as a guide to practical issues and because of its interesting mathematical properties. The well-posedness of the initial-value problem for the BO-equation and its generalizations, in various function classes, has been studied in a number of papers (see e.g. [2, 8, 24, 26, 31, 43, 44, 47] and the references contained therein).

The BO-equation (1.3) features a balance between nonlinear and dispersive effects. When damping is taken into account, an additional dissipative term needs to be appended to the wave equation (1.3). In [23], equation (1.3) with a dissipative term appended was proposed to describe wave motion supported by intense magnetic flux tubes in the solar atmosphere. Similar nonlinear dispersive wave equations with dissipation arise as models in many physical contexts [14, 15, 16, 27, 28, 33, 35, 36, 39]. For instance, the Korteweg-de Vries-Burgers (KdV-B) equation

$$u_t + u_x + uu_x - \nu u_{xx} + u_{xxx} = 0 \tag{1.4}$$

and the BBM or regularized long-wave-Burgers (RLW-B) equation

$$u_t + u_x + uu_x - \nu u_{xx} - u_{xxt} = 0 \tag{1.5}$$

were considered when the need to account for damping arose in problems involving bore propagation and other wave phenomena on the surface of water.

When dissipative effects are included in the model, the nature of the large-time asymptotics of solutions changes. For the BO-equation itself, reasonably large classes of initial data appear to resolve into a finite sequence of solitary waves followed by a dispersive tail (cf. [1]). Once the nonlinearity $P(u)$ grows near infinity at a critical or supercritical rate, small initial data leads to solutions that disperse, though of course energy is preserved (see [25]). Large initial data may blow up in finite time however. When even a small level of dissipation is appended, there are no longer exact solitary-wave solutions, as energy is constantly removed from the system. In the absence of external forcing, it is expected that solutions emanating from a finite-energy initial disturbance will eventually decay to the quiescent state $u \equiv 0$. This issue was first considered by Dix [21, 22] in the BO-case where $P(z) = z^2$. It is our purpose here to investigate in more detail the decay just posited. Because other than quadratic nonlinearities occasionally arise in practice, we focus on the more general version of (1.3) displayed in (1.1).

Logically prior to such an investigation is the question of whether or not the initial-value problem (1.1)-(1.2) has a solution defined for all $t \geq 0$. This issue will also be studied. The numerical simulations in [9, 10, 11] pertaining to the generalized KdV-B equation (1.4) with uu_x replaced by $u^p u_x$ and (1.1) for $P(u) = \frac{1}{p+1}u^{p+1}$, $p \geq 1$ an integer, indicates that global existence for such equations is not a forgone conclusion. Writing $\Lambda(r)$ for the primitive of the nonlinearity $P(r)$ normalized by the condition $\Lambda(0) = 0$, a theory of global well-posedness is developed based on growth properties of Λ as $r \to +\infty$. If $\Lambda = \Lambda_+ - \Lambda_-$ is broken into its positive and negative parts, then in essence we require that Λ_- grow at most exponentially at $\pm\infty$ whilst Λ_+ is restricted to have at most quartic growth.

To get an idea of what might be true, as far as decay is concerned, it is useful to recall results for the locally-defined KdV-B equation (1.4) and the RLW-B equation (1.5). It was shown in [3] that if u is a solution of either of these equations corresponding to initial data in $L_2(\mathbb{R})$, then the $L_2(\mathbb{R})$-norm of u decays at the rate $t^{-1/4}$ as $t \to +\infty$. This is exactly the rate that obtains via Fourier analysis for the equations in which the nonlinear term does not appear. Similar and more detailed results hold for (1.4) and (1.5) with more general nonlinearities of the form $(P(u))_x$ as in (1.1) (see [3, 6, 7, 12, 13, 20, 21, 37, 42, 45]). Moreover, if the initial data f has a Fourier transform $\hat{f}(y)$ that vanishes at the origin like $|y|^\alpha$ for some $\alpha > 0$, then the $L_2(\mathbb{R})$-norm of the solution u decays at the enhanced rate $t^{-1/4-\alpha/2}$.

Overlapping with and extending the theory developed by Dix in [21, 22], analogous results concerning the $L_2(\mathbb{R})$-norm and other norms are obtained for solutions of the generalized Benjamin-Ono-Burgers equation (1.1). It is shown generally that solutions do indeed decay to zero as $t \to +\infty$ in various norms. Moreover, in case the nonlinearity is weak, meaning that for $|u|$ small, $|P'(u)| \le c|u|^p$ for some $p \ge 2$, we are able to show that the decay is that of the linear equation (1.1) with $P \equiv 0$. Furthermore, rather precise enhanced decay results are available for initial data whose Fourier transform vanishes suitably at the origin.

The paper is organized as follows. In the next section, notation is briefly reviewed and precise theorems stated to give focus to the ensuing development. Section 3 contains some preliminary technicalities together with the results on global well-posedness. Section 4 features some non-optimal decay results for solutions of (1.1) obtained via energy arguments in both the original and the Fourier-transformed variables. Sharp decay results are then shown to follow from the preliminary decay results in Section 5, while Section 6 gives consideration to the situation that obtains for initial data satisfying a zero-mean type condition.

2 Notation and statement of the main results. Throughout the exposition, functions will be real-valued. The L_p-norm of a function f which is pth-power absolutely integrable on $\mathbb{R}$ is denoted by $|f|_p$ for $1 \le p < \infty$, and similarly $|f|_\infty = \|f\|_{L_\infty}$. If $m \ge 0$ is an integer, $W^{m,p}(\mathbb{R})$ will be the Sobolev space consisting of those $L_p(\mathbb{R})$-functions whose first m generalized derivatives lie in $L_p(\mathbb{R})$, equipped with the usual norm,

$$\|f\|_{W^{m,p}(\mathbb{R})} = \sum_{k=0}^{m} |f^{(k)}|_p.$$

The case $p = 2$ appears frequently and so is given the special notation $H^m(\mathbb{R})$. The $H^m(\mathbb{R})$-norm of a function f in $H^m(\mathbb{R})$ will be noted simply $\|f\|_m$. If m is not integer, then the norm for $H^m(\mathbb{R})$ is

$$||f||_m^2 = \int_{-\infty}^{\infty} (1+k^2)^m |\hat{f}(k)|^2 dk,$$

where $\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)dx$ is the Fourier transform of f. The space $C_b^k(\mathbb{R})$ connotes the functions defined on $\mathbb{R}$ whose first k derivatives are bounded, continuous functions and $C_b^\infty = C_b^\infty(\mathbb{R}) = \cap_{k \ge 0} C_b^k(\mathbb{R})$. For $1 \le p \le +\infty$, $L_p(0, T; X)$ is the Banach space of all measurable functions u: $(0, T) \to X$, such that $t \to \|u(t)\|_X$

is in $L_p(0,T)$, with the norm

$$\|u\|_{L_p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p dt\right)^{\frac{1}{p}}, \qquad \text{if} \qquad 1 \le p < +\infty$$

and

$$\|u\|_{L_\infty(0,T;X)} = \text{essential supremum}_{0<t<T}(\|u(t)\|_X).$$

Similarly, $C(0,T;X)$ denotes the subspace of $L_\infty(0,T;X)$ of all continuous functions $u:[0,T]\to X$ with the norm

$$||u||_{C(0,T;X)} = \sup_{0\le t\le T} ||u(t)||_X.$$

If $T=\infty$, then $C_b(\mathbb{R}^+;X)$ denotes the bounded continuous mappings $u:\mathbb{R}^+\to X$. This, too, is a Banach space with the norm

$$||u||_{C_b(\mathbb{R}^+;X)} = \sup_{\mathbb{R}^+} ||u(t)||_X.$$

Finally, recall that the Hilbert transformation $\mathbb{H}$ has the following properties:

$$\mathbb{H}^2 u = -u, \tag{2.1}$$

$$\int_{-\infty}^{\infty} u\mathbb{H}v = -\int_{-\infty}^{\infty} v\mathbb{H}u, \tag{2.2}$$

$$\mathbb{H}(uv) = u\mathbb{H}v + v\mathbb{H}u + \mathbb{H}(\mathbb{H}u\cdot\mathbb{H}v), \tag{2.3}$$

all of which hold for arbitrary u and v in $L^2(\mathbb{R})$. If instead $u\in H^{\frac{1}{2}}(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} u\mathbb{H}u_x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} |y|\,|\hat{u}(y)|^2 dy. \tag{2.4}$$

Main Results

Let $\nu>0$, $\Lambda'(r)=P(r)$ with $\Lambda(0)=0$ and $P'(r)=Q(r)$. Without loss of generality, we take it that $P(0)=P'(0)=0$ also. (If $P'(0)\neq 0$, one simply alters the coefficient of the transport term u_x.) Assume $\Lambda=\Lambda^+-\Lambda^-$ and $Q=Q^+-Q^-$, respectively where Λ^+, Λ^-, Q^+ and Q^- are nonnegative functions. Consider initial data f that is suitably restricted in smoothness and evanescence as $x\to\pm\infty$. (In practice, this means f lies in $H^s(\mathbb{R})$ for appropriate values of s and also in $L_1(\mathbb{R})$.) Then there is a unique global solution u of (1.1) corresponding to the initial value f if Λ^+ and Q^+ satisfy the restrictions

$$\limsup_{r\to+\infty}\frac{\Lambda^+(r)}{r^4}=0 \qquad \text{and} \qquad \limsup_{r\to+\infty}\frac{Q^+(r)}{r^2}\le C, \tag{2.5}$$

respectively, for some constant C, while Λ^- is assumed to be smooth but is otherwise unrestricted, and Q^- is such that for any $\epsilon>0$, there is a constant C_ϵ such that

$$\limsup_{r\to+\infty}\frac{Q^-(r)}{e^{\epsilon r^2}}\le C_\epsilon. \tag{2.6}$$

In (2.6), the value of ϵ actually used to infer suitable *a priori* bounds depends upon the $H^{\frac{1}{2}}$-norm of the initial data in a way to be made precise presently. When the growth of P is larger than the above restrictions require, there still exists a global solution for (1.1)-(1.2) if the initial data is small in L_2-norm, or if ν is sufficiently large. In all cases where global existence obtains, u decays to zero as $t \to +\infty$ in L_∞-norm. In particular, if P is a weak nonlinearity ($|P'(s)| \le c|u|^p$ near $u = 0$ where c a positive number and $p \ge 2$), then there are constants C_j, $1 \le j \le 3$, such that

$$|u(\cdot,t)|_2 \le C_1(1+t)^{-\frac{1}{4}}, \quad |u(\cdot,t)|_\infty \le C_2(1+t)^{-\frac{1}{2}}, \quad |u_x(\cdot,t)|_2 \le C_3(1+t)^{-\frac{3}{4}},$$

for all $t \ge 0$, and

$$\lim_{t\to+\infty} t^{\frac{1}{2}}|u(\cdot,t)|_2^2 = \lim_{t\to+\infty} t^{\frac{1}{2}}|w(\cdot,t)|_2^2, \tag{2.7}$$

where w is the solution of the linearized equation (1.1) in which the nonlinear term is simply dropped. In addition, if the initial data f satisfies $|\hat{f}| \le C|y|^\alpha$ for all small values of y, where C and α are positive constants and $0 \le \alpha \le 1$, then

$$\begin{aligned} |u(\cdot,t)|_2 \le C_1(1+t)^{-\frac{1+2\alpha}{4}}, \qquad |u(\cdot,t)|_\infty \le C_2(1+t)^{-\frac{1+\alpha}{2}}, \\ \text{and} \qquad |u_x(\cdot,t)|_2 \le C_3(1+t)^{-\frac{3+2\alpha}{4}}, \end{aligned} \tag{2.8}$$

for all $t \ge 0$. Indeed, if $0 \le \alpha < 1$, then

$$\lim_{t\to+\infty} t^{\frac{1}{2}+\alpha}|u(\cdot,t)|_2^2 = \lim_{t\to+\infty} t^{\frac{1}{2}+\alpha}|w(\cdot,t)|_2^2. \tag{2.9}$$

If $\alpha = 1$, however, one has

$$\lim_{t\to+\infty} t^{\frac{3}{2}}|u(\cdot,t) - w(\cdot,t)|_2^2 = \frac{1}{4\nu(8\nu\pi)^{\frac{1}{2}}}\Big(\int_0^\infty \int_{-\infty}^\infty P(u)dxdt\Big)^2. \tag{2.10}$$

If $\hat{f}(y) = iy\hat{g}(y)$ for some $g \in L_1(\mathbb{R})$, then

$$\lim_{t\to+\infty} t^{\frac{3}{2}}|u(\cdot,t)|_2^2 = \frac{1}{4\nu(8\nu\pi)^{\frac{1}{2}}}\Big(\int_{-\infty}^\infty g(x)dx - \int_0^\infty \int_{-\infty}^\infty P(u(x,t))dxdt\Big)^2. \tag{2.11}$$

Furthermore, if $xf(x) \in L_1(\mathbb{R})$ and $\frac{d}{dx}g(x) = f(x)$ with $xg(x) \to 0$ as $x \to \pm\infty$, it follows that

$$\lim_{t\to+\infty} t^{\frac{3}{2}}|u(\cdot,t)|_2^2 = \frac{1}{4\nu(8\nu\pi)^{\frac{1}{2}}}\Big(\int_{-\infty}^\infty xf(x)dx + \int_0^\infty \int_{-\infty}^\infty P(u(x,t))dxdt\Big)^2. \tag{2.12}$$

3 Properties of the linear BOB equation & global well-posedness. Some technical results are presented here connected with the linear semigroup corresponding to (1.1) without its nonlinear term and with the nonlinear well-posedness theory. All of these will find use later.

The linearized Benjamin-Ono-Burgers initial-value problem

$$w_t + w_x - \nu w_{xx} - \mathbb{H}w_{xx} = 0, \tag{3.1a}$$

$$w(x,0) = f(x), \tag{3.1b}$$

can be solved by formally taking the Fourier transform of equation (3.1a) with respect to the spatial variable x. One deduces at once that for $f \in H^1(\mathbb{R})$,

$$\hat{w}(y,t) = \exp\big(-\nu y^2 t - iyt + i|y|yt\big)\hat{w}(y,0), \tag{3.2}$$

and therefore that

$$w(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\big(-\nu y^2 t - iyt + i|y|yt + iyx\big)\hat{f}(y)dy. \tag{3.3}$$

The integral on the right-hand side of (3.3) will be denoted by $S(t)f(x)$. Here are some straightforward results about the decay of solutions of (3.1).

Lemma 3.1. *If $f \in H^1(\mathbb{R}) \cap L_1(\mathbb{R})$, then*

$$(a) \quad \lim_{t\to\infty} t^{\frac{1}{2}} \int_{-\infty}^{\infty} w^2(x,t)dx = \lim_{t\to\infty} t^{\frac{1}{2}} |S(t)f(x)|_2^2$$
$$= (8\nu\pi)^{-\frac{1}{2}} \Big(\int_{-\infty}^{\infty} f(x)dx\Big)^2 \quad \textit{and}$$
$$(b) \quad \lim_{t\to\infty} t^{\frac{3}{2}} \int_{-\infty}^{\infty} w_x^2(x,t)dx = (128\nu^3\pi)^{-\frac{1}{2}} \Big(\int_{-\infty}^{\infty} f(x)dx\Big)^2.$$

Proof. (a) By Parseval's theorem,

$$\int_{-\infty}^{\infty} w^2 dx = \int_{-\infty}^{\infty} |\hat{w}|^2 dy = \int_{-\infty}^{\infty} e^{-2\nu y^2 t} |\hat{f}(y)|^2 dy. \tag{3.4}$$

Let $\epsilon > 0$ be given and choose $\delta = \delta(\epsilon) \in (0,1)$ such that

$$\Big| |\hat{f}(y)|^2 - |\hat{f}(0)|^2 \Big| < \epsilon, \tag{3.5}$$

for all $|y| < \delta$. (The continuity of $\hat{f}$ follows from the Riemann-Lebesgue Lemma.) The integral in (3.4) may be written in the form

$$|w(\cdot,t)|_2^2 = \int_{-\delta}^{\delta} e^{-2\nu y^2 t} |\hat{f}(0)|^2 dy + \int_{|y|>\delta} e^{-2\nu y^2 t} |\hat{f}(y)|^2 dy$$
$$+ \int_{-\delta}^{\delta} e^{-2\nu y^2 t} \big[|\hat{f}(y)|^2 - |\hat{f}(0)|^2\big] dy. \tag{3.6}$$

The second term on the right-hand side of (3.6) is bounded via

$$\int_{|y|>\delta} e^{-2\nu y^2 t} |\hat{f}(y)|^2 dy \le e^{-2\nu\delta^2 t} \int_{|y|>\delta} |\hat{f}(y)|^2 dy \le e^{-2\nu\delta^2 t} |f|_2^2. \tag{3.7}$$

The final term in (3.6) may be bounded by means of the inequality

$$\epsilon \int_{-\delta}^{\delta} e^{-2\nu y^2 t} dy < \frac{\epsilon}{\sqrt{2\nu t}} \int_{-\infty}^{\infty} e^{-y^2} dy = \epsilon \sqrt{\frac{\pi}{2\nu t}}, \tag{3.8}$$

because of (3.5). Using (3.7) and (3.8) in (3.6) yields

$$\begin{aligned}\lim_{t\to+\infty} t^{\frac{1}{2}}|w(\cdot,t)|_2^2 &= 0(\epsilon) + \lim_{t\to+\infty} t^{\frac{1}{2}}|\hat{f}(0)|^2 \int_{-\delta}^{\delta} e^{-2\nu y^2 t} dy \\ &= 0(\epsilon) + \sqrt{\frac{\pi}{2\nu}}|\hat{f}(0)|^2,\end{aligned}$$

as $\epsilon \to 0^+$. Upon letting ϵ tend to zero, result (a) follows.

(b) Parseval's theorem gives

$$|w_x^2(\cdot,t)|_2^2 = \int_{-\infty}^{\infty} y^2|\hat{w}(y,t)|^2 dy = \int_{-\infty}^{\infty} y^2 e^{-2\nu y^2 t}|\hat{f}(y)|^2 dy. \tag{3.9}$$

Applying the same argument as appeared in part (a), we obtain that

$$\begin{aligned}\lim_{t\to+\infty} t^{\frac{3}{2}}|w(\cdot,t)|_2^2 &= 0(\epsilon) + \lim_{t\to+\infty} t^{\frac{3}{2}}|\hat{f}(0)|^2 \int_{-\delta}^{\delta} y^2 e^{-2\nu y^2 t} dy \\ &= 0(\epsilon) + \sqrt{\frac{\pi}{32\nu^3}}|\hat{f}(0)|^2,\end{aligned}$$

as $\epsilon \to 0^+$. Upon letting ϵ tend to zero, result (b) follows. □

Lemma 3.2. *Let ϕ be defined by its Fourier transform $\hat{\phi}$ as*

$$\hat{\phi}(y,r) = \exp\Big(\big(\nu y^2 + iy - i|y|y\big)r\Big).$$

Then it follows that

$$\sup_{t>0} \int_{-\infty}^{\infty} |\phi(x,-t)|dx < \infty. \tag{3.10}$$

If $\hat{\psi}$ is given by

$$\hat{\psi}(y,r) = y\exp\Big(\big(\nu y^2 + iy - i|y|y\big)r\Big),$$

then

$$\sup_{t>0} t^{\frac{1}{2}} \int_{-\infty}^{\infty} |\psi(x,-t)|dx < \infty. \tag{3.11}$$

Proof. The proofs follow the line of argument exposed in proving a similar lemma for the linearized BBM-Burgers equation in [3] (see also [12]). The proofs of (3.10) and (3.11) are very similar, and so we content ourselves with a demonstration of the former. The estimation of $|\phi(x,-t)|_1$ is made by breaking the range of integration into pieces, *viz.*

$$\int_{-\infty}^{\infty} |\phi(x,-t)|dx = \int_{|x|\le 1} |\phi(x,-t)|dx + \int_{|x|\ge 1} |\phi(x,-t)|dx. \tag{3.12}$$

Since the range of integration appearing in the first term on the right-hand side of (3.12) is bounded, a time-independent bound on $|\phi(x,-t)|$ implies boundedness of this term. By its definition, it is seen that

$$\begin{aligned}|\phi(x,-t)| &= \Big|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ixy}\exp\big(-\nu y^2 t - iyt + i|y|yt\big)dy\Big| \\ &\le \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\nu y^2 t}dy \le Ct^{-\frac{1}{2}},\end{aligned}$$

as $t \to \infty$.

To control the second term on the right-hand side of (3.12), write $h(y,t)$ for $\hat{\phi}(y,-t)$. Integration by parts shows that

$$\sqrt{2\pi}\phi(x,-t) = -\int_{-\infty}^{\infty}\frac{e^{ixy}}{ix}\partial_y h(y,t)dy = -\int_{-\infty}^{\infty}\frac{e^{ixy}}{x^2}\partial_y^2 h(y,t)dy.$$

If $|y| \ge 1$, and $t \ge 1$, say, it is straightforward to check that

$$\int_{|y|\ge 1}\big|\partial_y^2 h(y,t)\big|dy \le C\int_1^{+\infty}(1+y^2t)te^{-\nu y^2 t}dy \le Cte^{-\nu t},$$

and hence that

$$\begin{aligned}|\phi(x,-t)| &\le \Big|\int_{|y|\ge 1}\frac{e^{ixy}}{x^2}\partial_y^2 h(y,t)dy\Big| + \Big|\int_{|y|\le 1}\frac{e^{ixy}}{x^2}\partial_y^2 h(y,t)dy\Big| \\ &\le C\frac{t}{x^2}e^{-\nu t} + \Big|\int_{-1}^{1}\frac{e^{ixy}}{x^2}\partial_y^2 h(y,t)dy\Big|.\end{aligned}$$

Note that since

$$\partial_y h(y,t) = \big[-2\nu yt - it + 2i\mathrm{sign}(y)yt\big]\exp\big(-\nu y^2 t - iyt + i|y|yt\big), \tag{3.13}$$

it follows that

$$|\partial_y h(\pm 1,t)| \le Cte^{-\nu t},$$

and therefore that

$$|\phi(x,-t)| \le C\frac{t}{x^2}e^{-\nu t} + \Big|\int_{-1}^{1}\frac{ie^{ixy}}{x}\partial_y h(y,t)dy\Big|. \tag{3.14}$$

Define H by

$$H(x,t) = \int_{-1}^{1}\frac{ie^{ixy}}{x}\partial_y h(y,t)dy,$$

and then write it as

$$H(x,t) = \frac{i}{x}\int_{-1}^{1} e^{iy(x-t)}\big[-2\nu yt - it + 2i\mathrm{sgn}(y)yt\big]\exp\big(-\nu y^2 t + i|y|yt\big)dy. \tag{3.15}$$

Integrating by parts twice leads to the estimate

$$|H(x,t)| \le \frac{Ct^{\frac{1}{2}}}{|x(x-t)|} + \frac{Ct^{\frac{3}{2}}}{|x(x-t)^2|}.$$

Hence, to prove the lemma, it suffices to show that

$$\int_{|x|\geq 1} |H(x,t)|dx \leq C.$$

Divide the range of integration into four pieces, namely $(-\infty,-1)$, $(-1,t-\sqrt{t})$, $(t+\sqrt{t},\infty)$ and $(t-\sqrt{t},t+\sqrt{t})$ where we assume that $t\geq 1$. The arguments for bounding the integral over the first three intervals are similar, and therefore only one is worked out in detail.

$$\begin{aligned}\int_{-\infty}^{-1} |H(x,t)|dx &\leq C\int_{-\infty}^{-1}\Big(\frac{t^{\frac{1}{2}}}{|x(x-t)|}+\frac{t^{\frac{3}{2}}}{|x(x-t)^2|}\Big)dx\\ &= Ct^{-\frac{1}{2}}\int_{1/t}^{\infty}\Big(\frac{1}{y(y+1)}+\frac{1}{y(y+1)^2}\Big)dy \leq C\frac{\ln t}{t^{\frac{1}{2}}}\leq C,\end{aligned}$$

for values of t away from 0. To estimate the integral over $(t-\sqrt{t},t+\sqrt{t})$, use (3.15) to ascertain that

$$|H(x,t)| \leq \frac{C}{|x|}\int_0^1 (1+t)\exp\big(-\frac{1}{2}\nu y^2 t\big)dy \leq C\frac{t^{\frac{1}{2}}}{|x|},$$

whence

$$\int_{t-\sqrt{t}}^{t+\sqrt{t}} |H(x,t)|dx \leq Ct^{\frac{1}{2}}\Big(\ln(t+\sqrt{t})-\ln(t-\sqrt{t})\Big)\leq C$$

for $t\geq 1$. The proof of the lemma is complete. □

Local well-posedness results for the initial-value problem (1.2)-(1.3) can be found in [2, 24, 26, 31, 47] for example. For the problem (1.1)-(1.2), the reader may consult [46, 47]. A global well-posedness result obtains if the growth of nonlinearity P is limited appropriately.

Proposition 3.3. *Suppose $P:\mathbb{R}\to\mathbb{R}$ is C^∞ with $P(0)=P'(0)=0$. Define Λ and Q by $\Lambda'(r)=P(r)$ with $\Lambda(0)=0$ and $Q(r)=P'(r)$ for $r\in\mathbb{R}$, as above. Let a Sobolev exponent $s\geq 1$ be given. The initial-value problem (1.1)-(1.2) is globally well posed for arbitrary-sized data in $H^s(\mathbb{R})$ if there exists a constant C for which*

$$(a)\ \limsup_{r\to+\infty}\frac{\Lambda^+(r)}{r^4}=0,\ \ (b)\ \limsup_{r\to+\infty}\frac{Q^+(r)}{r^2}\leq C, \tag{3.16a}$$

and for every $\epsilon>0$, there is a constant C_ϵ such that

$$(c)\ \limsup_{r\to+\infty}\frac{Q^-(r)}{e^{\epsilon r^2}}\leq C_\epsilon. \tag{3.16b}$$

If $Q(u)$ grows faster than quadratically at infinity, the conclusion that $||u(\cdot,t)||_1$ is globally bounded remains without presuming (a)-(b)-(c) provided that $||f||_1$ is small enough.

In all these situations, the solution lies in $C_b(0,+\infty;H^1(\mathbb{R}))$ and in $C^k(\delta,T;H^s(\mathbb{R}))$ for any positive δ and T such that $0<\delta<T$ and for all $k\geq 0$.

If, the data happens to lie in $\in W^{k,1}(\mathbb{R}) \cap H^s(\mathbb{R})$, *then for every* $T > 0$, *the associated solution* u *and all its temporal derivatives lie in* $C(0,T;W^{k,1}(\mathbb{R}))$.

Remark. The simple condition

$$\limsup_{r\to+\infty} \frac{Q(r)}{r^2} \leq C, \tag{3.17a}$$

implies (b) and (c) of (3.16). In particular, condition (b) together with (c) implies that for any $\epsilon > 0$, there is a constant C_ϵ such that

$$\limsup_{r\to+\infty} \frac{Q(r)}{e^{\epsilon r^2}} \leq C_\epsilon. \tag{3.17b}$$

Proof. Local well-posedness follows readily from nonlinear semi-group theory or the contraction-mapping principle (see e.g. [46]). For example, if we take the Fourier transform of (1.1) in the spatial variable x, view the nonlinear term as known, solve the resulting ordinary differential equation using Duhamel's formula and take the inverse Fourier transform, there appears the integral equation

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi(x-y,-t)f(y)dy + \frac{1}{\sqrt{2\pi}}\int_0^t\int_{-\infty}^{\infty} \psi(x-y,\tau-t)P(u)dyd\tau \\ &= f_0(x,t) + \mathbb{A}(u)(x,t) = \mathbb{B}(u)(x,t), \end{aligned} \tag{3.18}$$

where ϕ and ψ are defined in Lemma 3.2. It is straightforward to ascertain that $\mathbb{B}$ is a contraction mapping on any ball B_R of radius R about zero in the space $C(0,T;H^s(\mathbb{R}))$ or $C(0,T;W^{k,1}(\mathbb{R}))$, for $s \geq 1$ and $k \geq 0$, provided R is taken large enough to encompass the initial data and $T = T(R) > 0$ is taken small enough. Note that we only require P to be C^2 for this part of the argument. The unique fixed point of $\mathbb{B}$ is straightforwardly inferred to be a solution of (1.1) on the time interval (0,T). Uniqueness and continuous dependence of the solution on the initial data follow because of the way the solution is obtained.

With a satisfactory local existence theory in hand, global well-posedness will follow as soon as supporting *a priori* bounds are established. Since continuous dependence is already established, locally in time, the calculations pursued below in search of appropriate *a priori* bounds can be justified by regularizing the relevant initial data making the computations with the solutions corresponding to the regularized data, and then passing to the limit as the regularization is allowed to evanesce (see e.g. [11]).

Multiplying (1.1) by $2u$ and then integrating the result over $\mathbb{R}\times[0,t)$, one obtains the equation

$$|u(\cdot,t)|_2^2 + 2\nu\int_0^t |u_x(\cdot,\tau)|_2^2 d\tau = |f|_2^2, \tag{3.19}$$

by using the elementary properties of the Hilbert transform to conclude that

$$\int_{-\infty}^{\infty} u\mathbb{H}u_{xx}dx = -\int_{-\infty}^{\infty} u_x\mathbb{H}u_x dx = 0.$$

If (1.1) is multiplied by the combination $\mathbb{H}u_x - P(u)$ and the result integrated over $\mathbb{R}\times[0,t)$, then after integrations by parts, there appears the formula

$$\begin{aligned} &\int_{-\infty}^{\infty} u\mathbb{H}u_x dx + \nu\int_0^t\int_{-\infty}^{\infty} u_x\mathbb{H}u_{xx}dxd\tau \\ &= \int_{-\infty}^{\infty} \Lambda(u)dx + \nu\int_0^t\int_{-\infty}^{\infty} P'(u)u_x^2 dxd\tau + \int_{-\infty}^{\infty}\Big[f\mathbb{H}f_x - \Lambda(f)\Big]dx. \end{aligned} \tag{3.20}$$

From this relation and (3.19), it is straightforward to ascertain that if $\Lambda(u)$ and $P'(u) = Q(u)$ satisfy the growth conditions (3.16), then for any $\delta > 0$, there is a constant C depending on δ and $||f||_{\frac{1}{2}}$ such that

$$\begin{aligned}
&\int_{-\infty}^{\infty} u\mathbb{H}u_x dx + \nu \int_0^t \int_{-\infty}^{\infty} u_x \mathbb{H}u_{xx} dx d\tau \\
&\le C(||f||_{\frac{1}{2}}) + \int_{-\infty}^{\infty} \Lambda^+(u) dx + \nu \int_0^t \int_{-\infty}^{\infty} Q^+(u) u_x^2 dx d\tau \\
&\le C(||f||_{\frac{1}{2}}, \delta) + C\delta |u(\cdot,t)|_4^4 + C \int_0^t \int_{-\infty}^{\infty} u^2(x,\tau) u_x^2(x,\tau) dx d\tau.
\end{aligned} \tag{3.21}$$

Applying a standard Sobolev embedding theorem and an interpolation result, it is deduced that

$$\begin{aligned}
&|u(\cdot,t)|_4^4 + C \int_0^t \int_{-\infty}^{\infty} u^2(x,\tau) u_x^2(x,\tau) dx d\tau \\
&\le C||u(\cdot,t)||_{\frac{1}{4}}^4 + C \int_0^t |u(\cdot,\tau)|_4^2 |u_x(\cdot,\tau)|_4^2 d\tau \\
&\le C||u(\cdot,t)||_{\frac{1}{2}}^2 |u(\cdot,t)|_2^2 + C \int_0^t |u(\cdot,\tau)|_2 ||u(\cdot,\tau)||_{\frac{1}{2}} |u_x(\cdot,\tau)|_2 ||u_x(\cdot,t)||_{\frac{1}{2}} d\tau.
\end{aligned} \tag{3.22}$$

Using (3.22) in the inequality (3.21) leads to

$$\begin{aligned}
\int_{-\infty}^{\infty} u\mathbb{H}u_x dx + \nu \int_0^t \int_{-\infty}^{\infty} u_x \mathbb{H}u_{xx} dx d\tau &\le C(||f||_{\frac{1}{2}}, \delta) + \delta C_1 ||u(\cdot,t)||_{\frac{1}{2}}^2 \\
&+ \int_0^t \left[\frac{\nu}{2} \int_{-\infty}^{\infty} u_x \mathbb{H}u_{xx} dx + C||u(\cdot,\tau)||_{\frac{1}{2}}^2 |u_x(\cdot,\tau)|_2^2 \right] d\tau.
\end{aligned} \tag{3.23}$$

The value C_1 depends on Sobolev imbedding and interpolation constants and on $|u(\cdot,t)|_2$, and hence from (3.19), may be chosen independently of t. Thus δ depends only on $|f|_2$. If δ is chosen so that

$$\delta C_1 \le \frac{1}{2},$$

it follows from (3.19) and (3.23) that

$$\begin{aligned}
&||u(\cdot,t)||_{\frac{1}{2}}^2 + \frac{\nu}{2} \int_0^t \int_{-\infty}^{\infty} u_x \mathbb{H}u_{xx} dx d\tau \\
&\le C(||f||_{\frac{1}{2}}) + C \int_0^t ||u(\cdot,\tau)||_{\frac{1}{2}}^2 |u_x(\cdot,\tau)|_2^2 d\tau.
\end{aligned} \tag{3.24}$$

Applying Gronwall's Lemma and noticing that for all $t \ge 0$, $\int_0^t |u_x(\cdot,\tau)|_2^2 d\tau \le \frac{1}{2\nu}|f|_2^2$, one derives the inequality

$$\begin{aligned}
&||u(\cdot,t)||_{\frac{1}{2}}^2 + \nu \int_0^t \int_{-\infty}^{\infty} |y|^3 |\hat{u}^2(y,\tau)| dy d\tau \\
&\le C(||f||_{\frac{1}{2}}) e^{C \int_0^t |u_x(\cdot,\tau)|_2^2 d\tau} \le C(||f||_{\frac{1}{2}}),
\end{aligned} \tag{3.25}$$

valid for all $t \geq 0$.

Multiply (1.1) by $2u_{xx}$ and integrate the result over $\mathbb{R} \times [0, t)$. After simplifying, there appears

$$\begin{aligned} |u_x(\cdot,t)|_2^2 + 2\nu \int_0^t |u_{xx}(\cdot,\tau)|_2^2 d\tau &= |f'|_2^2 + 2\int_0^t \int_{-\infty}^{\infty} u_{xx} P(u)_x dx d\tau \\ &\leq ||f||_1^2 + \nu \int_0^t |u_{xx}(\cdot,\tau)|_2^2 d\tau + \frac{1}{\nu}\int_0^t |P'(u(\cdot,\tau))|_\infty^2 |u_x(\cdot,\tau)|_2^2 d\tau. \end{aligned} \tag{3.26}$$

At this point, it is helpful to recall the inequality

$$|g|_\infty \leq C_0 ||g||_{\frac{1}{2}} \Big(1 + \log\Big(1 + ||g||_{\frac{3}{2}}\Big)\Big)^{\frac{1}{2}}, \tag{3.27}$$

where C_0 is an absolute constant. This is a special case in one space dimension of the logarithmic Sobolev inequalities of Brezis, Gallouët and Wainger [18, 19] (and see also Ozawa [40]). Apply the inequality (3.27) to the solution u emanating from f and use (3.25) for a time-independent bound on $||u(\cdot,t)||_{\frac{1}{2}}$ to infer existence of a constant $C_2 = C_2(||f||_{\frac{1}{2}})$ such that for all t for which the solution exists,

$$|u(\cdot,t)|_\infty^2 \leq C_2\Big(1 + \log\big(1 + ||u(\cdot,t)||_{\frac{3}{2}}\big)\Big). \tag{3.28}$$

Hence, from (b) and (c) of (3.16) (see (3.17b)), for any $\epsilon > 0$, there is a constant C_ϵ such that

$$\begin{aligned} |P'(u(\cdot,t))|_\infty^2 &\leq C_\epsilon^2 e^{2\epsilon |u(\cdot,t)|_\infty^2} \\ &\leq C_\epsilon^2 e^{2\epsilon C_2\left(1+\log\left(1+||u(\cdot,t)||_{\frac{3}{2}}\right)\right)} \leq C_3 + C_4||u(\cdot,t)||_{\frac{3}{2}}^2, \end{aligned} \tag{3.29}$$

provided ϵ is chosen small enough that

$$\epsilon C_2 \leq 1.$$

As $C_2 = C_2(||f||_{\frac{1}{2}})$ is time-independent, ϵ may also be chosen to depend only on $||f||_{\frac{1}{2}}$. The constants C_3 and C_4 depend on the fixed value of ϵ satisfying $\epsilon C_2 \leq 1$, through their dependence on C_ϵ and on C_2, and so they may likewise be taken to depend only on $||f||_{\frac{1}{2}}$. With these preliminary ruminations in hand, inequality (3.26) may be extended thusly;

$$\begin{aligned} &|u_x(\cdot,t)|_2^2 + \nu\int_0^t |u_{xx}(\cdot,\tau)|_2^2 d\tau \\ &\leq ||f||_1^2 + \frac{C_3}{\nu}\int_0^t |u_x(\cdot,\tau)|_2^2 d\tau + \frac{C_4}{\nu}\int_0^t |u_x(\cdot,\tau)|_2^2 ||u(\cdot,\tau)||_{\frac{3}{2}}^2 d\tau \\ &\leq ||f||_1^2 + \frac{C_3 + C_4 \sup_{t\geq 0}|u(\cdot,t)|_2^2}{\nu}\int_0^t |u_x(\cdot,\tau)|_2^2 d\tau \\ &\qquad + \frac{C_4}{\nu}\int_0^t |u_x(\cdot,\tau)|_2^2 \int_{-\infty}^{\infty} |y|^3 |\hat{u}(y,\tau)|^2 dy d\tau \\ &\leq C(||f||_1,\nu) + \frac{C_4}{\nu}\int_0^t |u_x(\cdot,\tau)|_2^2 \int_{-\infty}^{\infty} |y|^3|\hat{u}(y,\tau)|^2 dy d\tau. \end{aligned} \tag{3.30}$$

Gronwall's Lemma comes to our aid once more, yielding the time-independent bound

$$
\begin{aligned}
&|u_x(\cdot,t)|_2^2 + \nu \int_0^t |u_{xx}(\cdot,\tau)|_2^2 d\tau \\
&\leq C(||f||_1,\nu) e^{\frac{C_4}{\nu}\int_0^t \int_{-\infty}^{\infty} |y|^3 |\hat{u}(y,\tau)|^2 dy d\tau} \leq C(||f||_1)
\end{aligned} \tag{3.31}
$$

on account of (3.25), where the ν-dependence is ignored in the last step. Hence, in the presence of the growth restrictions (3.16), the H^1-norm of solutions is bounded, independently of t, on account of (3.19) and (3.31).

Assume now that Q grows super-quadratically, so that (3.16b) and (3.16c) are no longer valid. It may then be presumed that there is a constant C_3 for which

$$
|Q(z)| \leq C_3 + z^2 E(z^2),
$$

where E is a continuous, monotone increasing function with $\lim_{z\to+\infty} E(z) = +\infty$ and $E(0) = 0$. In consequence, it is seen that

$$
\begin{aligned}
|P'(u(\cdot,t))|_\infty^2 &\leq 2C_3^2 + 2|u(\cdot,t)|_\infty^4 E\Big(|u(\cdot,t)|_\infty^2\Big)^2 \\
&\leq 2C_4 + 2|u(\cdot,t)|_2^2 |u_x(\cdot,t)|_2^2 E\Big(|u(\cdot,t)|_2 |u_x(\cdot,t)|_2\Big)^2 \\
&\leq 2C_4 + 2|f|_2^2 ||u_x||^2_{C(0,\,t;\,L_2(\mathbb{R}))} E\Big(|f|_2 ||u_x||_{C(0,\,t;\,L_2(\mathbb{R}))}\Big)^2,
\end{aligned} \tag{3.32}
$$

because of (3.19). By re-estimating (3.26) for $|u_x(\cdot,t)|_2$, using (3.19) again and (3.32), there obtains the inequality

$$
\begin{aligned}
&||u_x||^2_{C(0,\,t;\,L_2(\mathbb{R}))} + \nu \int_0^t |u_{xx}(\cdot,\tau)|_2^2 d\tau \\
&\leq ||f||_1^2 + \frac{1}{\nu} ||P'(u)||^2_{C(0,\,t;\,L_\infty(\mathbb{R}))} \int_0^t |u_x(\cdot,\tau)|_2^2 d\tau \\
&\leq ||f||_1^2 + \frac{C_4}{\nu^2}|f|_2^2 + \frac{|f|_2^4}{\nu^2} E\Big(|f|_2 ||u_x||_{C(0,\,t;\,L_2(\mathbb{R}))}\Big)^2 ||u_x||^2_{C(0,\,t;\,L_2(\mathbb{R}))}.
\end{aligned} \tag{3.33}
$$

If $Y(t)$ and δ are defined by

$$
Y(t) = ||u_x||_{C(0,\,t;\,L_2(\mathbb{R}))} \qquad \text{and} \qquad \delta = ||f||_1,
$$

then (3.33) implies that

$$
p(Y(t),\delta) \leq C_5 \delta^2, \tag{3.34}
$$

where

$$
p(Y,\delta) = Y^2\Big[1 - \frac{\delta^4}{\nu^2} E\big(\delta Y\big)^2\Big].
$$

For any $\delta > 0$, $p(Y,\delta)$ is positive for Y near 0 since $E(0) = 0$, and bounded above, say $p(Y,\delta) \leq C_\delta$, for all $Y \geq 0$. Moreover, C_δ is a decreasing function of $\delta > 0$. Hence, there is a $\delta_0 > 0$, such that

$$
\sup_{Y>0} p(Y,\delta_0) = C_{\delta_0} = C_5 \delta_0^2. \tag{3.35}
$$

For any $\delta < \delta_0$, there are constants $Y_-(\delta) < Y_+(\delta)$ such that (3.34) implies that either $Y \le Y_-$ or $Y \ge Y_+$. As $Y = Y(t)$ is a continuous function of t, it follows that if $Y(0) < Y_-$, then $Y(t) \le Y_-$ for all $t \ge 0$, which is to say $|u_x(\cdot,t)|_2$ is uniformly bounded.

Once the H^1-norm of u is known to be bounded as a function of time, it is straightforward to deduce bounds on L_2-norms of higher derivatives, provided the initial data also possesses this regularity. For example, differentiating (3.18) twice with respect to x leads to the equation

$$\begin{aligned} u_{xx}(x,t) = &\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x-y,-t) f''(y) dy \\ &+ \frac{1}{\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} \psi(x-y,\tau-t) \Big[P''(u) u_x^2 + P'(u) u_{xx} \Big] dy d\tau. \end{aligned} \tag{3.36}$$

Fix $T > 0$. Since u is known to be uniformly bounded in $H^1(\mathbb{R})$ on $[0,T]$, it follows that $P''(u)$ and $P'(u)$ are bounded. Consequently, (3.36) implies that

$$\begin{aligned} |u_{xx}(\cdot,t)|_2 &\le C_6 |f''|_2 + C_7 \int_0^t |u_{xx}(\cdot,s)|_2^{\frac{1}{2}} ds + C_8 \int_0^t |u_{xx}(\cdot,s)|_2 ds \\ &\le C_6 |f''|_2 + C_9 + C_{10} \int_0^t |u_{xx}(\cdot,s)|_2 ds, \end{aligned}$$

where use has been made of Young's inequality, elementary Sobolev inequalities and the fact that $||u(\cdot,t)||_1$, and hence $|u(\cdot,t)|_\infty$, are uniformly bounded on $[0,T]$. Gronwall's Lemma then allows the conclusion that $u \in C(0,T;H^2(\mathbb{R}))$, as advertised. Similar considerations establish that, for any finite value of T, u is bounded in $C(0,T;H^k(\mathbb{R}))$ for larger values of k, provided the initial data also lies in $H^k(\mathbb{R})$. A more elaborate argument as in that starting with (3.32) reveals the H^k-norm to be uniformly bounded in time if $||f||_k$ is small and P vanishes to a suitably high order at 0, but we do not know such a result for large data.

Attention is now turned to the provision of L_1-bounds. Let $T > 0$ and let u be a solution of (1.1) on $[0,T]$ with initial data in $H^1(\mathbb{R}) \cap L_1(\mathbb{R})$. The formal path from (1.1) to (3.18) can be straightforwardly justified for a solution in $C(0,T;H^1(\mathbb{R}))$. Elementary estimates using (3.18) imply that

$$|u(\cdot,t)|_1 \le |\phi(\cdot,-t)|_1 |f|_1 + \int_0^t C |\psi(\cdot,\tau-t)|_1 |u(\cdot,\tau)|_1 d\tau,$$

where C depends only on the L_∞-norm of u on $\mathbb{R} \times [0,T]$, say. This latter quantity is bounded on $[0,T]$ on account of the H^1-bound that obtains on $[0,T]$. Gronwall's Lemma then provides the desired *a priori* L_1-bound.

This argument can be used to derive *a priori* bounds in $W^{1,1}(\mathbb{R})$ and then in $W^{2,1}(\mathbb{R})$. The $W^{2,1}$-bounds imply bounds in $W^{1,\infty}(\mathbb{R})$ and these in turn allow us to infer $W^{3,1}$-bounds. A continuation of this bootstrap-type argument provides $W^{k,1}$-bounds for any k such that $f \in W^{k,1}(\mathbb{R})$.

Finally, consideration is given to the temporal regularity. From (3.18), it follows that at least in the distributional sense,

$$\begin{aligned} u_t(x,t) = &\partial_t f_0(x,t) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x-y,-t) P(f(y)) dy \\ &+ \frac{1}{\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} \psi(x-y,\tau-t) P'(u(y,\tau)) u_\tau(y,\tau) dy d\tau. \end{aligned} \tag{3.37}$$

Since f_0 is the convolution of the initial data f with a kernel that has the same smoothing properties in Sobolev spaces as does the kernel arising in the solution of the initial-value problem for the heat diffusion equation, it follows that for any $\delta > 0$,

$$f_0(x,t) \in C_b^k(\delta, +\infty; H^r(\mathbb{R}))$$

for all $k = 0, 1, 2, \cdots$ and $r \geq 0$. Of course, the bound in this space depends inversely upon δ.

Now, fix δ with $0 < \delta < T$. As u lies in $C_b(\delta, T; H^s(\mathbb{R}))$ where $s \geq 1$, so does $P(u)$. Moreover, one deduces from (3.37) that

$$||u_t(\cdot,t)||_{H^s(\mathbb{R})} \leq C\Big(||\partial_t f_0(\cdot,t)||_{H^s(\mathbb{R})} + C(P'(||f||_{H^s(\mathbb{R})})\Big)\int_0^t ||u_\tau(\cdot,\tau)||_{H^s(\mathbb{R})}d\tau.$$

Then Gronwall's Lemma implies that $u_t \in C_b(\delta, T; H^s(\mathbb{R}))$. Because of this latter fact, the right-hand side of (3.37) is differentiable with respect to t, so it is therefore concluded that the left-hand side is also, and furthermore, $u_{tt} \in C_b(\delta, T; H^s(\mathbb{R}))$. An induction finishes the argument. □

Remark 3.4. In fact, as will appear later, the L_1-norm of solutions of the equation (1.1) with weak nonlinearity is bounded uniformly in t, which is a key point for the decay results in view.

The following corollary will be useful presently.

Corollary 3.5. *Let $f \in H^1(\mathbb{R})$ and P satisfy the conditions in Proposition 3.3. Then $u_x,\ u_{xx},\ u_t \in L_2(\mathbb{R}\times\mathbb{R}^+)$ and $u \in C_b(\mathbb{R}^+; H^1)$.*

Proof. By (3.19) and (3.31) it follows that u_x and u_{xx} are in $L_2(\mathbb{R}\times\mathbb{R}^+)$, and that $u \in C_b(\mathbb{R}^+; H^1)$. The fact that $u_t \in L_2(\mathbb{R}\times\mathbb{R}^+)$ follows from the equation (1.1) and the results just mentioned. □

The next decay result is a direct corollary of Proposition 3.3.

Corollary 3.6. *Let u be the solution of (1.1) corresponding to initial data $f \in H^1(\mathbb{R})$. Then, it follows that*

$$|u_x(\cdot,t)|_2 \to 0, \qquad as \qquad t \to +\infty,$$

and

$$|u(\cdot,t)|_\infty \to 0, \qquad as \qquad t \to +\infty.$$

Proof. Define $U\colon \mathbb{R}^+ \to \mathbb{R}^+$ by $U(t) = |u_x(\cdot,t)|_2^2$. Corollary 3.5 implies that $U(t) \in L_1(\mathbb{R}^+)$. Moreover, as we saw in (3.26),

$$\frac{dU(t)}{dt} = -2\nu\int_{-\infty}^{\infty} u_{xx}^2 dx + 2\int_{-\infty}^{\infty} P'(u)u_x u_{xx} dx. \tag{3.38}$$

Since $|u(\cdot,t)|_\infty$ is uniformly bounded, the right-hand side of (3.38) also lies in $L_1(\mathbb{R}^+)$ on account of Corollary 3.5. In consequence, $U \in W^{1,1}(\mathbb{R}^+)$ and so U is continuous and $U(t) \to 0$ as $t \to +\infty$.

The second conclusion now follows because

$$|u(\cdot,t)|_\infty^2 \leq |u(\cdot,t)|_2 |u_x(\cdot,t)|_2 \leq ||f||_1 |u_x(\cdot,t)|_2,$$

and the right-hand side of this inequality clearly tends to zero as t becomes large. The corollary is then proved. □

4 Non-optimal Decay Results. In the remainder of the paper, attention is given to the rate at which solutions of (1.1) decay as $t \to +\infty$. Throughout, it will be assumed that the nonlinearity P satisfies the conditions for globalcon well-posedness spelled out in Section 3. It will also be presumed that the nonlinearity is weak near the origin. Precisely, we assume there is a $p \geq 2$ and a constant c such that

$$|P'(u)| \leq c|u|^p \tag{4.1}$$

for u near 0.

In fact, we usually just take it that $P'(u) = cu^p$. That results derived under the latter assumption will lead to the same results for nonlinearities only satisfying (4.1) requires a moments thought. In fact, according to Corollary 3.6, if u solves (1.1), then there is a $T > 0$ such that $u(\cdot, t)$ is near zero for $t \geq T$. Thus, the inequality (4.1) applies uniformly for $t \geq T$, The asymptotic analysis only uses (4.1) and not the specific form $P(z) = cz^p$.

The analysis in this section will yield decay rates, albeit non-optimal ones. In Section 5, these rates are improved to their optimal values.

Lemma 4.1. *If u is the solution of equation (1.1) with $P'(u) = cu^p$ for some $p \geq 2$ corresponding to initial data $f \in H^1(\mathbb{R})$, then*

$$\sup_{t \in \mathbb{R}} \{t|u_x(\cdot, t)|_2^2\} < \infty.$$

Proof. If (1.1) is multiplied by $2u_{xx}$ and then the result integrated over $\mathbb{R}$, there appears

$$\begin{aligned} \frac{d}{dt}|u_x(\cdot, t)|_2^2 + 2\nu|u_{xx}(\cdot, t)|_2^2 &= 2\int_{-\infty}^{\infty} cu^p u_x u_{xx} dx \\ &\leq \nu|u_{xx}(\cdot, t)|_2^2 + \frac{|c|^2}{\nu}|u^p(\cdot, t)u_x(\cdot, t)|_2^2 \\ &\leq \nu|u_{xx}(\cdot, t)|_2^2 + \frac{|c|^2}{\nu}|u(\cdot, t)|_2^p |u_x(\cdot, t)|_2^{p+2}, \end{aligned} \tag{4.2}$$

where the inequality $|u|_\infty^2 \leq |u|_2|u_x|_2$ has been used in the last step. The differential inequality (4.2) leads to the related differential inequality

$$\frac{d}{dt}\big(t^2|u_x(\cdot, t)|_2^2\big) \leq 2t\Big(|u_x(\cdot, t)|_2^2 - \frac{\nu t}{2}|u_{xx}(\cdot, t)|_2^2\Big) + C_1 t^2 |u_x(\cdot, t)|_2^4. \tag{4.3}$$

By using Parseval's theorem and (3.19) in Proposition 3.3, the first term on the right-hand side of (4.3) can be bounded above, independently of $t \geq T$ for some $T > 0$. In fact, (4.3) reveals that

$$\begin{aligned} &t\Big(|u_x(\cdot, t)|_2^2 - \frac{\nu t}{2}|u_{xx}(\cdot, t)|_2^2\Big) \leq t\int_{|y| \leq \sqrt{\frac{2}{\nu t}}} y^2 |\hat{u}(y, t)|^2 dy \\ &\leq t\Big(\sqrt{\frac{2}{\nu t}}\Big)^2 \int_{|y| \leq \sqrt{\frac{2}{\nu t}}} |\hat{u}(y, t)|^2 dy \leq C\int_{-\infty}^{\infty} |\hat{u}(y, t)|^2 dy \\ &= C|u(\cdot, t)|_2^2 \leq C(|f|_2^2) = C_0. \end{aligned} \tag{4.4}$$

By using (4.4), (4.3) reduces to

$$\frac{d}{dt}\big(t^2|u_x(\cdot,t)|_2^2\big) \le C_0 + C_1 t^2 |u_x(\cdot,t)|_2^4, \tag{4.5}$$

holding for $t \ge T$, for some constants $T > 0$, C_0 and C_1.

Let $Y(t)$ be the solution of the equation

$$\begin{aligned} \frac{dY(t)}{dt} &= C_0 + C_1 m(t) Y(t) \\ Y(T) &= C, \end{aligned} \tag{4.6}$$

where $m(t) = |u_x(\cdot,t)|_2^2$. Then $t^2|u_x(\cdot,t)|_2^2 \le Y(t)$ for all $t \ge T$. Of course, $Y(t)$ can be found exactly as

$$Y(t) = \exp\Big(C_1 \int_T^t m(\tau)d\tau\Big)\Big[C + C_0 \int_T^t \exp\big(-C_1 \int_T^s m(\tau)d\tau\big)ds\Big]. \tag{4.7}$$

Note that $m(t) = |u_x(\cdot,t)|_2^2 \in L_1(\mathbb{R}^+)$ by Lemma 4.1. It follows from (4.7) that

$$t^2|u_x(\cdot,t)|_2^2 \le C(1 + C_0 t),$$

whence

$$|u_x(\cdot,t)|_2^2 \le C_0 t^{-1},$$

for all $t \ge T$. Thus the lemma is proved. □

Corollary 4.2. *If u is the solution of (1.1) corresponding to initial data f in $H^2(\mathbb{R}) \cap L_1(\mathbb{R})$, then there is a constant $\bar{C}$ such that for any $\epsilon > 0$, there is a $T = T(\epsilon) > 0$ for which $|u(\cdot,t)|_\infty \le \bar{C}\epsilon t^{-\frac{1}{4}}$ for all $t \ge T$.*

Proof. The decay of the L_∞-norm of solutions of (1.1) follows from the last result in Lemma 4.1 since

$$|u(\cdot,t)|_\infty^2 \le |u_x(\cdot,t)|_2 |u(\cdot,t)|_2 \le Ct^{-\frac{1}{2}}|u(\cdot,t)|_2 \le Ct^{-\frac{1}{2}}. \tag{4.8}$$

It is now shown that the constant C in (4.8) can be chosen to be small for large values of T. Take the Fourier transform of (1.1) with respect to the spatial variable x and solve the resulting ordinary differential equation to reach the integral equation

$$\begin{aligned} \hat{u}(y,t) = &\exp\big(-\nu y^2 t - iyt + iy|y|t\big)\hat{f}(y) \\ &- \frac{ci}{p+1}\int_0^t y \exp\big((-\nu y^2 - iy + iy|y|)(t-\tau)\big)\widehat{u^{p+1}}(y,\tau)d\tau, \end{aligned} \tag{4.9}$$

from whence it follows that

$$|\hat{u}(y,t)|^2 \le 2e^{-2\nu y^2 t}|\hat{f}(y)|^2 + \frac{2c^2}{(p+1)^2}\Big(\int_0^t |y|e^{-\nu y^2(t-\tau)}\big|\widehat{u^{p+1}}(y,\tau)\big|d\tau\Big)^2. \tag{4.10}$$

First note that for any $\epsilon > 0$, we may choose T_0 large enough so that

$$\int_{T_0}^\infty |u_x(\cdot,t)|_2^2 dt \le \epsilon.$$

This is possible because $u_x \in L_2(\mathbb{R} \times \mathbb{R}^+)$. If $t > T_0$ is large enough, say $t \geq \frac{T_0^{4/3}}{\epsilon^{2/3}}$, then for any $\gamma > 0$, one obtains

$$
\begin{aligned}
&\int_{|y| \leq \sqrt{\frac{\gamma}{t}}} y^2 \Big(\int_0^{T_0} e^{-\nu y^2 (t-\tau)} \big| \widehat{u^{p+1}}(y,\tau) \big| d\tau \Big)^2 dy \\
&\leq \frac{1}{2\pi} \int_{|y| \leq \sqrt{\frac{\gamma}{t}}} y^2 dy \Big(\int_0^{T_0} \big| u^{p+1}(\cdot,\tau) \big|_1 d\tau \Big)^2 \\
&\leq \frac{1}{3\pi} \Big(\sqrt{\frac{\gamma}{t}} \Big)^3 \Big(\int_0^{T_0} |u(\cdot,\tau)|_\infty^{p-1} |u(\cdot,\tau)|_2^2 d\tau \Big)^2 \\
&\leq C t^{-\frac{3}{2}} T_0^2 \leq C\epsilon.
\end{aligned} \tag{4.11}
$$

Note also that

$$
\begin{aligned}
&\int_{|y| \leq \sqrt{\frac{\gamma}{t}}} \Big(\int_{T_0}^{t} y e^{-\nu y^2 (t-\tau)} \big| \widehat{u^{p+1}}(y,\tau) \big| d\tau \Big)^2 dy \\
&\leq \frac{(p+1)^2}{2\pi} \int_{|y| \leq \sqrt{\frac{\gamma}{t}}} dy \Big(\int_{T_0}^{t} \big| u^p u_x(\cdot,\tau) \big|_1 d\tau \Big)^2 \\
&\leq C \sqrt{\frac{\gamma}{t}} \Big(\int_{T_0}^{t} |u_x(\cdot,\tau)|_2^{\frac{3}{2}} d\tau \Big)^2 \\
&\leq C t^{-\frac{1}{2}} (t - T_0)^{\frac{1}{2}} \Big(\int_{T_0}^{t} |u_x(\cdot,\tau)|_2^2 d\tau \Big)^{\frac{3}{2}} \\
&\leq C \epsilon^{\frac{3}{2}},
\end{aligned} \tag{4.12}
$$

where we have used that for $p \geq 2$,

$$
\begin{aligned}
|y\widehat{u^{p+1}}(y,\tau)| &\leq \frac{p+1}{\sqrt{2\pi}} |u^p u_x(\cdot,\tau)|_1 \\
&\leq \frac{p+1}{\sqrt{2\pi}} |u(\cdot,\tau)|_\infty^{p-1} |u_x(\cdot,\tau)|_2 |u(\cdot,\tau)|_2 \leq C |u_x(\cdot,\tau)|_2^{\frac{3}{2}}.
\end{aligned}
$$

Finally, note that for large t, say $t > \frac{1}{\epsilon^2}$,

$$
\int_{|y| \leq \sqrt{\frac{\gamma}{t}}} e^{-2\nu y^2 t} |\hat{f}(y)|^2 dy \leq C_f t^{-\frac{1}{2}} \leq C\epsilon. \tag{4.13}
$$

Combining (4.11), (4.12) and (4.13), one deduces from (4.10) that for t large enough,

$$
t \int_{|y| \leq \sqrt{\frac{\gamma}{t}}} y^2 |\hat{u}(y,t)|^2 dy \leq C\epsilon. \tag{4.14}
$$

Using the new estimate (4.14) in (4.4), gives a new version of the differential inequality (4.5) with C_0 replaced by $C\epsilon$. It is concluded from the new version of equation (4.6) that if $t > T_0$ for T_0 large enough and fixed, then

$$
|u_x(\cdot,t)|_2^2 \leq \bar{C}\epsilon t^{-1}. \tag{4.15}
$$

As $\epsilon > 0$ was arbitrary, the corollary follows from (4.8). $\square$

5. Decay rates for the GBOB equation. With the help of the non-optimal results derived in Section 4 and the preliminary results in Section 3, we are now ready to prove the main decay result concerning the BOB equation (1.1) with a weak nonlinearity. To simplify the argument, it is still assumed that $P' = cu^p$ for some $p \geq 2$. With this assumption, the following decay result obtains for generic initial data (1.2).

Theorem 5.1. *If $f \in H^1(\mathbb{R}) \cap L_1(\mathbb{R})$, then the solution of (1.1) corresponding to initial data f satisfies*

$$|u(\cdot, t)|_2 \leq C(1+t)^{-\frac{1}{4}}, \tag{5.1}$$

for all $t \geq 0$, where C is independent of t.

To prove Theorem 5.1, more information is needed about solutions of (1.1). The following lemma is a connection between the result advertised in Theorem 5.1 and the L_1-norms of solutions of (1.1).

Lemma 5.2. *Let $f \in H^1(\mathbb{R}) \cap L_1(\mathbb{R})$ and let u be the solution of (1.1) corresponding to initial data f. Suppose that*

$$\sup_{0 \leq t < \infty} |u(\cdot, t)|_1 < \infty. \tag{5.2}$$

Then, it must be the case that

$$\sup_{0 \leq t < \infty} t^{\frac{1}{2}} |u(\cdot, t)|_2^2 < \infty.$$

Proof. Note first that

$$\big|\hat{u}(y,t)\big| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \big|u(x,t)\big| dx = \frac{1}{\sqrt{2\pi}} |u(\cdot, t)|_1. \tag{5.3}$$

Suppose that (5.2) holds. The use of (3.19), Parseval's theorem and then (5.3) shows that

$$\begin{aligned}
\frac{d}{dt}\Big(t|u(\cdot,t)|_2^2\Big) &= |u(\cdot,t)|_2^2 - 2\nu t|u_x(\cdot,t)|_2^2 \\
&= \int_{-\infty}^{\infty} |\hat{u}(y,t)|^2 dy - 2\nu t \int_{-\infty}^{\infty} y^2 |\hat{u}(y,t)|^2 dy \\
&\leq \int_{|y| \leq \frac{1}{\sqrt{2\nu t}}} |\hat{u}(y,t)|^2 dy \leq \frac{2}{\sqrt{2\nu t}} |\hat{u}(\cdot,t)|_\infty^2 \\
&\leq C_1 t^{-\frac{1}{2}} |u(\cdot,t)|_1^2 \leq C_2 t^{-\frac{1}{2}},
\end{aligned}$$

where C_2 depends on the constant provided by (5.2). Integrating the inequality with respect to t over $[0, t)$, gives the desired result. $\square$

Lemma 5.2 shows that the decay result stated in Theorem 5.1 follows from deriving the time-independent, L_1-bound (5.2) for the solution of (1.1). This can be achieved when the nonlinearity P is weak.

Proof (of Theorem 5.1). Proceeding as in the derivation of (3.18) leads to the following formula for the solution u of (1.1);

$$\begin{aligned} u(x,t) =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x-y,-t) f(y) dy \\ & - \frac{ci}{\sqrt{2\pi}(p+1)} \int_0^t \int_{-\infty}^{\infty} \psi(x-y,\tau-t) u^{p+1}(y,\tau) dy d\tau, \end{aligned} \tag{5.4}$$

where ϕ and ψ are defined in Lemma 3.2. The first term on the right-hand side of (5.4) is in $L_1(\mathbb{R})$. In fact, the use of (3.10) in Lemma 3.2 shows that

$$\int_{-\infty}^{\infty} |\phi(x,-t)| dx \int_{-\infty}^{\infty} |f(y)| dy \leq C(|f|_1). \tag{5.5}$$

Similarly, let

$$g(x,t) = \frac{ic}{(p+1)\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} \psi(x-y,\tau-t) u^{p+1}(y,\tau) dy d\tau \tag{5.6}$$

connote the second term on the right-hand side of (5.4). Then, one sees that

$$\int_{-\infty}^{\infty} |g(x,t)| dx \leq C \int_0^t |\psi(\cdot,\tau-t)|_1 \int_{-\infty}^{\infty} |u^{p+1}(y,\tau)| dy d\tau. \tag{5.7}$$

By Lemma 3.2, there is a constant C such that

$$|\psi(\cdot,-r)|_1 \leq C r^{-\frac{1}{2}},$$

for all $r > 0$. Note also that because of Corollary 4.2, there is a constant $\bar{C}$ for which, for any $\epsilon > 0$, there is a $T > 0$ such that for all $\tau \geq T$,

$$\int_{-\infty}^{\infty} |u^{p+1}(y,\tau)| dy \leq |u(\cdot,\tau)|_\infty^p |u(\cdot,\tau)|_1 \leq \bar{C}^2 \epsilon^2 \tau^{-\frac{1}{2}} |u(\cdot,\tau)|_1, \tag{5.8}$$

since $p \geq 2$. Hence, the left-hand side of (5.7) can be bounded above as follows;

$$\int_{-\infty}^{\infty} |g(x,t)| dx \leq C \int_0^T \frac{|u^{p+1}(\cdot,\tau)|_1}{\sqrt{t-\tau}} d\tau + \bar{C}\epsilon \int_T^t \frac{1}{\sqrt{t-\tau}} \frac{|u(\cdot,\tau)|_1}{\sqrt{\tau}} d\tau \tag{5.9}$$

for T large enough and all $t \geq T$. Note the convention in force here is that the second integral does not appear if $t \leq T$. The use of (5.5) and (5.9) gives

$$|u(\cdot,t)|_1 \leq C_T + \bar{C}\epsilon \int_T^t \frac{|u(\cdot,\tau)|_1}{\sqrt{(t-\tau)\tau}} d\tau. \tag{5.10}$$

If $\epsilon > 0$ is chosen to be small enough that

$$\bar{C}\epsilon \int_0^1 \frac{1}{\sqrt{(1-r)r}} dr \leq \frac{1}{2},$$

say, then (5.10) implies that

$$\sup_{t\geq 0} |u(\cdot,t)|_1 \leq C_T + \frac{1}{2} \sup_{t \geq 0} |u(\cdot,t)|_1,$$

and the theorem is thereby established. □

Corollary 5.4. *If $f \in H^1(\mathbb{R}) \cap L_1(\mathbb{R})$, then the solution of the initial-value problem for equation (1.1) with initial data f satisfies*

$$t^{\frac{3}{4}} |u_x(\cdot,t)|_2 \le C \quad \text{and} \quad t^{\frac{1}{2}} |u(\cdot,t)|_\infty \le C,$$

for all $t \ge 0$, where the constants are independent of t.

Proof. Using (4.2), the inequality

$$\begin{aligned} \frac{d}{dt}\big(t^{\frac{5}{2}} |u_x(\cdot,t)|_2^2\big) &\le \frac{5}{2} t^{\frac{3}{2}} \Big(|u_x(\cdot,t)|_2^2 - \frac{2\nu t}{5} |u_{xx}(\cdot,t)|_2^2 \Big) + C t^{\frac{5}{2}} |u^p(\cdot,t) u_x(\cdot,t)|_2^2 \\ &\le \frac{5}{2} t^{\frac{3}{2}} \int_{|y| \le \sqrt{\frac{5}{2\nu t}}} y^2 |\hat{u}(y,t)|^2 dy + C t^{\frac{5}{2}} |u(\cdot,t)|_\infty^{2p} |u_x(\cdot,t)|_2^2 \\ &\le \frac{5}{3} t^{\frac{3}{2}} \Big(\sqrt{\frac{5}{2\nu t}} \Big)^3 |\hat{u}(\cdot,t)|_\infty^2 + C t^{\frac{5}{2}} |u(\cdot,t)|_2^p |u_x(\cdot,t)|_2^{2+p} \\ &\le C\big(1 + t^{\frac{5}{2}} |u_x(\cdot,t)|_2^4\big), \end{aligned} \tag{5.11}$$

follows since $p \ge 2$ and

$$|\hat{u}(\cdot,t)|_\infty \le |u(\cdot,t)|_1 \le C, \;\; |u(\cdot,t)|_2 \le C \;\text{ and }\; |u_x(\cdot,t)|_2 \le C.$$

Applying the arguments around (4.6) and (4.7) in Lemma 4.2 to the present considerations, one determines that

$$t^{\frac{5}{2}} |u_x(\cdot,t)|_2^2 \le C(1+t),$$

and so

$$|u_x(\cdot,t)|_2^2 \le C(1+t)^{-\frac{3}{2}}, \tag{5.12}$$

for all $t \ge 0$.

To see the validity of the second result, note that

$$\begin{aligned} |u(\cdot,t)|_\infty^2 &\le |u(\cdot,t)|_2 |u_x(\cdot,t)|_2 \\ &\le C(1+t)^{-\frac{1}{4}} (1+t)^{-\frac{3}{4}} = C(1+t)^{-1}, \end{aligned}$$

where the decay estimates (5.1) and (5.12) have been used. It thus follows that

$$|u(\cdot,t)|_\infty \le C(1+t)^{-\frac{1}{2}},$$

and the corollary is proved. □

6 More results on decay rates for the GBOB equation. In this section, further decay results are obtained. It will be shown that if the initial datum f has the special property that its Fourier transform vanishes at the origin like $|y|^\alpha$ as $y \to 0$, then the decay rate of the corresponding solutions of equation (1.1) will increase by $\frac{\alpha}{2}$ over what can be expected of generic initial data. As in Section 4 and Section 5, we still require the nonlinearity $P(u)$ to vanish at high order for u near zero. More precisely, it is required that P' vanishes at least quadratically at the origin. We begin with further results about the linear BOB equation (3.1).

Lemma 6.1. *If $f \in H^r(\mathbb{R}) \cap L_1(\mathbb{R})$, where $r \geq 1$ and*

$$|\hat{f}(y)| \leq C|y|^\alpha, \tag{6.1}$$

for small values of y, where $\alpha \geq 0$ and C is a positive constant, then the solution w of equation (3.1) satisfies

$$\sup_{0 \leq t \leq \infty} t^{\alpha+i+\frac{1}{2}} \int_{-\infty}^{\infty} [\partial_x^i w(x,t)]^2 dx < \infty, \tag{6.2}$$

for $0 \leq i \leq r$. In particular, if

$$|\hat{f}(y)| = |y|^\alpha |\hat{g}(y)|, \tag{6.3}$$

for some $g \in L_1(\mathbb{R})$, then

$$\lim_{t\to+\infty} t^{\alpha+i+\frac{1}{2}} \int_{-\infty}^{\infty} [\partial_x^i w(x,t)]^2 dx = \frac{\Gamma\big(i+\alpha+\frac{1}{2}\big)}{(2\nu)^{\alpha+i+\frac{1}{2}}} |\hat{g}(0)|^2. \tag{6.4}$$

where Γ denotes the Gamma function. Specifically, if

$$\int_{-\infty}^{\infty} |x|^j |f(x)| dx < \infty, \quad \text{for} \quad 0 \leq j \leq k,$$

with

$$\int_{-\infty}^{\infty} x^j f(x) dx = 0, \tag{6.5}$$

for $0 \leq j \leq k-1$, then for $0 \leq i \leq r$,

$$\begin{aligned} &\lim_{t\to\infty} t^{k+i+\frac{1}{2}} \int_{-\infty}^{\infty} [\partial_x^i w(x,t)]^2 dx \\ &= \frac{1\cdot 3\cdot 5 \cdots (2(k+i)-1)}{(8\nu\pi)^{\frac{1}{2}} (4\nu)^{k+i}} \Big(\int_{-\infty}^{\infty} x^k f(x) dx \Big)^2. \end{aligned} \tag{6.6}$$

Proof. The proof of the lemma is similar to those provided in the context of the generalized KdV-Burgers equation (1.7) and the generalized regularized long wave-Burgers equation (1.8) which can be found in [13, Lemma 2.1] (see also [22]). Accordingly, the proof is omitted. □

Lemma 6.2. *Let $f \in H^1(\mathbb{R}) \cap L_1(\mathbb{R})$ and suppose that $|\hat{f}(y)| \leq C|y|^\alpha$ for small values of y, where $0 \leq \alpha \leq 1$ and C is a constant. Then for any fixed $\gamma > 0$, the solution u of equation (1.1) corresponding to the initial data f satisfies*

$$\int_{|y|\leq\sqrt{\frac{\gamma}{t}}} |\hat{u}(y,t)|^2 dy \leq \begin{cases} C_f t^{-(\alpha+\frac{1}{2})}, & \text{if } 0 \leq \alpha < 1, \\ \Big[C_f + C_N\big(\log(1+t)\big)^2\Big] t^{-\frac{3}{2}}, & \text{if } \alpha = 1, \end{cases} \tag{6.7}$$

for all $t \geq \gamma$, where both C_f and C_N are independent of t. The constant C_f depends only on the initial data f while C_N depends on p.

Proof. As before, take the Fourier transform of (1.1) with respect to the spatial variable x and solve the resulting ordinary differential equation to reach the integral equation

$$\begin{aligned}\hat{u}(y,t) = &\exp\big(-\nu y^2 t - iyt + iy|y|t\big)\hat{f}(y)\\ &-\frac{ci}{p+1}\int_0^t y\exp\big((-\nu y^2 - iy + iy|y|)(t-\tau)\big)\widehat{u^{p+1}}(y,\tau)d\tau,\end{aligned} \tag{6.8}$$

from which one deduces that

$$|\hat{u}(y,t)|^2 \le 2e^{-2\nu y^2 t}|\hat{f}(y)|^2 + \frac{2c^2}{(p+1)^2}\Big(\int_0^t |y|e^{-\nu y^2(t-\tau)}\big|\widehat{u^{p+1}}(y,\tau)\big|d\tau\Big)^2. \tag{6.9}$$

Note that for any $\gamma > 0$ fixed and $t > 0$ large, the inequality

$$\begin{aligned}\int_{|y|\le\sqrt{\frac{\gamma}{t}}} e^{-2\nu y^2 t}|\hat{f}(y)|^2 dy &\le C^2\int_{|y|\le\sqrt{\frac{\gamma}{t}}} |y|^{2\alpha}e^{-2\nu y^2 t}dy\\ &\le C^2\Big(\sqrt{\frac{\gamma}{t}}\Big)^{2\alpha}\int_{-\infty}^{+\infty} e^{-2\nu y^2 t}dy\\ &\le C_f t^{-\frac{1+2\alpha}{2}}\end{aligned} \tag{6.10}$$

holds because of the hypothesis (6.1) on the initial data. Note also that for any $t \ge \gamma$, it is clear that

$$\begin{aligned}&\int_{|y|\le\sqrt{\frac{\gamma}{t}}}\Big(\int_0^t |y|e^{-\nu y^2(t-\tau)}\big|\widehat{u^{p+1}}(y,\tau)\big|d\tau\Big)^2 dy\\ &\le C\int_{|y|\le\sqrt{\frac{\gamma}{t}}} y^2 dy\Big(\int_0^t |u^{p+1}(\cdot,\tau)|_1 d\tau\Big)^2\\ &\le \frac{2C}{3}\Big(\sqrt{\frac{\gamma}{t}}\Big)^3\Big(\int_0^t \frac{1}{(1+\tau)^{\frac{p}{2}}}d\tau\Big)^2 \le \begin{cases} C_N t^{-\frac{3}{2}}, & \text{if } p > 2,\\ C_N t^{-\frac{3}{2}}\big(\log(t+1)\big)^2, & \text{if } p = 2,\end{cases}\end{aligned} \tag{6.11}$$

where use has been made of the inequality

$$|\widehat{u^{p+1}}(y,\tau)| \le \frac{1}{\sqrt{2\pi}}|u^{p+1}(\cdot,\tau)|_1 \le \frac{1}{\sqrt{2\pi}}|u(\cdot,\tau)|_\infty^p|u(\cdot,\tau)|_1$$

and the fact, garnered from Corollary 5.4, that

$$|u(\cdot,\tau)|_\infty \le C(1+\tau)^{-\frac{1}{2}}.$$

Combining (6.9), (6.10) and (6.11), there obtains

$$\int_{|y|\le\sqrt{\frac{\gamma}{t}}} |\hat{u}(y,t)|^2 dy \le \begin{cases} C_f t^{-(\alpha+\frac{1}{2})} + C_N t^{-\frac{3}{2}}, & \text{if } 0 \le \alpha < 1,\\ C_f t^{-\frac{3}{2}} + C_N(\log(1+t))^2 t^{-\frac{3}{2}}, & \text{if } \alpha = 1,\end{cases} \tag{6.12}$$

where C_f and C_N are independent of t. The constant C_f depends only on the initial data f, or more precisely on the function $\hat{w}(y,t) = \exp\big(-\nu y^2 t - iyt + iy|y|t\big)\hat{f}(y)$. The lemma is proved . □

With Lemma 6.1 and Lemma 6.2 in hand, it will be shown that the decay rate of the L_2-norm of solutions of (1.1) increases by order $\frac{\alpha}{2}$ if the initial data f satisfies condition (6.1) with $0 \le \alpha \le 1$. First, we have following lemma, valid for the restricted range $0 \le \alpha < 1$.

Lemma 6.3. *Let $f \in H^1(\mathbb{R}) \cap L_1(\mathbb{R})$ be such that $|\hat{f}(y)| \le C|y|^\alpha$ for small values of y, where $0 \le \alpha < 1$ and C is a constant. Then there are constants $C = C_\alpha$ such that the solution u of equation (1.1) satisfies*

$$|u(\cdot,t)|_2 \le C_\alpha t^{-\frac{1+2\alpha}{4}} \quad \text{and} \quad |u_x(\cdot,t)|_2 \le C_\alpha t^{-\frac{3+2\alpha}{4}}, \tag{6.13}$$

for $t \ge T$, where T is suitably large and C_α depends only on f.

Proof. From (4.2), it follows that

$$\begin{aligned} &\frac{d}{dt}\Big(t^{2+\alpha}|u_x(\cdot,t)|_2^2\Big) \\ &\le t^{1+\alpha}\Big((2+\alpha)|u_x(\cdot,t)|_2^2 - \nu t|u_{xx}(\cdot,t)|_2^2\Big) + Ct^{2+\alpha}|u^p(\cdot,t)u_x(\cdot,t)|_2^2. \end{aligned} \tag{6.14}$$

Using Parseval's theorem, the first term on the right-hand side of (6.14) can be estimated as follows;

$$\begin{aligned} &t^{1+\alpha}\big((2+\alpha)|u_x(\cdot,t)|_2^2 - \nu t|u_{xx}(\cdot,t)|_2^2\big) \\ &\le t^{1+\alpha}\int_{|y|\le\sqrt{\frac{(2+\alpha)}{\nu t}}} y^2|\hat{u}(y,t)|^2 dy \\ &\le t^{1+\alpha}\Big(\sqrt{\frac{(2+\alpha)}{\nu t}}\Big)^2 \int_{|y|\le\sqrt{\frac{(2+\alpha)}{\nu t}}} |\hat{u}(y,t)|^2 dy \le C_\alpha t^{-\frac{1}{2}}, \end{aligned} \tag{6.15}$$

where Lemma 6.2 has been applied in the last step.

The second term on the right-hand side of (6.14) has the upper bound

$$t^{2+\alpha}|u^p(\cdot,t)u_x(\cdot,t)|_2^2 \le t^{2+\alpha}|u(\cdot,t)|_\infty^{2p}|u_x(\cdot,t)|_2^2 \le C_N t^{-(p-1-\alpha+\frac{1}{2})}, \tag{6.16}$$

because of the estimates for $|u(\cdot,t)|_\infty$ and $|u_x(\cdot,t)|$ in Corollary 5.4. Using (6.15) and (6.16), (6.14) reduces to

$$\frac{d}{dt}\Big(t^{2+\alpha}|u_x(\cdot,t)|_2^2\Big) \le C_\alpha t^{-\frac{1}{2}} + C_N t^{-(p-1-\alpha+\frac{1}{2})},$$

from which it follows immediately that

$$|u_x(\cdot,t)|_2^2 \le C_\alpha t^{-(\alpha+\frac{3}{2})}.$$

The use of equation (3.19) and Parseval's formula yields

$$\begin{aligned} \frac{d}{dt}\Big(t^{\alpha+1}|u(\cdot,t)|_2^2\Big) &= (1+\alpha)t^\alpha|u(\cdot,t)|_2^2 - 2\nu t^{1+\alpha}|u_x(\cdot,t)|_2^2 \\ &= t^\alpha\Big((1+\alpha)|\hat{u}(\cdot,t)|_2^2 - 2\nu t\int_{-\infty}^{\infty} y^2|\hat{u}(y,t)|^2 dy\Big) \\ &\le t^\alpha \int_{|y|<\sqrt{\frac{1+\alpha}{2\nu t}}} |\hat{u}(y,t)|^2 dy \le C_\alpha t^{-\frac{1}{2}}, \end{aligned} \tag{6.17}$$

where C_α is a constant obtained by an application of Lemma 6.2. Integrating this inequality with respect to t leads to

$$t^{1+\alpha}|u(\cdot,t)|_2^2 \le C + C_\alpha t^{\frac{1}{2}}. \tag{6.18}$$

From (6.18) one easily obtains the first result in (6.13). The lemma is proved. □

The previous results are now extended to the case $\alpha = 1$.

Lemma 6.4. *Let $f \in H^1(\mathbb{R}) \cap L_1(\mathbb{R})$ and suppose that*

$$|\hat{f}(y)| \leq C|y|^\alpha,$$

for small values of y, where $0 \leq \alpha \leq 1$ and C is a constant. Then the solution of (1.1) with initial data f satisfies

$$\begin{aligned} &|u(\cdot,t)|_2 \leq C_\alpha^1 (1+t)^{-\frac{1+2\alpha}{4}} \qquad \textit{and} \\ &|u_x(\cdot,t)|_2 \leq C_\alpha^2 (1+t)^{-\frac{3+2\alpha}{4}}, \end{aligned} \tag{6.19}$$

for all $t \geq 0$, where the constants C_α^1 and C_α^2 are independent of t and of the form

$$\begin{cases} C_f, & \textit{if } 0 \leq \alpha < 1, \\ C_f + C_N, & \textit{if } \alpha = 1. \end{cases} \tag{6.20}$$

Corollary 6.5. *If f satisfies the conditions in Lemma 6.4, then the corresponding solution of (1.1) satisfies*

$$|u(\cdot,t)|_\infty \leq C_\alpha (1+t)^{-\frac{1+\alpha}{2}}, \tag{6.21}$$

for all $t \geq 0$, where C_α is a constant which is independent of t.

Proof. The inequality (6.21) follows immediately from those in (6.19) because

$$|u(\cdot,t)|_\infty^2 \leq |u(\cdot,t)|_2 |u_x(\cdot,t)|_2 \leq C_\alpha (1+t)^{-(1+\alpha)}.$$

Hence the corollary is proved. □

Proof (of Lemma 6.4). If $\alpha = 0$ or $0 < \alpha < 1$ the lemma follows from Theorem 5.1 or Corollary 5.4 and Lemma 6.3, respectively. Suppose $\alpha = 1$, then (6.19) certainly holds for any $\alpha_0 \leq \frac{1}{2}$, and of course (6.21) holds for the same range of α_0. Hence the solution u of equation (1.1) with the initial data f satisfies

$$|u(\cdot,t)|_\infty \leq C(1+t)^{-\frac{3}{4}}, \tag{6.22}$$

for all $t \geq 0$. It follows that

$$|\widehat{u^{p+1}}(y,t)| \leq \frac{1}{\sqrt{2\pi}} |u(\cdot,t)|_\infty^p |u(\cdot,t)|_1 \leq C(1+t)^{-\frac{3p}{4}}. \tag{6.23}$$

Hence, for $t \geq \gamma$, where $\gamma > 0$ is fixed, one has

$$\begin{aligned} &\int_{|y| \leq \sqrt{\frac{\gamma}{t}}} \Big(\int_0^t |y| \exp\big(-\nu y^2 (t-\tau)\big) \big|\widehat{u^{p+1}}(y,\tau)\big| d\tau \Big)^2 dy \\ &\leq C \int_{|y| \leq \sqrt{\frac{\gamma}{t}}} y^2 dy \Big(\int_0^t |u^{p+1}(\cdot,\tau)|_1 d\tau \Big)^2 \\ &\leq \frac{2C}{3} \Big(\sqrt{\frac{\gamma}{t}} \Big)^3 \Big(\int_0^t \frac{d\tau}{(1+\tau)^{\frac{3p}{4}}} \Big)^2 \leq C_N t^{-\frac{3}{2}}. \end{aligned} \tag{6.24}$$

For $\alpha = 1$ and the same value of γ, it follows at once that

$$\int_{|y|\le\sqrt{\frac{\gamma}{t}}} e^{-2\nu y^2 t}|\hat{f}(y)|^2 dy \le \int_{|y|\le\sqrt{\frac{\gamma}{t}}} |y|^2 e^{-2\nu y^2 t} dy \le C_f t^{-\frac{3}{2}}. \tag{6.25}$$

Using (6.9), (6.24) and (6.25) leads to the conclusion

$$\int_{|y|\le\sqrt{\frac{\gamma}{t}}} |\hat{u}(y,t)|^2 dy \le C_f t^{-\frac{3}{2}} + C_N t^{-\frac{3}{2}} \le C_\alpha t^{-\frac{3}{2}}. \tag{6.26}$$

If one chooses $\gamma = \frac{3}{\nu}$, then upon applying (6.26), one obtains that

$$\begin{aligned} & t^2\Big(3|u_x(\cdot,t)|_2^2 - \nu t|u_{xx}(\cdot,t)|_2^2\Big) \\ & \le t^2 \int_{|y|\le\sqrt{\frac{3}{\nu t}}} y^2|\hat{u}(y,t)|^2 dy \\ & \le t^2\Big(\sqrt{\frac{3}{\nu t}}\Big)^2 \int_{|y|\le\sqrt{\frac{3}{\nu t}}} |\hat{u}(y,t)|^2 dy \le C_\alpha t^{-\frac{1}{2}}. \end{aligned} \tag{6.27}$$

From (6.22) and the inequality $|u_x(\cdot,t)|_2^2 \le C_\alpha t^{-2}$, it is straightforward to see

$$t^3|u^p(\cdot,t)u_x(\cdot,t)|_2^2 \le t^3|u(\cdot,t)|_\infty^{2p}|u_x(\cdot,t)|_2^2 \le C_N t^{-\frac{3p-2}{2}}. \tag{6.28}$$

By using (6.27) and (6.28), the differential inequality (6.14) in Lemma 6.3 for the new value $\alpha = 1$ may be seen to imply that

$$\frac{d}{dt}\Big(t^3|u_x(\cdot,t)|_2^2\Big) \le C_\alpha t^{-\frac{1}{2}} + C_N t^{-\frac{3p-2}{2}}, \tag{6.29}$$

whence

$$|u_x(\cdot,t)|_2^2 \le C_\alpha^2 t^{-\frac{5}{2}}. \tag{6.30}$$

Finally, following the line of argument leading to (6.17) and (6.18), but using (6.26) and (6.30) with the new value $\alpha = 1$, it is concluded at once that if $\alpha = 1$, then

$$t^2|u(\cdot,t)|_2^2 \le C + C_\alpha^1 t^{\frac{1}{2}}, \quad \text{or } |u(\cdot,t)|_2^2 \le C_\alpha^1 t^{-\frac{3}{2}},$$

for all t and suitable constants C and C_α^1. The lemma is proved. □

When $p \ge 2$, the decay behavior of solutions of equation (1.1) as $t \to \infty$ is exactly the same as the decay behavior of solutions of the corresponding linear BOB equation. To see this scattering result, let u be the solution of equation (1.1) and w be the solution of the linear equation (3.1) with the same initial data f. If $U = u - w$, then U satisfies the initial-value problem

$$U_t + U_x - \nu U_{xx} - \mathbb{H}U_{xx} + cu^p u_x = 0, \tag{6.31a}$$

$$U(x,0) = 0. \tag{6.31b}$$

Attention is turned to the decay of the difference between the solution of (1.1) and the solution of the corresponding linear equation (3.1a). The outcome of this study will provide a decay rate for U in various norms. As a corollary, it is shown that the leading order, long-time asymptotics of solutions of equation (1.1) is in various ways the same as that of the solutions of the linear BOB equation.

Corollary 6.6. *Let $f \in H^1(\mathbb{R}) \cap L_1(\mathbb{R})$ and $p \geq 2$. Suppose*

$$|\hat{f}(y)| \leq C|y|^\alpha$$

for small values of y, where $0 \leq \alpha \leq 1$ and C is a positive constant. Then the solution $U = u - w$ of (6.31a) and (6.31b) has the properties that for $l = 0, 1$,

$$|\partial_x^l (u-w)(\cdot,t)|_2^2 \leq \begin{cases} Ct^{-(\frac{3}{2}+l)}, & \text{if } p > 2; \text{ or } p = 2 \text{ and } \alpha \neq 0, \\ Ct^{-(\frac{3}{2}+l)}\big(\log(1+t)\big)^2, & \text{if } p = 2 \text{ and } \alpha = 0. \end{cases} \tag{6.32}$$

Proof. Note first that from (6.31a), $\hat{u} - \hat{w}$ has the representation

$$\hat{u} - \hat{w} = -\frac{ci}{p+1}\int_0^t y \exp\big((-\nu y^2 - iy + iy|y|)(t-\tau)\big)\widehat{u^{p+1}}(y,\tau)d\tau. \tag{6.33}$$

Note also that by Lemma 6.4, one has

$$|u(\cdot,t)|_2^2 \leq Ct^{-(\frac{1}{2}+\alpha)} \quad \text{and} \quad |u_x(\cdot,t)|_2^2 \leq Ct^{-(\frac{3}{2}+\alpha)}. \tag{6.34}$$

Straightforward interpolation then implies

$$|u(\cdot,t)|_\infty \leq Ct^{-\frac{1+\alpha}{2}}. \tag{6.35}$$

By using (6.33), (6.34) and (6.35), one demonstrates that

$$\begin{aligned} \int_{|y|\leq\sqrt{\frac{\gamma}{t}}} |\hat{u}-\hat{w}|^2 dy &\leq C\int_{|y|\leq\sqrt{\frac{\gamma}{t}}} y^2\Big(\int_0^t e^{-\nu y^2(t-\tau)}\widehat{u^{p+1}}(y,\tau)d\tau\Big)^2 dy \\ &\leq C\int_{|y|\leq\sqrt{\frac{\gamma}{t}}} y^2 dy\Big(\int_0^t |u(\cdot,\tau)|_\infty^{p-1}|u(\cdot,\tau)|_2^2 d\tau\Big)^2 \\ &\leq Ct^{-\frac{3}{2}}\Big(\int_0^t (1+\tau)^{-(\frac{1}{2}+\alpha+\frac{(1+\alpha)(p-1)}{2})}d\tau\Big)^2 \\ &\leq \begin{cases} Ct^{-\frac{3}{2}}, & \text{if } \frac{1}{2}+\alpha+\frac{(1+\alpha)(p-1)}{2} > 1, \\ Ct^{-\frac{3}{2}}\big(\log(1+t)\big)^2, & \text{if } \frac{1}{2}+\alpha+\frac{(1+\alpha)(p-1)}{2} = 1, \end{cases} \\ &= \begin{cases} Ct^{-\frac{3}{2}}, & \text{if } p > 2; \text{ or } p = 2 \text{ and } \alpha \neq 0, \\ Ct^{-\frac{3}{2}}\big(\log(1+t)\big)^2, & \text{if } p = 2 \text{ and } \alpha = 0, \end{cases} \\ &= Ct^{-(\frac{1}{2}+\alpha+\delta)}, \end{aligned} \tag{6.36}$$

where

$$\delta = \begin{cases} 1-\alpha, & \text{if } p > 2; \text{ or } p = 2 \text{ and } \alpha \neq 0, \\ 1^-, & \text{if } p = 2 \text{ and } \alpha = 0, \end{cases}$$

and $\delta = 1^-$ is defined in such a way that $t^{-(\frac{1}{2}+\delta)} = t^{-\frac{3}{2}}\big(\log(1+t)\big)^2$.

The differential inequality

$$\frac{d}{dt}|(u-w)(\cdot,t)|_2^2 + \nu|\partial_x(u-w)(\cdot,t)|_2^2 \leq \frac{c^2}{\nu(p+1)^2}|u^{p+1}(\cdot,t)|_2^2, \tag{6.37}$$

is easily derived from (6.31a). Inequality (6.37) is equivalent to

$$\begin{aligned}\frac{d}{dt}\Big(t^{1+\alpha+\delta}|(u-w)(\cdot,t)|_2^2\Big) &\le \frac{c^2t^{1+\alpha+\delta}}{\nu(p+1)^2}|u^{p+1}(\cdot,t)|_2^2\\ &\quad+t^{\alpha+\delta}\big[(1+\alpha+\delta)|(u-w)(\cdot,t)|_2^2-\nu t|\partial_x(u-w)(\cdot,t)|_2^2\big],\end{aligned}\tag{6.38}$$

and the right-hand side of (6.38) is bounded above by

$$\begin{aligned}&C_1t^{\alpha+\delta}\int_{|y|\le\sqrt{\frac{1+\alpha+\delta}{t\nu}}}|(\hat{u}-\hat{w})(y,t)|^2dy+C_2t^{1+\alpha+\delta}|u(\cdot,t)|_2^2|u(\cdot,t)|_\infty^{2p}\\ &\le C_1t^{-\frac{1}{2}}+C_2t^{-(\frac{1}{2}+p-1+p\alpha-\delta)}.\end{aligned}$$

The estimate (6.34) and (6.36) have been used in the derivation of inequality (6.38). Note that $p-1+p\alpha-\delta\ge 0$ by the definition of δ. It follows immediately from (6.38) that

$$t^{\frac{1}{2}+\alpha+\delta}|(u-w)(\cdot,t)|_2^2\le C.$$

Again, direct appeal to (6.31a) leads to

$$\frac{d}{dt}|\partial_x(u-w)(\cdot,t)|_2^2+\nu|\partial_x^2(u-w)(\cdot,t)|_2^2\le\frac{c^2}{\nu}|u^pu_x(\cdot,t)|_2^2,\tag{6.39}$$

which is equivalent to

$$\begin{aligned}\frac{d}{dt}\Big(t^{2+\alpha+\delta}|\partial_x(u-w)(\cdot,t)|_2^2\Big) &\le \frac{t^{2+\alpha+\delta}c^2}{\nu}|u^pu_x(\cdot,t)|_2^2\\ &\quad+t^{1+\alpha+\delta}\big[(2+\alpha+\delta)|\partial_x(u-w)(\cdot,t)|_2^2-\nu t|\partial_x^2(u-w)(\cdot,t)|_2^2\big].\end{aligned}\tag{6.40}$$

By using (6.36), the first two terms on the right-hand side of (6.40) can be bounded above by

$$\begin{aligned}&t^{1+\alpha+\delta}\big[(2+\alpha+\delta)|\partial_x(u-w)(\cdot,t)|_2^2-\nu t|\partial_x^2(u-w)(\cdot,t)|_2^2\big]\\ &\qquad\le C_1t^{1+\alpha+\delta}\int_{|y|\le\sqrt{\frac{2+\alpha+\delta}{t\nu}}}y^2|(\hat{u}-\hat{w})(y,t)|^2dy\le C_1t^{-\frac{1}{2}}.\end{aligned}\tag{6.41}$$

Then, using (6.34) and (6.35), the last term on the right-hand side of (6.40) can be bounded as follows;

$$\begin{aligned}\frac{t^{2+\alpha+\delta}}{\nu}|u^pu_x(\cdot,t)|_2^2 &\le C_2t^{2+\alpha+\delta}|u_x(\cdot,t)|_2^2|u(\cdot,t)|_\infty^{2p}\\ &\le C_2t^{-(\frac{1}{2}+p-1+p\alpha-\delta)}.\end{aligned}\tag{6.42}$$

Applying (6.41) and (6.42) reduces (6.40) to the simple inequality

$$\frac{d}{dt}\Big(t^{2+\alpha+\delta}|\partial_x(u-w)(\cdot,t)|_2^2\Big)\le C_1t^{-\frac{1}{2}}+C_2t^{-(\frac{1}{2}+p-1+p\alpha-\delta)}.\tag{6.43}$$

Since $p-1+p\alpha-\delta\ge 0$, (6.43) yields

$$t^{\frac{3}{2}+\alpha+\delta}|\partial_x(u-w)(\cdot,t)|_2^2\le C,\tag{6.44}$$

and the corollary is established. □

Corollary 6.7. *Let f satisfy the conditions specified in Corollary 6.6. Then the solution u of equation (1.1) and the solution w of equation (3.1) corresponding to the initial data f have the properties*

$$\lim_{t\to+\infty} t^{\frac{1}{2}+l+\alpha}\Big||\partial_x^l u(\cdot,t)|_2^2-|\partial_x^l w(\cdot,t)|_2^2\Big| =\begin{cases} 0, & \text{if } 0\le\alpha<1,\\ C_N^l, & \text{if } \alpha=1,\end{cases} \tag{6.45}$$

where C_N^l is a constant and $l=0,1$.

Proof. By the triangle inequality,

$$\big||\partial_x^l u(\cdot,t)|_2^2-|\partial_x^l w(\cdot,t)|_2^2\big| \le |\partial_x^l(u-w)(\cdot,t)|_2\Big[|\partial_x^l u(\cdot,t)|_2+|\partial_x^l w(\cdot,t)|_2\Big]. \tag{6.46}$$

From Lemma 6.4, when $p\ge 2$, one has

$$\lim_{t\to+\infty} t^{\frac{1+2(l+\alpha)}{4}}|\partial_x^l u(\cdot,t)|_2 = C \tag{6.47}$$

where C is a positive constant, and the same result is easily seen to hold for w. It follows from (4.46) that

$$\begin{aligned}&\lim_{t\to+\infty} t^{\frac{1}{2}+l+\alpha}\Big||\partial_x^l u(\cdot,t)|_2^2-|\partial_x^l w(\cdot,t)|_2^2\Big|\\ &\le \lim_{t\to+\infty} t^{\frac{1+2(l+\alpha)}{4}}|\partial_x^l(u-w)(\cdot,t)|_2 \lim_{t\to+\infty}\Big(t^{\frac{1+2(l+\alpha)}{4}}\big[|\partial_x^l u(\cdot,t)|_2+|\partial_x^l w(\cdot,t)|_2\big]\Big).\end{aligned} \tag{6.48}$$

Corollary 6.6 implies the first limit on the right-hand side of (6.48) to be 0 when $0\le\alpha<1$, whereas it is a positive constant if $\alpha=1$. The result follows. ☐

Remark 6.8. Corollary 6.7 shows that the asymptotic behavior of solutions of equation (1.1) is exactly the same as that of solutions of the linear equation (3.1) when equation (1.1) features higher-order nonlinearity.

When $\alpha=1$, the L_2-norm of the solution u of equation (1.1) and the solution w of equation (3.1) both decay like $t^{-\frac{3}{4}}$, but it turns out their asymptotic states $\lim_{t\to+\infty} t^{\frac{3}{4}}|\cdot|_2$ are different. This was also noticed for the GKdV-Burgers equation (1.7) and GRLW-Burgers equation (1.8) in [13]. Next, we compute the limits of $u-w$ and u in L_2-norm when $\alpha=1$. The results imply that further decay of solutions of equation (1.1) depends on the nonlinear term.

Corollary 6.9. *Let f satisfy the conditions in Corollary 6.6 and suppose $\alpha=1$. Then the difference between the solution u of equation (1.1) with $P'=cu^p$ for $p\ge 2$ and the solution w of equation (3.1), both with initial value f, has the property*

$$\lim_{t\to+\infty} t^{\frac{3}{2}}|u(\cdot,t)-w(\cdot,t)|_2^2 = \frac{c^2}{4\nu(8\nu\pi)^{\frac{1}{2}}(p+1)^2}\Big(\int_0^\infty\int_{-\infty}^\infty u^{p+1}dxdt\Big)^2.$$

If $\hat f(y)=iy\hat g(y)$ for some $g\in L_1(\mathbb{R})$, then

$$\lim_{t\to+\infty} t^{\frac{3}{2}}|u(\cdot,t)|_2^2 = \frac{1}{4\nu(8\nu\pi)^{\frac{1}{2}}}\Big(\int_{-\infty}^\infty g(x)dx-\int_0^\infty\int_{-\infty}^\infty \frac{cu^{p+1}(x,t)}{p+1}dxdt\Big)^2.$$

In particular, if $xf(x) \in L_1(\mathbb{R})$ *and* $\frac{d}{dx}g(x) = f(x)$ *with* $xg(x) \to 0$ *as* $x \to \pm\infty$, *one has*

$$\lim_{t\to+\infty} t^{\frac{3}{2}}|u(\cdot,t)|_2^2 = \frac{1}{4\nu(8\nu\pi)^{\frac{1}{2}}}\Big(\int_{-\infty}^{\infty} xf(x)dx + \int_0^\infty \int_{-\infty}^{\infty} \frac{cu^{p+1}(x,t)}{p+1}dxdt\Big)^2.$$

Proof. First, write the equations (1.1) and (3.1a) in a form that is convenient for the analysis in view. Let u be the solution of equation (1.1) and w be the solution of equation (3.1) with the same initial data f. If $U(x,t) = u(x+t,t)$ and $W(x,t) = w(x+t,t)$, then $V = U - W$ satisfies the initial-value problem

$$V_t - \nu V_{xx} - \mathbb{H}V_{xx} + cU^pU_x = 0, \tag{6.49a}$$

$$V(x,0) = 0. \tag{6.49b}$$

Note that for any $j = 0, 1, \cdots$, the relevant norms

$$|\partial_x^j U(\cdot,t)|_2 = |\partial_x^j u(\cdot,t)|_2 \quad \text{and} \quad |\partial_x^j W(\cdot,t)|_2 = |\partial_x^j w(\cdot,t)|_2 \tag{6.50}$$

are finite. Hence U and u, and W and w have the same L_2-norm decay rates. Note also that

$$|V(\cdot,t)|_2 = |(U-W)(\cdot,t)|_2 = |(u-w)(\cdot,t)|_2.$$

Thus, the asymptotic behavior of $u - w$ in L_2-norm is exactly the asymptotic behavior of V in L_2-norm.

Take the Fourier transform of equation (6.49a) with respect to the spatial variable x and solve the resulting ordinary differential equation to reach the integral equation

$$\hat{U}(y,t) - \hat{W}(y,t) = -\frac{ci}{p+1}\int_0^t y\exp\big((-\nu y^2 + iy|y|)(t-\tau)\big)\widehat{U^{p+1}}(y,\tau)d\tau. \tag{6.51}$$

Since $\alpha = 1$, Lemma 6.4 asserts that

$$|u(\cdot,t)|_2 \le C(1+t)^{-\frac{3}{4}}, \quad \text{and} \quad |u_x(\cdot,t)|_2 \le C(1+t)^{-\frac{5}{4}},$$

for $t \ge 0$. Hence, one has

$$|U^{p+1}(\cdot,t)|_1 \le |u(\cdot,t)|_\infty^{p-1}|u(\cdot,t)|_2^2 \le C(1+t)^{-\frac{3}{2}}. \tag{6.52}$$

In consequence, $\int_0^{+\infty}\int_{-\infty}^{\infty} u^{p+1}(x,\tau)dxd\tau$ is a finite number. For $\epsilon > 0$ small and $t \ge 1$, say,

$$\begin{aligned}\Big|\int_{t^{1-\epsilon}}^t e^{-(\nu y^2 - iy|y|)(t-\tau)}\widehat{U^pU_x}(y,\tau)d\tau\Big| &\le \frac{1}{\sqrt{2\pi}}\int_{t^{1-\epsilon}}^t |U^pU_x(\cdot,\tau)|_1 d\tau \\ &\le C\int_{t^{1-\epsilon}}^t (1+\tau)^{-2}d\tau \le C(1+t^{1-\epsilon})^{-1} \le Ct^{\epsilon-1},\end{aligned} \tag{6.53}$$

where the inequality

$$\begin{aligned}|U^pU_x(\cdot,\tau)|_1 &\le |u(\cdot,\tau)|_\infty^{p-1}|u(\cdot,\tau)|_2|u_x(\cdot,\tau)|_2 \\ &\le C(1+\tau)^{-2}\end{aligned} \tag{6.54}$$

has been used in the second step. Remark also the elementary relation

$$\left|\exp\big((-\nu y^2+iy|y|)(t-\tau)\big)\right|_1 = \frac{C}{\sqrt{t-\tau}}. \tag{6.55}$$

With the preceding information in hand, one determines that

$$\begin{aligned}
&\lim_{t\to+\infty} t^{\frac{3}{2}}\Big|\int_{t^{1-\epsilon}}^{t} e^{(-\nu y^2+iy|y|)(t-\tau)}\widehat{U^pU_x}(y,\tau)d\tau\Big|_2^2\\
&\le \lim_{t\to+\infty} Ct^{\frac{3}{2}}t^{\epsilon-1}\Big|\int_{t^{1-\epsilon}}^{t} e^{(-\nu y^2+iy|y|)(t-\tau)}\widehat{U^pU_x}(y,\tau)d\tau\Big|_1\\
&\le \lim_{t\to+\infty} Ct^{\frac{1}{2}+\epsilon}\int_{t^{1-\epsilon}}^{t}\Big|e^{(-\nu y^2+iy|y|)(t-\tau)}\Big|_1|U^pU_x(\cdot,\tau)|_1d\tau\\
&\le \lim_{t\to+\infty} Ct^{\frac{1}{2}+\epsilon}\int_{t^{1-\epsilon}}^{t}\frac{1}{\sqrt{t-\tau}(1+\tau)^2}d\tau\\
&\le \lim_{t\to+\infty} \frac{Ct^{\frac{1}{2}+\epsilon}}{(1+t^{1-\epsilon})^{\frac{3}{2}}}\int_{t^{1-\epsilon}}^{t}\frac{1}{\sqrt{(t-\tau)(1+\tau)}}d\tau\\
&\le \lim_{t\to+\infty} Ct^{-1+\frac{5\epsilon}{2}}\int_0^1\frac{1}{\sqrt{(1-\tau)\tau}}d\tau = 0,
\end{aligned} \tag{6.56}$$

if, say, $0<\epsilon<\frac{2}{5}$, where the estimate (6.53) has been used at the first step, while (6.54) and (6.55) have been used at the third step. Henceforth, the positive parameter ϵ is fixed in the range $(0,\frac{2}{5})$.

For similar small positive values of ϵ, one may also compute that

$$\begin{aligned}
&\lim_{t\to+\infty} t^{\frac{3}{2}}\Big|\int_0^{t^{1-\epsilon}}\exp\big((-\nu y^2+iy|y|)(t-\tau)\big)\widehat{U^pU_x}(y,\tau)d\tau\Big|_2^2\\
&=\lim_{t\to+\infty} t^{\frac{3}{2}}\Big|\frac{i}{p+1}\int_0^{t^{1-\epsilon}} y\exp\big((-\nu y^2+iy|y|)(t-\tau)\big)\widehat{U^{p+1}}(y,\tau)d\tau\Big|_2^2\\
&=\lim_{t\to+\infty} t^{\frac{3}{2}}\int_{-\infty}^{\infty}\frac{y^2}{(p+1)^2}\Big|\int_0^{t^{1-\epsilon}} e^{(-\nu y^2+iy|y|)(t-\tau)}\widehat{U^{p+1}}(y,\tau)d\tau\Big|^2dy\\
&=\lim_{t\to+\infty}\int_{-\infty}^{\infty}\frac{s^2e^{-2\nu s^2}}{(p+1)^2}\Big|\int_0^{t^{1-\epsilon}} e^{(\nu s^2-is|s|)\frac{\tau}{t}}\widehat{U^{p+1}}(\frac{s}{\sqrt{t}},\tau)d\tau\Big|^2ds\\
&=\frac{1}{(p+1)^2}\int_{-\infty}^{\infty}s^2e^{-2\nu s^2}\Big|\int_0^{\infty}\widehat{U^{p+1}}(0,\tau)d\tau\Big|^2ds\\
&=\frac{1}{(p+1)^2}\int_{-\infty}^{\infty}s^2e^{-2\nu s^2}ds\Big(\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\int_{-\infty}^{\infty}U^{p+1}(x,\tau)dxd\tau\Big)^2\\
&=\frac{1}{2\pi(p+1)^2(2\nu)^{\frac{3}{2}}}\int_0^{\infty}s^{\frac{1}{2}}e^{-s}ds\Big(\int_0^{\infty}\int_{-\infty}^{\infty}u^{p+1}(x,\tau)dxd\tau\Big)^2\\
&=\frac{1}{4\nu(8\nu\pi)^{\frac{1}{2}}(p+1)^2}\Big(\int_0^{\infty}\int_{-\infty}^{\infty}u^{p+1}(x,\tau)dxd\tau\Big)^2,
\end{aligned} \tag{6.57}$$

because

$$\exp\Big((\nu s^2-is|s|)\frac{\tau}{t}\Big)\to 1 \tag{6.58}$$

as $t \to +\infty$, for any fixed s and $\tau \in [0, t^{1-\epsilon}]$. Note that the substitution $s = y\sqrt{t}$ has been used at the third step in (6.57) and that

$$\int_{-\infty}^{\infty} U^{p+1}(x,t)dx = \int_{-\infty}^{\infty} u^{p+1}(x,t)dx.$$

The use of (6.56) and (6.57) shows that if

$$\theta(y,t,\tau) = e^{-(\nu y^2 - iy|y|)(t-\tau)} \widehat{U^p U_x}(y,\tau),$$

then, by the Cauchy-Schwarz inequality,

$$\begin{aligned} &\lim_{t\to+\infty} t^{\frac{3}{2}} \Big| \int_{-\infty}^{\infty} Re\bigg(\Big(\int_0^{t^{1-\epsilon}} \theta(y,t,\tau)d\tau \Big) \Big(\int_{t^{1-\epsilon}}^{t} \theta(y,t,\tau)d\tau \Big) \bigg) dy \Big| \\ &\qquad \le \lim_{t\to+\infty} t^{\frac{3}{2}} \Big| \int_0^{t^{1-\epsilon}} \theta(y,t,\tau)d\tau \Big|_2 \Big| \int_{t^{1-\epsilon}}^{t} \theta(y,t,\tau)d\tau \Big|_2 = 0, \end{aligned} \tag{6.59}$$

where Re(z) connotes the real part of z.

Apply Parseval's Theorem to (6.51), and then use (6.56), (6.57) and (6.59) to obtain

$$\begin{aligned} &\lim_{t\to+\infty} t^{\frac{3}{2}} |u(\cdot,t) - w(\cdot,t)|_2^2 = \lim_{t\to+\infty} t^{\frac{3}{2}} |U(\cdot,t) - W(\cdot,t)|_2^2 \\ &= \lim_{t\to+\infty} t^{\frac{3}{2}} |\hat{U}(y,t) - \hat{W}(y,t)|_2^2 \\ &= \lim_{t\to+\infty} t^{\frac{3}{2}} \Big| \int_0^t \exp\big(-(\nu y^2 - iy|y|)(t-\tau) \big) \widehat{cU^p U_x}(y,\tau) d\tau \Big|_2^2 \\ &= \lim_{t\to+\infty} c^2 t^{\frac{3}{2}} \int_{-\infty}^{\infty} \Big| \Big(\int_0^{t^{1-\epsilon}} + \int_{t^{1-\epsilon}}^{t} \Big) e^{-(\nu y^2 - iy|y|)(t-\tau)} \widehat{U^p U_x}(y,\tau) d\tau \Big|^2 dy \\ &= \lim_{t\to+\infty} c^2 t^{\frac{3}{2}} \int_{-\infty}^{\infty} \Big| \int_0^{t^{1-\epsilon}} e^{-(\nu y^2 - iy|y|)(t-\tau)} \widehat{U^p U_x}(y,\tau) d\tau \Big|^2 dy \\ &\quad + \lim_{t\to+\infty} c^2 t^{\frac{3}{2}} \int_{-\infty}^{\infty} \Big| \int_{t^{1-\epsilon}}^{t} e^{-(\nu y^2 - iy|y|)(t-\tau)} \widehat{U^p U_x}(y,\tau) d\tau \Big|^2 dy \\ &\quad + \lim_{t\to+\infty} c^2 t^{\frac{3}{2}} \int_{-\infty}^{\infty} 2Re\Big(\Big(\int_0^{t^{1-\epsilon}} \theta(y,t,\tau)d\tau \Big) \Big(\int_{t^{1-\epsilon}}^{t} \theta(y,t,\tau)d\tau \Big) \Big) dy \\ &= \lim_{t\to+\infty} \int_{-\infty}^{\infty} \frac{c^2 s^2 e^{-2\nu s^2}}{(p+1)^2} \Big| \int_0^{t^{1-\epsilon}} \exp\Big((\nu s^2 - is|s|) \frac{\tau}{t} \Big) \widehat{U^{p+1}}\big(\frac{s}{\sqrt{t}}, \tau\big) d\tau \Big|^2 ds \\ &= \frac{c^2}{(p+1)^2} \int_{-\infty}^{\infty} s^2 e^{-2\nu s^2} \Big| \int_0^{\infty} \widehat{U^{p+1}}(0,\tau) d\tau \Big|^2 ds \\ &= \frac{c^2}{4\nu (8\nu\pi)^{\frac{1}{2}} (p+1)^2} \Big(\int_0^{\infty} \int_{-\infty}^{\infty} u^{p+1}(x,\tau) dx d\tau \Big)^2. \end{aligned} \tag{6.60}$$

If $\hat{f}(y) = iy\hat{g}(y)$ for some $g \in L_1(\mathbb{R})$, then by using the representation (6.8) for the solution u of (1.1) and following the line of argument laid out from (6.53) to

(6.60), one concludes that

$$
\begin{aligned}
&\lim_{t\to+\infty} t^{\frac{3}{2}}|u(\cdot,t)|_2^2 = \lim_{t\to+\infty} t^{\frac{3}{2}}|U(\cdot,t)|_2^2 = \lim_{t\to+\infty} t^{\frac{3}{2}}|\hat{U}(\cdot,t)|_2^2 \\
&= \lim_{t\to+\infty} t^{\frac{3}{2}}\Big|\hat{W}(y,t) - \int_0^t ce^{-(\nu y^2-iy|y|)(t-\tau)}\widehat{U^pU_x}(y,\tau)d\tau\Big|_2^2 \\
&= \lim_{t\to+\infty} t^{\frac{3}{2}}\Big|\hat{f}(y)e^{-(\nu y^2-iy|y|)t} - \Big(\int_0^{t^{1-\epsilon}} + \int_{t^{1-\epsilon}}^t ce^{-(\nu y^2-iy|y|)(t-\tau)}\widehat{U^pU_x}(y,\tau)d\tau\Big)\Big|_2^2 \\
&= \lim_{t\to+\infty} t^{\frac{3}{2}}\Big|iy\hat{g}(y)e^{-(\nu y^2-iy|y|)t} - \int_0^{t^{1-\epsilon}} \frac{iyc\, e^{-(\nu y^2-iy|y|)(t-\tau)}}{(1+p)}\widehat{U^{p+1}}(y,\tau)d\tau\Big|_2^2 \qquad (6.61) \\
&= \lim_{t\to+\infty} \int_{-\infty}^{\infty} s^2e^{-2\nu s^2}\Big|\hat{g}(\frac{s}{\sqrt{t}}) - \int_0^{t^{1-\epsilon}} \frac{ce^{(\nu s^2-is|s|)\frac{\tau}{t}}\widehat{U^{p+1}}(\frac{s}{\sqrt{t}},\tau)}{(1+p)}d\tau\Big|^2 ds \\
&= \int_{-\infty}^{\infty} s^2e^{-2\nu s^2}ds\Big|\hat{g}(0) - \int_0^{\infty} \frac{c\widehat{U^{p+1}}(0,\tau)}{(p+1)}d\tau\Big|^2 \\
&= \frac{1}{4\nu(8\nu\pi)^{\frac{1}{2}}}\Big(\int_{-\infty}^{\infty} g(x)dx - \int_0^{\infty}\int_{-\infty}^{\infty} \frac{cu^{p+1}(x,t)}{p+1}dxdt\Big)^2,
\end{aligned}
$$

where again we have used that $e^{(\nu s^2-is|s|)\frac{\tau}{t}} \to 1$ as $t \to +\infty$, for any fixed s and $\tau \in [0, t^{1-\epsilon}]$.

Furthermore, if $xf(x) \in L_1(\mathbb{R})$ and $\frac{d}{dx}g(x) = f(x)$ with $xg(x) \to 0$ as $x \to \pm\infty$, then

$$\int_{-\infty}^{\infty} g(x)dx = -\int_{-\infty}^{\infty} xf(x)dx, \qquad (6.62)$$

and so (6.61) becomes

$$\lim_{t\to+\infty} t^{\frac{3}{2}}|u(\cdot,t)|_2^2 = \frac{1}{4\nu(8\nu\pi)^{\frac{1}{2}}}\Big(\int_{-\infty}^{\infty} xf(x)dx + \int_0^{\infty}\int_{-\infty}^{\infty} \frac{cu^{p+1}(x,t)}{p+1}dxdt\Big)^2.$$

The corollary is proved. □

Remark 6.10. If the initial data f satisfies $|\hat{f}(y)| \le C|y|^\alpha$ for $\alpha > 1$, then the decay of solutions of (1.1) depends only on $\int_0^\infty \int_{-\infty}^\infty u^{p+1}dxdt$. This is because, by Lemma 6.1, the solution w of (3.1) decays in L_2-norm faster than $t^{-\frac{3}{4}}$. The same observation applies to the decay of solutions to the GKdV-Burgers equation (1.7) and GRLW-Burgers equation (1.8) as well (see [13]).

Remark 6.11. The results of Corollary 6.9 subsist upon the sharp decay results already obtained for solutions corresponding to generic initial data. Similar results are available for solutions of the equations (1.4) and (1.5) for the same reason of having in hand suitable decay results for generic initial data, and these hold even for nonlinearities only quadratic near zero (see [3]).The results of Corollary 6.9 are expected to hold even for quadratic nonlinearities, but this waits upon extending the results of Section 5 to this level.

Conclusion. The generalized Benjamin-Ono-Burgers equation (1.1) has been the object of the present investigation. Attention has been given to the pure initial-value problem in which the solution is specified for all $x \in \mathbb{R}$ at some given instant

$t = 0$, say, of time and inquiry is made into its development for $t > 0$ under the Benjamin-Ono-Burgers evolution. While a local well-posedness result is set forth, our focus has been solutions' behavior for large time.

A global well-posedness theory is put forward corresponding to growth restrictions on the generalized nonlinearity P. These are less stringent than those needed in the absence of dissipation. Moreover, taking advantage of the dispersion, the growth restriction is one-sided (see (2.5)-(2.6)).

In situations where global existence obtains, the long-time decay rates of solutions corresponding to initial data of finite energy are studied. Using the ideas already appearing in the earlier works [12] and [13] together with some new inequalities, decay estimates are obtained for nonlinearities $P(u)$ that are at least cubic near $u = 0$. Rates are obtained that are sharp for generic data. Moreover, in case the initial data has some extra structure (its Fourier transform vanishes at the origin in some particular way), enhanced, but still sharp, decay rates are derived.

A natural successor to the present work is similar theoretical considerations for the more general class of model equations of the form

$$u_t + u_x + P(u)_x + \mathbb{M}u - \mathbb{L}u_x = 0. \tag{**}$$

Here, both $\mathbb{M}$ and $\mathbb{L}$ are Fourier-multiplier operators defined in the term of their symbols α and β by

$$\widehat{\mathbb{L}u}(k) = \alpha(k)\hat{u}(k) \qquad \text{and} \qquad \widehat{\mathbb{M}u}(k) = \beta(k)\hat{u}(k)$$

for $k \in \mathbb{R}$. Typically, α and β are both real-valued, with $\beta > 0$, though in some situations, u might be complex-valued. The symbols α and β represent the effects of frequency dispersion and dissipation, respectively. Such models arise in a variety of contexts as models for wave propagation (see e.g. [2, 5, 8, 14, 35]).

Questions similar to those addressed here arise for the regularized version

$$u_t + u_x + P(u)_x + \mathbb{M}u + \mathbb{L}u_t = 0$$

of equation (**) (see [29, 34]).

There are also questions relating directly to the present line of development worthy of study. For example, can the growth conditions on the nonlinearity P be weakened whilst still retaining global well-posedness? And, what can one say about the long-time asymptotics of solutions corresponding to large initial data when P vanishes only to second order at the origin (e.g. quadratic nonlinearities)?

Acknowledgments. The work was partially supported by the National Science Foundation and the W. M. Keck Foundation, USA and by EPSRC, UK. Both authors would like to thank Professor Thanasis Fokas for his interest. LL thanks Professor Rumei Luo for his encouragement.

References

[1] M. J. Ablowitz and A. S. Fokas, *The inverse scattering transform for the Benjamin-Ono equation, A pivot to multidimensional problems*, Stud. Appl. Math., **68** (1983), 1–10.

[2] L. Abdelouhab, J. L. Bona, M. Felland and J.-C. Saut, *Nonlocal models for nonlinear dispersive waves*, Phys. D, **40** (1989), 360–392.

[3] C. J. Amick, J. L. Bona and M. E. Schonbek, *Decay of solutions of some nonlinear wave equations*, J. Differential Equations, **81** (1989), 1–49.

[4] T. B. Benjamin, *Internal waves of permanent form in fluids of great depth*, J. Fluid Mech., **29** (1967), 559–592.

[5] T. B. Benjamin, J. L. Bona and J. J. Mahony, *Model equations for long waves in nonlinear dispersive systems*, Philos. Trans. Royal Soc. London Ser. A, **272** (1972), 47–78.

[6] P. Biler, *Asymptotic behavior in time of some equations generalizing Korteweg-de Vries-Burgers*, Bull. Polish Acad. Sci., **32** (1984), 275–281.

[7] ——, *Large-time behavior of periodic solutions of two dispersive equations of Korteweg-de Vries-Burgers type*, Bull. Polish Acad. Sci., **32** (1984), 401–405.

[8] J. L. Bona, *On solitary waves and their role in the evolution of long waves*, Applications of Nonlinear Analysis in the Physical Sciences (ed. H. Amann, N. Bazley & K. Kirchgässner) Pitman, London (1981), 183–205.

[9] J. L. Bona, V. A. Dougalis, O. A. Karakashian and W. R. McKinney, *Computations of blow-up and decay for periodic solutions of the generalized Korteweg-de Vries-Burgers equation*, Applied Numerical Math., **10** (1992), 335–355.

[10] ——, *Conservative high-order numerical schemes for the generalized Korteweg-de Vries equation*, Philos. Trans. Royal Soc. London, Series A, **351** (1995), 107–164.

[11] J. L. Bona and H. Kalisch, *Singularity formation in the generalized Benjamin-Ono equation*, Discrete Cont. Dyn. Systems Series A, **11** (2004), 27–45.

[12] J. L. Bona and L. Luo, *Decay of the solutions to nonlinear, dispersive wave equations*, Diff. & Int. Equations, **6** (1993), 961–980.

[13] ———, *More results on the decay of solutions to nonlinear, dispersive wave equations*, Discrete Cont. Dyn. Systems, **1** (1995), 151–193.

[14] J. L. Bona, W. G. Pritchard and L. R. Scott, *An evaluation of a model equation for water waves*, Philos. Trans. Royal Soc. London Ser. A, **302** (1981), 457–510.

[15] J. L. Bona, S. Rajopadhye and M. E. Schonbek, *Models for propagation of bores I. Two-dimensional theory*, Differential & Int. Equations, **7** (1994), 699–734.

[16] J. L. Bona and M. E. Schonbek, *Travelling-wave solutions of the Korteweg-de Vries-Burgers equation*, Proc. Royal Soc. Edinburgh A, **101** (1985), 207–226.

[17] J. L. Bona and R. Smith, *The initial-value problem for the Korteweg-de Vries equation*, Philos. Trans. Royal Soc. London Ser A, **278** (1975), 555–601.

[18] H. Brezis and T. Gallouët, *Nonlinear Schrödinger evolution equations*, Nonlinear Anal., **4** (1980), 677–681.

[19] H. Brezis and S. Wainger, *A note on limiting cases of Sobolev embeddings and convolution inequalities*, Comm. P.D.E, **5** (1980), 773–789.

[20] D. Derks "Coherent Structures in the Dynamics of Perturbed Hamiltonian Systems", Ph.D. Thesis, University of Twente, Dec. 1992.

[21] D. B. Dix, *The dissipation of nonlinear dispersive waves: The case of asymptotically weak nonlinearity*, Comm. P.D.E, **17** (1992), 1665–1693.

[22] ——, *Temporal asymptotic behavior of solutions of the Benjamin-Ono -Burgers equation*, J. Differential Equations, **90** (1991), 238–287.

[23] P. M. Edwin and B. Roberts, *The Benjamin-Ono-Burgers equation: An application in solar physics*, Wave Motion, **8** (1986), 151–158.

[24] J. Ginibre and G. Velo, *Smoothing properties and existence of solutions for the generalized Benjamin-Ono equation*, J. Differential Equations, **93** (1991), 150–212.

[25] N. Hayashi and P. I. Naumkin, *Large time asymptotics of solutions to the generalized Benjamin-Ono equation*, Trans. American Math. Soc., **351** (1999), 109–130.

[26] R. J. Iório, *On the Cauchy problem for the Benjamin-Ono equation*, Comm. P.D.E., **11** (1986), 1031–1081.

[27] R. S. Johnson, *A nonlinear equation incorporating damping and dispersion*, J. Fluid Mech., **42** (1970), 49–60.

[28] ——, *Shallow water waves on a viscous fluid —The undular bore*, Phys. of Fluids, **15** (1972), 1693–1699.

[29] H. Kalisch and J. L. Bona, *Models for internal waves in deep water*, Discrete Cont. Dyn. Systems, **6** (2000), 1–20.

[30] T. Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, Studies in Applied Math., Advances in Math. Suppl. Studies, **8** (1983), 93–128.

[31] C. E. Kenig, G. Ponce and L. Vega, *On the generalized Benjamin-Ono equation*, Trans. American Math. Soc., **342** (1994), 155–172.

[32] D. J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, Philos. Mag., **39** (1895), 422–443.

[33] J. J. Mahony and W. G. Pritchard, *Wave reflexion from beaches*, J. Fluid Mech., **101** (1980), 809–832.

[34] Y. Mammeri, *On the decay in time of solutions of some generalized regularized long wave equations*, Comm. Pure Appl. Anal., **7** (2008), 513–532.

[35] C. C. Mei and L. F. Liu, *The damping of surface gravity waves in a bounded liquid*, J. Fluid Mech., **59** (1973), 239–256.

[36] J. W. Miles, *Surface-wave damping in a closed basin*, Proc. Royal Soc. London Ser. A, **297** (1967), 459–475.

[37] P. I. Naumkin and I. A. Shishmarev, "Nonlinear Nonlocal Equations in the Theory of Waves", Translation of Mathematical Monographs, **133** Amer. Math. Soc., Providence, 1994.

[38] H. Ono, *Algebraic solitary waves in stratified fluids*, J. Phys. Soc. Japan, **39** (1975), 1082–1091.

[39] E. Ott and R. N. Sudan, *Damping of solitary waves*, Phys. of Fluids, **13** (1970), 1432–1434.

[40] T. Ozawa, *On critical cases of Sobolev's inequality*, J. Functional Anal., **127** (1995), 259–269.

[41] D. H. Peregrine, *Calculations of the development of an undular bore*, J. Fluid Mech., **25** (1966), 321–330.

[42] E. Schechter, "Well-Behaved Evolutions and the Trotter Product Formulas", Ph.D. Thesis, University of Chicago, 1978.

[43] M. M. Tom, *Smoothing properties of some weak solutions of the Benjamin-Ono equation*, Differential & Int. Equations, **3** (1990), 683–694.

[44] S. Vento, *Well-posedness for the generalized Benjamin-Ono equation with arbitrary large initial data in the critical space*, Inter. Math. Res. Notices, (2009) (to appear).

[45] L. Zhang, *Decay of solutions to generalized Benjamin-Bona-Mahony-Burgers equations in n-space dimensions*, Nonlinear Analysis, **25** (1995), 1343–1369.

[46] ——, *Initial value problem for a nonlinear parabolic equation with singular integral-differential term*, ACTA Math. Appl. Sinica, **8** (1992), 367–376.

[47] Y. Zhou and B. Guo, *Initial value problems for a nonlinear singular integral-differential equation of deep water*, Lecture notes in Mathematics, **1306** (1986), 278–290.

Received March 2009; revised November 2009.

E-mail address: bona@math.uic.edu
E-mail address: lluo@nyit.edu