

# GLOBAL WELL-POSEDNESS FOR A SYSTEM OF KDV-TYPE EQUATIONS WITH COUPLED QUADRATIC NONLINEARITIES

JERRY L. BONA, JONATHAN COHEN, AND GANG WANG

**Abstract.** In this paper, coupled systems

$$\begin{aligned}u_t + u_{xxx} + P(u, v)_x &= 0, \\v_t + v_{xxx} + Q(u, v)_x &= 0\end{aligned}$$

of Korteweg–de Vries type are considered, where  $u = u(x, t), v = v(x, t)$  are real-valued functions and where  $x, t \in \mathbb{R}$ . Here, subscripts connote partial differentiation and

$$P(u, v) = Au^2 + Buv + Cv^2 \quad \text{and} \quad Q(u, v) = Du^2 + Euv + Fv^2$$

are quadratic polynomials in the variables  $u$  and  $v$ . Attention is given to the pure initial-value problem in which  $u(x, t)$  and  $v(x, t)$  are both specified at  $t = 0$ , namely,

$$u(x, 0) = u_0(x) \quad \text{and} \quad v(x, 0) = v_0(x),$$

for  $x \in \mathbb{R}$ . Under suitable conditions on  $P$  and  $Q$ , global well-posedness of this problem is established for initial data in the  $L^2$ -based Sobolev spaces  $H^s(\mathbb{R}) \times H^s(\mathbb{R})$  for any  $s > -3/4$ .

## CONTENTS

1. Introduction . . . . .	68
2. Local and global well-posedness: The case $s \geq 0$ . . . . .	73
3. Preliminaries and the theorem for $-3/4 < s < 0$ . . . . .	93
4. Global well-posedness: The case $-3/4 < s < 0$ . . . . .	99
5. Components of the modified energy . . . . .	104
6. Estimates of multilinear functionals . . . . .	118
7. The $\Lambda^5$ terms . . . . .	139
References . . . . .	147

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## §1. Introduction

Considered here is a coupled system

$$(1.1) \quad \begin{cases} u_t + u_{xxx} + A(u^2)_x + B(uv)_x + C(v^2)_x = 0, \\ v_t + v_{xxx} + D(u^2)_x + E(uv)_x + F(v^2)_x = 0 \end{cases}$$

of two Korteweg–de Vries (KdV)-type equations, posed for  $x \in \mathbb{R}$  and  $t \geq 0$ , with specified initial data

$$(1.2) \quad u(x, 0) = u_0(x) \quad \text{and} \quad v(x, 0) = v_0(x).$$

Here,  $A, B, \dots, F$  are constants, and the abbreviations

$$(1.3) \quad \begin{aligned} P(u, v) &= Au^2 + Buv + Cv^2, \\ Q(u, v) &= Du^2 + Euv + Fv^2 \end{aligned}$$

will be employed when convenient. Such systems arise as models for wave propagation in physical systems where both nonlinear and dispersive effects are important (see, e.g., [5], [6], and [25]). Here  $u$  and  $v$  are real-valued functions of  $(x, t)$ , and subscripts connote partial differentiation. The goal of this article is to give conditions on the coefficients  $A, B, \dots, F$  implying that the initial-value problem (1.1)–(1.2) is globally well posed in the  $L^2$ -based Sobolev classes  $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ . The outcome of our analysis is conditions on the coefficients  $A, B, \dots, F$  which, when satisfied, imply that the initial-value problem for (1.1) is globally well posed for any  $s > -3/4$ .

The work improves in several ways upon the global existence results in [3] and [8] pertaining to the Gear–Grimshaw system of equations (see [17]) which arise in the study of the interaction of internal waves on neighboring pycnoclines. Our theory also extends the results obtained by Oh in [26, Theorem 1.5] for the Majda–Biello system (see [25]):

$$(1.4) \quad \begin{cases} u_t + u_{xxx} + \frac{1}{2}(v^2)_x = 0, \\ v_t + \alpha v_{xxx} + (uv)_x = 0 \end{cases}$$

in the case  $\alpha = 1$ . (This system arises as a model for the interaction of barotropic and baroclinic equatorial Rossby waves. The parameter  $\alpha$  depends upon the Rossby wave in question. It typically has a value near 1.)

Oh [26, Section 3] showed global existence for the system (1.4) with initial data in the Sobolev space  $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ , for  $0 > s > -3/4$ . The system (1.4) with  $\alpha = 1$  is a special case of the general system (1.1) with

$A = 0, B = 0, C = 1/2, D = 0, E = 1, F = 0$ . Oh also obtained results in the periodic case where  $\alpha \neq 1$  for a different range of  $s$ , but those systems are not specializations of the systems considered in this paper.

To be more precise about the conditions on the coefficients that come to the fore in our global well-posedness theory, define the matrix  $M$  in terms of  $A, B, \dots, F$  to be

$$(1.5) \quad M = \begin{bmatrix} 2B & E - 2A & -4D \\ 4C & 2F - B & -2E \end{bmatrix}.$$

Then for  $s > -3/4$ , the KdV system with initial values in  $H^s(\mathbb{R}) \times H^s(\mathbb{R})$  has global solutions if  $\text{rank } M = 2$  and

$$(1.6) \quad \begin{aligned} &2EC(E - 2A)^2 + 2BD(2F - B)^2 \\ &- [4CD + BE](E - 2A)(2F - B) > (4CD - BE)^2, \end{aligned}$$

or  $\text{rank } M = 1$  and either

$$(1.7) \quad (2A - E)^2 + 8BD > 0 \quad \text{or} \quad (2F - B)^2 + 8EC > 0.$$

Substitution of the coefficients of the Majda–Biello system in the matrix  $M$  yields

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}.$$

In this case, the rank of  $M$  is 2, and the inequality (1.6) becomes the valid statement  $1 > 0$ . So, for the case  $\alpha = 1$ , there is global existence of solutions to the Majda–Biello system corresponding to data in  $H^s(\mathbb{R}) \times H^s(\mathbb{R})$  for any  $s > -3/4$ .

It is interesting to note that global existence for systems of the form (1.1) is not assured. Indeed, the system

$$(1.8) \quad \begin{cases} u_t + u_{xxx} + (\frac{1}{2}u^2 - \frac{1}{2}v^2)_x = 0, \\ v_t + v_{xxx} + (uv)_x = 0 \end{cases}$$

possesses solutions corresponding to smooth data that are exponentially decaying to zero at  $\pm\infty$  that blow up in finite time (see [12]). In this case,  $A = 1/2, B = 0, C = -1/2, D = 0, E = 1, F = 0$ , so that the associated matrix

$$M = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & -2 \end{bmatrix}$$

is a matrix of rank 1. Substituting the coefficients into the criterion (1.7), we find that

$$(2A - E)^2 + 8BD = 0 \quad \text{and} \quad (2F - B)^2 + 8EC = -4.$$

Thus, neither of the criteria in (1.7) is satisfied. This example might lead one to conjecture that the sufficient conditions for global well-posedness are in fact necessary. This is also not the case. The example

$$(1.9) \quad \begin{cases} u_t + u_{xxx} + (\frac{1}{2}u^2)_x, \\ v_t + v_{xxx} + (uv)_x = 0 \end{cases}$$

yields a matrix  $M$  that has rank 1 with both

$$(2A - E)^2 + 8BD = 0 \quad \text{and} \quad (2F - B)^2 + 8EC = 0.$$

Thus, this is a system that fails the test in (1.7). However, the first equation is simply the KdV equation, and substitution of the solution of the first equation into the second yields a linear equation for  $v$  that is clearly globally well posed.

As is common in such endeavors, our theory consists of a local well-posedness result together with the derivation of a priori bounds. The local theory relies upon the bilinear estimates of Kenig, Ponce, and Vega in [20] and [21]. The global theory for  $s \geq 0$  makes use of energy-type inequalities, nonlinear operator interpolation, and commutator estimates. For rough data where  $-3/4 < s < 0$ , the proof of global well-posedness owes its inspiration to the I-method of Colliander, Keel, Staffilani, Takaoka, and Tao in [16] that was used in their analysis of the initial-value problem for the KdV equation itself.

Several interesting points arise in the forthcoming development. First, it will be noted that the system (1.1) always has a Hamiltonian structure, no matter what the values of the parameters. This structure does not necessarily provide the a priori bounds that are so helpful in the analysis of a single KdV equation and some of its relatives. The first step in the global theory is to deduce conditions on the coefficients  $A, B, \dots, F$  so that the Hamiltonian structure does indeed yield helpful a priori bounds on solutions. For parameter values  $A, B, \dots, F$  that satisfy either (1.6) or (1.7), global well-posedness when  $s = 0$  does not present especial difficulty because the restrictions on the coefficients imply that the  $(L^2(\mathbb{R}) \times L^2(\mathbb{R}))$ -norm of the

solution to (1.1)–(1.2) is uniformly bounded in time as long as the solution exists. This allows the local theory to be iterated indefinitely to produce a global solution.

However, for other values of  $s$ , the  $(H^s(\mathbb{R}) \times H^s(\mathbb{R}))$ -norm is not necessarily uniformly bounded as a function of time, though this is certainly true if  $s = 1$ . For  $s \geq 2$ , time-dependent differential inequalities are deduced by use of commutator estimates. The resulting bounds show solutions to be bounded in  $H^s(\mathbb{R}) \times H^s(\mathbb{R})$  on bounded time intervals and hence lead to a result of global well-posedness in this range of  $s$ . When  $0 < s < 2, s \neq 1$ , nonlinear interpolation theory together with the bounds in  $L^2$ ,  $H^1$ , and  $H^2$  and the Lipschitz estimates provided by the local theory are used to deduce global existence.

For  $-3/4 < s < 0$ , the smoothing and approximate energy argument developed in [15] and [16] proves to be telling, and global well-posedness follows. In our adaptation of this energy argument, the aforementioned multilinear functionals that appear involve both  $u$  and  $v$ . These functionals do not possess the same symmetries that were so useful in the attack on the KdV equation. This has no bearing upon the arguments in favor of local well-posedness but has significant implications for the global well-posedness theory. In addition to developing techniques applicable to this more complicated situation, we present a detailed, and perhaps more transparent, argument to pass from local to global well-posedness in these large spaces.

The fundamental argument of the I-method revolves around a set of inequalities involving several parameters. These inequalities suffice to show that for any size initial data, a solution to the initial-value problem (1.1)–(1.2) exists at least on the time interval  $[0, 1]$ . The trick is to choose the parameters so that the inequalities can be simultaneously satisfied. The parameters involved are a threshold parameter  $\epsilon_0$  for the initial data that guarantees the existence of a solution for a time duration of at least 1, a scaling parameter  $\lambda$  that is used to replace initial data of any size with a scaled version that is below the threshold, and a positive number  $N$  that is used to split the Fourier transform of the solution into a smooth  $L^2$ -piece and a rougher  $H^s$ -piece. These inequalities are all dependent on the value of  $s$ . The present development incorporates the precise dependence of the various estimates on the value of  $s$ . If one keeps track of how things depend upon  $s$ , it is seen that a choice of parameters  $\epsilon_0, \lambda$ , and  $N$  that satisfies the full set of inequalities can hold only if either  $s < -3/2$  or  $-9/10 < s < 0$ . This restriction on  $s$  is an artifact of the I-method. In particular, this means that

the modified energies utilized in this paper, and also in the original paper [16], have nothing to say about the global well-posedness for  $s \geq 0$ . (This point is elucidated in more detail in Remark 8 in Section 4.)

For  $s \leq -3/4$ , the local well-posedness theory of Kenig, Ponce, and Vega [21] no longer obtains via a contraction-mapping argument applied in the Bourgain spaces defined in Section 2. Hence, no conclusion about global well-posedness is warranted for this range of  $s$ , despite the fact that the conditions for inferring a priori bounds still hold. Note, however, that both local and global well-posedness for the original KdV equation at the critical regularity  $s = -3/4$  has been established by Guo [18] and Kishimoto [22] by different methods.

It is worth emphasizing that the basic I-method is not an iteration of a local energy estimate. Indeed, thinking of it that way can cause confusion since growth would be compounded, resulting in a sufficiently rapidly increasing bound that the method would not reach time 1. A crucial part of the proof of global well-posedness is that if the solution starts out below the threshold, it can be continued on an interval of length  $N^{-5s}$  without the size of the solution exceeding the threshold. This argument differs from that of the original paper [16] in that it makes explicit the dependence of the size of the interval on the Sobolev index  $s$ , and it also explains the estimate of the size as a telescoping of the evolution of a quintilinear term rather than an iteration of the local theory.

The plan of the paper is as follows. Section 2 is concerned with local well-posedness, explication of the conservation laws, and the global well-posedness for  $s \geq 0$ . In particular, persistence of regularity is established for the system (1.1) via an  $H^1$  conservation law, a Gronwall-type inequality for  $H^2$ , commutator estimates for  $H^s$  when  $s > 2$ , and nonlinear interpolation for  $s \in (0, 1) \cup (1, 2)$ . The estimates for the systems considered in this paper do not follow from similar estimates for the single KdV equation because they need the full strength of the coefficient conditions described in Section 2. Section 3 introduces a modified energy, defines the relevant multilinear functionals, states the important inequalities for these functionals, and concludes with the statement of the main result for the case  $-3/4 < s < 0$ . This result is proved in Section 4, subject to the proofs of technical points concerning the multilinear functionals. Section 5 is devoted to finding explicit formulas for trilinear and quadrilinear correction terms that, when subtracted from the modified energy, yield a functional whose time derivative is controlled by  $N^{5s}$  times a quintilinear term. An explicit

formula for the symbol in the quintilinear functional is also found in this section. Section 6 is devoted to a pointwise estimate for the symbol of the quintilinear functional. The results of this section overlap with similar results obtained by Oh [26] for the specific case of the Majda–Biello system. An independent proof of the pointwise estimates for the symbol of the relevant functional is included. In Section 7, the estimate for the quintilinear functional is obtained from the pointwise estimates of Section 6, and this concludes the proof of the main theorem.

## §2. Local and global well-posedness: The case $s \geq 0$

In the present section, a local well-posedness theory for (1.1)–(1.2) is sketched following what are, by now, standard lines using Bourgain-type spaces (see [13] and [14]). The initial data  $(u_0, v_0)$  are presumed to lie in  $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ , where  $s > -3/4$ . Interest lies primarily in the case  $s < 0$ , but all values of  $s > -3/4$  fall within the range of the local theory. Such results have in fact been obtained previously in similar contexts, for example, in the works of Kenig, Ponce, and Vega (see [20] or [21]) and Alvarez-Samaniego and Carvajal [2]. Consequently, we content ourselves with a very brief outline of the details, as some of them will be relevant to the subsequent analysis. A conservation law will then be derived which plays a critical role in the further developments.

### 2.1. Notation

The notation used is more or less standard. The usual  $L^2$ -norm of a function of one variable is denoted  $|f|_{L^2}$  or sometimes just  $|f|_2$ , and the  $(L^2 \times L^2)$ -norm of a pair  $(f, g)$  of such functions is written  $\|(f, g)\|_{L^2 \times L^2} = |f|_2 + |g|_2$ . When the distinction needs to be made, we use the notation  $L^2_\xi, L^2_\tau$ , and so forth, to denote the  $L^2$ -norm taken with respect to the variables  $\xi, \tau$ , and so on. As already mentioned,  $H^s = H^s(\mathbb{R})$  is the usual Sobolev class of Schwartz distributions  $f$  whose Fourier transform  $\hat{f}(\xi)$  is a measurable function, square integrable with respect to the measure  $(1 + |\xi|)^{2s} d\xi$ . We will usually use simply  $H^s$  rather than  $H^s(\mathbb{R})$  unless emphasis on the domain of definition of the functions is needed. The norm in  $H^s$  is denoted  $\|\cdot\|_{H^s}$  or simply  $\|\cdot\|_s$ .

For  $s, b \in \mathbb{R}$ , the Bourgain space  $X_{s,b}$  is the completion of the Schwartz space  $\mathcal{S}(\mathbb{R}^2) \times \mathcal{S}(\mathbb{R}^2)$  of infinitely differentiable, rapidly decreasing functions

with respect to the norm

$$(2.1) \quad \begin{aligned} & \| (u, v) \|_{X_{s,b}}^2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (|\widehat{u}(\xi, \tau)|^2 + |\widehat{v}(\xi, \tau)|^2) (1 + |\xi|)^{2s} (1 + |\tau - \xi^3|)^{2b} d\xi d\tau, \end{aligned}$$

where

$$\widehat{u} = \widehat{u}(\xi, \tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, t) e^{-i(x\xi + t\tau)} dx dt$$

is the Fourier transform of  $u$ . For  $\alpha < \beta$ , the function class  $X_{s,b}^{[\alpha,\beta]}$  is the space of restrictions to  $[\alpha, \beta]$  of the elements of  $X_{s,b}$  with its usual quotient norm

$$(2.2) \quad \| (u, v) \|_{X_{s,b}^{[\alpha,\beta]}} = \inf \{ \| (u_1, v_1) \|_{X_{s,b}} : (u_1, v_1)|_{[\alpha,\beta]} = (u, v)|_{[\alpha,\beta]} \}.$$

When the interval  $[\alpha, \beta] = [0, \delta]$ , this restriction space is denoted simply by  $X_{s,b}^\delta$ . These are Cartesian products of the spaces used by Kenig, Ponce, and Vega [20] in their analysis of the generalized KdV equations. If  $X$  is any Banach space, and  $I \subset \mathbb{R}$  is a closed bounded interval, then  $C(I; X)$  is the set of continuous maps of  $I$  into  $X$  with the sup-norm

$$\| u \|_{C(I; X)} = \sup_{t \in I} \| u(t) \|_X.$$

A fundamental result is that if  $b > 1/2$ , then  $X_{s,b}^I$  is continuously embedded in  $C(I; H^s(\mathbb{R}))$  (see, e.g., [29, Corollary 2.10, p. 101]).

For the convenience of the readers and the authors, we often write

$$\int f \quad \text{for} \quad \int_{-\infty}^{\infty} f(x) dx$$

when no confusion is likely to result.

### 2.2. Local well-posedness

The local well-posedness theory consists of applying the bilinear estimates in [20] and [21] to the system (1.1)–(1.2). Here is the principal result.

**THEOREM 1.** *For any  $s > -3/4$ , there exists  $b \in (1/2, 1)$  such that the following holds. For any  $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ , there exists a  $\delta > 0$  and a solution  $(u, v)$  of the system (1.1) on the interval  $[0, \delta]$  with  $(u_0, v_0)$  as initial data such that  $(u, v) \in C([0, \delta] : H^s \times H^s)$ , and, moreover, the solution  $(u, v)$  also lies in  $X_{s,b}^\delta$ . This solution, which is unique within  $X_{s,b}^\delta$ , depends continuously in this function class on variations of  $(u_0, v_0)$  in  $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ .*



*Proof.* The proof follows the arguments in [20] and [21]. Introducing the notation

$$U = \begin{bmatrix} u \\ v \end{bmatrix}, \quad U_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \quad \text{and}$$

$$M(u, v) = \begin{bmatrix} A(u^2)_x + B(uv)_x + C(v^2)_x \\ D(u^2)_x + E(uv)_x + F(v^2)_x \end{bmatrix},$$

equations (1.1) can be written in the form

$$U_t + U_{xxx} = -M,$$

$$U(x, 0) = U_0(x).$$

If  $M = 0$ , the system is a linear homogeneous system whose solution is

$$W(t)U_0 = \int_{-\infty}^{\infty} e^{i(t\xi^3 + x\xi)} \widehat{U}_0(\xi) d\xi.$$

A distributional solution of (1.1)–(1.2) on the time interval  $[0, T]$  is thus supplied by a function in the space  $X_{s,b}^T$  satisfying the integral equation

$$(2.3) \quad U(t) = W(t)U_0 - \int_0^t W(t - \tau)M(x, \tau) d\tau$$

obtained by applying Duhamel’s formula. To solve this equation in the space  $X_{s,b}$ , introduce  $C^\infty$  cutoff functions  $\theta$  and  $\psi$  mapping  $\mathbb{R}$  into  $[0, 1]$ , such that  $\theta$  is supported in  $(-2, 2)$  and is identically 1 on the interval  $[-1, 1]$  and  $\psi$  is supported in  $(-2, 2)$  and is identically 1 on the support of  $\theta$ . Define an operator  $\Phi$  by

$$(2.4) \quad \Phi(U) = \theta(t)W(t)U_0 - \theta(t) \int_0^t W(t - \tau)\psi^2(\tau)M(x, \tau) d\tau.$$

The question of local existence is thus reduced to showing the existence of a fixed point of  $\Phi$ , and this in turn follows, for sufficiently small initial data, from the inequalities in [21, Theorems 1.1, 1.5] and the contraction mapping principle. As the details are nearly identical to those in [21], they are omitted here. The outcome of the analysis is that there is an  $r > 0$  such that if  $\|(u_0, v_0)\|_{H^s \times H^s} \leq r$ , then  $\Phi$  is a contraction mapping of the ball  $B_{s,b}(2r)$  of radius  $2r$  about 0 in  $X_{s,b}$  into itself. This implies that there is

a point  $U \in B_{s,b}(2r)$  such that  $\Phi(U) = U$  for  $t \in [-1, 1]$ . For  $t \in [0, 1]$ , the cutoff functions  $\theta$  and  $\psi$  are identically 1, and so the solution of (2.4) is also a solution of (2.3). Using a straightforward scaling argument as in the proof of [21, Theorem 1.5] yields a local solution corresponding to any initial value in  $H^s \times H^s$ , albeit with a possibly reduced time interval  $[0, \delta]$  of inferred existence. In the case  $s = 0$ , this argument already appears in [3] for the initial-value problem (1.1)–(1.2).  $\square$

### 2.3. Hamiltonian structure and conservation laws

The local well-posedness theory does not depend on assumptions about the coefficients  $A, B, \dots, F$ . However, to pass to a theory that is global in time, the techniques developed here require a priori information about the growth of spatial norms of the solutions as a function of time. When systems of the form (1.1) arise in practice, they often have a Hamiltonian structure which may then imply helpful further information about solutions. It will turn out that (1.1) always has a Hamiltonian structure. The goal of the present section is to introduce hypotheses on  $A, B, \dots, F$  which imply that the Hamiltonian structure yields information helpful to establishing global well-posedness.

There are several ways one can approach this issue. As what is ultimately central in our theory is a temporal bound on solutions in the space  $L^2 \times L^2$ , we pursue this property directly, noting that the outcome is indeed the existence of a helpful Hamiltonian structure for the system.

REMARK 1. It should be noted that one can make a preliminary, unitary change of the dependent variables to effect what appears to be a simpler system of equations. The subsequent analysis of the initial-value problem is not especially aided by this transformation, and so we have eschewed it here. When contemplating questions about the stability or lack thereof of traveling-wave solutions, such changes of variables are much more helpful.

Introduce the quadratic functional

$$(2.5) \quad \Omega(u, v) = \int (au^2 + buv + cv^2) dx,$$

where the real numbers  $a, b, c$  will be determined presently. A formal calculation made assuming that  $u$  and  $v$  are solutions and that we may integrate by parts with impunity with no contribution from the boundary terms at

$x = \pm\infty$  yields the formula

$$\begin{aligned}
 -\frac{d}{dt}\Omega &= -\int (2auu_t + bu_tv + bu_tv_t + 2cvv_t) dx \\
 &= \int [2auu_{xxx} + b(uv_{xxx} + vu_{xxx}) + 2cvv_{xxx}] dx \\
 (2.6) \quad &+ \int [u^2u_x(4aA + 2bD) + v^2v_x(2bC + 4cF)] dx \\
 &+ \int [vuu_x(2aB + bE + 2bA + 4cD) + u^2v_x(2aB + bE)] dx \\
 &+ \int [uvv_x(4aC + 2bF + 2cE + bB) + v^2u_x(2cE + bB)] dx \\
 &= I + II + III + IV.
 \end{aligned}$$

The integrals  $I$  and  $II$  have integrands that are  $x$ -derivatives, and hence, for smooth solutions tending to zero at  $\pm\infty$ , they vanish without further assumptions. The integrals  $III$  and  $IV$  vanish without recourse to further assumptions on  $(u, v)$  if and only if

$$(2.7) \quad 2aB + bE = 2bA + 4cD$$

and

$$(2.8) \quad bB + 2cE = 2bF + 4aC.$$

Equations (2.7) and (2.8) can be restated as a matrix equation:

$$\begin{bmatrix} 2B & E - 2A & -4D \\ 4C & 2F - B & -2E \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

As this comprises two equations in three unknowns, it always possesses a nontrivial solution.

Suppose that the coefficient matrix

$$(2.9) \quad M = \begin{bmatrix} 2B & E - 2A & -4D \\ 4C & 2F - B & -2E \end{bmatrix}$$

of the set of equations (2.7)–(2.8) has rank equal to 2, the generic case. Then, the solution space for (2.7)–(2.8) is 1-dimensional, and, up to an arbitrary

nonzero scalar multiple,

$$(2.10) \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} E(E-2A) - 2D(2F-B) \\ 8CD - 2BE \\ 2C(E-2A) - B(2F-B) \end{pmatrix}.$$

If instead the rank of  $M$  is 1, then the solutions  $(a, b, c)$  of (2.7)–(2.8) comprise the 2-dimensional space which is the orthogonal complement  $W = V^\perp$  in  $\mathbb{R}^3$  of the vector  $V = V_1$  or  $V = V_2$ , where

$$(2.11) \quad V_1 = \begin{pmatrix} 2B \\ E - 2A \\ -4D \end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix} 4C \\ 2F - B \\ -2E \end{pmatrix},$$

one of which could be the zero vector, in which case  $W$  is the orthogonal complement of the other one. (The remaining case where  $M$  has rank 0 can occur only when all the coefficients  $A, B, \dots, F$  are zero, a linear, uncoupled situation of no interest to the present discussion.) Indeed, it transpires that even the rank 1 case has trivial aspects, but this will not concern us here.

Presuming now that a nontrivial triple  $a, b, c$  has been chosen so that (2.7)–(2.8) holds for the given values of  $A, B, \dots, F$ , (2.6) then implies that

$$(2.12) \quad \Omega(u(\cdot, t), v(\cdot, t)) = \Omega(u_0, v_0)$$

for sufficiently smooth solutions  $(u, v)$  of (1.1), which, along with their first couple of partial derivatives with respect to  $x$ , vanish at  $\pm\infty$ . Thus, if  $(u, v)$  lies in  $C([0, T]; H^3(\mathbb{R}) \times H^3(\mathbb{R}))$ , for example, then (2.12) is valid at least for  $t \in [0, T]$ .

For any  $s \geq 0$ , the quadratic functional  $\Omega : H^s \times H^s \rightarrow \mathbb{R}$  is continuous. It thus follows from the continuous dependence result in Theorem 1 that for  $s \geq 0$ , the conservation property (2.12) continues to hold for the  $(H^s(\mathbb{R}) \times H^s(\mathbb{R}))$ -solutions  $(u, v)$  whose existence was proven in Theorem 1, and not only for smooth solutions. Thus, the functional  $\Omega(u, v)$  is a constant of the motion generated by (1.1).

Once  $a, b$ , and  $c$  are chosen so that (2.7)–(2.8) holds, it appears that the quadratic form

$$(2.13) \quad q(X, Y) = aX^2 + bXY + cY^2$$

is well adapted to the system (1.1).

Assuming once again that we may integrate by parts with impunity without contributions at  $x = \pm\infty$ , it is determined that

$$\begin{aligned}
 \frac{d}{dt}\Omega(u_x, v_x) &= \frac{d}{dt} \int_{-\infty}^{\infty} (au_x^2 + bu_xv_x + cv_x^2) dx \\
 (2.14) \qquad &= \int_{-\infty}^{\infty} (2aP(u, v) + bQ(u, v))u_t \\
 &\quad + \int_{-\infty}^{\infty} (bP(u, v) + 2cQ(u, v))v_t.
 \end{aligned}$$

The restrictions (2.7) and (2.8) are exactly the conditions implying that

$$(2.15) \qquad \frac{\partial}{\partial v}(2aP + bQ) = \frac{\partial}{\partial u}(bP + 2cQ).$$

It follows from (2.15) that there is a cubic polynomial  $R(u, v)$  such that

$$\frac{\partial R}{\partial u} = 2aP + bQ \qquad \text{and} \qquad \frac{\partial R}{\partial v} = bP + 2cQ.$$

Thus, (2.14) may be written

$$\begin{aligned}
 \frac{d}{dt} \int_{-\infty}^{\infty} (au_x^2 + bu_xv_x + cv_x^2) dx &= \int_{-\infty}^{\infty} \left( \frac{\partial R}{\partial u}u_t + \frac{\partial R}{\partial v}v_t \right) dx \\
 &= \frac{d}{dt} \int_{-\infty}^{\infty} R(u, v) dx
 \end{aligned}$$

or, what is the same,

$$\frac{d}{dt} \int_{-\infty}^{\infty} [au_x^2 + bu_xv_x + cv_x^2 - R(u, v)] dx = 0.$$

Thus, the functional

$$(2.16) \qquad \Theta(u, v) = \int_{-\infty}^{\infty} [au_x^2 + bu_xv_x + cv_x^2 - R(u, v)] dx$$

is also an invariant of the temporal evolution of smooth solutions of (1.1). Indeed,  $\Theta$  serves as a Hamiltonian for the system (1.1), but this point is not pursued here. However, it is worth emphasizing again that (1.1) always possesses a Hamiltonian structure. Indeed, as long as

$$4ac - b^2 \neq 0,$$

then the system (1.1) may be written in the Hamiltonian form

$$\frac{\partial}{\partial t} \nabla_{(u,v)} \Omega(u, v) = \frac{\partial}{\partial x} \nabla_{(u,v)} \Theta(u, v),$$

where  $\nabla_{(u,v)}$  connotes the Euler derivative.

The conserved quantities  $\Omega$  in (2.5) and  $\Theta$  in (2.16) are useful for obtaining a priori bounds on solutions of (1.1) when the quadratic form  $q$  in (2.13) vanishes only at the origin, and this is the case precisely when

$$(2.17) \quad 4ac - b^2 > 0.$$

The inequality (2.17) implies that both  $a$  and  $c$  are nonzero; without loss of generality, we may take it that  $a > 0$  so that  $c > 0$  and the quadratic form  $q$  is positive definite.

In case the matrix  $M$  in (2.9) has rank 2, the conditions (2.7)–(2.8) and (2.17) are all satisfied when  $a, b$ , and  $c$  are chosen as in (2.10) (or chosen as any nonzero scalar multiple of this vector) and the inequality

$$(2.18) \quad \begin{aligned} & [E(E - 2A) - 2D(2F - B)] [2C(E - 2A) - B(2F - B)] \\ & = 2EC(E - 2A)^2 - [4CD + BE](E - 2A)(2F - B) \\ & \quad + 2BD(2F - B)^2 \\ & > (4CD - BE)^2 \end{aligned}$$

is satisfied. If  $M$  has rank 1 and  $B = C = D = E = 0$ , the system is uncoupled, and one can choose  $b = 0$  and  $a = c = 1$  to obtain an  $\Omega$  that is positive definite and invariant under the flow generated by (1.1). If, for example,  $M$  has rank 1 and  $B \neq 0$ , one quickly deduces that  $a, b$ , and  $c$  can be chosen satisfying (2.7), (2.8), and the discriminant inequality (2.17) if and only if

$$(2A - E)^2 > -8BD.$$

In this situation, the choices

$$\begin{aligned} a &= 2(2A - E)^2 + 8BD, \\ b &= 4(2A - E)B, \\ c &= 4B^2 \end{aligned}$$

satisfy (2.7), (2.8), and (2.17) and yield a positive definite quadratic form  $q$  as in (2.13). (This choice is not unique, however.) Similar conditions can

be given if  $C \neq 0, D \neq 0$ , or  $E \neq 0$  in the rank 1 case. Once  $a, b$ , and  $c$  are found satisfying (2.7) and (2.8), it is straightforward to determine that the cubic polynomial  $R(u, v)$  is

$$(2.19) \quad R(u, v) = \frac{\alpha}{3}u^3 + \beta u^2v + \gamma uv^2 + \frac{\delta}{3}v^3,$$

where

$$(2.20) \quad \begin{aligned} \alpha &= 2aA + bD, & \beta &= bA + 2cD, \\ \gamma &= 2aC + bF, & \delta &= bC + 2cF. \end{aligned}$$

The polynomial  $R$  will appear again in the next section.

**2.4. A priori estimates and global well-posedness of smooth solutions**

Assuming conditions (2.7), (2.8), and (2.17), the quadratic form in the integrand defining  $\Omega$  is positive definite, and hence there is a  $\lambda > 0$  such that

$$(2.21) \quad \begin{aligned} \int_{-\infty}^{\infty} [u(x, t)^2 + v(x, t)^2] dx &\leq \lambda \int_{-\infty}^{\infty} [au^2 + buv + cv^2] dx \\ &= \lambda \Omega(u_0, v_0) = M_0^2. \end{aligned}$$

Thus, the  $(L^2 \times L^2)$ -norm of solution pairs  $(u(\cdot, t), v(\cdot, t))$  of (1.1) is uniformly bounded in time, and hence the local well-posedness result for  $s = 0$  can be extended to conclude existence of globally defined solutions which, for each  $T > 0$ , lie in  $X_{0,b}^T$ . Within these Bourgain classes, the solutions are unique, and they depend continuously there on variations of the initial data in  $L^2 \times L^2$ .

*2.4.1.  $H^1$ -bounds.* Suppose now that  $(u_0, v_0) \in H^1 \times H^1$ , and let  $(u(\cdot, t), v(\cdot, t))$  be the local solution of (1.1)–(1.2) corresponding to these initial data. It follows from the calculations in Section 2.2 for smooth solutions together with the continuous dependence result that the Hamiltonian  $\Theta(u(\cdot, t), v(\cdot, t))$  is constant in time as long as the solution exists. In consequence, it is seen

that

$$\begin{aligned}
 \int_{-\infty}^{\infty} [u_x^2 + v_x^2] dx &\leq \lambda \Omega(u_x(\cdot, t), v_x(\cdot, t)) \\
 (2.22) \qquad &= \lambda \Theta(u(\cdot, t), v(\cdot, t)) + \lambda \int_{-\infty}^{\infty} R(u(x, t), v(x, t)) dx \\
 &= \lambda \Theta(u_0, v_0) + \lambda \int_{-\infty}^{\infty} R(u(x, t), v(x, t)) dx,
 \end{aligned}$$

where  $\lambda > 0$  is related to the quadratic form  $q$  as in (2.21). Clearly,

$$(2.23) \qquad \lambda \Theta(u_0, v_0) \leq c_2 (1 + |u_0|_2 + |v_0|_2) (\|u_0\|_{H^1}^2 + \|v_0\|_{H^1}^2)$$

for some constant  $c_2$ . Moreover, there is a constant  $c_3$  depending only on  $A, B, \dots, F$  and  $a, b, c$  such that

$$\begin{aligned}
 \int_{-\infty}^{\infty} R(u, v) dx &\leq c_3 \int_{-\infty}^{\infty} (|u|^3 + |v|^3) dx \\
 (2.24) \qquad &\leq c_3 (|u|_{L^\infty} |u|_{L^2}^2 + |v|_{L^\infty} |v|_{L^2}^2) \\
 &\leq c_3 (|u|_{L^2}^{5/2} |u_x|_{L^2}^{1/2} + |v|_{L^2}^{5/2} |v_x|_{L^2}^{1/2}) \\
 &\leq c_3 M_0^{5/2} (|u_x|_{L^2}^{1/2} + |v_x|_{L^2}^{1/2}),
 \end{aligned}$$

where  $M_0$  is the time-independent bound on the  $(L^2 \times L^2)$ -norm of solutions put forward in (2.21). Combining (2.21), (2.22), (2.23), and (2.24), it is concluded that the  $(H^1 \times H^1)$ -norm of  $(u(\cdot, t), v(\cdot, t))$  is bounded, independently of  $t$ . Consequently, the local solution  $(u(\cdot, t), v(\cdot, t))$  emanating from  $(u_0, v_0)$  may be continued indefinitely, thereby obtaining globally defined solutions lying in  $X_{1,b}^T$ , for each  $T > 0$ , as just explained for the case when the initial data lie in  $L^2 \times L^2$ . A point that will find use presently is that (2.22)–(2.24) imply that for any  $T > 0$ ,

$$(2.25) \qquad \|(u, v)\|_{C(0,T;H^1 \times H^1)} \leq c_1 (|(u_0, v_0)|_2) \|(u_0, v_0)\|_{H^1 \times H^1},$$

where  $c_1 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  may be taken to be a continuous, nondecreasing function which, in this case, is independent of  $T$ .



2.4.2. *H<sup>2</sup>-bounds.* Let  $(u, v)$  again be a solution of (1.1) whose first few partial derivatives lie in  $L^2 \times L^2$ . Consider the real-valued function

$$\Omega(u_{xx}(\cdot, t), v_{xx}(\cdot, t))$$

of time, where  $\Omega$  is as in (2.5) with  $a, b, c$  determined by (2.7)–(2.8), and suppose that (2.17) holds. Differentiate this function with respect to  $t$ . After suitable integrations by parts, and using the equations to evaluate temporal derivatives as before, it is found that

$$\begin{aligned} \frac{d}{dt}\Omega(u_{xx}, v_{xx}) &= \int [2aP + bQ]u_{xxxxx} + [bP + 2cQ]v_{xxxxx} \\ &= \alpha \int u^2u_{xxxxx} + \beta \int u^2v_{xxxxx} + 2uvu_{xxxxx} \\ &\quad + \gamma \int v^2u_{xxxxx} + 2uvv_{xxxxx} + \delta \int v^2v_{xxxxx}, \end{aligned}$$

where  $\alpha, \beta, \gamma, \delta$  are the coefficients of the cubic polynomial  $R(u, v)$  in (2.20). Integrations by parts then leads to the equation

$$\begin{aligned} \frac{d}{dt}\Omega(u_{xx}, v_{xx}) &= -5\alpha \int u_x u_{xx}^2 - 5\beta \int v_x v_{xx}^2 - 10\beta \int u_x u_{xx} v_{xx} \\ (2.26) \quad &\quad - 5\gamma \int u_x v_{xx}^2 - 10\gamma \int v_x v_{xx} u_{xx} - 5\delta \int v_x v_{xx}^2. \end{aligned}$$

By itself, the latter equation does not provide global  $H^2$ -bounds on  $u$  and  $v$ . (A local bound may be inferred on a time interval of the form

$$T \leq \frac{C}{\|u_0\|_{H^2}^{1/2} + \|v_0\|_{H^2}^{1/2}},$$

but this can be improved upon, as we shall see.)

Because (2.26) is not definitive on its own, further ruminations are indicated. Guided by the computations made in [19] and [1] for the generalized KdV equation, we enter into a series of calculations, the upshot of which is the collection of formulas

$$\begin{aligned} \frac{d}{dt} \int u_x^2 u &= -3 \int u_x u_{xx}^2 + \int [2u_{xx} u P_x - u_x^2 P_x], \\ \frac{d}{dt} \int v_x^2 v &= -3 \int v_x v_{xx}^2 + \int [2v_{xx} v Q_x - v_x^2 Q_x], \end{aligned}$$

$$\begin{aligned}
 (2.27) \quad & \frac{d}{dt} \int v_x^2 u = -3 \int v_{xx}^2 u_x - \int [P_x v_x^2 - 2u_x v_x Q_x - 2uv_{xx} Q_x], \\
 & \frac{d}{dt} \int u_x^2 v = -3 \int u_{xx}^2 v_x - \int [Q_x u_x^2 - 2u_x v_x P_x - 2vu_{xx} P_x], \\
 & \frac{d}{dt} \int u_x v_x u = -3 \int v_{xx} u_{xx} u_x - \int [u_x v_x P_x + uv_x P_{xx} + u_x u Q_{xx}], \\
 & \frac{d}{dt} \int u_x v_x v = -3 \int v_{xx} u_{xx} v_x - \int [u_x v_x Q_x + u_x v Q_{xx} + v_x v P_{xx}].
 \end{aligned}$$

These formulas are obtained by using the equations (1.1) to evaluate temporal derivatives followed by suitable integrations by parts.

Define  $h : \mathbb{R}^4 \rightarrow \mathbb{R}$  by

$$h(u, v, u_x, v_x) = \alpha u_x^2 u + \beta v_x^2 v + 2\beta u_x v_x u + \delta v_x^2 v + \gamma v_x^2 u + 2\gamma u_x v_x v.$$

Combining (2.26) and (2.27) gives rise to the formula

$$\begin{aligned}
 (2.28) \quad & \frac{d}{dt} \int \left[ q(u_{xx}, v_{xx}) - \frac{5}{3} h(u, v, u_x, v_x) \right] \\
 & = \frac{5}{3} \int \left\{ \alpha [-u_x^2 P_x - 2u_{xx} u P_x] \right. \\
 & \quad + \beta [u_x^2 Q_x - 2vu_{xx} P_x + 2uv_x P_{xx} + 2u_x u Q_{xx}] \\
 & \quad + \gamma [v_x^2 P_x - 2uv_{xx} Q_x + 2vu_x Q_{xx} + 2v_x v P_{xx}] \\
 & \quad \left. + \delta [-v_x^2 Q_x - 2v_{xx} v Q_x] \right\}.
 \end{aligned}$$

While (2.28) does not yield a conservation law, as in (2.12) and (2.16), there is nevertheless sufficient information to conclude that the  $H^2$ -norms of  $u$  and  $v$  are bounded on bounded time intervals, as is now indicated.

Integrating (2.28) with respect to  $t$  over the time interval  $[0, T]$  yields

$$\begin{aligned}
 (2.29) \quad & \Omega(u_{xx}(\cdot, T), v_{xx}(\cdot, T)) = \Omega(u_{xx}(\cdot, 0), v_{xx}(\cdot, 0)) \\
 & \quad + \frac{5}{3} \int h(u(\cdot, T), v(\cdot, T), u_x(\cdot, T), v_x(\cdot, T)) \\
 & \quad - \frac{5}{3} \int h(u(\cdot, 0), v(\cdot, 0), u_x(\cdot, 0), v_x(\cdot, 0)) \\
 & \quad + \frac{5}{3} \int_0^T \int L(x, t) dx dt,
 \end{aligned}$$

where  $L(x, t)$  is the integrand on the right-hand side of (2.28). The function  $L$  is a polynomial in its four variables. The general monomial term in  $L$  has one of the two forms  $rw_yx_zxx$  or  $rw_x y_x z_x$ , where  $r, w, y, z$  stand for  $u$  or  $v$ . If  $M_0$  and  $M_1$  denote the previously derived time-independent bounds

$$(2.30) \quad |u(\cdot, t)|_2 + |v(\cdot, t)|_2 \leq M_0 \quad \text{and} \quad \|u(\cdot, t)\|_{H^1} + \|v(\cdot, t)\|_{H^1} \leq M_1,$$

respectively, where  $M_0$  depends only on  $|u_0|_2 + |v_0|_2$  and  $M_1$  depends only on  $\|u_0\|_{H^1} + \|v_0\|_{H^1}$ , then it is clear that

$$(2.31) \quad \begin{aligned} \left| \int r w y_x z_{xx} \right| &\leq |r|_\infty |w|_\infty |y_x|_2 |z_{xx}|_2 \\ &\leq |r|_2^{1/2} |w|_2^{1/2} |r_x|_2^{1/2} |w_x|_2^{1/2} |y_x|_2 |z_{xx}|_2 \\ &\leq M_0 M_1^2 |z_{xx}|_2 \end{aligned}$$

and

$$(2.32) \quad \begin{aligned} \left| \int r w_x y_x z_x \right| &\leq |r|_\infty |w_x|_\infty |y_x|_2 |z_x|_2 \\ &\leq |r|_2^{1/2} |r_x|_2^{1/2} |w_x|_2^{1/2} |w_{xx}|_2^{1/2} |y_x|_2 |z_x|_2 \\ &\leq M_0^{1/2} M_1^3 |w_{xx}|_2^{1/2}. \end{aligned}$$

The inequalities (2.31) and (2.32), when applied to (2.29), imply that

$$\begin{aligned} \frac{1}{\lambda} (|u_{xx}(\cdot, T)|_2^2 + |v_{xx}(\cdot, T)|_2^2) &\leq \Omega(u_{xx}(\cdot, T), v_{xx}(\cdot, T)) \\ &\leq \Omega(u_{xx}(\cdot, 0), v_{xx}(\cdot, 0)) \\ &\quad + \frac{5}{3} \int h(u(\cdot, T), v(\cdot, T), u_x(\cdot, T), v_x(\cdot, T)) \\ &\quad - \frac{5}{3} \int h(u(\cdot, 0), v(\cdot, 0), u_x(\cdot, 0), v_x(\cdot, 0)) \\ &\quad + C_0 + C_1 \int_0^T (|u_{xx}|_2 + |v_{xx}|_2) dt, \end{aligned}$$

where  $\lambda > 0$  is as in (2.21), associated to the positive definiteness of the quadratic form  $q$  in (2.13), and  $C_0, C_1$  depend only on  $M_0$  and  $M_1$ . The monomials that make up  $h$  have the form  $rs_x w_x$ , where  $r, s$ , and  $w$  are

either  $u$  or  $v$ . Because of the already established  $H^1$ -bound on  $u$  and  $v$ ,

$$(2.33) \quad \left| \int r s_x w_x dx \right| \leq M_0^{1/2} M_1^{5/4},$$

where  $M_0$  and  $M_1$  are as before (see again (2.30)). As a consequence of these calculations, it must be the case that

$$(2.34) \quad |u_{xx}(\cdot, T)|_2^2 + |v_{xx}(\cdot, T)|_2^2 \leq \tilde{C}_0 + C_1 \int_0^T (|u_{xx}(\cdot, t)|_2 + |v_{xx}(\cdot, t)|_2) dt$$

for suitable time-independent constants  $\tilde{C}_0$  and  $C_1$ . Solving the integral inequality (2.34) leads to the a priori bound

$$(2.35) \quad |u_{xx}(\cdot, T)|_2 + |v_{xx}(\cdot, T)|_2 \leq D_0 (|u_{xx}(\cdot, 0)|_2 + |v_{xx}(\cdot, 0)|_2) + D_1 T,$$

where  $D_0$  and  $D_1$  depend only on  $\tilde{C}_0$  and  $C_1$  and hence only on the  $(H^1 \times H^1)$ -norm of the initial data on account of (2.30). Thus, while the  $(H^2 \times H^2)$ -norm of solutions of (1.1) is not necessarily uniformly bounded, it grows no faster than linearly with time  $t$ .

Presently, we will need a bound on the  $(H^2 \times H^2)$ -norm of the solution  $(u, v)$  with a slightly different structure. The result in view is straightforwardly obtained by revisiting the inequalities (2.31) and (2.32). Instead of estimating as above, proceed as follows:

$$(2.36) \quad \begin{aligned} \left| \int r w y_x z_{xx} \right| &\leq |r|_\infty |w|_\infty |y_x|_2 |z_{xx}|_2 \\ &\leq |r|_2^{1/2} |r_x|_2^{1/2} |w|_2^{1/2} |w_x|_2^{1/2} |y_x|_2 |z_{xx}|_2 \\ &\leq |r|_2^{1/2} |r|_2^{1/4} |r_{xx}|_2^{1/4} |w|_2^{1/2} |w|_2^{1/4} |w_{xx}|_2^{1/4} |y|_2^{1/2} |y_{xx}|_2^{1/2} |z_{xx}|_2 \\ &\leq M_0^2 (|r_{xx}|_2^2 + |w_{xx}|_2^2 + |y_{xx}|_2^2 + |z_{xx}|_2^2) \end{aligned}$$

and

$$(2.37) \quad \begin{aligned} \left| \int r w_x y_x z_x \right| &\leq |r|_\infty |w_x|_\infty |y_x|_2 |z_x|_2 \\ &\leq |r|_2^{1/2} |r_x|_2^{1/2} |w_x|_2^{1/2} |w_{xx}|_2^{1/2} |y|_2^{1/2} |y_{xx}|_2^{1/2} |z|_2^{1/2} |z_{xx}|_2^{1/2} \\ &\leq |r|_2^{3/4} |r_{xx}|_2^{1/4} |w|_2^{1/4} |w_{xx}|_2^{3/4} |y|_2^{1/2} |y_{xx}|_2^{1/2} |z|_2^{1/2} |z_{xx}|_2^{1/2} \\ &\leq M_0^2 (|r_{xx}|_2^2 + |w_{xx}|_2^2 + |y_{xx}|_2^2 + |z_{xx}|_2^2). \end{aligned}$$

The elementary inequality

$$|f_x|_2^2 \leq |f|_2 |f_{xx}|_2$$

has been used repeatedly in these estimates.

Using the inequalities (2.36) and (2.37) rather than (2.31) and (2.32) leads to the integral inequality

$$(2.38) \quad \begin{aligned} &|u_{xx}(\cdot, T)|_2^2 + |v_{xx}(\cdot, T)|_2^2 \\ &\leq C_2 + C_3 \int_0^T (|u_{xx}(\cdot, t)|_2^2 + |v_{xx}(\cdot, t)|_2^2) dt, \end{aligned}$$

where  $C_2$  depends on  $M_0, M_1$ , and  $|u_{xx}(\cdot, 0)|_2^2 + |v_{xx}(\cdot, 0)|_2^2$ , while  $C_3$  depends only on  $M_0$ . A Gronwall-type argument then implies immediately that

$$(2.39) \quad \begin{aligned} &|u_{xx}(\cdot, T)|_2^2 + |v_{xx}(\cdot, T)|_2^2 \\ &\leq (|u_{xx}(\cdot, 0)|_2^2 + |v_{xx}(\cdot, 0)|_2^2) e^{C_3 T} + \frac{C_2}{C_3} (e^{C_3 T} - 1). \end{aligned}$$

From this inequality, the inequality (2.33), and the previously deduced bounds on  $(u, v)$  in  $H^1 \times H^1$ , it follows that for any  $T > 0$ ,

$$(2.40) \quad \begin{aligned} &\|u(\cdot, T)\|_{H^2} + \|v(\cdot, T)\|_{H^2} \\ &\leq c_2^T (\|u(\cdot, 0)\|_{H^1} + \|v(\cdot, 0)\|_{H^1}) (\|u(\cdot, 0)\|_{H^2} + \|v(\cdot, 0)\|_{H^2}), \end{aligned}$$

where the function  $c_2^T(z)$  is a continuous function that may grow exponentially in both  $z$  and  $T$ . This version of an  $H^2$ -bound, while inferior to that derived in (2.35) in terms of its growth rate, will find use in the next section.

2.4.3.  $H^s$ -bounds,  $s \geq 0$ . First, it is shown using standard commutator estimates that as soon as  $H^2$ -bounds are in hand, bounds on Sobolev norms of all higher orders follow. To see this, fix an  $s > 2$  and calculate formally as indicated below.

For nonnegative real numbers  $s$ , the operator  $D^s$  is defined by its action

$$\widehat{D^s f}(\xi) = |\xi|^s \widehat{f}$$

on any tempered distribution  $f$ , where the circumflex adorning a distribution connotes that distribution's Fourier transform. Of course,  $D^s$  is self-adjoint as an unbounded operator on the Sobolev spaces  $H^r$ , and, for  $s \geq 0$ ,

the  $H^s$ -norm of a function  $f$  is equivalent to  $|f|_2 + |D^s f|_2$ . A straightforward calculation using the equations (1.1) reveals that

$$\begin{aligned}
 & \frac{d}{dt} \Omega(D^s u, D^s v) \\
 &= \int (2aP + bQ) D^{2s} u_x + (bP + 2cQ) D^{2s} v_x \\
 (2.41) \quad &= \alpha \int D^s(u^2) D^s u_x + \beta \int [D^s(u^2) D^s v_x + 2D^s u_x D^s(vu)] \\
 & \quad + \gamma \int [D^s(v^2) D^s u_x + 2D^s(uv) D^s v_x] + \delta \int D^s(v^2) D^s v_x \\
 &= -\alpha \int 2D^s(u_x u) D^s u - \beta \int 2[D^s(u_x u) D^s v + D^s u D^s(u_x v + v_x u)] \\
 & \quad - \gamma \int 2[D^s(v_x v) D^s u + D^s(u_x v + v_x u) D^s v] - \delta \int 2D^s(v_x v) D^s v,
 \end{aligned}$$

where  $\alpha, \beta, \gamma, \delta$  are, as before, the coefficients of the cubic polynomial  $R(u, v)$  in (2.20). Write the integrand in the first integral on the right-hand side of the last formula in the form

$$\begin{aligned}
 (2.42) \quad D^s(u_x u) D^s u &= [D^s(u_x u) - u D^s u_x] D^s u + u D^s u_x D^s u \\
 &= D^s u [D^s, u] u_x + \frac{1}{2} u ((D^s u)^2)_x,
 \end{aligned}$$

where the commutator  $[D^s, u]$  is defined via its action on a function  $v$  by

$$[D^s, u]v = D^s(uv) - u D^s v.$$

Upon integrating by parts, the first integral on the right-hand side of (2.41) is expressed in the form

$$(2.43) \quad \int D^s(u_x u) D^s u = \int D^s u [D^s, u] u_x - \int \frac{1}{2} u_x (D^s u)^2.$$

The standard commutator estimate

$$(2.44) \quad |[D^s, f]g|_2 = |D^s(fg) - f D^s g|_2 \leq C(\|f'\|_A \|g\|_{s-1} + \|f'\|_{s-1} \|g\|_A),$$

where  $C$  is a constant dependent only upon the value of  $s \geq 1$ , comes to our rescue in trying to estimate the right-hand side of the last integral (see,

e.g., Kato [19, Appendix A]). Here, the  $A$ -norm is simply the  $L^1$ -norm of the Fourier transform, namely,

$$\|f\|_A = \|\widehat{f}\|_{L^1}.$$

Elementary considerations reveal that (2.44) implies the further inequality

$$(2.45) \quad |[D^s, u]u_x|_2 \leq C\|u\|_2\|u\|_s$$

provided that  $s > 3/2$ . Applying (2.45) to (2.43) yields

$$\begin{aligned} \left| \int D^s(u_x u) D^s u \right| &\leq C(\|u\|_2\|u\|_s^2 + |u_x|_\infty |D^s u|_2^2) \\ &\leq C\|u\|_s^2, \end{aligned}$$

with the constant  $C$  depending only on the already established  $H^2$ -bound on solutions. For the second integral, a reorganization of terms combined with two integrations by parts leads to the formula

$$\begin{aligned} &\int D^s(u_x u) D^s v + D^s u D^s(u_x v + v_x u) \\ &= \int [D^s, u]u_x D^s v + \int [D^s, u]v_x D^s u \\ &\quad + \int [D^s, v]u_x D^s u - \int u_x D^s u D^s v - \frac{1}{2} \int v_x (D^s u)^2. \end{aligned}$$

Applying the commutator estimate (2.44) repeatedly, it transpires that the latter integral is bounded above by the quantity  $C(\|u\|_s^2 + \|v\|_s^2)$ , where  $C$  again depends only on the  $H^2$ -norms of  $u$  and  $v$ . The third integral has the same structure as the second, and the fourth integral is handled as was the first. The upshot of all this is the differential inequality

$$\begin{aligned} \frac{d}{dt} [\Omega(D^s u, D^s v) + \Omega(u, v)] &\leq C(\|u\|_s^2 + \|v\|_s^2) \\ &\leq C_1(\Omega(D^s u, D^s v) + \Omega(u, v)), \end{aligned}$$

where the constant  $C$  depends only on  $s$  and the  $H^2$ -norms of  $u$  and  $v$ , while  $C_1$  depends on  $C$  and the value of the parameter  $\lambda$  introduced in (2.21) that derives from the positive definiteness of the quadratic form  $q$ . As the  $H^2$ -norm of the solution pair is bounded on bounded time intervals, it follows

from Gronwall’s lemma that  $|D^s u|_2 + |D^s v|_2$  is bounded on bounded time intervals.

These bounds and those in the preceding section imply that if the initial data  $(u_0, v_0)$  lie in  $H^s \times H^s$ , where  $s \geq 2$ , then the solution  $(u, v)$  emanating therefrom, obtained via Theorem 1, can be extended globally in time. Moreover, the problem is well posed in that, for any  $T > 0$ , the solution depends in  $X_{s,b}^T$  continuously on variations of  $(u_0, v_0)$  in  $H^s \times H^s$ , and it is unique in  $X_{s,b}^T$ .

Next, nonlinear interpolation together with the already established bounds in  $L^2 \times L^2$ ,  $H^1 \times H^1$ , and  $H^2 \times H^2$  is shown to yield a priori bounds in  $H^s \times H^s$  for  $0 < s < 2$  and hence to lead to global well-posedness in these spaces.

The principal ingredient in the analysis to follow is an interpolation theorem, taken from [10] and [30]. The relevant result is quoted here for the reader’s convenience. In the original statements of this result,  $B_0$  and  $B_1$  are two Banach spaces such that  $B_1 \subset B_0$ , with the inclusion mapping being continuous. For values  $(\theta, p)$  such that  $0 < \theta < 1$  and  $1 \leq p < \infty$ ,

$$[B_0, B_1]_{\theta,p} = B_{\theta,p}$$

is the  $K$ -method interpolation space between  $B_0$  and  $B_1$  (see, e.g., [23], [24], and [27]). Thus,  $B_1 \subset B_{\theta,p} \subset B_0$  with continuous inclusions. If  $(\theta_1, p_1)$  and  $(\theta_2, p_2)$  are two pairs as above, then

$$(\theta_1, p_1) \preceq (\theta_2, p_2)$$

means

$$\theta_1 < \theta_2 \quad \text{or} \quad \theta_1 = \theta_2 \quad \text{and} \quad p_1 > p_2.$$

If  $(\theta_1, p_1) \preceq (\theta_2, p_2)$ , then  $B_{\theta_1,p_1} \supset B_{\theta_2,p_2}$  with continuous inclusion. A concrete example of this situation is that for  $0 \leq \theta \leq 1, \theta \neq 1/2$ , and  $r > s$ ,

$$[H^s, H^r]_{\theta,2} = H^\mu,$$

where  $\mu = \theta r + (1 - \theta)s$ , and similarly, for  $T > 0$  fixed,

$$[C([0, T]; H^s), C([0, T]; H^r)]_{\theta,2} = C([0, T]; H^\mu).$$

While the theory of nonlinear operator interpolation to be used presently applies to mappings defined on  $L^p$ -based Sobolev spaces, where  $1 \leq p < \infty$ , we find use here only for the theory in the  $L^2$ -based Hilbert spaces  $H^s$ . In consequence, the proposition quoted below, which is a considerable specialization of [10, Theorems 1, 2], suffices for the present application.



**THEOREM 2.** *Let  $r$  and  $s$ , with  $r > s$ , be two nonnegative real numbers. Suppose that for some  $T > 0$ , the operator  $A$  is defined on both  $H^r \times H^r$  and  $H^s \times H^s$  and maps these spaces continuously into  $C([0, T]; H^r \times H^r)$  and  $C([0, T]; H^s \times H^s)$ , respectively. Suppose in addition that  $A$  respects the inequalities,*

$$(i) \quad \|Af - Ag\|_{C([0, T]; H^s \times H^s)} \leq c_0 (\|f\|_{H^s \times H^s} + \|g\|_{H^s \times H^s}) \|f - g\|_{H^s \times H^s}$$

and

$$(ii) \quad \|Ah\|_{C([0, T]; H^r \times H^r)} \leq c_1 (\|h\|_{H^s \times H^s}) \|h\|_{H^r \times H^r}$$

for some continuous functions  $c_0$  and  $c_1$ . Then, for any  $b \in [s, r]$ ,  $A$  maps  $H^b \times H^b$  continuously into  $C([0, T]; H^b \times H^b)$  and

$$\|Af\|_{C([0, T]; H^b \times H^b)} \leq c_b (\|f\|_{H^s \times H^s}) \|f\|_{H^b \times H^b}$$

where, for  $\gamma > 0$ ,  $c_b(\gamma)$  may be taken in the form  $c_b(\gamma) = 4c_0(4\gamma)^{1-\theta} c_1(3\gamma)^\theta$  with  $c_0$  and  $c_1$  as in (i) and (ii) and  $\theta = (b - s)/(r - s)$ .

**REMARK 2.** In fact, the theorem above does not apply exactly as announced at the midpoint  $\widehat{b} = (1/2)(r + s)$  corresponding to the index  $\theta$  taking the value  $1/2$ . This is because the interpolation space  $[H^s, H^r]_{1/2, 2}$  is not the obvious space (see Lions and Magenes [23]). However, in our case, the interpolation is made between  $L^2$  and  $H^1$  for the values  $0 < s < 1$  and between  $H^1$  and  $H^2$  when  $1 < s < 2$ . Once the advertised result is in hand for values of  $s \neq 1/2, 3/2$ , a simple iteration of the interpolation argument settles the issue for these isolated values as well.

Let  $A$  connote the solution map for the system (1.1). That is,  $A$  associates to initial data  $(u_0, v_0)$  the solution  $(u, v)$  of the system. The local existence theory together with the bounds obtained earlier in integer-order Sobolev classes assures that for any  $T > 0$ ,

$$A : H^s \times H^s \rightarrow C([0, T]; H^s \times H^s),$$

for, say,  $s = 0, 1$ , and  $2$ . Moreover, the solution mapping is Lipschitz in  $L^2 \times L^2$ , which is to say that for any  $T > 0$ , there is a continuous function  $c_0^T$  such that

$$(2.46) \quad \begin{aligned} & \|A(u_0, v_0) - A(\tilde{u}_0, \tilde{v}_0)\|_{C([0, T]; H^0 \times H^0)} \\ & \leq c_0^T (\|u_0\|_{L^2} + \|v_0\|_{L^2} + \|\tilde{u}_0\|_{L^2} + \|\tilde{v}_0\|_{L^2}) \\ & \quad \times \|(u_0 - \tilde{u}_0, v_0 - \tilde{v}_0)\|_{L^2 \times L^2}. \end{aligned}$$

In the first instance, this result is local and follows directly from the proof of local well-posedness via the contraction mapping theorem. However, as the solution evolves in a fixed ball in  $L^2 \times L^2$  on account of the time-independent bounds, it follows that the Lipschitz bound can be iterated, thereby establishing (2.46). Of course, the function  $c_0^T$  may grow as fast as exponentially in time, but this makes no difference to the issue of global existence. (It seems likely that the Lipschitz constant grows linearly with  $T$ , in fact; see, e.g., the theory and simulations reported in [9], [7], and [4].)

Exactly the same considerations reveal that  $A$  is Lipschitz as a mapping of  $H^1 \times H^1$  into  $C([0, T]; H^1 \times H^1)$ . That is, given any  $T > 0$ , there is a continuous function  $c_1^T$  such that

$$\begin{aligned}
 & \|A(u_0, v_0) - A(\tilde{u}_0, \tilde{v}_0)\|_{C([0, T]; H^1 \times H^1)} \\
 (2.47) \quad & \leq c_1^T (\|u_0\|_{H^1} + \|v_0\|_{H^1} + \|\tilde{u}_0\|_{H^1} + \|\tilde{v}_0\|_{H^1}) \\
 & \quad \times \|(u_0 - \tilde{u}_0, v_0 - \tilde{v}_0)\|_{H^1 \times H^1}.
 \end{aligned}$$

Of course,  $c_1^T$  may grow with  $T$ , but  $T$  is fixed in the discussion.

Apply Theorem 2, first to  $A$  considered as a mapping of  $L^2 \times L^2$  into the space  $C([0, T]; L^2 \times L^2)$  and of  $H^1 \times H^1$  into  $C([0, T]; H^1 \times H^1)$ . Note that the hypotheses needed to draw the conclusion of the theorem are exactly the inequalities (2.46) and (2.25). Thus, it follows that for any fixed  $T > 0$  and  $s$  with  $0 < s < 1, s \neq 1/2$ ,  $A$  maps  $H^s \times H^s$  continuously into  $C([0, T]; H^s \times H^s)$  and respects the inequality

$$\begin{aligned}
 & \|A(u_0, v_0)\|_{C([0, T]; H^s \times H^s)} \\
 (2.48) \quad & \leq c_s^T (\|u_0\|_{L^2} + \|v_0\|_{L^2}) \|(u_0, v_0)\|_{H^s \times H^s},
 \end{aligned}$$

where  $c_s^T(z) = 4c_0^T(4z)^{1-\theta}c_1(3z)^\theta$  and  $c_0^T$  and  $c_1$  are as in formulas (2.46) and (2.25). Once this result is in hand, the fact that the same conclusion holds for  $s = 1/2$  follows by reinterpolating between  $s = 0$  and  $s = 3/4$ , for example. The same arguments apply for  $1 < s < 2$  by an application of Theorem 2 and the inequalities (2.47) and (2.40).

Of course, we already know from the local well-posedness theory in Theorem 1 that  $A$  maps  $H^s \times H^s$  continuously into the smaller space  $X_{s,b}^T$ , at least for sufficiently small values of  $T$ . What is new are the a priori bounds in (2.48). These bounds allow the local well-posedness theory to be iterated at least out to time  $T$ . As  $T > 0$  was arbitrary, global well-posedness follows.

The theory thus far extant is summarized here.

**THEOREM 3.** *Let  $T > 0$  be fixed. Suppose that the conditions (2.7), (2.8), and (2.17) on the coefficients  $A, B, \dots, F$  hold for the system (1.1). Then, for any  $s \geq 0$  and any  $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ , there is a pair of functions  $(u, v) \in C([0, \infty) : H^s(\mathbb{R}) \times H^s(\mathbb{R}))$  starting at  $(u_0, v_0)$  when  $t = 0$  that satisfies the system (1.1). For any  $T > 0$ , there is a  $b > 1/2$  such that this solution pair  $(u, v)$  lies in  $X_{s,b}^T$  and is unique within this class. Moreover,  $(u, v)$  depends continuously in  $X_{s,b}^T$  on variations of  $(u_0, v_0)$  in  $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ , and hence continuously in  $C([0, \infty) : H^s(\mathbb{R}) \times H^s(\mathbb{R}))$ , with its Fréchet-space topology, on such variations.*

**REMARK 3.** Indeed, as is apparent from the proofs offered earlier, the dependence of the solution  $(u, v)$  on  $(u_0, v_0)$  is Lipschitz, no matter the value of  $s \geq 0$ .

**REMARK 4.** The results developed in [11] appertaining to the KdV equation can be carried over without essential change to the present context. When carried out, this theory provides a stronger version of uniqueness than that stated above. Roughly speaking, the strengthened theory asserts uniqueness within the class of functions  $C(0, T : H^s \times H^s)$  as soon as solutions in this class can be approximated by smooth solutions. That is to say, uniqueness holds within the class of mild solutions. We pass over this development here. Of course, elementary energy arguments allow us to conclude uniqueness in  $C(0, T : H^s \times H^s)$  whenever  $s > 3/2$  (see, e.g., Saut [28]).

### §3. Preliminaries and the theorem for $-3/4 < s < 0$

Attention is now turned to the more challenging case wherein  $-3/4 < s < 0$ . The present section lays out some preliminary, technical results that will find use in the development of global well-posedness theory in these larger  $H^s$ -spaces. In Section 3.1, the multilinear functionals on which energy inequalities are based are introduced and studied. Once these functionals are sufficiently investigated, a modified “energy” is introduced that plays a central role in the global theory to come in Section 4. It is worth emphasizing that parts of the theory to follow rely upon the presumption that  $s < 0$ . To provide a clear goal for the remainder of this article, the main theorem for the case  $-3/4 < s < 0$  is stated.

#### 3.1. Multilinear functionals and multipliers

The integrals that appear in the energy-type estimates to follow are all multilinear functionals. The assertion of “almost conservation of energy”

depends upon being able to bound these integrals appropriately. This task will occupy most of the rest of the paper. The following simple fact will be useful throughout the discussion:

$$\begin{aligned}
 & \int_{-\infty}^{\infty} f_1(x)f_2(x)\cdots f_n(x) dx \\
 &= \widehat{f_1 f_2 \cdots f_n}(0) \\
 &= \left(\frac{1}{2\pi}\right)^{n-1} \widehat{f_1} * \widehat{f_2} * \cdots * \widehat{f_n}(0) \\
 &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \widehat{f_1}(\xi_1) \cdots \widehat{f_{n-1}}(\xi_{n-1}) \\
 &\quad \times \widehat{f_n}(-\xi_1 - \cdots - \xi_{n-1}) d\mu(\xi_1) \cdots d\mu(\xi_{n-1}) \\
 &= \int_{\xi_1 + \cdots + \xi_n = 0} \widehat{f_1}(\xi_1) \widehat{f_2}(\xi_2) \cdots \widehat{f_n}(\xi_n),
 \end{aligned}$$

where  $d\mu(\xi) = (1/2\pi) d\xi$ . From now on, the notation

$$\begin{aligned}
 (3.1) \quad & \int_{\xi_1 + \cdots + \xi_n = 0} f(\xi_1, \xi_2, \dots, \xi_n) \\
 &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(\xi_1, \xi_2, \dots, -\xi_1 - \cdots - \xi_{n-1}) d\mu(\xi_1) \cdots d\mu(\xi_{n-1})
 \end{aligned}$$

will be used to denote integration over the hyperplane  $\xi_1 + \cdots + \xi_n = 0$ .

The analysis will also feature weighted multilinear functionals. It is convenient to use the notation for these introduced in [16]. Let  $u$  and  $v$  be functions of  $x$  (eventually,  $(u, v)$  will be a solution pair for (1.1) with the temporal variable appearing as a parameter), and let  $\widehat{u}, \widehat{v}$  be their Fourier transforms. Let  $k > 0$  be an integer, and for each pair of nonnegative integers  $(i, j)$  with  $i + j = k$ , suppose that  $f_{ij} : \mathbb{R}^k \rightarrow \mathbb{R}$  are given real-valued functions and that  $a_{ij}$  are specified real constants. The notation  $\mathbf{f} = \{f_{ij}(\xi_1, \xi_2, \dots, \xi_k) : i + j = k \text{ and } i, j \geq 0\}$  for the whole collection of functions will be used. Define  $k$ -linear functionals  $\Lambda^k(\mathbf{f})$  by

$$(3.2) \quad \Lambda^k(\mathbf{f}) = \sum_{i+j=k} a_{ij} \Lambda_{ij}^k(f_{ij}),$$

where

$$(3.3) \quad \Lambda_{ij}^k(h) = \int_{\xi_1 + \cdots + \xi_k = 0} h(\xi_1, \xi_2, \dots, \xi_k) \widehat{u}(\xi_1) \cdots \widehat{u}(\xi_i) \widehat{v}(\xi_{i+1}) \cdots \widehat{v}(\xi_k).$$

Notice that if  $f_{ij} = g$ , say, for all  $i, j$ , then

$$\Lambda^k(\mathbf{f}) = \sum_{i+j=k} a_{ij} \Lambda_{ij}^k(g).$$

In particular, for the special case  $f_{ij}(\xi_1, \xi_2, \dots, \xi_k) = 1$  for all the relevant  $i, j$ ,

$$\Lambda^k(1) = \int_{-\infty}^{\infty} p(u, v) dx,$$

where

$$p(u, v) = \sum_{i+j=k} a_{ij} u(x)^i v(x)^j.$$

**3.2. The modified energy  $H(t)$**

In this section, we introduce the function  $Iu$ , which agrees with  $u$  in the low frequencies but is attenuated by a factor of  $|\xi|^s$  in the high frequencies. Specifically,  $I$  is a Fourier-multiplier operator defined by  $\widehat{Iu}(\xi) = m(\xi)\widehat{u}(\xi)$  whose symbol  $m$  is such that

$$(3.4) \quad m(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq N, \\ N^{-s}|\xi|^s & \text{if } |\xi| \geq 2N. \end{cases}$$

The symbol  $m$  is chosen to be smooth, positive, even, monotone decreasing for  $\xi > 0$  and to satisfy the inequalities

$$(3.5) \quad |m'(\xi)| \leq C \frac{m(\xi)}{|\xi|}$$

and

$$(3.6) \quad |m''(\xi)| \leq C \frac{m(\xi)}{|\xi|^2}$$

for some constant  $C > 0$ . If it is presumed that  $N \geq 1$ , then it is straightforward to ascertain that

$$(3.7) \quad \|(Iu_0, Iv_0)\|_{L_x^2 \times L_x^2} \leq CN^{-s} \|(u_0, v_0)\|_{H^s \times H^s}.$$

For technical reasons arising subsequently,  $N$  is taken to be a dyadic number  $2^r$  for some integer  $r \geq 0$ .

Note that the function  $m(\xi)$  also satisfies the halving condition

$$(3.8) \quad m(\xi) \leq 2^{-s}m(2\xi),$$

which holds for all  $\xi > 0$ . This is clear from the definition (3.4) if  $\xi \geq N$  or  $0 \leq \xi \leq N/2$ . If  $N/2 \leq \xi \leq N$ , then  $m(\xi) = 1 = 2^{-s}m(2N) \leq 2^{-s}m(2\xi)$ .

For a pair of real-valued functions  $(u, v)$ , define the modified energy  $H(t)$  to be

$$(3.9) \quad H(t) = \Omega(Iu, Iv)^{1/2} = \left\{ \int a_{20}(Iu)^2 + a_{11}IuIv + a_{02}(Iv)^2 \right\}^{1/2},$$

where the coefficients  $a = a_{20}, b = a_{11}$ , and  $c = a_{02}$  are assumed to satisfy the equations (2.7) and (2.8), which in the present notation are

$$(3.10) \quad 2a_{20}B + a_{11}E = 2a_{11}A + 4a_{02}D$$

and

$$(3.11) \quad 4a_{20}C + 2a_{11}F = 2a_{02}E + a_{11}B,$$

and the positive-definiteness condition (2.17), namely,

$$(3.12) \quad 4a_{20}a_{02} - a_{11}^2 > 0.$$

As before, it may be assumed without loss of generality that  $a_{20} > 0$ , whence  $a_{02} > 0$  as well. Because the integrand is nonnegative, the positive square root defines  $H$  as a nonnegative function. Since  $u$  and  $v$  are real-valued functions, it transpires that  $\widehat{u}(\xi_1) = \widehat{u}(-\xi_1)$ ,  $\widehat{v}(\xi_1) = \widehat{v}(-\xi_1)$ , and thus, by Plancherel's formula,

$$\begin{aligned} H(t)^2 &= \int a_{20}(Iu)^2 + a_{11}IuIv + a_{02}(Iv)^2 \\ &= \frac{1}{2\pi} \int a_{20}m(\xi_1)\widehat{u}(\xi_1)\overline{m(\xi_1)\widehat{u}(\xi_1)} + a_{11}m(\xi_1)\widehat{u}(\xi_1)\overline{m(\xi_1)\widehat{v}(\xi_1)} \\ &\quad + a_{02}m(\xi_1)\widehat{v}(\xi_1)\overline{m(\xi_1)\widehat{v}(\xi_1)} d\xi_1 \\ (3.13) \quad &= \int_{\xi_1+\xi_2=0} m(\xi_1)m(\xi_2) [a_{20}\widehat{u}(\xi_1)\widehat{u}(\xi_2) \\ &\quad + a_{11}\widehat{u}(\xi_1)\widehat{v}(\xi_2) + a_{02}\widehat{v}(\xi_1)\widehat{v}(\xi_2)] \\ &= a_{20}\Lambda_{20}^2(m_2) + a_{11}\Lambda_{11}^2(m_2) + a_{02}\Lambda_{02}^2(m_2) = \Lambda^2(m_2), \end{aligned}$$

where the  $\Lambda_{ij}^2$  are as defined in the last section and  $M_2(\xi_1, \xi_2) = m(\xi_1)m(\xi_2) = m(\xi_1)^2 = m(\xi_2)^2$  because  $\xi_1 = -\xi_2$  and  $m$  is an even function.

REMARK 5. Since  $H(t)^2 = \int a_{20}(Iu)^2 + a_{11}IuIv + a_{02}(Iv)^2 \approx \|(Iu, Iv)\|_{L_x^2 \times L_x^2}^2$ , one can replace  $\|(Iu, Iv)\|_{L_x^2 \times L_x^2}$  by  $H(t)$  in (3.7). (The notation  $A \approx B$  means that there are positive constants  $c_1$  and  $c_2$  such that  $c_1B \leq A \leq c_2B$ .) In particular,  $\|(Iu_0, Iv_0)\|_{L_x^2 \times L_x^2} \approx H(0)$ .

### 3.3. The main theorem

While the modified energy is not preserved as a function of time, its rate of growth is slow. In fact, it will be shown that by choosing  $N$  in (3.4) large enough, one can iterate Theorem 1 on intervals of length 1 sufficiently often to obtain solutions defined on intervals of arbitrary length.

Establishing control of the growth of  $H(t)$  relies on the following two lemmas.

LEMMA 1. *Let the pair  $(u, v)$  be a solution of equations (1.1) on some time interval  $[t_0, t_1]$  in  $C([t_0, t_1]; H^s \times H^s)$ , and let  $H(t)$  be the modified energy defined in (3.13). Then there exist trilinear functionals  $\Lambda^3$  and  $\tilde{\Lambda}^3$ , quadrilinear functionals  $\Lambda^4$  and  $\tilde{\Lambda}^4$ , and a quintilinear functional  $\Lambda^5$  satisfying the equations*

$$(3.14) \quad \begin{aligned} \frac{d}{dt}H(t)^2 &= i\Lambda^3(t), \\ \frac{d}{dt}\{H(t)^2 - \tilde{\Lambda}^3\} &= i\Lambda^4(t), \\ \frac{d}{dt}\{H(t)^2 - \tilde{\Lambda}^3 - \tilde{\Lambda}^4\} &= i\Lambda^5(t). \end{aligned}$$

*Proof.* The proof of this lemma is deferred until Section 5, where explicit expressions for  $\Lambda^3, \tilde{\Lambda}^3, \Lambda^4$ , and  $\tilde{\Lambda}^4$  will be forthcoming.  $\square$

LEMMA 2. *(Suppose that the index  $s$  lies in  $(-3/4, 0)$ .) For a pair of functions  $(u, v)$  in  $C([t_0, t_1]; H^s \times H^s)$ , satisfying*

$$(3.15) \quad \begin{cases} u_t + u_{xxx} + A(u^2)_x + B(uv)_x + C(v^2)_x = 0, \\ v_t + v_{xxx} + D(u^2)_x + E(uv)_x + F(v^2)_x = 0 \end{cases}$$

*with initial data*

$$(3.16) \quad u(x, t_0) = u_0(x) \quad \text{and} \quad v(x, t_0) = v_0(x).$$

*Let  $H(t)$  be the corresponding modified energy. If  $\tilde{\Lambda}^3$  and  $\tilde{\Lambda}^4$  are the functionals referred to in Lemma 1, then there are constants  $c_3$  and  $c_4$  independent of  $t_0$  and  $t_1$  such that for  $t \in [t_0, t_1]$ ,*

$$(3.17) \quad |\tilde{\Lambda}^3(t)| \leq c_3H(t)^3 \quad \text{and} \quad |\tilde{\Lambda}^4(t)| \leq c_4H(t)^4.$$

Moreover, there is a  $0 < \delta < t_1 - t_0$  such that for  $t \in [t_0, t_0 + \delta]$ ,

$$(3.18) \quad \left| \int_{t_0}^t \frac{d}{dt} \{H(t)^2 - \tilde{\Lambda}^3 - \tilde{\Lambda}^4\} \right| \leq CN^{5s} \|(Iu, Iv)\|_{X_{0,(1/2)^+}^{[t_0,t]}}^5 \\ \leq c_5 N^{5s} \|(Iu_0, Iv_0)\|_{L^2 \times L^2},$$

where  $C$  is independent of  $\delta, t_0$ , and  $t_1$ . The constant  $c_5$  in the last inequality is independent of  $t$ .

REMARK 6. The estimates of  $\tilde{\Lambda}^3$  and  $\tilde{\Lambda}^4$  are established in [16, Section 6]. In [16, proof of Lemma 6.1, pp. 724–725], the estimate for  $\tilde{\Lambda}_3$  in (3.17) assumes that  $-3/4 < s < -1/2$ . This leaves a gap in the global existence theory since the proof of global existence depends on this estimate. A slight modification of the proof of [16, Lemma 6.1] is provided at the end of Section 7, which shows that the inequality (3.17) still holds for the full range  $-3/4 < s < 0$ .

The first inequality in (3.18) is a more precise version of estimate [16, (5.6)] and is proved here in Section 7. The second inequality follows from Proposition 1 in the next section. More precisely,

$$\|(Iu, Iv)\|_{X_{0,(1/2)^+}^{[t_0,t]}} \leq \|(Iu, Iv)\|_{X_{0,(1/2)^+}^{[t_0,t_0+\delta]}}$$

for  $t \in [t_0, t_0 + \delta]$ , and by Proposition 1,

$$\|(Iu, Iv)\|_{X_{0,(1/2)^+}^{[t_0,t_0+\delta]}} \leq \tilde{C} \|(Iu_0, Iv_0)\|_{L^2 \times L^2},$$

so  $c_5$  is a product of  $C$  in the lemma and  $\tilde{C}$ . It is used to estimate the growth of  $H(t)^2 - \tilde{\Lambda}^3 - \tilde{\Lambda}^4$  (see (4.5) and (4.6) in Section 4).

Here is the principal result of the remainder of the paper. A proof is provided in the next section, subject to a technical point which will be discussed in Section 5.

THEOREM 4. Fix an  $s$  in the range  $-3/4 < s < 0$ , and let real numbers  $A, B, \dots, F$  be given. Define the matrix  $M$  by

$$(3.19) \quad M = \begin{bmatrix} 2B & E - 2A & -4D \\ 4C & 2F - B & -2E \end{bmatrix}.$$

Then the KdV system

$$(3.20) \quad \begin{cases} u_t + u_{xxx} + A(u^2)_x + B(uv)_x + C(v^2)_x = 0, \\ v_t + v_{xxx} + D(u^2)_x + E(uv)_x + F(v^2)_x = 0 \end{cases}$$



posed for  $x \in \mathbb{R}$  and  $t \geq 0$ , with specified initial data

$$(3.21) \quad u(x, 0) = u_0(x) \quad \text{and} \quad v(x, 0) = v_0(x),$$

in  $H^s(\mathbb{R}) \times H^s(\mathbb{R})$  has global solutions if rank  $M = 2$  and

$$(3.22) \quad \begin{aligned} &2EC(E - 2A)^2 + 2BD(2F - B)^2 \\ &- [4CD + BE](E - 2A)(2F - B) > (4CD - BE)^2 \end{aligned}$$

or rank  $M = 1$  and

$$(3.23) \quad (2A - E)^2 + 8BD > 0 \quad \text{or} \quad (2F - B)^2 + 8EC > 0.$$

Moreover, there is a real number  $b > 1/2$  such that for all  $T > 0$ ,  $(u, v)$  lies in the Bourgain space  $X_{s,b}^T \times X_{s,b}^T$ . The solution is unique in this latter class and depends continuously in this space on variations of  $(u_0, v_0)$  in  $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ .

**§4. Global well-posedness: The case  $-3/4 < s < 0$**

This section indicates how the inequalities in Lemma 2 can be used to prove that for any  $s$  with  $0 > s > -3/4$ ,  $T > 0$ , and any initial data  $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ , there is a solution  $(u, v)$  to equation (1.1) on the interval  $[0, T]$ . Note that if it can be established that a solution  $(u, v)$  to the initial-value problem (1.1)–(1.2) exists on  $[0, 1]$  for arbitrarily large initial data  $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ , then this result may be iterated to extend the solution to the interval  $[1, 2]$ , and so on. Thus, without loss of generality, we may assume that  $T = 1$ .

The proof has two major steps. The first shows there is an  $H_0 > 0$  so that for all initial data  $(u_0, v_0)$  with modified energy  $H(0) < H_0$ , there is a solution to the initial-value problem (1.1)–(1.2) valid on the entire interval  $[0, 1]$ . The second uses the following scaling argument to complete the proof. If  $\lambda > 0$ , then  $(u(x, t), v(x, t))$  satisfies the system (1.1) if and only if  $\lambda^2(u(\lambda x, \lambda^3 t), v(\lambda x, \lambda^3 t))$  satisfies the system (1.1). Hence, the pair  $(u(x, t), v(x, t))$  satisfies the initial-value problem (1.1)–(1.2) on  $[0, 1]$  with initial data  $(u_0, v_0)$  if there exists  $(u_\lambda(x, t), v_\lambda(x, t))$  satisfying (1.1)–(1.2) on interval  $[0, \lambda^3]$  with  $(u_\lambda(x, 0), v_\lambda(x, 0)) = (1/\lambda^2)(u_0(x/\lambda), v_0(x/\lambda))$ . It will be shown that no matter how large is  $(u_0, v_0)$ , one can choose  $\lambda$  and the parameter  $N$  appearing in the definition of the multiplier  $Iu$  so that the initial data  $(1/\lambda^2)(u_0(x/\lambda), v_0(x/\lambda))$  has  $H(0) < H_0$  and, moreover, that the local

solution process can be iterated  $\lambda^3$  times without violating the condition  $H(t) < H_0$ .

To carry out the first step in the proof, it is useful to reframe the initial-value problem (1.1)–(1.2) so that the size of the initial condition is expressed in terms of the modified energy  $H(t)$  rather than via the  $(H^s(\mathbb{R}) \times H^s(\mathbb{R}))$ -norm. The following alternative version of local well-posedness is convenient for this purpose.

**PROPOSITION 1.** *If  $s > -3/4$ , the initial-value problem (1.1)–(1.2) is locally well posed for data  $(u_0, v_0)$  satisfying  $(Iu_0, Iv_0) \in L^2 \times L^2$ . The solution exists on a time interval  $[0, \delta]$ , and there are positive constants  $C$  and  $\alpha$ , independent of  $N$ , such that  $\delta$  respects the lower bound*

$$C \|(Iu_0, Iv_0)\|_{L^2 \times L^2}^{-\alpha} \leq \delta.$$

*In addition, there is another positive constant  $C$ , also independent of  $N$ , such that for all  $t \in [0, \delta]$ ,*

$$\|(Iu, Iv)\|_{X_{0, (1/2)^+}^{[0, t]}} \leq C \|(Iu_0, Iv_0)\|_{L^2 \times L^2}.$$

This is [16, Proposition 2], adapted to the system of equations studied here. The estimates for the Duhamel terms are exactly the same as in the latter reference because they are all of the form

$$\int_0^t W(t-t') \frac{\partial}{\partial x} (u(x, t')v(x, t')) dt'.$$

It is now shown how these local results can be iterated sufficiently often to infer existence of a solution on the time interval  $[0, 1]$ , provided that the original data are sufficiently small.

### 4.1. Energy growth

From Proposition 1, the following remark is evident.

**REMARK 7.** There exists an  $\epsilon_1 > 0$  small enough that if the initial data have  $H(0) < \epsilon_1$ , then there is a solution to (1.1)–(1.2) defined at least on the interval  $[0, 1]$ . The parameter  $\epsilon_1$  is independent of  $N$ .

The next lemma is useful in controlling the growth of  $H(t)$  when  $H(0)$  is sufficiently small.

LEMMA 3. *Suppose that  $(u, v)$  is a solution of the initial-value problem (1.1)–(1.2) on at least the time interval  $[0, t_0]$  and that  $\lceil t_0 \rceil + 1 \leq N^{-5s}$ . Let  $H(t)$  be the modified energy associated to  $(u, v)$ . There is a constant  $\epsilon_0$  independent of values of  $t_0 \leq N^{-5s}$ , such that if  $H(0) < \epsilon_0$  and  $H(t) < 2\epsilon_0$  for  $t \in [0, t_0]$ , then  $(u, v)$  can be extended to a solution of (1.1)–(1.2) on the longer time interval  $[0, t_0 + 1]$ . In addition, it is still the case that  $H(t) < 2\epsilon_0$  for  $t \in [0, t_0 + 1]$ .*

*Proof.* To begin with, choose  $\epsilon_0$  so that  $0 < \epsilon_0 < \min\{1, (1/2)\epsilon_1\}$ , where  $\epsilon_1$  is defined as in Remark 7. Since  $H(t_0) < 2\epsilon_0 \leq \epsilon_1$ , it follows that if (1.1)–(1.2) is posed with initial data  $(u(\cdot, t_0), v(\cdot, t_0))$ , then there is a solution  $(u_1, v_1)$  defined at least for a time interval of length 1. Setting

$$\tilde{u}(x, t) = \begin{cases} u(x, t) & \text{if } 0 \leq t \leq t_0, \\ u_1(x, t) & \text{if } t_0 \leq t \leq t_0 + 1, \end{cases}$$

and similarly for  $v$ , yields an extension of  $(u, v)$  to  $(\tilde{u}, \tilde{v})$  defined for  $t \in [0, t_0 + 1]$ .

Attention is now turned to providing the bound  $H(t) < 2\epsilon_0$  on  $[0, t_0 + 1]$ . This is the crux of the matter, in fact. For convenience, define  $\Gamma^4$  by

$$(4.1) \quad \Gamma^4(t) = H(t)^2 - \tilde{\Lambda}^3(t) - \tilde{\Lambda}^4(t),$$

and write

$$(4.2) \quad H(t)^2 = \tilde{\Lambda}^3(t) + \tilde{\Lambda}^4(t) + [\Gamma^4(t) - \Gamma^4(0)] - [\tilde{\Lambda}^3(0) + \tilde{\Lambda}^4(0)] + H(0)^2.$$

At least on the interval  $0 \leq t \leq t_0 + 1$ , Lemma 2 implies that

$$(4.3) \quad |\tilde{\Lambda}^3(t) + \tilde{\Lambda}^4(t)| \leq c_3 H(t)^3 + c_4 H(t)^4$$

and that

$$(4.4) \quad |\tilde{\Lambda}^3(0) + \tilde{\Lambda}^4(0)| \leq c_3 H(0)^3 + c_4 H(0)^4.$$

Furthermore, from Lemma 2 and the fundamental theorem of calculus, if  $0 \leq t \leq t_0$  and  $0 \leq \delta \leq 1$ , then

$$(4.5) \quad \begin{aligned} & |\Gamma^4(t + \delta) - \Gamma^4(t)| \\ &= |[H(t + \delta)^2 - \tilde{\Lambda}^3(t + \delta) - \tilde{\Lambda}^4(t + \delta)] - [H(t)^2 - \tilde{\Lambda}^3(t) - \tilde{\Lambda}^4(t)]| \\ &\leq c_5 N^{5s} H(t)^5. \end{aligned}$$

This latter inequality can be improved as follows. Since, by assumption,  $H(t) < 2\epsilon_0$  for  $0 \leq t \leq t_0$ , it transpires that for  $t_0 \leq t \leq t_0 + 1$ ,

$$\begin{aligned}
 |\Gamma^4(t) - \Gamma^4(0)| &\leq |\Gamma^4(t) - \Gamma^4(t_0)| \\
 &\quad + \sum_{j=1}^{\lfloor t_0 \rfloor} |\Gamma^4(t_0 - j + 1) - \Gamma^4(t_0 - j)| \\
 &\quad + \Gamma^4(t_0 - \lfloor t_0 \rfloor) - \Gamma^4(0) \\
 (4.6) \qquad &\leq c_5 N^{5s} (2\epsilon_0)^5 + \sum_{j=1}^{\lfloor t_0 \rfloor} c_5 N^{5s} (2\epsilon_0)^5 + c_5 N^{5s} (\epsilon_0)^5 \\
 &\leq 32c_5 n N^{5s} \epsilon_0^5,
 \end{aligned}$$

where  $n = \lfloor t_0 \rfloor + 2 = \lceil t_0 \rceil + 1$ . Substituting in (4.2) and using inequalities (4.3), (4.4), and (4.6), it transpires that for  $t \in [t_0, t_0 + 1]$ ,

$$H(t)^2 < H(0)^2 + c_3 H(0)^3 + c_4 H(0)^4 + 32c_5 n N^{5s} \epsilon_0^5 + c_3 H(t)^3 + c_4 H(t)^4.$$

Because  $H(0) < \epsilon_0$  and  $\lceil t_0 \rceil \leq N^{-5s}$ , the right-hand side of the last inequality can be further bounded above, leading to the inequality

$$(4.7) \qquad H(t)^2 < \epsilon_0^2 + c_3 \epsilon_0^3 + c_4 \epsilon_0^4 + 32c_5 \epsilon_0^5 + c_3 H(t)^3 + c_4 H(t)^4,$$

which applies for all  $t \in [t_0, t_0 + 1]$ . As  $H(t_0) < 2\epsilon_0$  and  $H$  is continuous, either  $H(t) < 2\epsilon_0$  for all  $t \in [t_0, t_0 + 1]$  or else there is a  $t^* > t_0$  such that  $H(t^*) = 2\epsilon_0$ . Evaluating the inequality in the last display at  $t = t^*$  yields

$$4\epsilon_0^2 < \epsilon_0^2 + c_3 \epsilon_0^3 + c_4 \epsilon_0^4 + 32c_5 \epsilon_0^5 + 8c_3 \epsilon_0^3 + 16c_4 \epsilon_0^4$$

or, since  $\epsilon_0 < 1$ ,

$$1 < \frac{1}{3} \epsilon_0 (9c_3 + 17c_4 + 32c_5).$$

This plainly cannot occur for small values of  $\epsilon_0$ .

To recapitulate, if  $\epsilon_0 > 0$  is such that

$$(4.8) \qquad \epsilon_0 < 1, \qquad \epsilon_0 < \frac{1}{2} \epsilon_1 \qquad \text{and} \qquad \epsilon_0 < \frac{3}{9c_3 + 17c_4 + 32c_5},$$

then for initial data  $(u_0, v_0)$  whose energy satisfies  $H(0) < \epsilon_0$ , it must be the case that  $H(t) < 2\epsilon_0$  on the entire interval  $[0, t_0 + 1]$ . The lemma is thus established. □

**4.2. Proof of Theorem 4**

Let  $(u_0, v_0) \in H^s \times H^s$  be arbitrary initial data. As mentioned earlier, it suffices to show that (1.1)–(1.2) has an  $(H^s \times H^s)$ -solution emanating from  $(u_0, v_0)$  that exists at least for  $t \in [0, 1]$ . Let  $\lambda \geq 1$ , and consider the scaled initial data

$$(u_{0\lambda}, v_{0\lambda}) = \frac{1}{\lambda^2} \left( u_0 \left( \frac{x}{\lambda} \right), v_0 \left( \frac{x}{\lambda} \right) \right).$$

An elementary calculation reveals that the modified energy  $H = H_\lambda$  associated to  $(u_{0\lambda}, v_{0\lambda})$  is bounded in terms of  $\lambda$  and the energy  $H_1 = H(0)$ , namely,

$$\begin{aligned} H_\lambda(0) &= \left( \int a_{20}(Iu_{\lambda,0})^2 + a_{11}Iu_{\lambda,0}Iv_{\lambda,0} + a_{02}(Iv_{\lambda,0})^2 \right)^{1/2} \\ (4.9) \quad &\leq C \|(Iu_{0\lambda}, Iv_{0\lambda})\|_{L^2 \times L^2} \\ &\leq C' N^{-s} \|(u_{0\lambda}, v_{0\lambda})\|_{H^s \times H^s} \\ &\leq \tilde{C} \lambda^{-(3+2s)/2} N^{-s} \|(u_0, v_0)\|_{H^s \times H^s}, \end{aligned}$$

provided that  $s \leq 0$  and  $\lambda, N \geq 1$ , say.

Let  $s$  be given with  $-3/4 < s \leq 0$ . Existence of a solution of (1.1)–(1.2) in  $C([0, 1]; H^s \times H^s)$  starting at  $(u_0, v_0)$  is equivalent to the existence of a solution  $(u_\lambda, v_\lambda)$  starting at  $(u_{0\lambda}, v_{0\lambda})$  on the interval  $[0, \lambda^3]$ . Select  $\epsilon_1 > 0$  to satisfy the condition specified in Remark 7, and then select  $\epsilon_0 > 0$  satisfying the strictures of (4.8). Choose  $N \geq 1$  such that

$$\begin{aligned} N^{-5s} &\geq N^{-6s/(3+2s)} \left[ \frac{\tilde{C} \|(u_0, v_0)\|_{H^s \times H^s}}{\epsilon_0} \right]^{6/(3+2s)} \\ &= \left( \left( N^{-s} \left[ \frac{\tilde{C} \|(u_0, v_0)\|_{H^s \times H^s}}{\epsilon_0} \right] \right)^{2/(3+2s)} \right)^3. \end{aligned}$$

This is possible precisely when  $-5s + (6s/3 + 2s) > 0$ , which occurs if and only if  $s < -3/2$  or  $-9/10 < s < 0$ . Since the local theory works only for  $s > -3/4$ , the values of  $s$  are restricted to the interval  $-3/4 < s < 0$ . In particular, this argument fails when  $s \geq 0$ .

REMARK 8. An estimate similar to (3.18) occurs in [16] with the exponent  $-15/4+$  instead of  $5s$ . If  $s = -3/4 + \epsilon/5$ , then  $5s = -15/4 + \epsilon$ . This gives the impression that the above argument could work for any  $s > -3/4$ , which as just shown fails for  $s \geq 0$ . The failure of this argument for  $s > 0$  explains the use of very different methods for the  $H^s$  theory for positive  $s$ .

Once  $\epsilon_0$  and  $N$  are fixed, let

$$\lambda = \max \left\{ 1, \left( N^{-s} \left[ \frac{\tilde{C} \|(u_0, v_0)\|_{H^s \times H^s}}{\epsilon_0} \right] \right)^{2/(3+2s)} \right\},$$

so that

$$N^{-5s} \geq \lambda^3 \quad \text{and} \quad \tilde{C} \lambda^{-(3+2s)/2} N^{-s} \|(u_0, v_0)\|_{H^s \times H^s} \leq \epsilon_0.$$

With the above choices of  $\lambda$  and  $N$ , we may apply Lemma 3  $N^{-5s}$  times to obtain a solution  $(u_\lambda, v_\lambda)$  of (1.1), with initial data  $(u_{0\lambda}, v_{0\lambda})$ , on the time interval  $[0, N^{-5s}]$ . A solution pair  $(u(x, t), v(x, t)) = (\lambda^2 u_\lambda(\lambda x, \lambda^3 t), \lambda^2 v_\lambda(\lambda x, \lambda^3 t))$  of (1.1) is thereby generated which emanates from the initial data  $(u_0, v_0)$  and which is defined at least on the interval  $[0, \lambda^{-3} N^{-5s}] \supseteq [0, 1]$ . Theorem 4 is proved.

### §5. Components of the modified energy

The analysis leading to a proof of Lemma 1 is carried out in Sections 5.1 and 5.2. Recalling the definition of the modified energy (3.4) and using the Plancherel identity, it is seen that

$$\begin{aligned} \Lambda^2(t) &= H(t)^2 \\ &= \int a_{20}(Iu)^2 + a_{11}IuIv + a_{02}(Iv)^2 \\ &= \int_{\xi_1+\xi_2=0} m(\xi_1)m(\xi_2) [a_{20}\hat{u}(\xi_1)\hat{u}(\xi_2) + a_{11}\hat{u}(\xi_1)\hat{v}(\xi_2) + a_{02}\hat{v}(\xi_1)\hat{v}(\xi_2)] \\ &= \Lambda^2(M_2). \end{aligned}$$

Assume that the pair  $(u, v)$  satisfies the KdV system (1.1). Given  $k \geq 2$  and function  $h$ , there is an associated vector  $\mathbf{h} = \{h_{rs} : r + s = k + 1\}$  of multipliers such that we have the following useful equation:

$$\begin{aligned} &\frac{d}{dt} \Lambda_{rs}(h) \\ &= i \int_{\xi_1+\dots+\xi_k=0} (\xi_1^3 + \dots + \xi_k^3) h(\xi_1, \dots, \xi_k) \\ &\quad \times \hat{u}(\xi_1) \cdots \hat{u}(\xi_r) \hat{v}(\xi_{r+1}) \cdots \hat{v}(\xi_{r+s}) \\ (5.1) \quad &+ \sum_{i+j=k+1} a_{ij} \Lambda_{ij}(h_{ij}) \end{aligned}$$

$$\begin{aligned}
 &= i \int_{\xi_1 + \dots + \xi_k = 0} \alpha_k h(\xi_1, \dots, \xi_k) \widehat{u}(\xi_1) \cdots \widehat{u}(\xi_r) \widehat{v}(\xi_{r+1}) \cdots \widehat{v}(\xi_{r+s}) \\
 &\quad + \sum_{i+j=k+1} a_{ij} \Lambda_{ij}(h_{ij}) \\
 &= i \Lambda_{rs}(\alpha_k h) + \sum_{i+j=k+1} a_{ij} \Lambda_{ij}(h_{ij}),
 \end{aligned}$$

where

$$\alpha_k = \alpha_k(\xi_1, \dots, \xi_k) = \xi_1^3 + \dots + \xi_k^3$$

and  $\{a_{ij}, h_{ij} : i + j = k + 1\}$  depends on  $h$  and the coefficients  $A, B, \dots, F$  in the KdV system (1.1). We will often write simply  $\alpha_k$ , omitting the explicit dependence on the independent variables  $\xi_1, \dots, \xi_k$  unless they are needed. In general, finding the exact form of  $\{a_{ij}, h_{ij} : i + j = k + 1\}$  from  $h$  involves a fair amount of calculation. Explicit formulas for the  $a_{ij}$  and  $h_{ij}$  are derived in the following two sections for the cases  $k = 3$  and  $k = 4$ .

**5.1. The functional  $\Lambda^3$**

LEMMA 4. For  $H(t)$  defined as in (3.13), we have  $\frac{d}{dt}H(t)^2 = i\Lambda^3$ , where

$$(5.2) \quad \Lambda^3 = \sum_{i+j=3} a_{ij} \Lambda_{ij}(M_3),$$

with coefficients  $a_{ij}$  given by

$$(5.3) \quad \begin{aligned}
 a_{30} &= \frac{1}{3}(2a_{20}A + a_{11}D), & a_{21} &= a_{11}A + 2a_{02}D, \\
 a_{12} &= a_{11}F + 2a_{20}C, & a_{03} &= \frac{1}{3}(2a_{02}F + a_{11}C)
 \end{aligned}$$

and the multiplier  $M_3$  defined by

$$(5.4) \quad M_3(\xi_1, \xi_2, \xi_3) = \sum_{i=1}^3 \xi_i m(\xi_i)^2.$$

*Proof.* Recalling the Fourier representation of  $H(t)$  in (3.4), namely,

$$H(t)^2 = \int_{\xi_1 + \xi_2 = 0} m(\xi_1)m(\xi_2) [a_{20}\widehat{u}(\xi_1)\widehat{u}(\xi_2) + a_{11}\widehat{u}(\xi_1)\widehat{v}(\xi_2) + a_{02}\widehat{v}(\xi_1)\widehat{v}(\xi_2)],$$

it is straightforward to compute

$$\begin{aligned} \frac{d}{dt} H(t)^2 &= \int_{\xi_1+\xi_2=0} m(\xi_1)m(\xi_2) [2a_{20}\widehat{u}(\xi_1)\widehat{u}_t(\xi_2) + a_{11}\widehat{v}(\xi_1)\widehat{u}_t(\xi_2) \\ &\quad + a_{11}\widehat{u}(\xi_1)\widehat{v}_t(\xi_2) + 2a_{02}\widehat{v}(\xi_1)\widehat{v}_t(\xi_2)]. \end{aligned}$$

Upon using the equations (1.1) to express the temporal derivatives, this is seen to equal

$$\begin{aligned} & - \int_{\xi_1+\xi_2=0} m(\xi_1)m(\xi_2)2a_{20}\widehat{u}(\xi_1) [\widehat{u_{xxx}} + A(\widehat{u^2})_x + B(\widehat{uv})_x + C(\widehat{v^2})_x](\xi_2) \\ & - \int_{\xi_1+\xi_2=0} m(\xi_1)m(\xi_2)a_{11}\widehat{v}(\xi_1) [\widehat{u_{xxx}} + A(\widehat{u^2})_x + B(\widehat{uv})_x + C(\widehat{v^2})_x](\xi_2) \\ & - \int_{\xi_1+\xi_2=0} m(\xi_1)m(\xi_2)a_{11}\widehat{u}(\xi_1) [\widehat{v_{xxx}} + D(\widehat{u^2})_x + E(\widehat{uv})_x + F(\widehat{v^2})_x](\xi_2) \\ & - \int_{\xi_1+\xi_2=0} m(\xi_1)m(\xi_2)2a_{02}\widehat{v}(\xi_1) [\widehat{v_{xxx}} + D(\widehat{u^2})_x + E(\widehat{uv})_x + F(\widehat{v^2})_x](\xi_2). \end{aligned}$$

Notice that

$$\begin{aligned} (5.5) \quad & \int_{\xi_1+\xi_2=0} m(\xi_1)m(\xi_2)\widehat{u}(\xi_1)(\widehat{u^2})_x(\xi_2) \\ & = \int_{\xi_1+\xi_2=0} m(\xi_1)m(\xi_2)\widehat{u}(\xi_1)i\xi_2 \int \widehat{u}(\xi_2 - \xi_3)\widehat{u}(\xi_3) d\mu(\xi_3), \end{aligned}$$

and thus setting  $\tau = \xi_2 - \xi_3$  and then replacing  $\tau$  by  $\xi_2$  and using  $\xi_1 + \xi_2 + \xi_3 = 0$  to replace  $\xi_2 + \xi_3$  by  $-\xi_1$ , the latter integral equals

$$\begin{aligned} & \int_{\xi_1+\xi_2+\xi_3=0} i(\xi_2 + \xi_3)m(\xi_1)m(\xi_2 + \xi_3)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3) \\ & = -i \int_{\xi_1+\xi_2+\xi_3=0} \xi_1 m(\xi_1)^2 \widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3). \end{aligned}$$

In a similar fashion, the identities

$$\begin{aligned} & \int_{\xi_1+\xi_2=0} m(\xi_1)m(\xi_2)\widehat{u}(\xi_1)(\widehat{uv})_x(\xi_2) \\ & = -i \int_{\xi_1+\xi_2+\xi_3=0} \xi_1 m(\xi_1)^2 \widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{v}(\xi_3), \end{aligned}$$



$$\begin{aligned} & \int_{\xi_1+\xi_2=0} m(\xi_1)m(\xi_2)\widehat{u}(\xi_1)\widehat{(v^2)_x}(\xi_2) \\ &= -i \int_{\xi_1+\xi_2+\xi_3=0} \xi_1 m(\xi_1)^2 \widehat{u}(\xi_1)\widehat{v}(\xi_2)\widehat{v}(\xi_3), \\ & \int_{\xi_1+\xi_2=0} m(\xi_1)m(\xi_2)\widehat{v}(\xi_1)\widehat{(u^2)_x}(\xi_2) \\ &= -i \int_{\xi_1+\xi_2+\xi_3=0} \xi_1 m(\xi_1)^2 \widehat{v}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3), \\ & \int_{\xi_1+\xi_2=0} m(\xi_1)m(\xi_2)\widehat{v}(\xi_1)\widehat{(uv)_x}(\xi_2) \\ &= -i \int_{\xi_1+\xi_2+\xi_3=0} \xi_1 m(\xi_1)^2 \widehat{v}(\xi_1)\widehat{u}(\xi_2)\widehat{v}(\xi_3), \\ & \int_{\xi_1+\xi_2=0} m(\xi_1)m(\xi_2)\widehat{v}(\xi_1)\widehat{(v^2)_x}(\xi_2) \\ &= -i \int_{\xi_1+\xi_2+\xi_3=0} \xi_1 m(\xi_1)^2 \widehat{v}(\xi_1)\widehat{v}(\xi_2)\widehat{v}(\xi_3), \end{aligned}$$

are seen to hold. These observations allow us to adduce the functionals

$$\begin{aligned} J &= i \int_{\xi_1+\xi_2=0} m(\xi_1)m(\xi_2) [2a_{20}\xi_1^3\widehat{u}(\xi_1)\widehat{u}(\xi_2) + 2a_{02}\xi_1^3\widehat{v}(\xi_1)\widehat{v}(\xi_2)] \\ &+ i \int_{\xi_1+\xi_2=0} m(\xi_1)m(\xi_2)a_{11} [\xi_1^3\widehat{u}(\xi_1)\widehat{v}(\xi_2) + \xi_2^3\widehat{u}(\xi_1)\widehat{v}(\xi_2)] \end{aligned}$$

and

$$L = i\Lambda^3 = i[a_{30}\Lambda_{30} + a_{21}\Lambda_{21} + a_{12}\Lambda_{12} + a_{03}\Lambda_{03}],$$

for which

$$\frac{d}{dt}H(t)^2 = J + L.$$

The terms appearing in the definition of  $L$  are

$$\begin{aligned} a_{30}\Lambda_{30} &= \int_{\xi_1+\xi_2+\xi_3=0} [2a_{20}A + a_{11}D]\xi_1 m^2(\xi_1)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3), \\ a_{21}\Lambda_{21} &= \int_{\xi_1+\xi_2+\xi_3=0} a_{11}A\xi_3 m^2(\xi_3)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{v}(\xi_3) \\ &+ \int_{\xi_1+\xi_2+\xi_3=0} a_{11}E\xi_1 m^2(\xi_1)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{v}(\xi_3) \end{aligned}$$

$$\begin{aligned}
 & + \int_{\xi_1+\xi_2+\xi_3=0} 2a_{20}B\xi_1m^2(\xi_1)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{v}(\xi_3) \\
 & + \int_{\xi_1+\xi_2+\xi_3=0} 2a_{02}D\xi_3m^2(\xi_3)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{v}(\xi_3), \\
 a_{12}\Lambda_{12} = & \int_{\xi_1+\xi_2+\xi_3=0} 2a_{20}C\xi_1m^2(\xi_1)\widehat{u}(\xi_1)\widehat{v}(\xi_2)\widehat{v}(\xi_3) \\
 & + \int_{\xi_1+\xi_2+\xi_3=0} a_{11}B\xi_3m^2(\xi_3)\widehat{u}(\xi_1)\widehat{v}(\xi_2)\widehat{v}(\xi_3) \\
 & + \int_{\xi_1+\xi_2+\xi_3=0} a_{11}F\xi_1m^2(\xi_1)\widehat{u}(\xi_1)\widehat{v}(\xi_2)\widehat{v}(\xi_3) \\
 & + \int_{\xi_1+\xi_2+\xi_3=0} 2a_{02}E\xi_3m^2(\xi_3)\widehat{u}(\xi_1)\widehat{v}(\xi_2)\widehat{v}(\xi_3)
 \end{aligned}$$

and

$$a_{03}\Lambda_{03} = \int_{\xi_1+\xi_2+\xi_3=0} [2a_{02}F + a_{11}C]\xi_1m^2(\xi_1)\widehat{v}(\xi_1)\widehat{v}(\xi_2)\widehat{v}(\xi_3).$$

Notice the symmetrization principles

$$\begin{aligned}
 & \int_{\xi_1+\xi_2+\xi_3=0} \xi_1m^2(\xi_1)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3) \\
 & = \int_{\xi_1+\xi_2+\xi_3=0} \xi_2m^2(\xi_2)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3) \\
 (5.6) \quad & = \int_{\xi_1+\xi_2+\xi_3=0} \xi_3m^2(\xi_3)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3) \\
 & = \int_{\xi_1+\xi_2+\xi_3=0} \frac{\xi_1m^2(\xi_1) + \xi_2m^2(\xi_2) + \xi_3m^2(\xi_3)}{3}\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3)
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_{\xi_1+\xi_2+\xi_3=0} \xi_1m^2(\xi_1)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{v}(\xi_3) \\
 & = \int_{\xi_1+\xi_2+\xi_3=0} \xi_2m^2(\xi_2)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{v}(\xi_3) \\
 & = \int_{\xi_1+\xi_2+\xi_3=0} \frac{\xi_1m^2(\xi_1) + \xi_2m^2(\xi_2)}{2}\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{v}(\xi_3).
 \end{aligned}$$

Applying these formulas to the terms in  $J$  and using the fact that  $\xi_1 + \xi_2 = 0$  implies that  $\xi_1^3 + \xi_2^3 = 0$ , it follows that

$$\begin{aligned} & \int_{\xi_1+\xi_2=0} m(\xi_1)m(\xi_2) [2a_{20}\xi_1^3\widehat{u}(\xi_1)\widehat{u}(\xi_2) + 2a_{02}\xi_1^3\widehat{v}(\xi_1)\widehat{v}(\xi_2)] \\ &= \int_{\xi_1+\xi_2=0} m(\xi_1)m(\xi_2) [a_{20}(\xi_1^3 + \xi_2^3)\widehat{u}(\xi_1)\widehat{u}(\xi_2) + a_{02}(\xi_1^3 + \xi_2^3)\widehat{v}(\xi_1)\widehat{v}(\xi_2)] \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} & \int_{\xi_1+\xi_2=0} m(\xi_1)m(\xi_2)a_{11} [\xi_1^3\widehat{u}(\xi_1)\widehat{v}(\xi_2) + \xi_2^3\widehat{u}(\xi_1)\widehat{v}(\xi_2)] \\ &= \int_{\xi_1+\xi_2=0} m(\xi_1)m(\xi_2)a_{11} [\xi_1^3 + \xi_2^3]\widehat{u}(\xi_1)\widehat{v}(\xi_2) \\ &= 0. \end{aligned}$$

Consideration is now given to the trilinear integrals arising from the quadratic terms in the system (1.1). First, it is seen that

$$\begin{aligned} a_{30}\Lambda_{30} &= \int_{\xi_1+\xi_2+\xi_3=0} \frac{1}{3}(2a_{20}A + a_{11}D) \\ &\quad \times [\xi_1m^2(\xi_1) + \xi_2m^2(\xi_2) + \xi_3m^2(\xi_3)]\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3). \end{aligned}$$

Using the condition (3.10) on the coefficients  $a_{02}, a_{11}, a_{20}$ , we conclude that

$$\begin{aligned} a_{20}\Lambda_{21} &= \int_{\xi_1+\xi_2+\xi_3=0} \{ [a_{11}A + 2a_{02}D]\xi_3m^2(\xi_3) + [a_{11}E + 2a_{20}B]\xi_1m^2(\xi_1) \} \\ &\quad \times \widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{v}(\xi_3) \\ &= \int_{\xi_1+\xi_2+\xi_3=0} [a_{11}A + 2a_{02}D] [\xi_3m^2(\xi_3) + 2\xi_1m^2(\xi_1)] \\ &\quad \times \widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{v}(\xi_3) \\ &= \int_{\xi_1+\xi_2+\xi_3=0} [a_{11}A + 2a_{02}D] [\xi_1m^2(\xi_1) + \xi_2m^2(\xi_2) + \xi_3m^2(\xi_3)] \\ &\quad \times \widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{v}(\xi_3). \end{aligned}$$

Using the other restriction (3.11) on the coefficients  $a_{02}, a_{11}, a_{20}$  allows us to derive the formula

$$\begin{aligned} a_{12}\Lambda_{12} &= \int_{\xi_1+\xi_2+\xi_3=0} \{ [a_{11}B + 2a_{02}E]\xi_3m^2(\xi_3) + [a_{11}F + 2a_{20}C]\xi_1m^2(\xi_1) \} \\ &\quad \times \widehat{u}(\xi_1)\widehat{v}(\xi_2)\widehat{v}(\xi_3) \\ &= \int_{\xi_1+\xi_2+\xi_3=0} [a_{11}F + 2a_{20}C][2\xi_3m^2(\xi_3) + \xi_1m^2(\xi_1)] \\ &\quad \times \widehat{u}(\xi_1)\widehat{v}(\xi_2)\widehat{v}(\xi_3) \\ &= \int_{\xi_1+\xi_2+\xi_3=0} [a_{11}F + 2a_{20}C][\xi_1m^2(\xi_1) + \xi_2m^2(\xi_2) + \xi_3m^2(\xi_3)] \\ &\quad \times \widehat{u}(\xi_1)\widehat{v}(\xi_2)\widehat{v}(\xi_3). \end{aligned}$$

Finally, note that

$$\begin{aligned} a_{03}\Lambda_{03} &= \int_{\xi_1+\xi_2+\xi_3=0} \frac{1}{3}[2a_{02}F + a_{11}C][\xi_1m^2(\xi_1) + \xi_2m^2(\xi_2) + \xi_3m^2(\xi_3)] \\ &\quad \times \widehat{v}(\xi_1)\widehat{v}(\xi_2)\widehat{v}(\xi_3). \end{aligned}$$

If  $M_3$  is defined by  $M_3(\xi_1, \xi_2, \xi_3) = \xi_1m^2(\xi_1) + \xi_2m^2(\xi_2) + \xi_3m^2(\xi_3)$ , it appears that

$$\Lambda^3 = a_{30}\Lambda_{30}(M_3) + a_{21}\Lambda_{21}(M_3) + a_{12}\Lambda_{12}(M_3) + a_{03}\Lambda_{03}(M_3),$$

where

$$\begin{aligned} \Lambda_{30} &= \int_{\xi_1+\xi_2+\xi_3=0} M_3(\xi_1, \xi_2, \xi_3)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3), \\ \Lambda_{21} &= \int_{\xi_1+\xi_2+\xi_3=0} M_3(\xi_1, \xi_2, \xi_3)\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{v}(\xi_3), \\ \Lambda_{12} &= \int_{\xi_1+\xi_2+\xi_3=0} M_3(\xi_1, \xi_2, \xi_3)\widehat{u}(\xi_1)\widehat{v}(\xi_2)\widehat{v}(\xi_3), \\ \Lambda_{03} &= \int_{\xi_1+\xi_2+\xi_3=0} M_3(\xi_1, \xi_2, \xi_3)\widehat{v}(\xi_1)\widehat{v}(\xi_2)\widehat{v}(\xi_3), \end{aligned}$$

with the  $a_{ij}$  given in (5.3) in the statement of the lemma. □

REMARK 9. The calculation provided in (5.5) also shows that

$$\begin{aligned} & \int_{\xi_1+\dots+\xi_{r+s}=0} m_{rs}(\xi_1, \dots, \xi_{r+s}) \widehat{u}(\xi_1) \cdots \widehat{(u^2)_x}(\xi_l) \cdots \widehat{u}(\xi_r) \widehat{v}(\xi_{r+1}) \cdots \widehat{v}(\xi_{r+s}) \\ &= i \int_{\xi_1+\dots+\xi_{r+s+1}=0} (\xi_l + \xi_{l+1}) m_{rs}(\xi_1, \dots, \xi_l + \xi_{l+1}, \dots, \xi_{r+s+1}) \\ & \quad \times \widehat{u}(\xi_1) \cdots \widehat{u}(\xi_l) \widehat{u}(\xi_{l+1}) \cdots \widehat{u}(\xi_{r+1}) \widehat{v}(\xi_{r+2}) \cdots \widehat{v}(\xi_{r+s+1}), \\ & \int_{\xi_1+\dots+\xi_{r+s}=0} m_{rs}(\xi_1, \dots, \xi_{r+s}) \widehat{u}(\xi_1) \cdots \widehat{(uv)_x}(\xi_l) \cdots \widehat{u}(\xi_r) \widehat{v}(\xi_{r+1}) \cdots \widehat{v}(\xi_{r+s}) \\ &= i \int_{\xi_1+\dots+\xi_{r+s+1}=0} (\xi_l + \xi_{l+1}) m_{rs}(\xi_1, \dots, \xi_l + \xi_{l+1}, \dots, \xi_{r+s+1}) \\ & \quad \times \widehat{u}(\xi_1) \cdots \widehat{u}(\xi_l) \widehat{v}(\xi_{l+1}) \cdots \widehat{u}(\xi_{r+1}) \widehat{v}(\xi_{r+2}) \cdots \widehat{v}(\xi_{r+s+1}) \end{aligned}$$

and

$$\begin{aligned} & \int_{\xi_1+\dots+\xi_{r+s}=0} m_{rs}(\xi_1, \dots, \xi_{r+s}) \widehat{u}(\xi_1) \cdots \widehat{(v^2)_x}(\xi_l) \cdots \widehat{u}(\xi_r) \widehat{v}(\xi_{r+1}) \cdots \widehat{v}(\xi_{r+s}) \\ &= i \int_{\xi_1+\dots+\xi_{r+s+1}=0} (\xi_l + \xi_{l+1}) m_{rs}(\xi_1, \dots, \xi_l + \xi_{l+1}, \dots, \xi_{r+s+1}) \\ & \quad \times \widehat{u}(\xi_1) \cdots \widehat{v}(\xi_l) \widehat{v}(\xi_{l+1}) \cdots \widehat{u}(\xi_{r+1}) \widehat{v}(\xi_{r+2}) \cdots \widehat{v}(\xi_{r+s+1}). \end{aligned}$$

If the symbol  $m_{rs}$  exhibits suitable symmetry, then the integrals above may also have equivalent forms based on that symmetry, just as in (5.6).

### 5.2. The calculation of $\Lambda_4$

Recall that

$$M_3(\xi_1, \xi_2, \xi_3) = \xi_1 m(\xi_1)^2 + \xi_2 m(\xi_2)^2 + \xi_3 m(\xi_3)^2$$

and that

$$\alpha_3 = \xi_1^3 + \xi_2^3 + \xi_3^3.$$

Define the functionals  $\widetilde{\Lambda}^3$  and  $\Gamma^3$  by

$$(5.7) \quad \widetilde{\Lambda}^3 = a_{30} \Lambda_{30} \left( \frac{M_3}{\alpha_3} \right) + a_{21} \Lambda_{21} \left( \frac{M_3}{\alpha_3} \right) + a_{12} \Lambda_{12} \left( \frac{M_3}{\alpha_3} \right) + a_{03} \Lambda_{03} \left( \frac{M_3}{\alpha_3} \right)$$

and

$$(5.8) \quad \Gamma^3 = H(t)^2 - \widetilde{\Lambda}^3.$$

Note that  $\xi_1 + \xi_2 + \xi_3 = 0$  implies that  $\xi_1^3 + \xi_2^3 + \xi_3^3 = 3\xi_1\xi_2\xi_3$ , and so it is easily seen that the singularity of  $M_3/\alpha_3$  is removable.

LEMMA 5. For the functional  $\Gamma^3$  defined in (5.8),  $\frac{d}{dt}\Gamma^3 = i\Lambda^4$ , where

$$\Lambda^4 = \tilde{a}_{22}\Lambda_{22}(M_4^2) + \sum_{i+j=4} a_{ij}\Lambda_{ij}(M_4^1),$$

with coefficients given by

$$\begin{aligned} 6a_{40} &= \frac{3a_{30}A + a_{21}D}{3}, & 3a_{31} &= \frac{3a_{30}B + a_{21}E}{3} = \frac{2a_{21}A + 2a_{12}D}{3}, \\ 4a_{22} &= \frac{2a_{21}B + 2a_{12}E}{3}, \\ \tilde{a}_{22} &= \frac{3a_{03}D + a_{12}A}{3} - a_{22} = \frac{3a_{30}C + a_{21}F}{3} - a_{22}, \\ 3a_{13} &= \frac{3a_{03}E + a_{12}B}{3} = \frac{2a_{21}C + 2a_{12}F}{3}, \\ 6a_{04} &= \frac{3a_{03}F + a_{12}C}{3}, \end{aligned}$$

and multipliers

$$\begin{aligned} M_4^1 &= \frac{\alpha_4}{3\xi_1\xi_2\xi_3\xi_4} [m^2(\xi_1) + m^2(\xi_2) + m^2(\xi_3) + m^2(\xi_4) \\ &\quad - m^2(\xi_1 + \xi_2) - m^2(\xi_1 + \xi_3) - m^2(\xi_1 + \xi_4)] \\ &\quad - \left[ \frac{m^2(\xi_1)}{\xi_1} + \frac{m^2(\xi_2)}{\xi_2} + \frac{m^2(\xi_3)}{\xi_3} + \frac{m^2(\xi_4)}{\xi_4} \right] \end{aligned}$$

and

$$\begin{aligned} M_4^2 &= \frac{m^2(\xi_1)}{\xi_2} + \frac{m^2(\xi_2)}{\xi_1} + \frac{m^2(\xi_3)}{\xi_4} + \frac{m^2(\xi_4)}{\xi_3} \\ &\quad - \frac{m^2(\xi_1 + \xi_2)}{\xi_1} - \frac{m^2(\xi_1 + \xi_2)}{\xi_2} - \frac{m^2(\xi_1 + \xi_2)}{\xi_3} - \frac{m^2(\xi_1 + \xi_2)}{\xi_4}. \end{aligned}$$

*Proof.* It is clear from (5.1) and (3.14) that

$$\begin{aligned} \frac{d}{dt}\Gamma^3 &= i\Lambda^3 - i\Lambda^3 + i \sum_{r+s=4} a_{rs}\Lambda_{rs}(m_{rs}, u, v) \\ &= i \sum_{r+s=4} a_{rs}\Lambda_{rs}(m_{rs}, u, v). \end{aligned}$$

The right-hand side of the last formula is composed of a sum of quartics, namely,

$$\begin{aligned}
 & i \int_{\xi_1+\xi_2+\xi_3+\xi_4=0} [3a_{30}A + a_{21}D] \frac{M_3(\xi_1 + \xi_4, \xi_2, \xi_3)}{3\xi_2\xi_3} \widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3)\widehat{u}(\xi_4) \\
 & + i \int_{\xi_1+\xi_2+\xi_3+\xi_4=0} [3a_{30}B + a_{21}E] \frac{M_3(\xi_1 + \xi_4, \xi_2, \xi_3)}{3\xi_2\xi_3} \widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3)\widehat{v}(\xi_4) \\
 & + i \int_{\xi_1+\xi_2+\xi_3+\xi_4=0} [2a_{21}A + 2a_{12}D] \frac{M_3(\xi_1 + \xi_3, \xi_2, \xi_4)}{3\xi_2\xi_4} \widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3)\widehat{v}(\xi_4) \\
 & + i \int_{\xi_1+\xi_2+\xi_3+\xi_4=0} [2a_{21}B + 2a_{12}E] \frac{M_3(\xi_1 + \xi_4, \xi_2, \xi_3)}{3\xi_2\xi_3} \widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{v}(\xi_3)\widehat{v}(\xi_4) \\
 & + i \int_{\xi_1+\xi_2+\xi_3+\xi_4=0} [3a_{03}D + a_{12}A] \frac{M_3(\xi_1 + \xi_2, \xi_3, \xi_4)}{3\xi_3\xi_4} \widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{v}(\xi_3)\widehat{v}(\xi_4) \\
 & + i \int_{\xi_1+\xi_2+\xi_3+\xi_4=0} [3a_{30}C + a_{21}F] \frac{M_3(\xi_1, \xi_2, \xi_3 + \xi_4)}{3\xi_1\xi_2} \widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{v}(\xi_3)\widehat{v}(\xi_4) \\
 & + i \int_{\xi_1+\xi_2+\xi_3+\xi_4=0} [3a_{03}E + a_{12}B] \frac{M_3(\xi_1 + \xi_4, \xi_2, \xi_3)}{3\xi_2\xi_3} \widehat{u}(\xi_1)\widehat{v}(\xi_2)\widehat{v}(\xi_3)\widehat{v}(\xi_4) \\
 & + i \int_{\xi_1+\xi_2+\xi_3+\xi_4=0} [2a_{21}C + 2a_{12}F] \frac{M_3(\xi_1, \xi_2 + \xi_3, \xi_4)}{3\xi_1\xi_4} \widehat{u}(\xi_1)\widehat{v}(\xi_2)\widehat{v}(\xi_3)\widehat{v}(\xi_4) \\
 & + i \int_{\xi_1+\xi_2+\xi_3+\xi_4=0} [3a_{03}F + a_{12}C] \frac{M_3(\xi_1 + \xi_4, \xi_2, \xi_3)}{3\xi_2\xi_3} \widehat{v}(\xi_1)\widehat{v}(\xi_2)\widehat{v}(\xi_3)\widehat{v}(\xi_4).
 \end{aligned}$$

The compatibility equations (3.10) and (3.11) together with the definition of the trilinear coefficients (5.3) imply that

$$\begin{aligned}
 3a_{30}B + a_{21}E &= [2a_{20}A + a_{11}D]B + [a_{11}A + 2a_{02}D]E \\
 &= 2a_{20}AB + a_{11}AE + 2a_{02}DE + a_{11}DB \\
 &= [2a_{20}B + a_{11}E]A + [2a_{02}E + a_{11}B]D
 \end{aligned}$$

and

$$\begin{aligned}
 2a_{21}A + 2a_{12}D &= 2[a_{11}A + 2a_{02}D]A + 2[a_{11}F + 2a_{20}C]D \\
 &= [2a_{20}B + a_{11}E]A + [2a_{02}E + a_{11}B]D,
 \end{aligned}$$

from which it is concluded that

$$3a_{30}B + a_{21}E = 2a_{21}A + 2a_{12}D.$$

We note next that

$$3a_{03}D + a_{12}A = [2a_{02}F + a_{11}C]D + [a_{11}F + 2a_{20}C]A,$$

whereas

$$\begin{aligned} 3a_{30}C + a_{21}F &= [2a_{20}A + a_{11}D]C + [a_{11}A + 2a_{02}D]F \\ &= 2a_{02}DF + a_{11}DC + a_{11}AF + 2a_{20}AC \\ &= [2a_{02}F + a_{11}C]D + [a_{11}F + 2a_{20}C]A. \end{aligned}$$

It follows that

$$3a_{30}C + a_{21}F = 3a_{03}D + a_{12}A.$$

Finally, it is seen that

$$\begin{aligned} 3a_{03}E + a_{12}B &= [2a_{02}F + a_{11}C]E + [a_{11}F + 2a_{20}C]B \\ &= 2a_{20}CB + a_{11}CE + 2a_{02}FE + a_{11}FB \\ &= [2a_{20}B + a_{11}E]C + [2a_{02}E + a_{11}B]F \end{aligned}$$

and

$$\begin{aligned} 2a_{12}F + 2a_{21}C &= 2[a_{11}F + 2a_{20}C]F + 2[a_{11}A + 2a_{02}D]C \\ &= [2a_{20}B + a_{11}E]C + [2a_{02}E + a_{11}B]F, \end{aligned}$$

so that

$$3a_{03}E + a_{12}B = 2a_{12}F + 2a_{21}C.$$

Since  $M_3(\xi_1 + \xi_4, \xi_2, \xi_3) = (\xi_1 + \xi_4)m^2(\xi_1 + \xi_4) + \xi_2m^2(\xi_2) + \xi_3m^2(\xi_3)$ , it transpires that on the hyperplane  $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$ ,  $\xi_1 + \xi_4 = -(\xi_2 + \xi_3)$ , and therefore

$$\frac{M_3(\xi_1 + \xi_4, \xi_2, \xi_3)}{\xi_2\xi_3} = \frac{\xi_2m^2(\xi_2) + \xi_3m^2(\xi_3) - (\xi_2 + \xi_3)m^2(\xi_2 + \xi_3)}{\xi_2\xi_3},$$

a quantity which will be called  $M^{23}$  henceforth.

It is clear that all of the quadrilinear terms involve permutations of the variables, and it is therefore convenient to define the functions  $M^{12}, M^{13}$ ,



$M^{14}, M^{24}$ , and  $M^{34}$  analogously. With this notation,  $\Lambda^4$  is written

$$\begin{aligned}
 \Lambda^4 &= 6a_{40}\Lambda_{40}(M^{23}) + 3a_{31}\Lambda_{31}(M^{23} + M^{24}) \\
 (5.9) \quad &+ 4a_{22}\Lambda_{22}(M^{23}) + (\tilde{a}_{22} + a_{22})\Lambda_{22}(M^{12} + M^{34}) \\
 &+ 4a_{13}\Lambda_{13}(M^{23} + M^{14}) + 6a_{04}\Lambda_{04}(M^{23}),
 \end{aligned}$$

with coefficients given exactly by the formulas displayed in the statement of the lemma. Using the symmetrization principles, the functionals can be represented more usefully as follows:

$$\begin{aligned}
 \Lambda_{40}(M^{23}) &= \Lambda_{40}\left(\frac{M^{12} + M^{13} + M^{14} + M^{23} + M^{24} + M^{34}}{6}\right), \\
 \Lambda_{31}(M^{23} + M^{24}) &= \Lambda_{31}\left(\frac{M^{12} + M^{13} + M^{14} + M^{23} + M^{24} + M^{34}}{3}\right), \\
 \Lambda_{22}(M^{23}) &= \Lambda_{22}\left(\frac{M^{13} + M^{23} + M^{24} + M^{14}}{4}\right), \\
 \Lambda_{13}(M^{23} + M^{14}) &= \Lambda_{13}\left(\frac{M^{12} + M^{13} + M^{14} + M^{23} + M^{24} + M^{34}}{3}\right), \\
 \Lambda_{04}(M^{23}) &= \Lambda_{04}\left(\frac{M^{12} + M^{13} + M^{14} + M^{23} + M^{24} + M^{34}}{6}\right).
 \end{aligned}$$

Defining multipliers  $M_4^1 = M^{12} + M^{13} + M^{14} + M^{23} + M^{24} + M^{34}$  and  $M_4^2 = M^{12} + M^{34}$ , the  $\Lambda_{22}$  terms of (5.9) can be rewritten as

$$\begin{aligned}
 &4a_{22}\Lambda_{22}(M^{23}) + (\tilde{a}_{22} + a_{22})\Lambda_{22}(M^{12} + M^{34}) \\
 &= a_{22}\Lambda_{22}(M^{13} + M^{23} + M^{24} + M^{14}) + a_{22}\Lambda_{22}(M^{12} + M^{34}) \\
 &\quad + \tilde{a}_{22}\Lambda_{22}(M^{12} + M^{34}) \\
 &= a_{22}\Lambda_{22}(M^{13} + M^{23} + M^{24} + M^{14} + M^{34} + M^{12}) \\
 &\quad + \tilde{a}_{22}\Lambda_{22}(M^{12} + M^{34}) \\
 &= a_{22}\Lambda_{22}(M_4^1) + \tilde{a}_{22}\Lambda_{22}(M_4^2).
 \end{aligned}$$

Convenient pointwise descriptions of  $M_4^1$  and  $M_4^2$  can be found as follows:

$$\begin{aligned}
 &M^{12} + M^{13} + M^{14} + M^{23} + M^{24} + M^{34} \\
 &= \frac{1}{\xi_1\xi_2\xi_3\xi_4} \{m^2(\xi_1)[\xi_1\xi_3\xi_4 + \xi_1\xi_2\xi_4 + \xi_1\xi_2\xi_3]\}
 \end{aligned}$$

$$\begin{aligned}
 &+ m^2(\xi_2)[\xi_2\xi_3\xi_4 + \xi_1\xi_2\xi_4 + \xi_1\xi_2\xi_3] \\
 &+ m^2(\xi_3)[\xi_2\xi_3\xi_4 + \xi_1\xi_3\xi_4 + \xi_1\xi_2\xi_3] + m^2(\xi_4)[\xi_2\xi_3\xi_4 + \xi_1\xi_3\xi_4 + \xi_1\xi_2\xi_4] \\
 &- (\xi_1 + \xi_2)m^2(\xi_1 + \xi_2)\xi_3\xi_4 - (\xi_1 + \xi_3)m^2(\xi_1 + \xi_3)\xi_2\xi_4 \\
 &- (\xi_1 + \xi_4)m^2(\xi_1 + \xi_4)\xi_2\xi_3 \\
 &- (\xi_2 + \xi_3)m^2(\xi_2 + \xi_3)\xi_1\xi_4 - (\xi_2 + \xi_4)m^2(\xi_2 + \xi_4)\xi_1\xi_3 \\
 &- (\xi_3 + \xi_4)m^2(\xi_3 + \xi_4)\xi_1\xi_2 \}.
 \end{aligned}$$

Using the facts that  $m$  is even and that  $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$ , the latter sum is seen to equal

$$\begin{aligned}
 &\frac{1}{\xi_1\xi_2\xi_3\xi_4} \{ m^2(\xi_1)[\xi_1\xi_3\xi_4 + \xi_1\xi_2\xi_4 + \xi_1\xi_2\xi_3 + \xi_2\xi_3\xi_4 - \xi_2\xi_3\xi_4] \\
 &\quad + m^2(\xi_2)[\xi_2\xi_3\xi_4 + \xi_1\xi_2\xi_4 + \xi_1\xi_2\xi_3 + \xi_1\xi_3\xi_4 - \xi_1\xi_3\xi_4] \\
 &\quad + m^2(\xi_3)[\xi_2\xi_3\xi_4 + \xi_1\xi_3\xi_4 + \xi_1\xi_2\xi_3 + \xi_1\xi_2\xi_4 - \xi_1\xi_2\xi_4] \\
 &\quad + m^2(\xi_4)[\xi_2\xi_3\xi_4 + \xi_1\xi_3\xi_4 + \xi_1\xi_2\xi_4 + \xi_1\xi_2\xi_3 - \xi_1\xi_2\xi_3] \\
 &\quad - [\xi_1\xi_3\xi_4 + \xi_2\xi_3\xi_4]m^2(\xi_1 + \xi_2) - [\xi_1\xi_2\xi_4 + \xi_2\xi_3\xi_4]m^2(\xi_1 + \xi_3) \\
 &\quad - [\xi_1\xi_2\xi_3 + \xi_2\xi_3\xi_4]m^2(\xi_1 + \xi_4) - [\xi_1\xi_2\xi_4 + \xi_1\xi_3\xi_4]m^2(\xi_1 + \xi_4) \\
 &\quad - [\xi_1\xi_2\xi_3 + \xi_1\xi_3\xi_4]m^2(\xi_1 + \xi_3) - [\xi_1\xi_2\xi_3 + \xi_1\xi_2\xi_4]m^2(\xi_1 + \xi_2) \}.
 \end{aligned}$$

Since  $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$ , it transpires that  $\alpha_4 = \xi_1^3 + \xi_2^3 + \xi_3^3 + \xi_4^3 = 3(\xi_1\xi_2\xi_3 + \xi_1\xi_2\xi_4 + \xi_1\xi_3\xi_4 + \xi_2\xi_3\xi_4)$ , whence

$$\begin{aligned}
 &M^{12} + M^{13} + M^{14} + M^{23} + M^{24} + M^{34} \\
 &= \frac{\alpha_4}{3\xi_1\xi_2\xi_3\xi_4} [m^2(\xi_1) + m^2(\xi_2) + m^2(\xi_3) + m^2(\xi_4) \\
 &\quad - m^2(\xi_1 + \xi_2) - m^2(\xi_1 + \xi_3) - m^2(\xi_1 + \xi_4)] \\
 &\quad - \left[ \frac{m^2(\xi_1)}{\xi_1} + \frac{m^2(\xi_2)}{\xi_2} + \frac{m^2(\xi_3)}{\xi_3} + \frac{m^2(\xi_4)}{\xi_4} \right] = M_4^1.
 \end{aligned}$$

Then  $M_4^2$  is calculated to have the form

$$\begin{aligned}
 M_4^2 &= M^{34} + M^{12} \\
 &= \frac{M_3(\xi_1 + \xi_2, \xi_3, \xi_4)}{\xi_3\xi_4} + \frac{M_3(\xi_1, \xi_2, \xi_3 + \xi_4)}{\xi_1\xi_2}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\xi_1 \xi_2 \xi_3 \xi_4} [m^2(\xi_1) \xi_1 \xi_3 \xi_4 + m^2(\xi_2) \xi_2 \xi_3 \xi_4 \\
 &\quad + m^2(\xi_3) \xi_1 \xi_2 \xi_3 + m^2(\xi_4) \xi_1 \xi_2 \xi_4 \\
 (5.10) \quad &\quad - (\xi_1 + \xi_2) m^2(\xi_1 + \xi_2) \xi_3 \xi_4 - (\xi_3 + \xi_4) m^2(\xi_3 + \xi_4) \xi_1 \xi_2] \\
 &= \frac{m^2(\xi_1)}{\xi_2} + \frac{m^2(\xi_2)}{\xi_1} + \frac{m^2(\xi_3)}{\xi_4} + \frac{m^2(\xi_4)}{\xi_3} \\
 &\quad - \frac{(\xi_1 \xi_2 \xi_3 + \xi_1 \xi_2 \xi_4 + \xi_1 \xi_3 \xi_4 + \xi_2 \xi_3 \xi_4) m^2(\xi_1 + \xi_2)}{\xi_1 \xi_2 \xi_3 \xi_4} \\
 &= \frac{m^2(\xi_1)}{\xi_2} + \frac{m^2(\xi_2)}{\xi_1} + \frac{m^2(\xi_3)}{\xi_4} + \frac{m^2(\xi_4)}{\xi_3} \\
 &\quad - \frac{m^2(\xi_1 + \xi_2)}{\xi_1} - \frac{m^2(\xi_1 + \xi_2)}{\xi_2} - \frac{m^2(\xi_1 + \xi_2)}{\xi_3} - \frac{m^2(\xi_1 + \xi_2)}{\xi_4},
 \end{aligned}$$

where the assumption that  $m$  is an even function and the fact that  $(\xi_3 + \xi_4) = -(\xi_1 + \xi_2)$  have been used again. □

Observe that an estimate for  $M_4^2$  proves a similar estimate for  $M_4^1$  since it is simply a sum of three terms having the same structure as  $M_4^2$ .

It is clear from (5.1) and Lemma 5 that if we set

$$\tilde{\Lambda}^4 = \tilde{a}_{22} \Lambda_{22} \left( \frac{M_4^2}{\alpha_4} \right) + \sum_{i+j=4} a_{ij} \Lambda_{ij} \left( \frac{M_4^1}{\alpha_4} \right),$$

then

$$\frac{\partial}{\partial t} \{ H(t)^2 - \tilde{\Lambda}^3 - \tilde{\Lambda}^4 \} = i \Lambda^5,$$

where

$$\Lambda^5 = \sum_{i+j=5} a_{ij} \Lambda_{ij}(m_{ij})$$

for certain constants  $a_{ij}$  and symbols  $m_{ij}$ . Note that the functional  $\tilde{\Lambda}^4$  is well defined since  $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$  implies that  $\xi_1^4 + \xi_2^4 + \xi_3^4 + \xi_4^4 = 3(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_1 + \xi_4)$ , and hence the singularity of  $M_4/\alpha_4$  is removable. Thus, Lemma 1 is established.

**§6. Estimates of multilinear functionals**

In this section a Littlewood–Paley decomposition is used to estimate the functionals  $\Lambda_{22}(M_4^2)$  and  $\Lambda_{ij}(M_4^1)$ . These functionals have integral representations with integrands of the form

$$M_4^i(\xi_1, \xi_2, \xi_3, \xi_4) \widehat{w_{j_1}}(\xi_1) \widehat{w_{j_2}}(\xi_2) \widehat{w_{j_3}}(\xi_3) \widehat{w_{j_4}}(\xi_4),$$

where  $i = 1, 2$  and  $\widehat{w_{j_k}}(\xi_k)$  can be either  $\widehat{u_{j_k}}(\xi_k)$  or  $\widehat{v_{j_k}}(\xi_k)$ ,  $1 \leq k \leq 4$ . The subscripts  $j_k$  refer to terms of a Littlewood–Paley decomposition that will be defined in Section 6.2. Pointwise estimates of  $M_4^1$  and  $M_4^2$  are obtained that are the essential ingredient in the establishment of (3.18).

**6.1. Pointwise description of  $M_4^1$  and  $M_4^2$**

For  $1 \leq i \leq 4$ , let  $N_i = 2^{j_i}$  be the smallest dyadic number that is greater than or equal to  $|\xi_i|$  when  $|\xi_i| \geq 1$ , and let  $N_i = 1$  if  $|\xi_i| < 1$ . Define  $N_{ij}$  similarly in terms of  $\xi_i + \xi_j$ , where  $i \neq j$  and  $1 \leq i, j \leq 4$ . It is shown in [16, Lemma 4.5] that there is a constant  $c > 0$  such that if  $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$ , then the function  $M_4^1 = M^{12} + M^{13} + M^{14} + M^{23} + M^{24} + M^{34}$  satisfies the inequality

$$(6.1) \quad |M_4^1(\xi_1, \xi_2, \xi_3, \xi_4)| \leq \frac{c|\alpha_4|m^2(\min\{N_i, N_{jk}\})}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)},$$

and  $c$  does not depend on the  $\xi_i$  or the  $N_i$ .

The remainder of this section is devoted to establishing the related estimate

$$(6.2) \quad |M_4^2(\xi_1, \xi_2, \xi_3, \xi_4)| \leq \frac{c|\alpha_4|m^2(\min\{N_i, N_{jk}\})}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)}$$

for  $M_4^2$ , where  $c$  again does not depend on any of the variables or the parameters. As noted above, this estimate for  $M_4^2$  allows one to deduce the inequality (6.1) for  $M_4^1$  since the latter quantity is simply a sum of three terms having the same structure as  $M_4^2$ .

**6.2. Partitioning the Fourier domain**

To analyze the integral, a Littlewood–Paley decomposition is used. Define the functions  $u_j$  by

$$\widehat{u_j}(\xi) = \widehat{u}(\xi) \quad \text{for } 2^{j-1} < |\xi| \leq 2^j \text{ and zero otherwise,}$$

for  $j \geq 1$ , and

$$\widehat{u_0}(\xi) = \widehat{u}(\xi) \quad \text{for } 0 \leq |\xi| \leq 1 \text{ and zero otherwise.}$$

The same decomposition is employed for  $v$ . Naturally, it must be that

$$u = \sum_{j=0}^{\infty} u_j \quad \text{and} \quad v = \sum_{j=0}^{\infty} v_j.$$

The estimate of  $M_4^2(\xi_1, \xi_2, \xi_3, \xi_4)$  is made by a systematic study of its size throughout  $\mathbb{R}^4$  using the following partition of  $(\xi_1, \xi_2, \xi_3, \xi_4)$  into 16 pieces, namely,

$$\begin{aligned} R_{>>>>} &= \{(\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_1| \geq N, |\xi_2| \geq N, |\xi_3| \geq N, |\xi_4| \geq N\}, \\ R_{>>><} &= \{(\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_1| \geq N, |\xi_2| \geq N, |\xi_3| \geq N, |\xi_4| < N\}, \\ R_{>><>} &= \{(\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_1| \geq N, |\xi_2| \geq N, |\xi_3| < N, |\xi_4| \geq N\}, \\ &\vdots \\ R_{<<<<} &= \{(\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_1| < N, |\xi_2| < N, |\xi_3| < N, |\xi_4| < N\}. \end{aligned}$$

Recall from (5.10) that

$$\begin{aligned} M_4^2 &= \frac{m^2(\xi_1)}{\xi_2} + \frac{m^2(\xi_2)}{\xi_1} + \frac{m^2(\xi_3)}{\xi_4} + \frac{m^2(\xi_4)}{\xi_3} \\ &\quad - \frac{m^2(\xi_1 + \xi_2)}{\xi_1} - \frac{m^2(\xi_1 + \xi_2)}{\xi_2} - \frac{m^2(\xi_1 + \xi_2)}{\xi_3} - \frac{m^2(\xi_1 + \xi_2)}{\xi_4}. \end{aligned}$$

Thus,  $\xi_1$  and  $\xi_2$  are interchangeable, and  $\xi_3$  and  $\xi_4$  are interchangeable. Because of this symmetry, it may be assumed that  $|\xi_1| \geq |\xi_2|$ ,  $|\xi_3| \geq |\xi_4|$ , and  $|\xi_1| \geq |\xi_3|$ , which implies that  $N_1 \geq N_2, N_3 \geq N_4$ , and  $N_1 \geq N_3$ . Consequently, it is enough to consider the seven cases

$$\begin{aligned} R_{>>>>} &= \{(\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_1| \geq N, |\xi_2| \geq N, |\xi_3| \geq N, |\xi_4| \geq N\}, \\ R_{>>><} &= \{(\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_1| \geq N, |\xi_2| \geq N, |\xi_3| \geq N, |\xi_4| < N\}, \\ R_{>><<} &= \{(\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_1| \geq N, |\xi_2| \geq N, |\xi_3| < N, |\xi_4| < N\}, \\ R_{><>>} &= \{(\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_1| \geq N, |\xi_2| < N, |\xi_3| \geq N, |\xi_4| \geq N\}, \\ R_{><<<} &= \{(\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_1| \geq N, |\xi_2| < N, |\xi_3| \geq N, |\xi_4| < N\}, \end{aligned}$$

$$R_{><<<<} = \{(\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_1| \geq N, |\xi_2| < N, |\xi_3| < N, |\xi_4| < N\},$$

$$R_{<<<<} = \{(\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_1| < N, |\xi_2| < N, |\xi_3| < N, |\xi_4| < N\}.$$

Note that  $m(\xi) = 1$  for  $|\xi| \leq N$ , so  $M_4^2 = 0$  in  $R_{<<<<}$ . Thus, inequality (6.2) is trivial in  $R_{<<<<}$ . So, consideration need be given only to the remaining six cases.

In the following, frequent use will be made of the fact that  $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$  implies that

$$\begin{aligned} \alpha_4 &= \xi_1^3 + \xi_2^3 + \xi_3^3 + \xi_4^3 \\ (6.3) \quad &= 3(\xi_1 \xi_3 \xi_4 + \xi_1 \xi_2 \xi_4 + \xi_1 \xi_2 \xi_3 + \xi_2 \xi_3 \xi_4) \\ &= 3(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_1 + \xi_4). \end{aligned}$$

By symmetry, it is also the case that

$$\begin{aligned} \alpha_4 &= 3(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_1 + \xi_4) \\ &= 3(\xi_2 + \xi_1)(\xi_2 + \xi_3)(\xi_2 + \xi_4) \\ (6.4) \quad &= 3(\xi_3 + \xi_1)(\xi_3 + \xi_2)(\xi_3 + \xi_4) \\ &= 3(\xi_4 + \xi_1)(\xi_4 + \xi_2)(\xi_4 + \xi_3). \end{aligned}$$

For any  $a$ , define  $G_a(\xi) = (\xi + a)m^2(\xi)$ . Note that it follows from (3.5) and (3.6) that

$$\begin{aligned} (6.5) \quad |G'_a(\xi)| &\leq C \left(1 + \frac{|a|}{|\xi|}\right) m^2(\xi) \quad \text{and} \\ |G''_a(\xi)| &\leq C \left(1 + \frac{|a|}{|\xi|}\right) \frac{m^2(\xi)}{|\xi|}. \end{aligned}$$

When  $a = 0$ , we also write  $G(\xi) = G_a(\xi)$ .

**6.3. Pointwise estimates of  $M_4^2$**

Similar estimates were obtained by Oh [26, Section 3] to handle the specific version of the modified energy that arises in his study of the Majda–Biello system of equations.

*Case A:* The region is of the form

$$R_1 = R_{>>>>} = \{(\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_1| \geq N, |\xi_2| \geq N, |\xi_3| \geq N, |\xi_4| \geq N\}.$$

In this case, since  $N \geq 1$ , it follows that  $N_i/2 < |\xi_i| \leq N_i$ . Use will be made of the expression

$$M_4^2 = \frac{m^2(\xi_1)}{\xi_2} + \frac{m^2(\xi_2)}{\xi_1} + \frac{m^2(\xi_3)}{\xi_4} + \frac{m^2(\xi_4)}{\xi_3} - \frac{\alpha_4 m^2(\xi_1 + \xi_2)}{3\xi_1 \xi_2 \xi_3 \xi_4},$$

which follows from (5.10). Note that if  $(\xi_1, \xi_2, \xi_3, \xi_4) \in R_1$ , then

$$(6.6) \quad \frac{1}{|\xi_i|} \leq \frac{2}{N_i} \leq \frac{4}{N + N_i}.$$

Hence, by the halving property (3.8) of  $m$ , there is a constant  $C > 0$ , independent of the variables and the parameters, such that

$$(6.7) \quad \left| \frac{\alpha_4 m^2(\xi_1 + \xi_2)}{\xi_1 \xi_2 \xi_3 \xi_4} \right| \leq \frac{C |\alpha_4| m^2(\min_{1 \leq i, j \leq 4} \{N_i, N_{ij}\})}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)},$$

where as above,  $N_{ij}$  is the smallest power of 2 greater than or equal to  $|\xi_i + \xi_j|$ . Since the expression  $m^2(\min_{1 \leq i, j \leq 4} \{N_i, N_{ij}\})$  occurs frequently in what follows, the notation

$$\kappa = \min_{1 \leq i, j \leq 4} \{N_i, N_{ij}\}$$

will be useful. Because of (6.7), attention may be restricted to the sum of the first four terms in  $M_4^2$ , namely,

$$\widetilde{M}_4^2 = \frac{m^2(\xi_1)}{\xi_2} + \frac{m^2(\xi_2)}{\xi_1} + \frac{m^2(\xi_3)}{\xi_4} + \frac{m^2(\xi_4)}{\xi_3}.$$

Next as  $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$ , we have essentially four possibilities:

- Case I:  $\xi_1 > 0, \xi_2 < 0, \xi_3 < 0$ , and  $\xi_4 < 0$ ;
- Case II:  $\xi_1 > 0, \xi_2 > 0$ , and  $\xi_3 < 0, \xi_4 < 0$ ;
- Case III:  $\xi_1 > 0, \xi_3 > 0, \xi_2 < 0$ , and  $\xi_4 < 0$ ;
- Case IV:  $\xi_1 > 0, \xi_4 > 0, \xi_2 < 0$ , and  $\xi_3 < 0$ .

In fact, cases III and IV in most situations are the same, as  $\xi_3$  and  $\xi_4$  are interchangeable. We then further partition  $R_1$  into four regions according to the four cases I–IV:

$$\begin{aligned} S_{+----}^1 &= \{(\xi_1, \xi_2, \xi_3, \xi_4) \in R_1 : \xi_1 > 0, \xi_2 < 0, \xi_3 < 0, \xi_4 < 0\}, \\ S_{++--}^1 &= \{(\xi_1, \xi_2, \xi_3, \xi_4) \in R_1 : \xi_1 > 0, \xi_2 > 0, \xi_3 < 0, \xi_4 < 0\}, \\ S_{+-+-}^1 &= \{(\xi_1, \xi_2, \xi_3, \xi_4) \in R_1 : \xi_1 > 0, \xi_2 < 0, \xi_3 > 0, \xi_4 < 0\}, \\ S_{+--+}^1 &= \{(\xi_1, \xi_2, \xi_3, \xi_4) \in R_1 : \xi_1 > 0, \xi_2 < 0, \xi_3 < 0, \xi_4 > 0\}. \end{aligned}$$

*Subcase 1.* If  $(\xi_1, \xi_2, \xi_3, \xi_4) \in S^1_{+---}$ , then  $\xi_1 = -(\xi_2 + \xi_3 + \xi_4) > 0$ . So  $\max\{|\xi_3|, |\xi_4|\} \leq |\xi_3| + |\xi_4| = |\xi_1 + \xi_2| \leq |\xi_1|$ ,  $\max\{|\xi_2|, |\xi_4|\} \leq |\xi_2| + |\xi_4| = |\xi_1 + \xi_3| \leq |\xi_1|$ , and  $\max\{|\xi_2|, |\xi_3|\} \leq |\xi_2| + |\xi_3| = |\xi_1 + \xi_4| \leq |\xi_1|$ . Thus,

$$\begin{aligned} \frac{1}{|\xi_1 + \xi_2|} &\leq \min\left\{\frac{1}{|\xi_3|}, \frac{1}{|\xi_4|}\right\}, \\ \frac{1}{|\xi_1 + \xi_3|} &\leq \min\left\{\frac{1}{|\xi_2|}, \frac{1}{|\xi_4|}\right\}, \\ \frac{1}{|\xi_1 + \xi_4|} &\leq \min\left\{\frac{1}{|\xi_2|}, \frac{1}{|\xi_3|}\right\}. \end{aligned}$$

It follows that

$$\begin{aligned} \left|\frac{m^2(\xi_2)}{\xi_1}\right| &= \left|\frac{m^2(\xi_2)(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_1 + \xi_4)}{\xi_1(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_1 + \xi_4)}\right| \\ &= \left|\frac{\alpha_4 m^2(\xi_2)}{\xi_1(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_1 + \xi_4)}\right| \\ &\leq \left|\frac{\alpha_4 m^2(\xi_2)}{\xi_1 \xi_2 \xi_3 \xi_4}\right| \\ &\leq \frac{C|\alpha_4| m^2(\kappa)}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)}. \end{aligned}$$

Moreover, since

$$\begin{aligned} 2|\xi_1| &= |\xi_1 + \xi_2 + \xi_1 + \xi_3 + \xi_1 + \xi_4| \\ &= |\xi_1 + \xi_2| + |\xi_1 + \xi_3| + |\xi_1 + \xi_4|, \end{aligned}$$

we must have  $\max_{2 \leq i \leq 4}\{|\xi_1 + \xi_i|\} \geq (2/3)|\xi_1|$ . It then follows that

$$\begin{aligned} \left|\frac{m^2(\xi_1)}{\xi_2}\right| &= \left|\frac{m^2(\xi_1)(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_1 + \xi_4)}{\xi_2(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_1 + \xi_4)}\right| \\ &= \left|\frac{\alpha_4 m^2(\xi_1)}{\xi_2(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_1 + \xi_4)}\right| \\ &\leq \frac{3}{2} \left|\frac{\alpha_4 m^2(\xi_1)}{\xi_1 \xi_2 \xi_3 \xi_4}\right| \\ &\leq \frac{C|\alpha_4| m^2(\kappa)}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)}. \end{aligned}$$



Similar arguments show that this upper bound is also valid for the terms  $(m^2(\xi_3))/\xi_4$  and  $(m^2(\xi_4))/\xi_3$  appearing in the expression for  $M_4^2$ . Thus, in this case,

$$|\widetilde{M}_4^2(\xi_1, \xi_2, \xi_3, \xi_4)| \leq \frac{C|\alpha_4|m^2(\kappa)}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)}.$$

*Subcase 2.* If  $(\xi_1, \xi_2, \xi_3, \xi_4) \in S_{+--}$ , then  $|\xi_1 + \xi_2| \leq |\xi_1|$ ,  $|\xi_1 + \xi_3| \leq |\xi_1|$ , and  $\max_{1 \leq i \leq 4} \{|\xi_i|\} \leq |\xi_2| + |\xi_3| = |\xi_1 + \xi_4| \leq 2|\xi_1|$ . Because  $0 < \xi_4 = -\xi_2 - (\xi_1 + \xi_3) \leq -\xi_2$ , and  $0 < \xi_4 = -\xi_3 - (\xi_1 + \xi_2) \leq -\xi_3$ ,  $\min_{1 \leq i \leq 4} |\xi_i| = |\xi_4|$ . It follows that

$$\begin{aligned} \widetilde{M}_4^2(\xi_1, \xi_2, \xi_3, \xi_4) &= \frac{m^2(\xi_1)}{\xi_2} + \frac{m^2(\xi_2)}{\xi_1} + \frac{m^2(\xi_3)}{\xi_4} + \frac{m^2(\xi_4)}{\xi_3} \\ &= \frac{\xi_1 m^2(\xi_1) + \xi_2 m^2(\xi_2)}{\xi_1 \xi_2} + \frac{\xi_3 m^2(\xi_3) + \xi_4 m^2(\xi_4)}{\xi_3 \xi_4} \\ &= \frac{\xi_1 m^2(\xi_1) + \xi_2 m^2(\xi_2) + \xi_3 m^2(\xi_3) + \xi_4 m^2(\xi_4)}{\xi_1 \xi_2} \\ &\quad + [\xi_3 m^2(\xi_3) + \xi_4 m^2(\xi_4)] \left[ \frac{1}{\xi_3 \xi_4} - \frac{1}{\xi_1 \xi_2} \right] \\ &= I + II. \end{aligned}$$

Observing that  $\xi_2 = -(\xi_1 + \xi_3 + \xi_4)$ ,  $\xi_3 = -(\xi_1 + \xi_2 + \xi_4)$ , and  $\xi_1 + \xi_2 + \xi_4 + \xi_3 + \xi_4 = \xi_4$ , we can rewrite  $I$  as

$$\begin{aligned} I &= (\xi_1 m^2(\xi_1) - (\xi_1 + \xi_3 + \xi_4) m^2(\xi_1 + \xi_3 + \xi_4) \\ &\quad - (\xi_1 + \xi_2 + \xi_4) m^2(\xi_1 + \xi_2 + \xi_4) + \xi_4 m^2(\xi_4)) / (\xi_1 \xi_2) \\ &= \frac{1}{\xi_1 \xi_2} G''(\xi_1 - \theta_1[\xi_1 + \xi_2] - \theta_2[\xi_1 + \xi_3]) [\xi_1 + \xi_2] [\xi_1 + \xi_3], \end{aligned}$$

where  $0 < \theta_1 < 1$  and  $0 < \theta_2 < 1$ . The latter follows from the fact that for  $C^2$ -functions  $f$  and points  $x, h$ , and  $k$ ,  $f(x + h + k) - f(x + h) - f(x + k) + f(x) = f''(x + \theta_1 h + \theta_2 k) h k$  for some values of  $\theta_1, \theta_2 \in (0, 1)$ , a consequence of two applications of the mean-value theorem. Note the values of  $\xi_4 + \theta_1[\xi_1 + \xi_3] + \theta_2[\xi_1 + \xi_2]$  when  $\theta_1, \theta_2 \in (0, 1)$  are between  $\xi_1$  and  $\xi_4$ . The estimate (6.5) together with the halving property (3.8) of  $m$  yields the estimate

$$|G''(\xi_1 - \theta_1[\xi_1 + \xi_2] - \theta_2[\xi_1 + \xi_3])| \leq C \frac{m^2(\kappa)}{\xi_4}.$$

It then follows from (6.7) that

$$\begin{aligned}
 |I| &\leq C \left| \frac{m^2(\kappa)}{\xi_1 \xi_2 \xi_4} (\xi_1 + \xi_2)(\xi_1 + \xi_3) \right| \\
 &= C \left| \frac{m^2(\kappa)(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_1 + \xi_4)}{\xi_1 \xi_2 (\xi_1 + \xi_4) \xi_4} \right| \\
 &\leq C \left| \frac{\alpha_4 m^2(\kappa)}{\xi_1 \xi_2 \xi_3 \xi_4} \right| \\
 &\leq \frac{C |\alpha_4| m^2(\kappa)}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)}.
 \end{aligned}$$

To estimate  $II$ , note that since  $\xi_1 + \xi_3 + \xi_4 + \xi_2 = 0$ ,

$$\frac{1}{\xi_3 \xi_4} - \frac{1}{\xi_1 \xi_2} = - \frac{(\xi_1 + \xi_4)(\xi_1 + \xi_3)}{\xi_1 \xi_2 \xi_3 \xi_4}$$

and

$$\xi_3 m^2(\xi_3) + \xi_4 m^2(\xi_4) = - [(\xi_3 + \xi_1 + \xi_2) m^2(\xi_3 + \xi_1 + \xi_2) - \xi_3 m^2(\xi_3)].$$

Hence, using (6.5), (6.7), and the mean-value theorem, there is a  $\theta \in (0, 1)$  such that

$$\begin{aligned}
 |II| &= \left| [\xi_3 m^2(\xi_3) + \xi_4 m^2(\xi_4)] \left( \frac{1}{\xi_3 \xi_4} - \frac{1}{\xi_1 \xi_2} \right) \right| \\
 &\leq C |G'(\xi_3 + \theta(\xi_1 + \xi_2))(\xi_1 + \xi_2)| \left| \frac{(\xi_1 + \xi_4)(\xi_1 + \xi_3)}{\xi_1 \xi_2 \xi_3 \xi_4} \right| \\
 &\leq \frac{C |\alpha_4| m^2(\kappa)}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)}.
 \end{aligned}$$

Here we use the fact that values of  $\xi_3 + \theta(\xi_1 + \xi_2)(\xi_1 + \xi_2)$  when  $\theta \in (0, 1)$  are between  $\xi_3$  and  $-\xi_4$ . As  $\xi_3$  and  $-\xi_4$  are of the same sign, the values are at least  $\kappa$ .

*Subcase 3.* If  $(\xi_1, \xi_2, \xi_3, \xi_4) \in S_{+-+}$ , then switch  $\xi_3$  and  $\xi_4$  in the above, and the same proof applies in this case.

*Subcase 4.* If  $(\xi_1, \xi_2, \xi_3, \xi_4) \in S_{++-}$ , then  $|\xi_1 + \xi_3| \leq |\xi_1|$ ,  $|\xi_1 + \xi_4| \leq |\xi_1|$ , and  $\max_{1 \leq i \leq 4} \{|\xi_i|\} \leq |\xi_3| + |\xi_4| = |\xi_1 + \xi_2| \leq 2|\xi_1|$ . Because  $0 < \xi_2 = -\xi_3 - (\xi_1 + \xi_4) \leq -\xi_3$ , and  $0 < \xi_2 = -\xi_4 - (\xi_1 + \xi_3) \leq -\xi_4$ ,  $\min_{1 \leq i \leq 4} |\xi_i| = |\xi_2|$ . As  $\xi_3$  and  $\xi_4$  are interchangeable, we assume that  $|\xi_3| \geq |\xi_4| \geq |\xi_2|$ .

*Subcase 4.1.* ( $|\xi_2| \leq (1/2)|\xi_4|$ ) Then  $\max\{|\xi_3|, |\xi_4|\} \leq |\xi_3| + |\xi_4| = |\xi_1 + \xi_2| \leq 2|\xi_1|$ ,  $(1/2)\max\{|\xi_2|, |\xi_4|\} \leq |\xi_2 + \xi_4| = |\xi_1 + \xi_3| \leq |\xi_1|$ , and  $(1/2)\max\{|\xi_2|, |\xi_3|\} \leq |\xi_2 + \xi_3| = |\xi_1 + \xi_4| \leq |\xi_1|$ . Also because  $\max_{2 \leq i \leq 4}\{|\xi_1 + \xi_i|\} = |\xi_1 + \xi_2| \geq |\xi_1|$ , it essentially follows from subcase 1 that

$$\begin{aligned} &|\widetilde{M}_4^2(\xi_1, \xi_2, \xi_3, \xi_4)| \\ &\leq \frac{C|\alpha_4|m^2(\kappa)}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)}. \end{aligned}$$

*Subcase 4.2.* ( $|\xi_2| \geq (1/2)|\xi_4|$ ) Here it follows that

$$\begin{aligned} &-\widetilde{M}_4^2(\xi_1, \xi_2, \xi_3, \xi_4) \\ &= -\left(\frac{m^2(\xi_1)}{\xi_2} + \frac{m^2(\xi_2)}{\xi_1} + \frac{m^2(\xi_3)}{\xi_4} + \frac{m^2(\xi_4)}{\xi_3}\right) \\ &= -\left(\frac{\xi_4 m^2(\xi_2) + \xi_1 m^2(\xi_3)}{\xi_1 \xi_4} + \frac{\xi_3 m^2(\xi_1) + \xi_2 m^2(\xi_4)}{\xi_2 \xi_3}\right) \\ &= -\left(\frac{\xi_3 m^2(\xi_1) + \xi_4 m^2(\xi_2) + \xi_1 m^2(\xi_3) + \xi_2 m^2(\xi_4)}{\xi_1 \xi_4}\right) \\ &\quad - [\xi_3 m^2(\xi_1) + \xi_2 m^2(\xi_4)] \left[\frac{1}{\xi_2 \xi_3} - \frac{1}{\xi_1 \xi_4}\right] \\ &= ((\xi_1 + \xi_2 + \xi_4)m^2(\xi_1) - (-\xi_3 - \xi_2 - \xi_4)m^2(-\xi_3) \\ &\quad - (-\xi_4 + \xi_2 + \xi_4)m^2(-\xi_4) + (\xi_2 - \xi_2 - \xi_4)m^2(\xi_2))/(\xi_1 \xi_4) \\ &\quad + [(\xi_1 + \xi_2 + \xi_4)m^2(\xi_1) - (-\xi_4 + \xi_2 + \xi_4)m^2(-\xi_4)] \left[\frac{1}{\xi_2 \xi_3} - \frac{1}{\xi_1 \xi_4}\right] \\ &= ((\xi_1 + \xi_2 + \xi_4)m^2(\xi_1) - (-\xi_3 + \xi_2 + \xi_4)m^2(-\xi_3) \\ &\quad - (-\xi_4 + \xi_2 + \xi_4)m^2(-\xi_4) + (\xi_2 + \xi_2 + \xi_4)m^2(\xi_2))/(\xi_1 \xi_4) \\ &\quad + [(\xi_1 + \xi_2 + \xi_4)m^2(\xi_1) - (-\xi_4 + \xi_2 + \xi_4)m^2(-\xi_4)] \left[\frac{1}{\xi_2 \xi_3} - \frac{1}{\xi_1 \xi_4}\right] \\ &\quad + 2\frac{(\xi_2 + \xi_4)m^2(-\xi_3) - (\xi_2 + \xi_4)m^2(\xi_2)}{\xi_1 \xi_4} \\ &= I + II + III. \end{aligned}$$

As in subcase 2,

$$\begin{aligned} I &= \frac{1}{\xi_1 \xi_3} G''_{\xi_2 + \xi_4} (\xi_2 - \theta_1[\xi_2 + \xi_4] - \theta_2[\xi_2 + \xi_3]) [\xi_2 + \xi_3] [\xi_2 + \xi_4] \\ &= \frac{1}{\xi_1 \xi_3} G''_{\xi_2 + \xi_4} (\xi_2 - \theta_1[\xi_2 + \xi_4] - \theta_2[\xi_2 + \xi_3]) [\xi_1 + \xi_3] [\xi_1 + \xi_4], \end{aligned}$$

where  $0 < \theta_1 < 1$  and  $0 < \theta_2 < 1$ . Note that the values of  $\xi_2 - \theta_1[\xi_2 + \xi_4] - \theta_2[\xi_2 + \xi_3]$  when  $\theta_1, \theta_2 \in (0, 1)$  are between  $\xi_1$  and  $\xi_2$ . The estimate (6.5) together with the halving property (3.8) of  $m$  yields the estimate

$$\begin{aligned} |G''_{\xi_2 + \xi_4} (\xi_2 - \theta_1[\xi_2 + \xi_4] - \theta_2[\xi_2 + \xi_3])| &\leq C \left( 1 + \frac{|\xi_2 + \xi_4|}{|\xi_2|} \right) \frac{m^2(\kappa)}{|\xi_2|} \\ &\leq 4C \frac{m^2(\kappa)}{|\xi_2|} \end{aligned}$$

as  $|\xi_2 + \xi_4| \leq 3|\xi_2|$ . It then follows from (6.7) that

$$\begin{aligned} |I| &\leq C \left| \frac{m^2(\kappa)}{\xi_1 \xi_2 \xi_4} (\xi_1 + \xi_3)(\xi_1 + \xi_4) \right| \\ &= C \left| \frac{m^2(\kappa)(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_1 + \xi_4)}{\xi_1 \xi_2 (\xi_1 + \xi_2) \xi_4} \right| \\ &\leq C \left| \frac{\alpha_4 m^2(\kappa)}{\xi_1 \xi_2 \xi_3 \xi_4} \right| \\ &\leq \frac{C |\alpha_4| m^2(\kappa)}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)}. \end{aligned}$$

To estimate  $II$ , observe that since  $\xi_1 + \xi_3 + \xi_4 + \xi_2 = 0$ ,

$$\frac{1}{\xi_2 \xi_3} - \frac{1}{\xi_1 \xi_4} = -\frac{(\xi_1 + \xi_2)(\xi_1 + \xi_3)}{\xi_1 \xi_2 \xi_3 \xi_4},$$

and as in subcase 2, there is a  $\theta \in (0, 1)$  such that

$$\begin{aligned} |II| &\leq C \left| G'_{\xi_2 + \xi_4} (-\xi_4 + \theta(\xi_1 + \xi_4))(\xi_1 + \xi_4) \right| \left| \frac{(\xi_1 + \xi_2)(\xi_1 + \xi_3)}{\xi_1 \xi_2 \xi_3 \xi_4} \right| \\ &\leq \left( 1 + \frac{|\xi_2 + \xi_4|}{|\xi_2|} \right) \frac{C |\alpha_4| m^2(\kappa)}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)} \\ &\leq \frac{4C |\alpha_4| m^2(\kappa)}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)}. \end{aligned}$$

Here we use the facts that  $|\xi_2 + \xi_4| \leq 3|\xi_2|$  and that values of  $-\xi_4 + \theta(\xi_1 + \xi_4)$  when  $\theta \in (0, 1)$  are between  $\xi_1$  and  $-\xi_4$ . As  $\xi_1$  and  $-\xi_4$  are of the same sign, the values are at least  $\kappa$ .

Finally, to estimate *III*, as done in the above, it follows from the mean value theorem and (6.7) that

$$\begin{aligned} |III| &= 2 \left| \frac{(\xi_2 + \xi_4)m^2(-\xi_3) - (\xi_2 + \xi_4)m^2(\xi_2)}{\xi_1\xi_4} \right| \\ &\leq 2 \left| \frac{(\xi_2 + \xi_4)(\xi_2 + \xi_3)m^2(\kappa)}{\xi_1\xi_2\xi_4} \right| \\ &= 2 \left| \frac{(\xi_1 + \xi_4)(\xi_1 + \xi_4)m^2(\kappa)}{\xi_1\xi_2\xi_4} \right| \\ &= 2 \left| \frac{(\xi_1 + \xi_2)(\xi_1 + \xi_4)(\xi_1 + \xi_4)m^2(\kappa)}{\xi_1\xi_2\xi_4(\xi_1 + \xi_2)} \right| \\ &\leq 2 \left| \frac{\alpha_4 m^2(\kappa)}{\xi_1\xi_2\xi_3\xi_4} \right| \\ &\leq \frac{C|\alpha_4|m^2(\kappa)}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)}. \end{aligned}$$

REMARK 10. If the region  $R_1$  is replaced by

$$\widetilde{R}_1 = \left\{ (\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_1| \geq \frac{N}{C_1}, |\xi_2| \geq \frac{N}{C_2}, |\xi_3| \geq \frac{N}{C_3}, |\xi_4| \geq \frac{N}{C_4} \right\}$$

for some dyadic numbers  $C_i \geq 1, i = 1, \dots, 4$ , then with (6.6) being replaced by

$$\frac{1}{|\xi_i|} \leq \frac{2}{N_i} \leq \frac{2(C_i + 1)}{N + N_i}$$

and some other obvious changes in the above arguments, the same proof shows that inequality (6.2) holds when  $(\xi_1, \xi_2, \xi_3, \xi_4) \in \widetilde{R}_1$ .

Case B: The region has the form

$$R_2 = R_{><>>} = \left\{ (\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_1| \geq N, |\xi_2| < N, |\xi_3| \geq N, |\xi_4| \geq N \right\}.$$

In this case, the region is divided into the two subregions

$$S_1^2 = \left\{ (\xi_1, \xi_2, \xi_3, \xi_4) \in R_2 : |\xi_2| \geq \frac{N}{4} \right\}$$

and

$$S_2^2 = \left\{ (\xi_1, \xi_2, \xi_3, \xi_4) \in R_2 : |\xi_2| \leq \frac{N}{4} \right\}.$$

If  $(\xi_1, \xi_2, \xi_3, \xi_4) \in S_1^2$ , then

$$(\xi_1, \xi_2, \xi_3, \xi_4) \in \left\{ (\eta_1, \eta_2, \eta_3, \eta_4) : |\eta_1| \geq \frac{N}{4}, |\eta_2| \geq \frac{N}{4}, |\eta_3| \geq \frac{N}{4}, |\eta_4| \geq \frac{N}{4} \right\},$$

so inequality (6.2) holds for this region according to Remark 10.

If  $(\xi_1, \xi_2, \xi_3, \xi_4) \in S_2^2$ , then  $|\xi_2| \leq (1/4)N$ . Thus,  $N_2 \leq (1/4)N \leq N_i$ , for  $i \neq 2$ . For  $i \neq 2$ , the condition  $N_i \geq N \geq 1$  implies that  $|\xi_i| \geq (1/2)N_i$ . These inequalities have as a consequence that

$$|\xi_i + \xi_2| \geq |\xi_i| - |\xi_2| \geq \frac{N_i}{2} - \frac{N}{4} \geq \frac{N + N_i}{8},$$

whence

$$(6.8) \quad \frac{1}{|\xi_i + \xi_2|} \leq \frac{8}{N + N_i}, \quad \text{for } i \neq 2.$$

As in the previous case,

$$(6.9) \quad \frac{1}{|\xi_i|} \leq \frac{2}{N_i} \leq \frac{4}{N + N_i} \leq \frac{4}{N + N_2}, \quad i = 1, 3, 4.$$

Using the formula

$$\begin{aligned} M_4^2 &= \frac{m^2(\xi_1)}{\xi_2} + \frac{m^2(\xi_2)}{\xi_1} + \frac{m^2(\xi_3)}{\xi_4} + \frac{m^2(\xi_4)}{\xi_3} \\ &\quad - \frac{m^2(\xi_1 + \xi_2)}{\xi_2} - \frac{m^2(\xi_1 + \xi_2)}{\xi_1} - \frac{m^2(\xi_1 + \xi_2)}{\xi_4} - \frac{m^2(\xi_1 + \xi_2)}{\xi_3} \\ &= \left\{ \frac{m^2(\xi_1)}{\xi_2} - \frac{m^2(\xi_1 + \xi_2)}{\xi_2} \right\} + \frac{m^2(\xi_2)}{\xi_1} + \frac{m^2(\xi_3)}{\xi_4} + \frac{m^2(\xi_4)}{\xi_3} \\ &\quad - \frac{m^2(\xi_1 + \xi_2)}{\xi_1} - \frac{m^2(\xi_1 + \xi_2)}{\xi_4} - \frac{m^2(\xi_1 + \xi_2)}{\xi_3}, \end{aligned}$$

we may write

$$M_4^2 = I + II,$$

where

$$I = \frac{m^2(\xi_1)}{\xi_2} - \frac{m^2(\xi_1 + \xi_2)}{\xi_2}$$

and

$$II = \frac{m^2(\xi_2)}{\xi_1} + \frac{m^2(\xi_3)}{\xi_4} + \frac{m^2(\xi_4)}{\xi_3} - \frac{m^2(\xi_1 + \xi_2)}{\xi_1} - \frac{m^2(\xi_1 + \xi_2)}{\xi_4} - \frac{m^2(\xi_1 + \xi_2)}{\xi_3}.$$

The mean-value theorem implies that there is a  $\theta \in (0, 1)$  such that

$$\begin{aligned} I &= \frac{m^2(\xi_1)}{\xi_2} - \frac{m^2(\xi_1 + \xi_2)}{\xi_2} \\ &= \frac{m^2(\xi_1) - m^2(\xi_1 + \xi_2)}{\xi_2} \\ &= \frac{(m^2)'(\xi_1 + \theta\xi_2)\xi_2}{\xi_2} \\ &= (m^2)'(\xi_1 + \theta\xi_2). \end{aligned}$$

Since  $|\xi_2| \leq (1/4)N \leq (1/4)|\xi_1|$ , it follows that  $(3/4)|\xi_1| \leq |\xi_1 + \theta\xi_2| \leq (5/4)|\xi_1|$ , and therefore,

$$\begin{aligned} |I| &= \left| \frac{m^2(\xi_1)}{\xi_2} - \frac{m^2(\xi_1 + \xi_2)}{\xi_2} \right| \\ &\leq C \frac{m^2(\xi_1)}{|\xi_1|} \\ &= C \left| \frac{m^2(\xi_1)(\xi_1 + \xi_2)(\xi_3 + \xi_2)(\xi_4 + \xi_2)}{\xi_1(\xi_1 + \xi_2)(\xi_2 + \xi_3)(\xi_2 + \xi_4)} \right| \\ &\leq \frac{C|\alpha_4|m^2(\kappa)}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)} \end{aligned}$$

by (6.8) and (6.9), where we have used the fact that  $\alpha_4$  has the representation  $\alpha_4 = 3(\xi_1 + \xi_2)(\xi_3 + \xi_2)(\xi_4 + \xi_2)$  (see (6.4)).

To estimate  $II$ , note that for  $i \neq 2$ ,

$$\begin{aligned} \left| \frac{1}{\xi_i} \right| &\leq \frac{2}{N_i} \\ &= \left| \frac{2(\xi_1 + \xi_2)(\xi_3 + \xi_2)(\xi_4 + \xi_2)}{N_i(\xi_1 + \xi_2)(\xi_3 + \xi_2)(\xi_4 + \xi_2)} \right| \\ &\leq \frac{C|\alpha_4|}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)} \end{aligned}$$

by (6.8) and (6.9). Of course,  $m^2(\kappa) = 1$  since  $\kappa = \min\{N_i, N_{ij}\} \leq (1/4)N$ , and thus,

$$\begin{aligned} |II| &= \left| \frac{m^2(\xi_2)}{\xi_1} + \frac{m^2(\xi_3)}{\xi_4} + \frac{m^2(\xi_4)}{\xi_3} \right. \\ &\quad \left. - \frac{m^2(\xi_1 + \xi_2)}{\xi_1} - \frac{m^2(\xi_1 + \xi_2)}{\xi_4} - \frac{m^2(\xi_1 + \xi_2)}{\xi_3} \right| \\ &\leq \frac{C|\alpha_4|m^2(\kappa)}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)}, \end{aligned}$$

as each of the summands is bounded separately by the right-hand side of the inequality. Therefore, inequality (6.2) holds for this region.

Case C: The region has the form

$$R_3 = R_{>>><} = \{(\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_1| \geq N, |\xi_2| \geq N, |\xi_3| \geq N, |\xi_4| < N\}.$$

In this case, as in Case B, the region is divided into the two subregions

$$S_1^3 = \left\{ (\xi_1, \xi_2, \xi_3, \xi_4) \in R_3 : |\xi_4| \geq \frac{N}{4} \right\}$$

and

$$S_2^3 = \left\{ (\xi_1, \xi_2, \xi_3, \xi_4) \in R_3 : |\xi_4| \leq \frac{N}{4} \right\}.$$

If  $(\xi_1, \xi_2, \xi_3, \xi_4) \in S_1^3$ , then

$$(\xi_1, \xi_2, \xi_3, \xi_4) \in \left\{ (\eta_1, \eta_2, \eta_3, \eta_4) : |\eta_1| \geq \frac{N}{4}, |\eta_2| \geq \frac{N}{4}, |\eta_3| \geq \frac{N}{4}, |\eta_4| \geq \frac{N}{4} \right\},$$

so inequality (6.2) holds for this region according to Remark 10.

If  $(\xi_1, \xi_2, \xi_3, \xi_4) \in S_2^3$ , then  $|\xi_4| \leq (1/4)N$ . Thus  $N_4 \leq (1/4)N \leq N_i$ , for  $i \neq 4$ . For  $i \neq 4$ , the condition  $N_i \geq N \geq 1$  implies that  $|\xi_i| \geq (1/2)N_i$ . These inequalities have as a consequence that

$$|\xi_i + \xi_4| \geq |\xi_i| - |\xi_4| \geq \frac{N_i}{2} - \frac{N}{4} \geq \frac{N + N_i}{8},$$

whence

$$(6.10) \quad \frac{1}{|\xi_i + \xi_4|} \leq \frac{8}{N + N_i}, \quad \text{for } i \neq 4.$$



As in the previous case,

$$(6.11) \quad \frac{1}{|\xi_i|} \leq \frac{2}{N_i} \leq \frac{4}{N + N_i} \leq \frac{4}{N + N_4}, \quad i = 1, 2, 3.$$

Using the formula

$$\begin{aligned} M_4^2 &= \frac{m^2(\xi_1)}{\xi_2} + \frac{m^2(\xi_2)}{\xi_1} + \frac{m^2(\xi_3)}{\xi_4} + \frac{m^2(\xi_4)}{\xi_3} \\ &\quad - \frac{m^2(\xi_1 + \xi_2)}{\xi_2} - \frac{m^2(\xi_1 + \xi_2)}{\xi_1} - \frac{m^2(\xi_1 + \xi_2)}{\xi_4} - \frac{m^2(\xi_1 + \xi_2)}{\xi_3} \\ &= \left\{ \frac{m^2(\xi_3)}{\xi_4} - \frac{m^2(\xi_1 + \xi_2)}{\xi_4} \right\} + \frac{m^2(\xi_2)}{\xi_1} + \frac{m^2(\xi_1)}{\xi_2} + \frac{m^2(\xi_4)}{\xi_3} \\ &\quad - \frac{m^2(\xi_1 + \xi_2)}{\xi_1} - \frac{m^2(\xi_1 + \xi_2)}{\xi_2} - \frac{m^2(\xi_1 + \xi_2)}{\xi_3}, \end{aligned}$$

we may write

$$M_4^2 = I + II,$$

where

$$\begin{aligned} I &= \frac{m^2(\xi_3)}{\xi_4} - \frac{m^2(\xi_1 + \xi_2)}{\xi_4} \\ &= \frac{m^2(\xi_3)}{\xi_4} - \frac{m^2(\xi_3 + \xi_4)}{\xi_4} \end{aligned}$$

and

$$\begin{aligned} II &= \frac{m^2(\xi_2)}{\xi_1} + \frac{m^2(\xi_1)}{\xi_2} + \frac{m^2(\xi_4)}{\xi_3} \\ &\quad - \frac{m^2(\xi_1 + \xi_2)}{\xi_1} - \frac{m^2(\xi_1 + \xi_2)}{\xi_2} - \frac{m^2(\xi_1 + \xi_2)}{\xi_3}. \end{aligned}$$

The mean-value theorem implies that there is a  $\theta \in (0, 1)$  such that

$$\begin{aligned} I &= \frac{m^2(\xi_3)}{\xi_4} - \frac{m^2(\xi_3 + \xi_4)}{\xi_4} \\ &= - \frac{(m^2)'(\xi_3 + \theta\xi_4)\xi_4}{\xi_2} \\ &= (m^2)'(\xi_3 + \theta\xi_4). \end{aligned}$$

Since  $|\xi_4| \leq (1/4)N \leq (1/4)|\xi_3|$ , it follows that  $(3/4)|\xi_3| \leq |\xi_3 + \theta\xi_4| \leq (5/4)|\xi_3|$ , and therefore,

$$\begin{aligned} |I| &= \left| \frac{m^2(\xi_3)}{\xi_4} - \frac{m^2(\xi_3 + \xi_4)}{\xi_4} \right| \\ &\leq C \frac{m^2(\xi_3)}{|\xi_3|} \\ &= C \left| \frac{m^2(\xi_3)(\xi_1 + \xi_4)(\xi_2 + \xi_4)(\xi_3 + \xi_4)}{\xi_3(\xi_1 + \xi_4)(\xi_2 + \xi_4)(\xi_3 + \xi_4)} \right| \\ &\leq \frac{C|\alpha_4|m^2(\kappa)}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)} \end{aligned}$$

by (6.10) and (6.11), where we have used the fact that  $\alpha_4$  has the representation  $\alpha_4 = 3(\xi_1 + \xi_4)(\xi_2 + \xi_4)(\xi_3 + \xi_4)$  (see (6.4)).

To estimate  $II$ , note that for  $i \neq 4$ ,

$$\begin{aligned} \left| \frac{1}{\xi_i} \right| &\leq \frac{2}{N_i} \\ &= \left| \frac{2(\xi_1 + \xi_4)(\xi_2 + \xi_4)(\xi_3 + \xi_4)}{N_i(\xi_1 + \xi_4)(\xi_2 + \xi_4)(\xi_3 + \xi_4)} \right| \\ &\leq \frac{C|\alpha_4|}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)} \end{aligned}$$

by (6.10) and (6.11). Of course,  $m^2(\kappa) = 1$  since  $\kappa = \min\{N_i, N_{ij}\} \leq (1/4)N$ , and thus,

$$\begin{aligned} |II| &= \left| \frac{m^2(\xi_2)}{\xi_1} + \frac{m^2(\xi_1)}{\xi_2} + \frac{m^2(\xi_4)}{\xi_3} \right. \\ &\quad \left. - \frac{m^2(\xi_1 + \xi_2)}{\xi_1} - \frac{m^2(\xi_1 + \xi_2)}{\xi_2} - \frac{m^2(\xi_1 + \xi_2)}{\xi_3} \right| \\ &\leq \frac{C|\alpha_4|m^2(\kappa)}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)}, \end{aligned}$$

as each of the summands is bounded separately by the right-hand side of the inequality. Therefore, inequality (6.2) holds for this region.

REMARK 11. If  $S_2^2$  is replaced by

$$\widetilde{S}_2^2 = \left\{ (\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_1| \geq \frac{N}{C_1}, |\xi_2| \leq \frac{N}{C_2 D_2}, |\xi_3| \geq \frac{N}{C_3}, |\xi_4| \geq \frac{N}{C_4} \right\}$$

for some dyadic numbers  $C_i \geq 1$  and  $C_2 D_2 > 2C_i$ , then if one replaces (6.8) and (6.9) by

$$\frac{1}{|\xi_i + \xi_2|} \leq \frac{2C_2 D_2 (C_i + 1)}{(C_2 D_2 - 2C_i)(N + N_i)}$$

and

$$\frac{1}{|\xi_i|} \leq \frac{2}{N_i} \leq \frac{2(C_i + 1)}{N + N_i} \leq \frac{2(C_i + 1)}{N + N_2},$$

respectively, for  $i = 1, 3, 4$ , and makes some other obvious changes, the above arguments allow (6.2) to be verified when  $(\xi_1, \xi_2, \xi_3, \xi_4) \in \widetilde{S_2^2}$ . If these observations are combined with Remark 10, it is deduced that (6.2) holds for regions of the form

$$\left\{ (\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_2| \leq \frac{N}{C_2}, |\xi_j| \geq \frac{N}{C_j}, j \neq 2 \right\}.$$

The case

$$\left\{ (\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_4| \leq \frac{N}{C_4}, |\xi_j| \geq \frac{N}{C_j}, j \neq 4 \right\}$$

is also similarly proved.

*Case D:* The region has the form

$$R_4 = R_{><><} = \left\{ (\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_1| \geq N, |\xi_2| < N, |\xi_3| \geq N, |\xi_4| < N \right\}.$$

The region  $R_4$  may be divided into the two subregions

$$S_1^4 = \left\{ (\xi_1, \xi_2, \xi_3, \xi_4) \in R_4 : |\xi_2| \geq \frac{N}{4} \text{ or } |\xi_4| \geq \frac{N}{4} \right\}$$

and

$$S_2^4 = \left\{ (\xi_1, \xi_2, \xi_3, \xi_4) \in R_4 : |\xi_2| \leq \frac{N}{4}, |\xi_4| \leq \frac{N}{4} \right\}.$$

The case where  $(\xi_1, \xi_2, \xi_3, \xi_4) \in S_1^4$  is covered on account of Remarks 10 and 11. If  $(\xi_1, \xi_2, \xi_3, \xi_4) \in S_2^4$ , then  $|\xi_i| \leq N_i \leq (1/4)N \leq (1/4)|\xi_1|$  or  $(1/4)|\xi_3|$  for  $i = 2, 4$ . Hence,  $|\xi_1 + \xi_3| = |\xi_2 + \xi_4| \leq (1/2)N$  and  $(3/4)|\xi_1| \leq |\xi_1 + \xi_i| \leq (5/4)|\xi_1|$ ,  $i = 2, 4$ . Thus, for  $i = 2, 4$ ,  $N + N_i \leq (5/4)N$  and  $|\xi_1 + \xi_i| \geq (3/4)N$ , whence

$$(6.12) \quad \frac{1}{|\xi_1 + \xi_i|} \leq \frac{5}{3(N + N_i)}, \quad i = 2, 4.$$

Moreover, for  $i = 1, 3, N_i \geq N \geq 1$ , and so

$$(6.13) \quad \frac{1}{|\xi_1|} \leq \frac{1}{|\xi_i|} \leq \frac{2}{N_i} \leq \frac{4}{N + N_i}.$$

Without loss of generality, take it that  $|\xi_1| \geq |\xi_3|$  and  $|\xi_2| \geq |\xi_4|$ . From the facts that  $|\xi_1 + \xi_3| = |\xi_2 + \xi_4| \leq (1/2)N$  and that  $|\xi_1| \geq N \geq 4N_2 \geq 4|\xi_2|$ , it is clear that  $\xi_1$  and  $\xi_3$  must have different signs.

Write  $M_4^2$  in the form

$$\begin{aligned} M_4^2 &= \frac{m^2(\xi_1)}{\xi_2} + \frac{m^2(\xi_2)}{\xi_1} + \frac{m^2(\xi_3)}{\xi_4} + \frac{m^2(\xi_4)}{\xi_3} \\ &\quad - \frac{m^2(\xi_1 + \xi_2)}{\xi_2} - \frac{m^2(\xi_1 + \xi_2)}{\xi_1} - \frac{m^2(\xi_1 + \xi_2)}{\xi_4} - \frac{m^2(\xi_1 + \xi_2)}{\xi_3} \\ &= \frac{m^2(\xi_1)}{\xi_2} - \frac{m^2(\xi_1 + \xi_2)}{\xi_2} + \frac{m^2(\xi_3)}{\xi_4} - \frac{m^2(\xi_1 + \xi_2)}{\xi_4} \\ &\quad + \frac{m^2(\xi_2)}{\xi_1} - \frac{m^2(\xi_1 + \xi_2)}{\xi_1} + \frac{m^2(\xi_4)}{\xi_3} - \frac{m^2(\xi_1 + \xi_2)}{\xi_3} \\ &= I + II. \end{aligned}$$

There are two cases that arise in estimating  $I$ .

*Subcase 1:* ( $|\xi_2| \geq 2|\xi_1 + \xi_3|$ ). Let  $\xi_3 = -\xi_1 + \rho\xi_2$ . Then  $|\rho| \leq 1/2$  and  $\xi_4 = -(\xi_1 + \xi_2 + \xi_3) = -(\rho + 1)\xi_2$ . Thus,  $I$  may be written in the form

$$\begin{aligned} I &= \frac{m^2(\xi_1)}{\xi_2} - \frac{m^2(\xi_1 + \xi_2)}{\xi_2} - \frac{m^2(-\xi_3)}{(\rho + 1)\xi_2} + \frac{m^2(\xi_1 + \xi_2)}{(\rho + 1)\xi_2} \\ &= \frac{(\rho + 1)m^2(\xi_1) - \rho m^2(\xi_1 + \xi_2) - m^2(\xi_1 - \rho\xi_2)}{(\rho + 1)\xi_2}. \end{aligned}$$

If  $m^2$  is expanded in a second-order Taylor polynomial about  $\xi_1$ , it follows that for some  $\theta_1$  and  $\theta_2$  with  $0 < \theta_1 < 1$  and  $0 < \theta_2 < 1$ ,

$$m^2(\xi_1 + \xi_2) = m^2(\xi_1) + (m^2)'(\xi_1)\xi_2 + \frac{(m^2)''(\xi_1 + \theta_1\xi_2)\xi_2^2}{2}$$

and

$$m^2(\xi_1 - \rho\xi_2) = m^2(\xi_1) - (m^2)'(\xi_1)\rho\xi_2 + \frac{(m^2)''(\xi_1 - \rho\theta_2\xi_2)\rho^2\xi_2^2}{2}.$$

Thus, because of (3.5), (3.6), (6.12), (6.13), and the facts that  $(3/4)|\xi_1| \leq |\xi_1 + \theta_1 \xi_2| \leq (5/4)|\xi_1|$  and that  $(3/4)|\xi_1| \leq |\xi_1 - \rho \theta_2 \xi_2| \leq (5/4)|\xi_1|$ , the inequality

$$\begin{aligned} |I| &= \left| \frac{(\rho + 1)m^2(\xi_1) - \rho m^2(\xi_1 + \xi_2) - m^2(\xi_1 - \rho \xi_2)}{(\rho + 1)\xi_2} \right| \\ &\leq C \left| \frac{(m^2)''(\xi_1)\rho \xi_2^2}{(\rho + 1)\xi_2} \right| \\ &\leq C \left| \frac{m^2(\kappa)\rho \xi_2}{\xi_1^2} \right| \\ &= C \left| \frac{m^2(\kappa)(\xi_1 + \xi_3)(\xi_1 + \xi_2)(\xi_1 + \xi_4)}{\xi_1^2(\xi_1 + \xi_2)(\xi_1 + \xi_4)} \right| \\ &\leq C \frac{|\alpha_4|m^2(\kappa)}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)} \end{aligned}$$

is seen to be valid.

*Subcase 2:* ( $|\xi_2| \leq 2|\xi_1 + \xi_3|$ ). By the mean-value theorem, there are values  $\theta_1$  and  $\theta_2$  with  $0 < \theta_1 < 1$  and  $0 < \theta_2 < 1$  such that

$$\begin{aligned} I &= \frac{m^2(\xi_1)}{\xi_2} - \frac{m^2(\xi_1 + \xi_2)}{\xi_2} + \frac{m^2(\xi_3)}{\xi_4} - \frac{m^2(\xi_3 + \xi_4)}{\xi_4} \\ &= (m^2)'(\xi_1 + \theta_1 \xi_2) + (m^2)'(\xi_3 + \theta_2 \xi_4) \\ &= 2m(\xi_1 + \theta_1 \xi_2)m'(\xi_1 + \theta_1 \xi_2) + 2m(\xi_3 + \theta_2 \xi_4)m'(\xi_3 + \theta_2 \xi_4) \\ &= 2m(\xi_1 + \theta_1 \xi_2)m'(\xi_1 + \theta_1 \xi_2) - 2m(-\xi_3 - \theta_2 \xi_4)m'(-\xi_3 - \theta_2 \xi_4), \end{aligned}$$

where the last line follows since  $m$  is an even function and so  $m'$  is an odd function. Applying the mean-value theorem a second time yields

$$\begin{aligned} &2m(\xi_1 + \theta_1 \xi_2)m'(\xi_1 + \theta_1 \xi_2) - 2m(-\xi_3 - \theta_2 \xi_4)m'(-\xi_3 - \theta_2 \xi_4) \\ &= 2(mm')'(\eta)(\xi_1 + \theta_1 \xi_2 + \xi_3 + \theta_2 \xi_4), \end{aligned}$$

where  $\eta$  lies between  $\xi_1 + \theta_1 \xi_2$  and  $-(\xi_3 + \theta_2 \xi_4)$ . Because  $|\xi_1 + \xi_3| = |\xi_2 + \xi_4| \leq (1/2)N$ ,  $|\xi_1| \geq N \geq 4N_2 \geq 4|\xi_2|$ , and  $|\xi_3| \geq N \geq 4N_4 \geq 4|\xi_4|$ , it follows right away that  $\xi_1$  and  $\xi_3$  must have different signs and, second, that the signs of  $\xi_1 + \theta_1 \xi_2$  and  $-(\xi_3 + \theta_2 \xi_4)$  are the same. In consequence,  $(3/8)|\xi_1| \leq (3/4)|\xi_3| \leq |\eta| \leq (5/4)|\xi_1|$ . Note also that

$$(6.14) \quad |\xi_1 + \theta_1 \xi_2 + \xi_3 + \theta_2 \xi_4| \leq |\xi_1 + \xi_3| + \theta_2 |\xi_2 + \xi_4| + |\theta_2 - \theta_1| |\xi_2| \leq 4|\xi_1 + \xi_3|.$$

Since  $(2mm')' = 2mm'' + 2m'm'$ , the inequality

$$\begin{aligned} |I| &\leq C \frac{m^2(\eta)}{\eta^2} |\xi_1 + \xi_3| \\ &= C \left| \frac{m^2(\kappa)(\xi_1 + \xi_3)(\xi_1 + \xi_2)(\xi_1 + \xi_4)}{\xi_1^2(\xi_1 + \xi_2)(\xi_1 + \xi_4)} \right| \\ &\leq C \frac{|\alpha_4| m^2(\kappa)}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)} \end{aligned}$$

is a consequence of (6.12) and (6.13).

Attention is now turned to  $II$ . Using the fact that  $|\xi_2| \leq (1/4)N$  and  $|\xi_4| \leq (1/4)N$ , it follows from the definition of  $m$  that  $m^2(\xi_2) = m^2(\xi_4) = 1$ , whence

$$\begin{aligned} II &= \frac{m^2(\xi_2)}{\xi_1} - \frac{m^2(\xi_1 + \xi_2)}{\xi_1} + \frac{m^2(\xi_4)}{\xi_3} - \frac{m^2(\xi_1 + \xi_2)}{\xi_3} \\ &= \frac{1}{\xi_1} + \frac{1}{\xi_3} - m^2(\xi_1 + \xi_2) \left( \frac{1}{\xi_3} + \frac{1}{\xi_1} \right) \\ &= (1 - m^2(\xi_1 + \xi_2)) \left( \frac{1}{\xi_3} + \frac{1}{\xi_1} \right). \end{aligned}$$

Because  $\kappa = \min\{N_i, N_{ij}\} \leq (1/4)N$ ,  $m^2(\kappa) = 1$ . In consequence, it is ascertained that

$$\begin{aligned} |III| &= \left| \frac{m^2(\xi_2)}{\xi_1} - \frac{m^2(\xi_1 + \xi_2)}{\xi_1} + \frac{m^2(\xi_4)}{\xi_3} - \frac{m^2(\xi_1 + \xi_2)}{\xi_3} \right| \\ &\leq C \left| \frac{\xi_1 + \xi_3}{\xi_1 \xi_3} \right| = C \left| \frac{\xi_1 + \xi_3}{\xi_1 \xi_3} \right| m^2(\kappa) \\ &\leq C \left| \frac{\xi_1 + \xi_3}{\xi_1 \xi_3} \right| m^2(\kappa) \frac{|(\xi_1 + \xi_2)(\xi_1 + \xi_4)|}{|(\xi_1 + \xi_2)(\xi_1 + \xi_4)|} \\ &\leq C \frac{|\alpha_4| m^2(\kappa)}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)} \end{aligned}$$

by (6.12) and (6.13).

*Case E:* The region has the form

$$R_5 = R_{>><<} = \{(\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_1| \geq N, |\xi_2| \geq N, |\xi_3| < N, |\xi_4| < N\}.$$

The region  $R_5$  may be divided into the two subregions

$$S_1^5 = \left\{ (\xi_1, \xi_2, \xi_3, \xi_4) \in R_5 : |\xi_3| \geq \frac{N}{4} \text{ or } |\xi_4| \geq \frac{N}{4} \right\}$$

and

$$S_2^5 = \left\{ (\xi_1, \xi_2, \xi_3, \xi_4) \in R_5 : |\xi_3| \leq \frac{N}{4}, |\xi_4| \leq \frac{N}{4} \right\}.$$

The case where  $(\xi_1, \xi_2, \xi_3, \xi_4) \in S_1^5$  is covered on account of Remarks 10 and 11. If  $(\xi_1, \xi_2, \xi_3, \xi_4) \in S_2^5$ , then  $|\xi_i| \leq N_i \leq (1/4)N$  for  $i = 3, 4$ . Hence,  $|\xi_1 + \xi_2| = |\xi_3 + \xi_4| \leq (1/2)N$ . So  $m^2(\xi_3) = m^2(\xi_4) = m^2(\xi_1 + \xi_2) = 1$ . Consequently,

$$\begin{aligned} M_4^2 &= \frac{m^2(\xi_1)}{\xi_2} + \frac{m^2(\xi_2)}{\xi_1} + \frac{m^2(\xi_3)}{\xi_4} + \frac{m^2(\xi_4)}{\xi_3} \\ &\quad - \frac{m^2(\xi_1 + \xi_2)}{\xi_2} - \frac{m^2(\xi_1 + \xi_2)}{\xi_1} - \frac{m^2(\xi_1 + \xi_2)}{\xi_4} - \frac{m^2(\xi_1 + \xi_2)}{\xi_3} \\ &= \frac{m^2(\xi_1)}{\xi_2} + \frac{m^2(\xi_2)}{\xi_1} - \frac{1}{\xi_2} - \frac{1}{\xi_1} \\ &= \frac{\xi_1 m^2(\xi_1) + \xi_2 m^2(\xi_2)}{\xi_1 \xi_2} - \frac{\xi_1 + \xi_2}{\xi_1 \xi_2} \\ &= I - II. \end{aligned}$$

From the facts that  $|\xi_1 + \xi_2| = |\xi_3 + \xi_4| \leq (1/2)N$  and that  $|\xi_1| \geq N$ , it is clear that  $\xi_1$  and  $\xi_2$  must have different signs. Thus, by the mean-value theorem and (6.5),

$$\begin{aligned} |I| &= \left| \frac{\xi_1 m^2(\xi_1) + \xi_2 m^2(\xi_2)}{\xi_1 \xi_2} \right| \\ &= \left| \frac{\xi_1 m^2(\xi_1) - (-\xi_2) m^2(-\xi_2)}{\xi_1 \xi_2} \right| \\ &= \left| \frac{G'(\eta)(\xi_1 + \xi_2)}{\xi_1 \xi_2} \right| \\ &\leq C \left| \frac{\xi_1 + \xi_2}{\xi_1 \xi_2} \right|, \end{aligned}$$

where  $\eta$  lies between  $\xi_1$  and  $-\xi_2$ .

Because  $|\xi_i| \leq N_i \leq (1/4)N \leq (1/4)|\xi_1|$  or  $(1/4)|\xi_2|$  for  $i = 3, 4$ ,  $(3/4)|\xi_1| \leq |\xi_1 + \xi_i| \leq (5/4)|\xi_1|$ ,  $i = 3, 4$ . Thus, for  $i = 3, 4$ ,  $N + N_i \leq (5/4)N$  and  $|\xi_1 + \xi_i| \geq (3/4)N$ , whence

$$(6.15) \quad \frac{1}{|\xi_1 + \xi_i|} \leq \frac{5}{3(N + N_i)}, \quad i = 3, 4.$$

Moreover, for  $i = 1, 2, N_i \geq N \geq 1$ , and so

$$(6.16) \quad \frac{1}{|\xi_1|} \leq \frac{1}{|\xi_i|} \leq \frac{2}{N_i} \leq \frac{4}{N + N_i}.$$

Thus,

$$\begin{aligned} |M_4^2| &\leq |I| + |II| \\ &\leq C \left| \frac{\xi_1 + \xi_2}{\xi_1 \xi_2} \right| \\ &= 2 \left| \frac{(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_1 + \xi_4)}{\xi_1 \xi_2 (\xi_1 + \xi_3)(\xi_1 + \xi_4)} \right| \\ &\leq \frac{C|\alpha_4|m^2(\kappa)}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)} \end{aligned}$$

because of (6.15), (6.16), and  $m^2(\kappa) = 1$ .

REMARK 12. With some obvious changes, the same arguments show that the inequality (6.2) holds for regions of the form

$$\left\{ (\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_i| \leq \frac{N}{C_i}, i = 2, 4, |\xi_j| \geq \frac{N}{C_j}, j = 1, 3 \right\}$$

and for the regions

$$\left\{ (\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_i| \leq \frac{N}{C_i}, i = 3, 4, |\xi_j| \geq \frac{N}{C_j}, j = 1, 2 \right\}.$$

Case F: The region is of the form

$$R_6 = R_{><<<} = \{ (\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_1| \geq N, |\xi_2| < N, |\xi_3| < N, |\xi_4| < N \}.$$

We divide this region into the two subregions

$$S_1^6 = \left\{ (\xi_1, \xi_2, \xi_3, \xi_4) \in R_6 : |\xi_2| \geq \frac{N}{4} \text{ or } |\xi_3| \geq \frac{N}{4} \text{ or } |\xi_4| \geq \frac{N}{4} \right\}$$

and

$$S_2^6 = \left\{ (\xi_1, \xi_2, \xi_3, \xi_4) \in R_6 : |\xi_i| \leq \frac{N}{4}, i = 2, 3, 4 \right\}.$$

If  $(\xi_1, \xi_2, \xi_3, \xi_4) \in S_1^6$ , then by Remarks 10, 11, and 12, the situation is covered.



If  $(\xi_1, \xi_2, \xi_3, \xi_4) \in S_2^6$ , then  $|\xi_i| \leq N_i \leq (1/4)N \leq (1/4)|\xi_1|$ . Since

$$\begin{aligned} 0 &= |\xi_1 + \xi_2 + \xi_3 + \xi_4| \\ &\geq |\xi_1| - |\xi_2| - |\xi_3| - |\xi_4| \\ &\geq \frac{|\xi_1|}{4} \geq \frac{N}{4}, \end{aligned}$$

this is a contradiction. Thus,  $S_2^6 = \emptyset$ .

REMARK 13. With some obvious changes, the same arguments show that the inequality (6.2) holds for regions of the form

$$\left\{ (\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_1| \geq \frac{N}{C_1}, |\xi_i| \leq \frac{N}{C_i}, i = 2, 3, 4 \right\}.$$

Case G: The region is of the form

$$R_7 = R_{<<<<} = \{(\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_1| \leq N, |\xi_2| < N, |\xi_3| < N, |\xi_4| < N\}.$$

We divide this region into the two subregions

$$S_1^7 = \left\{ (\xi_1, \xi_2, \xi_3, \xi_4) \in R_7 : |\xi_1| \geq \frac{N}{4}, |\xi_2| < N, |\xi_3| < N, |\xi_4| < N \right\}$$

and

$$S_2^7 = \left\{ (\xi_1, \xi_2, \xi_3, \xi_4) \in R_7 : |\xi_i| \leq \frac{N}{4}, i = 1, 2, 3, 4 \right\}.$$

If  $(\xi_1, \xi_2, \xi_3, \xi_4) \in S_1^7$ , then by Remarks 10, 11, 12, and 13, the situation is covered. If  $(\xi_1, \xi_2, \xi_3, \xi_4) \in S_2^7$ , as  $\max_{1 \leq i \leq 4} \{|\xi_i|\} \leq (N/4) < N$  and  $|\xi_1 + \xi_2| \leq (N/2) < N$ ,  $M_4^2 = 0$  as  $m^2(\xi_i) = m^2(\xi_1 + \xi_2) = 1, i = 1, 2, 3, 4$ . So this is a trivial case.

This establishes estimate (6.2).

### §7. The $\Lambda^5$ terms

In [16, Section 6], an estimate of a quintilinear operator analogous to our  $\Lambda^5$  was developed that completed the control of the growth of the simpler version of the modified energy arising in the analysis of the KdV equation. In our analysis, the calculation and estimation of the time derivative of the quantity  $E^4$  that is applicable to the systems considered here are more complicated than the analogous computations in [16] owing to the appearance of quintilinear products involving both  $u$  and  $v$ . Another aspect that adds

difficulty is the appearance of two different multipliers  $M_4^1$  and  $M_4^2$  in the analysis. These are defined in terms of the  $M^{ij}$  by the formulas

$$M_4^2(\xi_1, \xi_2, \xi_3, \xi_4) = M^{12} + M^{34}$$

and

$$M_4^1(\xi_1, \xi_2, \xi_3, \xi_4) = M^{12} + M^{34} + M^{13} + M^{24} + M^{14} + M^{23}.$$

It is immediate that  $M_4^1(\xi_1, \xi_2, \xi_3, \xi_4)$  can be written as the partial symmetrization

$$M_4^1(\xi_1, \xi_2, \xi_3, \xi_4) = M_4^2(\xi_1, \xi_2, \xi_3, \xi_4) + M_4^2(\xi_1, \xi_3, \xi_2, \xi_4) + M_4^2(\xi_1, \xi_4, \xi_3, \xi_2)$$

of  $M_4^2(\xi_1, \xi_2, \xi_3, \xi_4)$ . Since the estimates for  $M_4^2(\xi_1, \xi_2, \xi_3, \xi_4)$  are symmetric with respect to the dyadic bands provided by the  $N_i$ , it is clear that bounds derived for  $M_4^2$  will also hold for  $M_4^1$ . (This remark provides a proof of [16, Lemma 4.4].)

Define the quantity  $\Gamma^4$  by

$$\Gamma^4 = \Gamma^3 - \tilde{a}_{22}\Lambda_{22}\left(\frac{M_4^2}{\alpha_4}\right) - \sum_{i+j=4} a_{ij}\Lambda_{ij}\left(\frac{M_4^1}{\alpha_4}\right).$$

From what has already been determined about the  $t$ -derivatives of the integrals defining the various  $\Lambda$ , it transpires that  $\frac{d}{dt}\Gamma^4$  can be written as a linear combination of terms of the form

$$\Lambda_{ij}(M_{\sigma, \tilde{\sigma}}) = \Lambda_{ij}([\xi_{\tilde{\sigma}(5)} + \xi_{\tilde{\sigma}(1)}]M_{\sigma}(\xi_{\tilde{\sigma}(1)} + \xi_{\tilde{\sigma}(5)}, \xi_{\tilde{\sigma}(2)}, \xi_{\tilde{\sigma}(3)}, \xi_{\tilde{\sigma}(4)})),$$

where  $i + j = 5$ ,  $\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 = 0$ ,  $\sigma$  is a permutation of  $\{1, 2, 3, 4\}$ ,  $\tilde{\sigma}$  is a permutation of  $\{1, 2, 3, 4, 5\}$ ,  $M_{\sigma}$  is defined by

$$M_{\sigma}(\xi_1, \xi_2, \xi_3, \xi_4) = M(\xi_{\sigma(1)}, \xi_{\sigma(2)}, \xi_{\sigma(3)}, \xi_{\sigma(4)}),$$

and  $M$  is either  $M_4^1/\alpha_4$  or  $M_4^2/\alpha_4$ . So, for example, if  $\sigma = \begin{pmatrix} 1234 \\ 3214 \end{pmatrix}$ , then

$$M_{\sigma}(\xi_1 + \xi_5, \xi_2, \xi_3, \xi_4) = M(\xi_3, \xi_2, \xi_1 + \xi_5, \xi_4).$$

The inequality (6.2) implies that

$$\begin{aligned} & \left| [\xi_{\tilde{\sigma}(5)} + \xi_{\tilde{\sigma}(1)}]M_{\sigma}(\xi_{\tilde{\sigma}(1)} + \xi_{\tilde{\sigma}(5)}, \xi_{\tilde{\sigma}(2)}, \xi_{\tilde{\sigma}(3)}, \xi_{\tilde{\sigma}(4)}) \right| \\ & \leq \frac{Cm^2(\kappa)(N + N_l)N_{l5}}{(N + N_1)(N + N_2)(N + N_3)(N + N_4)(N + N_{l5})} \end{aligned}$$

for any permutations  $\sigma, \tilde{\sigma}$ . This is the inequality that is essential for the proof of (3.18) in Lemma 2. What needs to be shown now is that for  $0 > s > -3/4$ ,

$$(7.1) \quad \left| \int_0^\delta \int_{\xi_1+\dots+\xi_5=0} M_{\sigma,\tilde{\sigma}}(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) \widehat{u}_1(\xi_1) \widehat{u}_2(\xi_2) \widehat{u}_3(\xi_3) \widehat{u}_4(\xi_4) \widehat{u}_5(\xi_5) dt \right| \\ \lesssim N^{-5s} \prod_{i=1}^5 \|Iu_i\|_{X_{0,(1/2)+}^\delta},$$

where the Bourgain spaces  $X_{s,b}^\delta$  are as defined earlier in (2.1) and (2.2). The proof of this latter inequality uses [16, Lemma 5.1], which is restated here for the reader’s convenience.

LEMMA 6. *Given functions  $w_i = w_i(x, t)$ , with  $1 \leq i \leq 5$ , the inequality*

$$\left| \int_0^\delta \int \prod_{i=1}^5 w_i(x, t) dx dt \right| \\ \lesssim \left( \prod_{i=1}^3 \|w_i(x, t)\|_{X_{(1/4),(1/2)+}^\delta} \right) \|w_4\|_{X_{-(3/4),(1/2)+}^\delta} \|w_5\|_{X_{-(3/4),(1/2)+}^\delta}$$

holds for any  $\delta > 0$ .

The stage is set to initiate a discussion of the important inequality (3.18) of Lemma 2. The proof of (3.18) is similar to the proof of [16, Lemma 5.2]. Consequently, we content ourselves with providing a few indications of the needed calculations and then refer the reader to the commentary in [16]. Begin by noting that with the right choices of  $u'_i$ , we can rewrite

$$\left| \int_0^\delta \Lambda_{ij}(M_{\sigma,\tilde{\sigma}}) dt \right| \\ = \left| \int_0^\delta \int_{\xi_1+\dots+\xi_5=0} M_{\sigma,\tilde{\sigma}}(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) \widehat{u}_1(\xi_1) \widehat{u}_2(\xi_2) \widehat{u}_3(\xi_3) \widehat{u}_4(\xi_4) \widehat{u}_5(\xi_5) dt \right| \\ = \left| \int_0^\delta \int_{\xi_1+\dots+\xi_5=0} M_{\sigma,\tilde{\sigma}}(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) \right. \\ \left. \times \frac{\widehat{Iu}_1(\xi_1)}{m(\xi_1)} \frac{\widehat{Iu}_2(\xi_2)}{m(\xi_2)} \frac{\widehat{Iu}_3(\xi_3)}{m(\xi_3)} \frac{\widehat{Iu}_4(\xi_4)}{m(\xi_4)} \frac{\widehat{Iu}_5(\xi_5)}{m(\xi_5)} dt \right|$$

$$= \left| \int_0^\delta \int_{\xi_1 + \dots + \xi_5 = 0} \Delta(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) \times \widehat{Iw}_1(\xi_1) \widehat{Iw}_2(\xi_2) \widehat{Iw}_3(\xi_3) \widehat{Iw}_4(\xi_4) \widehat{Iw}_5(\xi_5) dt \right|,$$

where

$$\Delta(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = e^{iE} \frac{M_{\sigma, \bar{\sigma}}(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)}{m(\xi_1)m(\xi_2)m(\xi_3)m(\xi_4)m(\xi_5)},$$

with

$$E = \sum_{j=1}^5 \arg \widehat{u}_j(\xi_j), \quad \widehat{w}_j(\xi_j) = e^{-i \arg \widehat{u}_j(\xi_j)} \widehat{u}_j(\xi_j),$$

and the operator  $I = I(m)$  the Fourier multiplier operator with positive symbol  $m$  defined in (3.4). The point here is that the exponential  $\exp(iE)$  is unimodular and the  $\widehat{w}_j, j = 1, \dots, 5$ , are all nonnegative. Since the symbol  $m$  is everywhere positive, it follows that the functions  $\widehat{Iw}_j, j = 1, \dots, 5$ , are also nonnegative. Consequently, if  $F$  is the nonnegative function

$$F(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = \widehat{Iw}_1(\xi_1) \widehat{Iw}_2(\xi_2) \widehat{Iw}_3(\xi_3) \widehat{Iw}_4(\xi_4) \widehat{Iw}_5(\xi_5),$$

then the absolute value of the integral with respect to time of  $\Lambda_{ij}(M_{\sigma, \bar{\sigma}})$  appearing above can be straightforwardly bounded above in terms of the integral of  $F$  over the hyperplane  $\xi_1 + \dots + \xi_5 = 0$ . To see this, proceed as follows:

$$\begin{aligned} & \left| \int_0^\delta \Lambda_{ij}(M_{\sigma, \bar{\sigma}}) dt \right| \\ & \leq \int_0^\delta \int_{\xi_1 + \dots + \xi_5 = 0} \frac{|M_{\sigma, \bar{\sigma}}(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)|}{m(\xi_1)m(\xi_2)m(\xi_3)m(\xi_4)m(\xi_5)} F(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) dt \\ & \leq C \int_0^\delta \int_{\xi_1 + \dots + \xi_5 = 0} \frac{|M_{\sigma, \bar{\sigma}}(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)|}{m(N_1)m(N_2)m(N_3)m(N_4)m(N_5)} F(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) dt \\ & \leq C \int_0^\delta \frac{m^2(\kappa)N_{45}}{(N+N_1)(N+N_2)(N+N_3)(N+N_{45})} \int_{\xi_1 + \dots + \xi_5 = 0} F(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) dt \\ & \leq C \int_0^\delta \mathfrak{M}(\kappa) \int_{\xi_1 + \dots + \xi_5 = 0} F(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) dt \\ & = C \int_0^\delta \int_{-\infty}^\infty \mathfrak{M}(\kappa) Iw_1(x, t) Iw_2(x, t) Iw_3(x, t) Iw_4(x, t) Iw_5(x, t) dx dt, \end{aligned}$$

where

$$\mathfrak{M}(\kappa) = \frac{m^2(\kappa)}{(N + N_1)m(N_1)(N + N_2)m(N_2)(N + N_3)m(N_3)m(N_4)m(N_5)}.$$

Since  $m^2(\kappa) = m^2(\min\{N_i, N_{ij}\}) \leq 1$ , an estimate for the last integral depends only on bounding the quantity

$$\frac{1}{(N + N_1)m(N_1)(N + N_2)m(N_2)(N + N_3)m(N_3)m(N_4)m(N_5)}.$$

Following the argument in [16, Lemma 5.2], assume for reasons of symmetry that  $N_1 \geq N_2 \geq N_3$  and that  $N_4 \geq N_5$ . For  $N_3 \geq N$ ,

$$\frac{1}{(N + N_j)m(N_j)} \leq cN^s N_j^{-1-s}$$

for  $j = 1, 2, 3$ , while for  $N_5 \geq N$ ,

$$\frac{1}{m(N_j)} \leq CN^s N_j^{-s}$$

for  $j = 4, 5$ . Since  $\widehat{u}_j$  is nonzero only on the dyadic band  $(1/2)N_j \leq |\xi_j| \leq N_j$ , it follows that  $N_j^{-s} |\widehat{w}_j(\xi_j)| \leq 2^{-s} |\xi_j|^{-s} |\widehat{w}_j(\xi_j)|$ . These last two inequalities imply that

$$\|N_j^{-s} Iw_j\|_{X_{-(3/4), (1/2)+}^\delta} \leq CN_j^{-(3/4)-s} \|Iw_j\|_{X_{0, (1/2)+}^\delta}$$

and that

$$\|N_j^{-1-s} Iw_j\|_{X_{(1/4), (1/2)+}^\delta} \leq CN_j^{-(3/4)-s} \|Iw_j\|_{X_{0, (1/2)+}^\delta}.$$

In summary, the preceding machinations allow the conclusion that

$$\begin{aligned} & \left| \int_0^\delta \Lambda_{ij}(M_{\sigma, \bar{\sigma}}) dt \right| \\ & \leq CN^{5s} \int_0^\delta \int_{-\infty}^\infty [N_1^{-1-s} Iw_1 N_2^{-1-s} Iw_2 N_3^{-1-s} Iw_3 \\ & \quad \times N_4^{-s} Iw_4 N_5^{-s} Iw_5](x, t) dx dt, \end{aligned}$$

which, upon applying [16, Lemma 5.1] and the fact that  $\|Iw_i\|_{X_{0,(1/2)+}^\delta} = \|Iu_i\|_{X_{0,(1/2)+}^\delta}$ , allows us to continue the inequality, namely,

$$\begin{aligned} &\leq CN^{5s} \left( \prod_{j=1}^3 \|N_i^{-1-s} Iw_j(x, t)\|_{X_{-(3/4),(1/2)+}^\delta} \right) \\ &\quad \times \|N_4^{-s} Iw_4\|_{X_{(1/4),(1/2)+}^\delta} \|N_5^{-s} Iw_5\|_{X_{(1/4),(1/2)+}^\delta} \\ &\leq CN^{5s} \prod_{j=1}^5 N_j^{-(3/4)-s} \|Iu_j\|_{X_{0,(1/2)+}^\delta}. \end{aligned}$$

Further details for the cases depending on the relative sizes of  $N$  and  $N_j$  are handled as in [16, proof of Lemma 5.2]. Noting that  $s > -3/4$  implies that  $-3/4 - s < 0$ , summing the dyadic pieces, and then summing over all the different quintilinear integrals, the desired estimate,

$$\begin{aligned} \int_0^\delta \frac{d}{dt} \{H(t)^2 - \tilde{\Lambda}^3 - \tilde{\Lambda}^4\} &\leq CN^{5s} \sum_{l=0}^5 \|Iu\|_{X_{0,(1/2)+}^\delta}^l \|Iv\|_{X_{0,(1/2)+}^\delta}^{5-l} \\ &\leq CN^{5s} (\|Iu\|_{X_{0,(1/2)+}^\delta} + \|Iv\|_{X_{0,(1/2)+}^\delta})^5 \\ &\leq CN^{5s} \|(Iu, Iv)\|_{X_{0,(1/2)+}^\delta}^5 \\ &\leq CN^{5s} H(0)^5, \end{aligned}$$

emerges.

PROPOSITION 2. *Let  $I$  be the multiplier defined in (3.4), and suppose that  $0 > s > -3/4$ . Let  $(u(x, t), v(x, t))$  be any pair of real-valued functions. Then for any  $t$ ,*

$$(7.2) \quad |\tilde{\Lambda}^3(t) + \tilde{\Lambda}^4(t)| \leq c_3 H(t)^3 + c_4 H(t)^4.$$

*Proof.* The trilinear terms are the same as the  $\Lambda_3(\sigma_3)$  of [16], and the quadrilinear terms, while not exactly the same as those in [16], have multipliers that satisfy the same estimates as the  $\Lambda_4(\sigma_4)$  term of [16]. The estimate [16, (6.8)] for  $\Lambda_4(\sigma_4)$  holds for the different quadrilinear terms that appear in  $\tilde{\Lambda}_4$ .

The proof of the estimate [16, (6.7)] uses an assumption that  $-3/4 < s < -1/2$ . Since we wish to conclude global existence for the full range

$-3/4 < s < 0$ , we need to modify the proof in [16]. As stated in [16, (6.9), p. 724], it suffices to show that

$$\left| \Lambda_3 \left( \frac{\xi_1 m^2(\xi_1) + \xi_2 m^2(\xi_2) + \xi_3 m^2(\xi_3)}{\xi_1 \xi_2 \xi_3 m(\xi_1) m(\xi_2) m(\xi_3)}; u_1, u_2, u_3 \right) \right| \leq \prod_{j=1}^3 \|u_j\|_2.$$

Let  $(1/2)N_j < |\xi_j| \leq N_j$ , where  $N_j$  is dyadic, and assume that  $N_1 \geq N_2 \geq N_3$ . The proof boils down to showing that

$$(7.3) \quad \left| \frac{\xi_1 m^2(\xi_1) + \xi_2 m^2(\xi_2) + \xi_3 m^2(\xi_3)}{\xi_1 \xi_2 \xi_3 m(\xi_1) m(\xi_2) m(\xi_3)} \right| \leq c N_1^{-(1/6)} N_2^{-(1/6)} N_3^{-(1/6)}.$$

The case  $N_1 \ll N$  is covered in [16, Lemma 6.1]. For the case  $N_3 > N$ , reason as follows. First of all,

$$|\xi_1 m^2(\xi_1) + \xi_2 m^2(\xi_2) + \xi_3 m^2(\xi_3)| \leq c N^{-2s} [N_1^{1+2s} + N_2^{1+2s} + N_3^{1+2s}].$$

If  $-(3/4) < s < -(1/2)$ , then  $1 + 2s < 0$  and

$$c N^{-2s} [N_1^{1+2s} + N_2^{1+2s} + N_3^{1+2s}] \leq 3c N^{-2s} N_3^{1+2s},$$

which is shown in [16] to lead to the desired bound (7.3). If, on the other hand,  $-1/2 < s < 0$ , then  $1 + 2s \geq 0$  and the argument is slightly more complicated. If  $N_1 \leq 16N_3$ , then

$$\begin{aligned} |\xi_1 m^2(\xi_1) + \xi_2 m^2(\xi_2) + \xi_3 m^2(\xi_3)| &\leq c N^{-2s} [N_1^{1+2s} + N_2^{1+2s} + N_3^{1+2s}] \\ &\leq c N^{-2s} (2 \cdot 32^{1+2s} + 1) N_3^{1+2s}. \end{aligned}$$

The desired estimate (7.3) follows once again with a constant in the estimate which depends on  $s$ . Next assume that  $N_1 > 16N_3$ . Then since  $\xi_1 + \xi_2 + \xi_3 = 0$  and  $m$  is an even function,

$$(7.4) \quad \begin{aligned} &\xi_1 m^2(\xi_1) + \xi_2 m^2(\xi_2) + \xi_3 m^2(\xi_3) \\ &= \xi_1 m^2(\xi_1) - (\xi_1 + \xi_3) m^2(\xi_1 + \xi_3) + \xi_3 m^2(\xi_3) \\ &= \xi_1 [m^2(\xi_1) - m^2(\xi_1 + \xi_3)] + \xi_3 [m^2(\xi_3) - m^2(\xi_1 + \xi_3)]. \end{aligned}$$

Applying the mean-value theorem to the first term on the right,

$$\xi_1 [m^2(\xi_1) - m^2(\xi_1 + \xi_3)] = 2\xi_1 \xi_3 m(\theta) m'(\theta),$$

where  $\theta$  is between  $\xi_1$  and  $\xi_1 + \xi_3$ . Since  $N_1 > 16N_3$ ,  $|\xi_1 + \xi_3| \geq (1/4)N_1$ , and so

$$(7.5) \quad |\xi_1 \xi_3 m(\theta) m'(\theta)| \leq c N_1 N_3 N^{-2s} N_1^{2s-1} = c N^{-2s} N_1^{2s} N_3.$$

The second term in (7.4) is estimated as

$$(7.6) \quad \begin{aligned} |\xi_3 [m^2(\xi_3) - m^2(\xi_1 + \xi_3)]| &\leq c N_3 \left[ N^{-2s} \left( \frac{N_3}{2} \right)^{2s} - N^{-2s} (2N_1)^{2s} \right] \\ &\leq c \frac{1 - 64^{2s}}{4^s} N^{-2s} N_3^{1+2s}. \end{aligned}$$

Thus,

$$\begin{aligned} \left| \frac{\xi_1 m^2(\xi_1) + \xi_2 m^2(\xi_2) + \xi_3 m^2(\xi_3)}{\xi_1 \xi_2 \xi_3 m(\xi_1) m(\xi_2) m(\xi_3)} \right| &\leq c_s \frac{N^{-2s} N_1^{2s} N_3 + N^{-2s} N_3^{1+2s}}{N^{-3s} N_1^{1+s} N_2^{1+s} N_3^{1+s}} \\ &\leq c_s N^s \left[ \frac{N_1^s}{N_1 N_2^{1+s} N_3^s} + \frac{N_3^s}{N_1^{1+s} N_2^{1+s}} \right] \\ &\leq c_s N^s \frac{1}{N_1^{(1/6)} N_2^{(1/6)} N_3^{2((5/6)+s)}} \\ &\leq c_s N^s N_1^{-(1/6)} N_2^{-(1/6)} N_3^{-(1/6)}, \end{aligned}$$

since  $-3/4 < s < 0 \implies 1/6 < 2(5/6 + s)$ . Now [16, Lemma 6.1] holds for  $-3/4 < s < 0$ , and so (7.2) then follows from [16, (6.7), (6.8)] as follows:

$$\begin{aligned} |\tilde{\Lambda}^3(t) + \tilde{\Lambda}^4(t)| &\leq c \sum_{i=0}^3 \|Iu(t)\|_{L_x^2}^i \|Iv(t)\|_{L_x^2}^{3-i} + \tilde{c} \sum_{i=0}^4 \|Iu(t)\|_{L_x^2}^i \|Iv(t)\|_{L_x^2}^{4-i} \\ &\leq c (\|Iu(t)\|_{L_x^2} + \|Iv(t)\|_{L_x^2})^3 + \tilde{c} (\|Iu(t)\|_{L_x^2} + \|Iv(t)\|_{L_x^2})^4 \\ &\leq c_1 \|(Iu(t), Iv(t))\|_{L_x^2 \times L_x^2}^3 + c_2 \|(Iu(t), Iv(t))\|_{L_x^2 \times L_x^2}^4 \\ &\leq c_3 H(t)^3 + c_4 H(t)^4, \end{aligned}$$

since

$$\|(Iu(t), Iv(t))\|_{L_x^2 \times L_x^2} \approx H(t).$$

Because the constants in the above inequalities are independent of  $t$ , the estimate in the proposition obtains.  $\square$

This concludes the development of the technical details needed to establish the inequalities in Lemma 2 and thus concludes the proof of the main theorem.



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## REFERENCES

- [1] J. P. Albert, J. L. Bona, and M. Felland, *A criterion for the formation of singularities for the generalized Korteweg–de Vries equation*, *Comput. Appl. Math.* **7** (1988), 3–11. [MR 0965674](#).
- [2] B. Alvarez-Samaniego and X. Carvajal, *On the local well-posedness for some systems of coupled KdV equations*, *Nonlinear Anal.* **69** (2008), 692–715. [MR 2426282](#). [DOI 10.1016/j.na.2007.06.009](#).
- [3] J. M. Ash, J. Cohen, and G. Wang, *On strongly interacting internal solitary waves*, *J. Fourier Anal. Appl.* **2** (1996), 507–517. [MR 1412066](#). [DOI 10.1007/s00041-001-4041-4](#).
- [4] J. L. Bona, H. Chen, O. A. Karakashian, and Y. Xing, *Conservative, discontinuous-Galerkin methods for the generalized Korteweg–de Vries equation*, *Math. Comp.* **82** (2013), 1401–1432. [MR 3042569](#).
- [5] J. L. Bona, M. Chen, and J.-C. Saut, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media, I: Derivation and linear theory*, *J. Nonlinear Sci.* **12** (2002), 283–318. [MR 1915939](#). [DOI 10.1007/s00332-002-0466-4](#).
- [6] ———, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media, II: The nonlinear theory*, *Nonlinearity* **17** (2004), 925–952. [MR 2057134](#). [DOI 10.1088/0951-7715/17/3/010](#).
- [7] J. L. Bona, V. A. Dougalis, O. A. Karakashian, and W. R. McKinney, *Conservative, high-order numerical schemes for the generalized Korteweg–de Vries equation*, *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **351** (1995), 107–164. [MR 1336983](#). [DOI 10.1098/rsta.1995.0027](#).
- [8] J. L. Bona, G. Ponce, J.-C. Saut, and M. M. Tom, *A model system for strong interaction between internal solitary waves*, *Comm. Math. Phys.* **143** (1992), 287–313. [MR 1145797](#).
- [9] J. L. Bona, W. G. Pritchard, and L. R. Scott, *Numerical schemes for a model for nonlinear dispersive waves*, *J. Comput. Phys.* **60** (1985), 167–186. [MR 0805869](#). [DOI 10.1016/0021-9991\(85\)90001-4](#).
- [10] J. L. Bona and R. Scott, *Solutions of the Korteweg–de Vries equation in fractional order Sobolev spaces*, *Duke Math. J.* **43** (1976), 87–99. [MR 0393887](#).

- [11] J. L. Bona, S.-M. Sun, and B.-Y. Zhang, *Conditional and unconditional well-posedness for nonlinear evolution equations*, Adv. Differential Equations **9** (2004), 241–265. MR 2100628.
- [12] J. L. Bona and F. B. Weissler, *Pole dynamics of interacting solitons and blowup of complex-valued solutions of KdV*, Nonlinearity **22** (2009), 311–349. MR 2475549. DOI 10.1088/0951-7715/22/2/005.
- [13] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, I: Schrödinger equations* Geom. Funct. Anal. **3** (1993), 107–156. MR 1209299. DOI 10.1007/BF01896020.
- [14] ———, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, II: The KdV-equation*, Geom. Funct. Anal. **3** (1993), 209–262. MR 1215780. DOI 10.1007/BF01895688.
- [15] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Global well-posedness for KdV in Sobolev spaces of negative index*, Electron. J. Differential Equations **2001**, no. 26, 7 pp. MR 1824796.
- [16] ———, *Sharp global well-posedness for KdV and modified KdV on  $\mathbb{R}$  and  $\mathbb{T}$* , J. Amer. Math. Soc. **16** (2003), 705–749. MR 1969209. DOI 10.1090/S0894-0347-03-00421-1.
- [17] J. A. Gear and R. Grimshaw, *Weak and strong interactions between internal solitary waves*, Stud. Appl. Math. **70** (1984), 235–258. MR 0742590.
- [18] Z. Guo, *Global well-posedness of Korteweg–de Vries equation in  $H^{-3/4}(\mathbb{R})$* , J. Math. Pures Appl. (9) **91** (2009), 583–597. MR 2531556. DOI 10.1016/j.matpur.2009.01.012.
- [19] T. Kato, “On the Cauchy problem for the (generalized) Korteweg–de Vries equation” in *Studies in Applied Mathematics*, Adv. Math. Suppl. Stud. **8**, Academic Press, New York, 1983, 93–128. MR 0759907.
- [20] C. E. Kenig, G. Ponce, and L. Vega, *The Cauchy problem for the Korteweg–de Vries equation in Sobolev spaces of negative indices*, Duke Math. J. **71** (1993), 1–21. MR 1230283. DOI 10.1215/S0012-7094-93-07101-3.
- [21] ———, *A bilinear estimate with applications to the KdV equation*, J. Amer. Math. Soc. **9** (1996), 573–603. MR 1329387. DOI 10.1090/S0894-0347-96-00200-7.
- [22] N. Kishimoto, *Well-posedness of the Cauchy problem for the Korteweg–de Vries equation at the critical regularity*, Differential Integral Equations **22** (2009), 447–464. MR 2501679.
- [23] J.-L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications, II*, Grundlehren Math. Wiss. **182**, Springer, New York, 1972. MR 0350178.
- [24] J.-L. Lions and J. Peetre, *Propriétés d’espaces d’interpolation*, C. R. Math. Acad. Sci. Paris **253** (1961), 1747–1749. MR 0133693.
- [25] A. J. Majda and J. A. Biello, *The nonlinear interaction of barotropic and equatorial baroclinic Rossby waves*, J. Atmospheric Sci. **60** (2003), 1809–1821. MR 1994152. DOI 10.1175/1520-0469(2003)060<1809:TNI0BA>2.0.CO;2.
- [26] T. Oh, *Diophantine conditions in global well-posedness for coupled KdV-type systems*, Electron. J. Differential Equations **2009**, no. 52, 48 pp. MR 2505110.
- [27] J. Peetre, *Espaces d’interpolation et théorème de Soboleff*, Ann. Inst. Fourier (Grenoble) **16** (1966), 279–317. MR 0221282.
- [28] J.-C. Saut, *Sur quelques généralisations de l’équation de Korteweg–de Vries*, J. Math. Pures Appl. (9) **58** (1979), 21–61. MR 0533234.
- [29] T. Tao, *Nonlinear Dispersive Equations: Local and Global Analysis*, CBMS Reg. Conf. Ser. Math. **106**, Amer. Math. Soc., Providence, 2006. MR 2233925.

- [30] L. Tartar, *Interpolation non linéaire et régularité*, J. Funct. Anal. **9** (1972), 469–489.  
MR 0310619.

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