Nonhomogeneous Boundary-Value Problems for One-Dimensional Nonlinear Schrödinger Equations

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Abstract

This paper is concerned with initial-boundary-value problems (IBVPs) for a class of nonlinear Schrödinger equations posed either on a half line $\mathbb{R}^+$ or on a bounded interval $(0, L)$ with nonhomogeneous boundary conditions. For any s with $0 \leq s < 5/2$ and $s \neq 3/2$, it is shown that the relevant IBVPs are locally well-posed if the initial data lie in the L^2–based Sobolev spaces $H^s(\mathbb{R}^+)$ in the case of the half line and in $H^s(0, L)$ on a bounded interval, provided the boundary data are selected from $H_{loc}^{(2s+1)/4}(\mathbb{R}^+)$ and $H_{loc}^{(s+1)/2}(\mathbb{R}^+)$, respectively. (For $s > \frac{1}{2}$, compatibility between the initial and boundary conditions is also needed.) Global well-posedness is also discussed when $s \geq 1$. From the point of view of the well-posedness theory, the results obtained reveal a significant difference between the IBVP posed on $\mathbb{R}^+$ and the IBVP posed on $(0, L)$. The former is reminiscent of the theory for the pure initial-value problem (IVP) for these Schrödinger equations posed on the whole line $\mathbb{R}$ while the theory on a bounded interval looks more like that of the pure IVP posed on a periodic domain. In particular, the regularity demanded of the boundary data for the IBVP on $\mathbb{R}^+$ is consistent with the temporal trace results that obtain for solutions of the pure IVP

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on $\mathbb{R}$, while the slightly higher regularity of boundary data for the IBVP on $(0, L)$ resembles what is found for temporal traces of spatially periodic solutions.

Keywords: Nonlinear Schrödinger equations, Initial boundary value problems

1 Introduction

Studied here are initial-boundary-value problems for nonlinear Schrödinger equations posed either on a half line $\mathbb{R}^+$, *viz.*

$$\begin{cases} iu_t + u_{xx} + \lambda |u|^{p-2} u = 0, & x \in \mathbb{R}^+,\ t \in \mathbb{R}, \\ u(x,0) = \phi(x), & u(0,t) = h(t), \end{cases} \tag{1.1}$$

or on a finite interval $(0, L)$,

$$\begin{cases} iu_t + u_{xx} + \lambda |u|^{p-2} u = 0, & x \in (0, L), \quad t \in \mathbb{R}\,, \\ u(x,0) = \phi(x), \qquad u(0,t) = h_1(t), & u(L,t) = h_2(t). \end{cases} \tag{1.2}$$

Here, the parameter λ is a non-zero real number and $p \geq 3$.[1] Note that, due to the symmetry of the equation with respect to the change of variables $x \to -x$, results established for (1.1) carry over *mutatis muntandis* to the quarter-plane problem where $\mathbb{R}^+$ is replaced by $\mathbb{R}^-$. (The situation regarding the quarter-plane problems posed on $\mathbb{R}^+$ and $\mathbb{R}^-$ for the Korteweg-de Vries equation are significantly different on the other hand.) In all cases where (1.1) and (1.2) arise in practice, the second-order derivative models dispersive effects, which is to say the tendency of waves to spread out due to the fact that different wavelengths propagate with different speeds, while the $|u|^{p-2}u$–term accounts for a variety of nonlinear effects.

Nonlinear Schrödinger equations are derived as models for a considerable range of applications. This includes propagation of light in fiber optics cables, certain types of shallow and deep surface water waves, Langmuir waves in a hot plasma and in more general forms in Bose-Einstein condensate theory. In the case of gravity waves on the surface of an inviscid liquid, the parameter λ depends upon the undisturbed depth of the water, becoming negative in water deep with respect to the wavelength of the wavetrain. A particularly interesting application of nonlinear Schrödinger (NLS henceforth) equations has been their use in attempting to explain the somewhat mysterious formation of rogue waves in the ocean and in optical propagation (see [6], [7] , [31] and [62]).

In many of the physical applications mentioned above, the independent variable x is a coordinate representing position in the medium of propagation, t is proportional to elapsed time and $u(x,t)$ is a velocity or an amplitude at the point x at time t. One configuration that arises naturally in making predictions of waves in water is to take $x \in \mathbb{R}^+ = \{x \,|\, x \geq 0\}$ and specify $u(0,t)$ for $t > 0$. This corresponds to a given wave-train generated at one end by a wave-maker and propagating into a region of the medium of propagation (see [1] for an example of this situation). The domain is natural since solutions of this wavemaker problem for the NLS-equation are an approximation of waves moving in the direction of increasing values of x. The semi-infinite

[1] Only the case $p \geq 3$ is considered here, but a substantial part of the theory goes through under the weaker hypothesis $p > 2$.

aspect of the domain is convenient in that no lateral boundary need be considered downstream of the wavemaker.

However, real domains are bounded, and in some cases it may be necessary to impose boundary conditions at both ends of the medium of propagation. Especially if one is interested in implementing a numerical scheme to calculate solutions of the half-line problem or localized solutions of the pure initial-value problem on the whole line, there arises the need to truncate the spatial domain. In such situations, the problem posed on a finite domain comes to the fore, and one must impose boundary conditions at both ends to specify solutions. Of course, when approximating localized solutions of the problem on all of $\mathbb{R}$, it is reasonable to take $u(0,t) = u(L,t) = 0$ for $0 \leq t \leq T$ and $L > 0$ large enough that essentially no disturbance reaches the boundary during the time interval $[0,T]$. However, the wavemaker problem and its finite domain counterpart demand non-homogeneous boundary conditions. Neumann conditions may also be appropriate in some circumstances.

In this paper, the discussion will center around the fundamental questions of existence and uniqueness of solutions corresponding to specified initial and boundary data. The issue of the solutions' dependence upon the auxiliary data is also examined, thereby completing Hadamard's basic idea of well posedness. The theory developed here will be for initial data in the L^2–based Sobolev spaces $H^s(\mathbb{R}^+)$ and $H^s(0,L)$. The spaces from which the boundary data will be drawn are dictated by these choices of initial data, as will become apparent presently. Theory will be developed wherein the time for existence depends upon the size of the auxiliary data. With more restrictive hypotheses, global well-posedness results will also be provided. Here and below, the notation is that which is current in the theory of partial differential equations.

Theory for the nonlinear Schrödinger equation in the form depicted in (1.1) and (1.2) has seen a lot of development in the last four decades, beginning with the pioneering work of Zakharov and his collaborators [74, 75]. For the most part, the mathematical theory for this equation has been concerned with either the pure initial-value problem posed on the entire real line $\mathbb{R}$ or the periodic initial-value problem posed on the one-dimensional torus $\mathbb{T}$. A large body of literature has been concerned with the fundamental questions of existence, uniqueness and continuous dependence of solutions corresponding to initial data drawn from Sobolev classes (again, well posedness *a la* Hadamard [41, 42]). Some highlights of the developments are [16, 18, 27, 29, 39, 40, 53, 54, 69, 72], for example. We caution that this is only a small sample of the extant work on these problems. The monograph of Cazenave [26] provides a good entry into the literature.

The study of the initial-boundary-value problems (IBVP henceforth) (1.1) and ((1.2)) with nonhomogeneous boundary conditions has not progressed to the same extent (see [30, 19, 20, 21, 22, 23, 45, 46, 52, 68, 69, 70, 71] and more recent work on the boundary-value problems of some other dispersive equations [55], and the references therein). In this paper, the goal is to advance the study of the IBVP's (1.1) and (1.2) to the same level as that obtaining for the relevant pure initial-value problems posed on all of $\mathbb{R}$. The local well-posedness theory constructed in the body of the paper is summarized in the following three theorems. In all of these results, we assume that the lowest order compatibility conditions

$$h(0) = \phi(0) \text{ for (1.1)} \quad \text{or} \quad h_1(0) = \phi(0),\ h_2(0) = \phi(L) \text{ for (1.2)} \tag{1.3}$$

are valid when $s > \frac{1}{2}$. These derive simply from the requirement that the solution be continuous at the corners of the relevant space-time domain, which are $(0,0)$ for the half-line problem and $(0,0)$ and $(L,0)$ for the finite interval problem. This point will be elaborated at the end of the next section. If $s > 0$ is large, we need to assume that $|u|^{p-2}u$ is differentiable, a requirement that imposes a relationship between s and p, *viz.*

$$\text{if } p \text{ is even, } s \text{ is arbitrary; if } p \text{ is odd, } s \leq p-1; \quad \text{otherwise, } \lfloor s \rfloor < p-2 \; , \tag{1.4}$$

where $\lfloor s \rfloor$ is the largest integer less than s. Furthermore, for the convenience of our discussion of the traces of functions in $H^s(\mathbb{R})$, it is always assumed that

$$s \neq n + \frac{1}{2} \qquad \text{for} \qquad n = 0, 1, 2, \cdots . \tag{1.5}$$

This aspect is not always recalled in the body of the paper.

For a given $s \in \mathbb{R}$ and Ω being $\mathbb{R}^+$ or a finite interval $(0, L)$, the space $H^s(\Omega)$ is defined as the restriction of the space $H^s(\mathbb{R})$ to Ω, *viz.*

$$H^s(\Omega) \; = \; \{f|_\Omega \; \big| \; f \in H^s(\mathbb{R})\}$$

endowed with the quotient norm

$$\|f\|_{H^s(\Omega)} \; = \; \inf \Big\{ \|\tilde{f}\|_{H^s(\mathbb{R})} \, | \, \tilde{f} \in H^s(\mathbb{R}), \tilde{f}\big|_\Omega = f \Big\}.$$

Other equivalent definitions of $H^s(\Omega)$ can be found in Chapter 1 of [59].

Theorem 1.1 *(i) Suppose $\frac{1}{2} < s < \frac{5}{2}$ with $3 \leq p < \infty$. Then, for $\phi \in H^s(\mathbb{R}^+)$ and $h \in H_{loc}^{\frac{2s+1}{4}}(\mathbb{R}^+)$, the IBVP (1.1) is locally well-posed in $H^s(\mathbb{R}^+)$.*

(ii) If $0 \leq s < \frac{1}{2}$ with $3 \leq p < \frac{6-4s}{1-2s}$, the IBVP (1.1) is (conditionally) locally well-posed in $H^s(\mathbb{R}^+)$.

In both (i) and (ii), what is meant precisely is that for any given $T > 0$ and $\gamma > 0$, there exists a T^* with $0 < T^* \leq T$ depending only on s, γ and T such that if $\phi \in H^s(\mathbb{R}^+)$ and $h \in H^{\frac{2s+1}{4}}(0, T)$ satisfy

$$\|\phi\|_{H^s(\mathbb{R}^+)} + \|h\|_{H^{\frac{2s+1}{4}}(0,T)} \leq \gamma,$$

then the IBVP (1.1) admits a solution $u \in C([0, T^*]; H^s(\mathbb{R}^+))$. In case (i), the solution $u \in C([0, T^*]; H^s(\mathbb{R}^+))$ is unique, while in (ii), the solution satisfies the auxiliary condition

$$\|u\|_{L^q(0,T^*;L^r(\mathbb{R}^+))} < +\infty, \tag{1.6}$$

where (q, r) is an admissible pair, and it is the only $C([0, T^*]; H^s(\mathbb{R}^+))$–solution with this property. Here, a pair (q, r) is admissible when $\frac{1}{q} + \frac{1}{2r} = \frac{1}{4}$. In both cases (i) and (ii), the corresponding solution map is Lipschitz continuous.

Theorem 1.2 *(i) If $\frac{1}{2} < s < \frac{5}{2}$ with $3 \leq p < \infty$, the IBVP (1.2) is locally well-posed in $H^s(0, L)$ for $\phi \in H^s(0, L)$ and h_1, $h_2 \in H_{loc}^{\frac{s+1}{2}}(\mathbb{R}^+)$.*

(ii) If $0 \leq s < \frac{1}{2}$ with $3 \leq p \leq 4$, then the IBVP (1.2) is (conditionally) locally well-posed in $H^s(0, L)$ for ϕ, h_1, h_2 in the same spaces.

For both cases, this means that for any $T>0$ and $\gamma>0$, there exists a T^* with $0<T^*\le T$ depending only on s, γ and T such that if $\phi\in H^s(0,L)$ and $h_1,\ h_2\in H^{\frac{s+1}{2}}(0,T)$ satisfy

$$\|\phi\|_{H^s(0,L)}+\|h_1\|_{H^{\frac{s+1}{2}}(0,T)}+\|h_2\|_{H^{\frac{s+1}{2}}(0,T)}\le\gamma,$$

the IBVP (1.2) admits a solution $u\in C([0,T^*];H^s(0,L))$. In case (i) this solution $u\in C([0,T^*];H^s(0,L))$ is unique, while in case (ii), the solution also satisfies

$$\|u\|_{L^4((0,T^*)\times(0,L))}<+\infty \tag{1.7}$$

and is the unique $C([0,T^*];H^s(0,L))$–solution with this property. In both cases, the corresponding solution map is Lipschitz continuous.

The issue of uniqueness could use some elaboration. In (i) of both Theorems 1.1 and 1.2, the uniqueness means that if there are two solutions $u,v\in C([0,T^*];H^s)$, then $u\equiv v$. However, for (ii) of both Theorems 1.1 and 1.2, the uniqueness means that if there are two solutions $u,v\in C([0,T^*];H^s)$ satisfying either (1.6) or (1.7), then $u\equiv v$. Therefore, when $0\le s<\frac{1}{2}$, the local well-posedness results presented in both Theorem 1.1 and Theorem 1.2 are *conditional* (see Kato [53] where this distinction was made in the context of general classes of equations) since (1.6) or (1.7) is needed to ensure the uniqueness. It is naturally of interest to know whether these conditions can be removed. If these auxiliary conditions can be removed, the corresponding results are called unconditional well-posedness, or simply well-posedness. In fact, a further argument allows the results for smaller values of s to be extended, so obtaining the following additional wrinkle appertaining to Theorems 1.1 and 1.2.

Theorem 1.3 (unconditional well-posedness) *Let $0\le s<\frac{1}{2}$ be given. Then, both (1.6) and (1.7) can be removed, so the corresponding well-posedness is unconditional.*

As mentioned, the preceding results are all local, which is to say the time interval $(0,T^*)$ over which the solution is guaranteed to exist depends on the size of initial and boundary data. If T^* can be chosen independently of the size of the initial and boundary data, then the result is termed global well-posedness. The following global well-posedness results for (1.1) and (1.2) are proved here.

Theorem 1.4

(i) Assume that either $p\ge 3$ if $\lambda<0$ or $3\le p\le 4$ if $\lambda>0$. The IBVP (1.1) is globally well-posed in $H^s(\mathbb{R}^+)$ for any $1\le s<\frac{5}{2}$ with auxiliary date (ϕ,h) drawn from $H^s(\mathbb{R}^+)\times H_{loc}^{\frac{s+1}{2}}(\mathbb{R}^+)$.

(ii) Assume that either $p\ge 3$ if $\lambda<0$ or $3\le p\le\frac{10}{3}$ if $\lambda>0$. The IBVP (1.2) is globally well-posed in $H^s(0,L)$ for any $1\le s<\frac{5}{2}$ with $\phi\in H^s(0,L)$ and $h_1,\ h_2\in H_{loc}^{\frac{s+1}{2}}(\mathbb{R}^+)$.

The rest of the paper is organized as follows. A general overview of the problems together with an outline of the strategy for analyzing them is provided in Section 2. The IBVP (1.1) takes center stage in Section 3 which consists of three subsections. In Subsection 3.1, explicit solution formulas are derived for associated linear problems. In Subsection 3.2, various Strichartz estimates are established using these solution formulas. The local well-posedness of the IBVP (1.1) on $\mathbb{R}^+$ is established in Subsection 3.3. In Section 4, local well-posedness for the IBVP (1.2) on the finite interval $(0, L)$ is studied. The global well-posedness of (1.1) and (1.2) will be investigated in Section 5. The paper concludes with an Appendix where a technical lemma needed in establishing Proposition 4.6 is proved and a telling counterexample, which concerns the optimality of the assumption $h_1, h_2 \in H^{(s+1)/2}(0,T)$ in Theorem 1.2, is presented.

2 Overview

We begin by reviewing the state of the art for the pure initial-value problems

$$iu_t + u_{xx} + \lambda |u|^{p-2}u = 0, \quad u(x,0) = \phi(x), \text{ for } x \in \mathbb{R}, \tag{2.8}$$

for the Schrödinger equations considered here. First discussed is the case of initial data ϕ that is localized on an unbounded domain, which is to say it evanesces at infinity in at least a weak sense.

Theorem A

(i) For $s > \frac{1}{2}$ with $3 \le p < \infty$ or $0 \le s < \frac{1}{2}$ with $3 \le p < \frac{6-4s}{1-2s}$, the initial-value problem (2.8) is locally well-posed in $H^s(\mathbb{R})$. That is, for any $r > 0$, there exists a $T > 0$ depending on r such that if $\|\phi\|_{H^s(\mathbb{R})} \le r$, then (2.8) admits a unique solution $u \in C([0,T]; H^s(\mathbb{R}))$ and the corresponding solution map is Lipschitz continuous. [2]

Moreover, for $0 \le s < \frac{1}{2}$, the solution also satisfies

$$\|u\|_{L^q_{loc}(0,T;B^s_{r,2}(\mathbb{R}))} < +\infty\,, \tag{2.9}$$

where $B^s_{r,2}(\mathbb{R})$ is the Besov space and $1/q + 1/(2r) = 1/4$. Uniqueness when $0 \le s < \frac{1}{2}$ requires that (2.9) holds.

(ii) If, in addition, $3 \le p < 6$, then the above local well-posedness results are global, i.e., T is independent of r and can be chosen arbitrarily large.

Next, the existing results obtained when ϕ is periodic are recalled.

Theorem B

(i) For $s > \frac{1}{2}$ with $3 \le p < \infty$ or $0 \le s < \frac{1}{2}$ with $3 \le p < \frac{6-4s}{1-2s}$, the IVP (2.8) is locally well-posed in $H^s(\mathbb{T})$, i.e., for any $r > 0$, there exists a $T > 0$ depending only on r such that if $\phi \in H^s(\mathbb{T})$ with $\|\phi\|_{H^s(\mathbb{T})} \le r$, then (2.8) admits a unique solution $u \in C([0,T]; H^s(\mathbb{T}))$ and the corresponding solution map is Lipschitz continuous. Moreover, for $0 \le s < \frac{1}{2}$, the solution u satisfies

$$\|u\|_{\mathbb{B}^T_{s,\frac{1}{2}}} < +\infty\,, \tag{2.10}$$

where $\mathbb{B}^T_{s,\frac{1}{2}}$ is the restricted Bourgain space associated to the Schrödinger equation (see [16]). As in Theorem A, uniqueness when $0 \le s < \frac{1}{2}$ is conditional and relies upon (2.10).

[2] For many years since the pioneering work in [54, 29], the solution map was only known to be continuous from $H^s(\mathbb{R})$ to $C([0,T]; H^{s-\epsilon}(\mathbb{R}))$. It was proved recently by Cazenave, et al. [27] to be continuous from $H^s(\mathbb{R})$ to $C([0,T]; H^s(\mathbb{R}))$.

(ii) If, in addition, $3 \le p < 6$, then the above local well-posedness results are global, i.e., T is independent of r and can be chosen arbitrarily large.

These results may be found in the previously cited references. We emphasize that at present, the uniqueness part of the well-posedness results in the parts (i) of Theorem A and Theorem B requires the extra conditions (2.9) and (2.10) when $s < \frac{1}{2}$. As mentioned, such well posedness was termed *conditional* by Kato [53]. If the auxiliary conditions can be removed, which is to say the solution is shown to be unique only assuming it lies in $C([0,T];H^s(\mathbb{R}))$, then the problem (2.8) is said to be unconditionally well-posed. According to the general discussion presented in [11], if $3 \le p < 6$, the conditional well-posedness results stated in parts (i) of Theorems A and B are, in fact, unconditional.[3]

The overall goal of the present essay is to bring the well-posedness theory for the IBVP's (1.1) and (1.2) into line with what is known for the pure initial-value problem (2.8).

The precise terminology used in the paper is now provided and motivation is developed for the choice of appropriate function spaces for the initial and boundary conditions. The main ideas and methodology for proving the results stated in the Introduction are also set forth.

The notion of well-posedness used for the problems (1.1) and (1.2) is detailed first.

Definition 2.1 *Let $s,\ s' \in \mathbb{R}$ and $T > 0$ be given.*

(i) The IBVP (1.1) is said to be (locally) well-posed in $H^s(\mathbb{R}^+) \times H^{s'}(0,T)$ if for $\phi \in H^s(\mathbb{R}^+)$ and $h \in H^{s'}(0,T)$ satisfying certain natural compatibility conditions, there exists a $T' \in (0,T]$ depending only on $\|\phi\|_{H^s(\mathbb{R}^+)} + \|h\|_{H^{s'}(0,T)}$ such that (1.1) admits a unique solution $u \in C([0,T'];H^s(\mathbb{R}^+))$. Moreover, the solution depends continuously on (ϕ,h) in the corresponding spaces.

(ii) The IBVP (1.2) is said to be (locally) well-posed in $H^s(0,L) \times H^{s'}(0,T)$ if for $\phi \in H^s(0,L)$ and $h_1,\ h_2 \in H^{s'}(0,T)$ satisfying certain natural compatibility conditions, there exists a $T' \in (0,T]$ depending only on $\|\phi\|_{H^s((0,L))} + \|h_1\|_{H^{s'}(0,T)} + \|h_2\|_{H^{s'}(0,T)}$ such that (1.2) admits a unique solution $u \in C([0,T'];H^s(0,L))$. Moreover, the solution depends continuously on (ϕ,h_1,h_2) in the corresponding spaces.

In either case, if T' can be chosen independently of r, the relevant IBVP is said to be globally well posed.

Completing this definition of well-posedness requires making precise what it means for u to be a *solution* of (1.1) or (1.2). The issue is important for small values of s, where the meaning of the derivatives and nonlinear term has to be addressed. The usual approach in the literature is to say that u solves the equation in the sense of Schwartz distributions. This, however, leads to a further question about how the nonlinear term $\lambda|u|^{p-2}u$ makes sense as a distribution, as well as how the solution u takes on the given initial and boundary values. In this paper, we will use the following definitions (see [11] for a general discussion) for the solutions of (1.1) and (1.2), respectively.

Definition 2.2 *Let $s \le 2$, $s' \le s$ and $T > 0$ be given.*

[3] The reader is referred to [73, 43] and the references therein for recent progress on the issue of unconditional well-posedness of nonlinear Schrödinger equations.

(a) *For* $\phi \in H^s(\mathbb{R}^+)$ *and* $h \in H^{s'}(0,T)$, *we say that* $u \in C([0,T]; H^s(\mathbb{R}^+))$ *is a solution of (1.1) if there exists a sequence*

$$u_n \in C([0,T]; H^2(\mathbb{R}^+)) \cap C^1([0,T]; L^2(\mathbb{R}^+)), \ n = 1, 2, 3, \cdots$$

such that

1) u_n *satisfies the equation in (1.1) in* $L^2(\mathbb{R}^+)$ *for* $0 \le t \le T$,
2) u_n *converges to* u *in* $C([0,T]; H^s(\mathbb{R}^+))$ *as* $n \to \infty$,
3) $\phi_n(x) = u_n(x,0)$ *converges to* $\phi(x)$ *in* $H^s(\mathbb{R}^+)$ *as* $n \to \infty$,
4) $h_n(t) = u_n(0,t)$ *is in* $H^{s'}(0,T)$ *and converges to* $h(t)$ *in* $H^{s'}(0,T)$ *as* $n \to \infty$.

(b) *For* $\phi \in H^s(0,L)$ *and* $h_1,\ h_2 \in H^{s'}(0,T)$, *we say that* $u \in C([0,T]; H^s(0,L))$ *is a solution of (1.2) if there exists a sequence*

$$u_n \in C([0,T]; H^2(0,L)) \cap C^1([0,T]; L^2(0,L)), \ n = 1, 2, 3, \cdots$$

such that

1) u_n *satisfies the equation of (1.2) in* $L^2(0,L)$ *for* $0 \le t \le T$,
2) u_n *converges to* u *in* $C([0,T]; H^s(0,L))$ *as* $n \to \infty$,
3) $\phi_n(x) = u_n(x,0)$ *converges to* $\phi(x)$ *in* $H^s(0,L)$ *as* $n \to \infty$,
4) $h_{1,n}(t) = u_n(0,t), h_{2,n}(t) = u_n(L,t)$ *are in* $H^{s'}(0,T)$ *and converge to* $h_1(t)$ *and* $h_2(t)$, *respectively, in* $H^{s'}(0,T)$ *as* $n \to \infty$.

Of course, if $s \ge 2$, then a solution in the above sense, sometimes called a mild solution, is a solution in the ordinary L^2–sense.

Attention is now turned to the relation between s' and s in the definition of well posedness. It is well known that the linear Schödinger equation

$$iv_t + v_{xx} = 0, \quad v(x,0) = \phi(x),$$

posed on the whole line $\mathbb{R}$ has the Kato smoothing property, which is to say $\phi \in H^s(\mathbb{R})$ implies $v \in L^2_{loc}(\mathbb{R}; H^{s+\frac{1}{2}}_{loc}(\mathbb{R}))$. In addition, the Schrödinger equation itself entails that $\partial_t \sim \partial_{xx}$, so suggesting that

$$s' = \frac{1}{2}\left(s + \frac{1}{2}\right) = \frac{2s+1}{4} \tag{2.11}$$

(see [45] for a more detailed discussion and [2, 3] for recent studies of this issue for Schrödinger equations). We are thus led to complete the definition of well posedness for the IBVP (1.1) with the stipulation (2.11).

For the IBVP (1.2), one might imagine that the correct value should also be $s' = \frac{2s+1}{4}$. However, as will be seen presently, this is not the case. Instead, the optimal relation between s and s' for the IBVP (1.2) is

$$s' = \frac{s+1}{2}. \tag{2.12}$$

Thus, a significant, albeit technical difference, emerges between the IBVP (1.1) (posed on an unbounded domain) and the IBVP (1.2) (posed on a finite domain).

To put our main theorems into context, we sketch previous work on such IBVP's. Carrolle and Bu in [30] studied (1.1) with $p = 4$ and showed that if $\phi \in H^2(\mathbb{R}^+)$ and $h \in C^2(\mathbb{R}^+)$ with $\phi(0) = h(0)$, then the problem admits a unique global solution

$$u \in C^1(\mathbb{R}^+; L^2(\mathbb{R}^+)) \cap C(\mathbb{R}^+; H^2(\mathbb{R}^+)).$$

This result was extended to the case $p \geq 3$ by Bu in [20] for the defocusing case ($\lambda < 0$). In [68], Strauss and Bu considered the problem

$$\begin{cases} u_t - \Delta u + \lambda |u|^{p-2} u = 0, \qquad x \in \Omega, \quad t \in \mathbb{R}\,, \\ \\ u(x,0) = \phi(x), \quad x \in \Omega, \qquad u(x,t) = q(x,t), \quad x \in \partial\Omega\,, \end{cases} \tag{2.13}$$

for the NLS equations posed on a smooth (bounded or unbounded) domain $\Omega \subset \mathbb{R}^n$. Assuming that $\lambda < 0$ and $p \geq 3$, they showed that for any $\phi \in H^1(\Omega)$ and $q \in C^3(\mathbb{R}^n \times (-\infty, \infty))$ with compact support satisfying the natural compatibility condition, the IBVP (2.13) admits a global solution

$$u \in L^\infty_{loc}((-\infty, \infty); H^1(\Omega) \cap L^p(\Omega)).$$

Bu, Tsutaya and Zhang [23] extended the above result to the case of $\lambda > 0$ assuming $3 \leq p \leq 2 + \frac{n}{2}$ and $n \geq 2$. In all this work, the third leg of Hadamard's conception, namely continuous dependence of solutions on the initial and boundary data, was not discussed. For small $s \geq 0$, Holmer [45] obtained the following result for the half-line problem (1.1).

Theorem 2.3 *(Holmer) Let $\frac{1}{2} < s < \frac{3}{2}$ with $3 \leq p < \infty$ or $0 \leq s < \frac{1}{2}$ with $3 \leq p < \frac{6-4s}{1-2s}$ be given. For any $r > 0$, there exists $T > 0$ such that if $\phi \in H^s(\mathbb{R}^+)$ and $h \in H^{\frac{2s+1}{4}}_{loc}(\mathbb{R}^+)$ satisfy*

$$\|\phi\|_{H^s(\mathbb{R}^+)} + \|h\|_{H^{\frac{2s+1}{4}}(0,T)} \leq r \qquad \Big(h(0) = \phi(0) \quad \textit{for} \quad s > \frac{1}{2}\Big)\,,$$

then the IBVP (1.1) admits a solution $u \in C([0,T]; H^s(\mathbb{R}^+))$ which depends continuously upon the auxiliary data in the relevant function classes. Moreover, for $\frac{1}{2} < s < \frac{3}{2}$, the solution u is unique.

This result is very similar to that obtained here for the quarter-plane problem (1.1). Our result, which is obtained by a different approach to be described presently, improves Theorem 2.3 in small ways (the issue of uniqueness for s in the range $0 \leq s < \frac{1}{2}$ is clarified and the range of values of s is extended). The boundary integral method used in this paper and in our earlier work [12] on the KdV equation, has other points to recommend it, however. First, one can read off from our representation of solutions a significant difference between the IBVPs for the KdV equation and the nonlinear Schrödinger equation. For the KdV equation, the imposition of a boundary condition at the left-hand end of $\mathbb{R}^+$ produces a strong dissipative smoothing mechanism, whereas no such dissipative smoothing appears from solving the same boundary-value problem for the nonlinear Schrödinger equation. This distinction is not so clearly seen using the earlier methods. (More detail concerning this distinction will be presented elsewhere.) Another point in favor of the boundary-integral method is that it generalizes immediately to higher space dimensions. This, also, is a project for future investigation.

The discussion is now turned in a slightly more technical direction. The first point we want to make is that at least for relatively small values of s, the case where the boundary data is homogeneous (*i.e.* $h \equiv 0$ or $h_1 = h_2 \equiv 0$) can be reduced to the situation described in Theorem A or Theorem B, respectively. (This is no longer true for larger values of s, however.) Thus, with essentially no effort, the following results obtain.

Theorem 2.4 *Assume that $h = 0$ in (1.1).*

(i) If $\frac{1}{2} < s < \frac{5}{2}$ with $3 \le p < \infty$ or $0 \le s < \frac{1}{2}$ with $3 \le p < \frac{6-4s}{1-2s}$, the IBVP (1.1) is locally well-posed in $H^s(\mathbb{R}^+)$.

(ii) If $3 \le p < 6$, then the IBVP (1.1) is (unconditionally) globally well-posed in $H^s(\mathbb{R}^+)$ for any s with $0 \le s < \frac{5}{2}$.

Theorem 2.5 *Assume that $h_1 = 0$ and $h_2 = 0$ in (1.2).*

(i) If $\frac{1}{2} < s < \frac{5}{2}$ with $p \ge 3$ or $0 \le s < \frac{1}{2}$ with $3 \le p < \frac{6-4s}{1-2s}$, the IBVP (1.2) is locally well-posed in $H^s(0, L)$.

(ii) If $3 \le p < 6$, then the IBVP (1.2) is (unconditionally) globally well-posed in $H^s(0, L)$ for any s with $0 \le s < \frac{5}{2}$.

For Theorem 2.4, the result follows by choosing as initial data the odd extension $\tilde{\phi}$ of ϕ, solving the equation on $\mathbb{R}$ with $\tilde{\phi}$ as initial data and then restricting the resulting solution to the half line. For Theorem 2.5, extend ϕ to $[-L, L]$ by taking the odd extension and then extend to all of $\mathbb{R}$ by $2L$–periodicity. Solve the resulting periodic initial-value problem and then restrict to $[0, L]$.

For the nonhomogeneous boundary-value problems that are the focus of attention here, such simple methods do not appear to give results. To deal with nonhomogeneous boundary data, a standard approach is to homogenize the boundary data by a change of the dependent variables. Define a new dependent variable by subtracting from the original dependent variable a known function that takes on the given boundary values. This new variable will satisfy a related equation, but with zero boundary conditions. While this works well in some cases, *e.g.* BBM-type equations (see [4] and the references therein), in the present context it requires that the boundary data must have stronger regularity than should be needed according to the heuristic analysis leading to the relation (2.11) between the function classes of the initial and the boundary data. For instance, this method, applied in a straightforward way for $p = 4$, say, ends up requiring for the quarter-plane problem (1.1) that $h \in H^1([0, T])$ to obtain the well-posedness of the IBVP (1.1) in $L^2(\mathbb{R}^+)$ rather than $h \in H^{\frac{1}{4}}([0, T])$ as advertised in Theorem 1.1, part (*ii*).

The initial-boundary-value problem

$$\begin{cases} u_t + u^k u_x + u_{xxx} = 0, \ x \in \mathbb{R}^+, \ t \in (0, T), \\ \\ u(x, 0) = \phi(x), \quad u(0, t) = h(t), \end{cases} \tag{2.14}$$

for the generalized Korteweg-de Vries (KdV) equation posed on a half line $\mathbb{R}^+$, is instructive. Colliander and Kenig [33] introduced a new method to analyze this problem by solving the pure

IVP

$$\begin{cases} w_t + w^k w_x + w_{xxx} = \delta(x) f(t), \;\; x \in \mathbb{R}, \;\; t \in (0,T)\,, \\ \\ w(x,0) = \psi(x), \end{cases} \tag{2.15}$$

of a forced, generalized KdV equation with an appropriate forcing function $f(t)$. Here, $\delta(x)$ denotes the Dirac mass at $x = 0$ and $\psi(x)$ is an extension of $\phi(x)$ from $\mathbb{R}^+$ to $\mathbb{R}$. It is demonstrated that an appropriate forcing function $f(t)$ can be chosen so that the corresponding solution w of (2.15) satisfies

$$w(0,t) = h(t), \quad \text{for } \; 0 < t < T.$$

Consequently, the restriction of $w(x,t)$ to the half line $\mathbb{R}^+$ is a solution of the IBVP (2.14). The IVP (2.15) is solved using the contraction mapping principle in a carefully constructed, Bourgain-type space $X_{s,T}$. The key step of this approach is to study the associated linear problem,

$$\begin{cases} v_t + v_{xxx} = \delta(x) f(t), \;\; x \in \mathbb{R}, \;\; t \in (0,T)\,, \\ \\ v(x,0) = \psi(x)\,, \end{cases} \tag{2.16}$$

and show that there exists a real number s' (depending only on s) such that for any given $\psi \in H^s(\mathbb{R})$,

(i) if $f \equiv 0$, the solution v of (2.16) satisfies

$$\sup_{x\in\mathbb{R}} \|v(x,\cdot)\|_{H^{s'}(0,T)} \leq C_{s,T} \|\psi\|_{H^s(\mathbb{R})}\,, \tag{2.17}$$

(ii) if $h \in H^{s'}(0,T)$, one can find a forcing function f such that the IVP (2.16) admits a solution $v \in X_{s,T}$ and

$$\|v\|_{X_{s,T}} \leq C_{s,T}\Big(\|\psi\|_{H^s(\mathbb{R})} + \|h\|_{H^{s'}(0,T)} \Big). \tag{2.18}$$

It turns out that for the IBVP (2.14),

$$s' = \frac{s+1}{3}. \tag{2.19}$$

The estimate (2.17) is, in fact, the sharp Kato smoothing property possessed by the solutions of the linearized KdV equation. The Riemann-Liouville fractional integral is the main tool used to establish the estimate (2.18).

There is another approach to deal with the IBVP (2.14) put forward by the present authors in [8]. A major constituent of this latter approach is the explicit solution formula

$$q(x,t) = [U_b(t)h](x) + \overline{[U_b(t)h](x)} \tag{2.20}$$

where

$$[U_b(t)h](x) = \frac{1}{2\pi} \int_1^\infty e^{i\mu^3 t - i\mu t} e^{-\left(\frac{\sqrt{3\mu^2-4}+i\mu}{2}\right)x} (3\mu^2 - 1) \int_0^\infty e^{-i(\mu^3+\mu)\xi} h(\xi)\, d\xi\, d\mu,$$

of the linear, nonhomogeneous boundary-value problem,

$$\begin{cases} q_t + q_x + q_{xxx} = 0, \quad x \in \mathbb{R}^+,\ t \in (0,T), \\ \\ q(x,0) = 0, \qquad q(0,t) = h(t). \end{cases} \tag{2.21}$$

This explicit formula, which is obtained by formally taking the Laplace transform in time, solving the resulting third-order problem and taking the inverse Laplace transform, enables one to establish directly various estimates needed for proving the well-posedness of the IBVP (2.14). Moreover, it clearly demonstrates that the solution $q(x,t)$ of (2.21) becomes infinitely smooth when $x > 0$ and $t > 0$. It has been further shown in [12] that

the solution $q(x,t)$ is the restriction to $\mathbb{R}^+ \times \mathbb{R}^+$ of a function $w(x,t)$ defined on $\mathbb{R} \times \mathbb{R}$ which is such that

$$\left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1+|\xi|)^{2s} (1+|\tau - \xi^3|)^{2b} |\hat{w}(\xi,\tau)|^2 d\xi d\tau \right)^{1/2} \leq C \|h\|_{H^{\frac{3b+s-1/2}{3}}(\mathbb{R}^+)}$$

where b is any value in $[0, \frac{1}{2} - \frac{s}{3})$ if $-\frac{3}{2} \leq s < \frac{3}{2}$, b is any value in $[0, \frac{5}{6} - \frac{s}{3}]$ if $-\frac{1}{2} < s < 1$ and C is a constant depending only on s and b.

It then follows that the IBVP (2.21) possesses the following strong dissipative smoothing property:

$$h \in H^{(s+1)/3}_{loc}(\mathbb{R}^+) \implies q \in L^2(0,T; H^{s+\frac{3}{2}}(\mathbb{R}^+)).$$

In [45], Holmer applied the Colliander–Kenig approach to study the IBVP (1.1) and obtained the results described in Theorem 2.3. However, as we will show in this paper, this approach may fail for the IBVP (1.2). More precisely, we show that for the solution u of the IBVP

$$\begin{cases} iu_t + u_{xx} = 0, \quad x \in (0,L),\ t \in (0,T), \\ \\ u(x,0) = 0, \quad u(0,t) = h_1(t), \quad u(L,t) = h_2(t), \end{cases} \tag{2.22}$$

for the linear Schrödinger equation posed on $(0,L)$, the estimate

$$\|u\|_{L^2((0,L)\times(0,T))} \leq C \left(\|h_1\|_{H^\alpha(0,T)} + \|h_2\|_{H^\alpha(0,T)} \right) \tag{2.23}$$

holds if $\alpha \geq \frac{1}{2}$, but fails if $\alpha < \frac{1}{2}$. (Indeed, Example A2 in the Appendix shows the optimality of the assumption $h_1, h_2 \in H^{1/2}(0,T)$ for this estimate to hold). By contrast, for solutions of the IBVP

$$\begin{cases} iv_t + v_{xx} = 0, \quad x \in \mathbb{R}^+,\ t \in (0,T), \\ \\ v(x,0) = 0, \quad v(0,t) = h(t), \end{cases} \tag{2.24}$$

for the linear Schrödinger equation posed on $\mathbb{R}^+$, it is indeed the case that

$$\|v\|_{L^2(\mathbb{R}^+\times(0,T))} \leq C \|h\|_{H^{\frac{1}{4}}(0,T)}. \tag{2.25}$$

And, solutions of the pure IVP

$$iw_t + w_{xx} = 0, \quad w(x,0) = \psi(x),\ x \in \mathbb{R}, \quad t \in (0,T) \tag{2.26}$$

for the linear Schrödinger equation posed on $\mathbb{R}$, comply with the inequality

$$\sup_{x\in\mathbb{R}} \|w(x,\cdot)\|_{H^{\frac{1}{4}}(0,T)} \le C\|\psi\|_{L^2(\mathbb{R})}. \tag{2.27}$$

Thus, while it is possible to solve the nonhomogeneous IBVP (1.1) by solving a forced IVP of the form

$$iu_t + \lambda|u|^{p-2}u + u_{xx} = \delta(x)f(t),\ u(x,0) = \psi(x), \quad x \in \mathbb{R},\ t \in (0,T),$$

with an appropriate forcing function $f(t)$, it may not be feasible to apply the same approach to the two-point IBVP (1.2).

In this paper, the approach developed earlier in [8] for studying nonhomogeneous boundary-value problems of the KdV equation will be used to establish local well-posedness results for (1.1) and (1.2). Analogous to the solution formula (2.20) in the KdV case, the nonhomogeneous, linear IBVP (2.24) has the explicit solution

$$v(x,t) = \frac{1}{\pi}\int_0^\infty e^{-i\beta^2 t}e^{i\beta x}\beta\int_0^\infty e^{i\beta^2\tau}h(\tau)d\tau d\beta + \frac{1}{\pi}\int_0^\infty e^{i\beta^2 t}e^{-\beta x}\beta\int_0^\infty e^{-i\beta^2\tau}h(\tau)d\tau d\beta. \tag{2.28}$$

Similarly, the solution formula for the nonhomogeneous, linear IBVP (2.22) is

$$u(x,t) = \sum_{n=1}^\infty 2in\pi e^{-i(n\pi)^2 t}\int_0^t e^{i(n\pi)^2\tau}\Big(h_1(\tau) - (-1)^n h_2(\tau)\Big)d\tau \sin n\pi x. \tag{2.29}$$

As in the case of the KdV equation, these formulas are derived by taking the Laplace transform of u in the temporal variable, solving the resulting, second-order, ordinary differential equation and taking the inverse Laplace transform of the result. The inequalities needed to advance the local well-posedness theory obtain directly from these explicit solution formulas. Moreover, from these formulas, one ascertains that, unlike the KdV equation, the imposition of boundary conditions brings no smoothing effect. For example, consider the IBVP (2.24). The second term

$$B(x,t) = \frac{1}{\pi}\int_0^\infty e^{i\beta^2 t}e^{-\beta x}\beta\int_0^\infty e^{-i\beta^2\tau}h(\tau)d\tau d\beta$$

on the right-hand side of the solution formula (2.28) becomes infinitely smooth as soon as $x > 0$ and $t > 0$. On the other hand, the first term on the right-hand side of (2.28) can be written as

$$A(x,t) = \frac{1}{\pi}\int_{-\infty}^\infty e^{-i\beta^2 t}e^{i\beta x}\hat{\psi}(\beta)d\beta,$$

where ψ is the function whose Fourier transform is

$$\hat{\psi}(\beta) = \begin{cases} \beta\int_0^\infty e^{i\beta^2\tau}h(\tau)d\tau & if\ \beta > 0, \\ 0 & if\ \beta < 0. \end{cases}$$

Thus, $A(x,t)$ solves the pure initial-value problem for the linear Schrödinger equation, posed on the whole line $\mathbb{R}$, with the initial value $\psi(x)$. It follows that $\psi \in H^s(\mathbb{R})$ if and only if $h \in H_0^{\frac{2s+1}{4}}(\mathbb{R}^+)$. Consequently, in contrast to the KdV equation, there is no boundary smoothing for the Schrödinger equation.

This section is concluded with remarks on higher-order regularity and global well-posedness. The theory outlined above, and which is developed in detail in the remainder of the essay, has upper limits on the regularity of the auxiliary data. As we will see momentarily, these restrictions are necessary. They can be relaxed only by asking for additional properties of the auxiliary data.

When equations like the Schrödinger equation are derived to describe physical phenomena, they often come as a simplification of a more complete model. Justifying the simpler model as an approximation of a more elaborate model typically requires smoothness of the solutions of both the full and the approximate models (see [5, 34, 66] for justification of the KdV equation as an approximation of the full water-wave problem, for instance). Without smoothness, the comparisons are not in fact valid. Thus, it is not only of academic interest to understand higher regularity solutions.

An example will illustrate the problem that arises when smoother solutions are in question. Take the classical case $p = 4$ so that the nonlinearity is cubic and smooth. Suppose that the quarter-plane problem (1.1) is locally well posed in $H^3(\mathbb{R}^+)$, say. Then, there is a $T > 0$ and a solution $u \in C(0, T; H^3(\mathbb{R}^+))$. Because $u_{xx} \in C(0, T; H^1(\mathbb{R}^+))$ and u satisfies the equation, it must be the case that $u \in C^1(0, T; H^1(\mathbb{R}^+))$. It follows that each term in the evolution equation is a continuous function of both space and time in $\mathbb{R}^+ \times [0, T]$. Evaluating the equation at the point $(0, 0)$ and using the initial and boundary conditions then yields

$$ih'(0) \,+\, \phi''(0) \,+\, \lambda|\phi(0)|^2\phi(0) \,=\, 0. \tag{2.30}$$

Thus, the auxiliary data necessarily satisfies a higher-order compatibility condition in addition to the lower-order condition (1.3) that has been assumed throughout the discussion. It is straightforward to calculate yet higher-order conditions on the auxiliary data that must obtain for well-posedness to hold in smaller Sobolev spaces. This issue also arises for the KdV equation posed on the half-line or on a bounded interval. In that case, higher-order regularity theory has been developed in the presence of higher-order compatibility conditions (see [13, 15]).

When the nonlinearity is smooth, *e.g.* when $p = 4, 6, 8, \cdots$, local well posedness in the presence of higher regularity and the associated compatibility conditions can be established by the methods put forward here. However, we eschew this task in the present script.

Finally, we come to the issue of global well-posedness. As is standard in the theory of evolution equations, local well-posedness coupled with suitable *a priori* bounds on solutions is the path to global well posedness. For the pure initial-value problem (2.8), the bounds provided by the conserved quantities

$$I(t) := \int_{-\infty}^{\infty} |u(x,t)|^2 dx \quad \text{and} \quad II(t) := \int_{-\infty}^{\infty} \left(|u_x(x,t)|^2 - \frac{2\lambda}{p}|u(x,t)|^p \right) dx \tag{2.31}$$

suffice for the global results mentioned earlier. However, corresponding to the quarter-plane problem (1.1), one has (cf. [30])

$$I'(t) := \frac{d}{dt}\int_0^{\infty} |u(x,t)|^2 dx = -2\,\mathrm{Im}\Big(u_x(0,t)\overline{h}(t)\Big) \tag{2.32}$$

and

$$II'(t) := \frac{d}{dt}\int_0^{\infty} \left(|u_x(x,t)|^2 - \frac{2\lambda}{p}|u(x,t)|^p \right) dx = -2\,\mathrm{Re}\Big(u_x(0,t)\overline{h}\,'(t)\Big) \tag{2.33}$$

while the two-point IBVP (1.2) has

$$I'(t) := \frac{d}{dt}\int_0^L |u(x,t)|^2 dx = 2\,\mathrm{Im}\Big(u_x(L,t)\overline{h}_2(t) - u_x(0,t)\overline{h}_1(t)\Big) \tag{2.34}$$

and

$$II'(t) := \frac{d}{dt}\int_0^L \Big(|u_x(x,t)|^2 - \frac{2\lambda}{p}|u(x,t)|^p\Big)\,dx = 2\,\mathrm{Re}\Big(u_x(L,t)\overline{h}\,'_2(t) - u_x(0,t)\overline{h}\,'_1(t)\Big), \tag{2.35}$$

for all $t \in \mathbb{R}$ for which the solutions exist. In case the boundary conditions are homogeneous, *viz.* $h \equiv 0$ or $h_1 = h_2 \equiv 0$, both $I(t)$ and $II(t)$ are formally conserved just as in the case of the pure initial-value problem (2.8). At least for small values of the Sobolev index s, global well-posedness results for the homogeneous IBVP's (1.1) and (1.2) then follow readily. For the nonhomogeneous cases, both $I(t)$ and $II(t)$ are no longer conserved and the task of obtaining global *a priori* estimates becomes interesting (see Section 5).

We turn now to the body of the paper where detailed analysis is given leading to the conclusions recounted in the Introduction. The explicit solution formulas (2.28) and (2.29) will play a central role in our development.

3 The Schrödinger equation posed on the half line $\mathbb{R}^+$

Considered first is the IBVP (1.1)

$$\begin{cases} iu_t + u_{xx} + \lambda u|u|^{p-2} = 0, \quad x \in \mathbb{R}^+,\ t \in \mathbb{R}, \\[2ex] u(x,0) = \phi(x), \quad u(0,t) = 0\,. \end{cases} \tag{3.1}$$

with a homogeneous boundary condition. It transpires that this can be reduced to the pure IVP

$$\begin{cases} iw_t + w_{xx} + \lambda w|w|^{p-2} = 0, \qquad x \in \mathbb{R},\ t \in \mathbb{R}, \\[2ex] w(x,0) = \psi(x) \end{cases} \tag{3.2}$$

of the NLS equation posed on the whole line $\mathbb{R}$. Indeed, observe that if $w = w(x,t)$ is a solution of (3.2) which is an odd function with respect to x, then its restriction

$$u(x,t) := w(x,t), \quad x \in \mathbb{R}^+,$$

to the half-line is a solution of (3.1) with $\phi(x) = \psi(x), x \in \mathbb{R}^+$. On the other hand, the IVP (3.2) possesses the following invariance property.

Lemma 3.1 *If ψ is an odd and smooth function, then for any $t \in \mathbb{R}$, the corresponding solution w of (3.2) is odd with respect to x.*

Proof: Consider first the associated linear problem

$$\begin{cases} iw_t + w_{xx} = f, \qquad x \in \mathbb{R},\ t \in \mathbb{R}, \\[2ex] w(x,0) = \psi(x). \end{cases} \tag{3.3}$$

Using the Fourier transform, its solution $w(x,t)$ is

$$w(x,t)=\int_{\mathbb{R}} e^{i\xi^2 t}e^{i\xi x}\hat{\psi}(\xi)d\xi+\int_0^t\int_{\mathbb{R}} e^{i\xi^2(t-\tau)}e^{i\xi x}\hat{f}(\xi,\tau)dx\xi d\tau\,.$$

It then follows directly that the solution $w(x,t)$ of (3.3) is odd with respect to x if ψ and f are odd in x. For the IBVP (3.2), suppose ψ is odd and consider the map $\Gamma : v \mapsto w$, where $v=v(x,t)$ is an odd function in x and w is the solution of

$$\begin{cases} iw_t+w_{xx}=-\lambda|v|^{p-2}v, \qquad x\in\mathbb{R},\ t\in\mathbb{R}, \\ \\ w(x,0)=\psi(x). \end{cases} \tag{3.4}$$

It follows from the previous remark about (3.3) that $\Gamma(v)$ is odd in x if v is odd in x. The classical contraction mapping principle provides the solution w of the nonlinear IVP (3.2). This solution is necessarily odd as a function of x if its initial value ψ is odd, as one determines by iterating Γ starting at $v=0$. $\Box$

Thus, the following well-posedness result for the IBVP (3.1) follows from the well-posedness of the IVP (3.2).

Theorem 3.2 *For any s satisfying either $\frac{1}{2}<s<\frac{5}{2}$ for $3\le p<\infty$, $\lfloor s\rfloor<p-2$ if $s\neq 1,2$, or $0\le s<\frac{1}{2}$ for $3\le p<\frac{6-4s}{1-2s}$, the IBVP (3.1) is locally well-posed in $H^s(\mathbb{R}^+)$ (for $\frac{1}{2}<s<\frac{5}{2}$, it is required that $\phi(0)=0$).*

Now, (1.1) is considered with nonhomogeneous boundary data. The analysis of this problem is carried out in several subsections.

3.1 Solution formulas for linear problems

Consideration is first given to the linear, nonhomogeneous, boundary-value problem

$$\begin{cases} iu_t+u_{xx}=0, \qquad x\in\mathbb{R}^+,\ t\in\mathbb{R}^+, \\ \\ u(x,0)=0, \qquad u(0,t)=h(t). \end{cases} \tag{3.5}$$

By taking the Laplace transform with respect to t of both sides of (3.5), the IBVP is converted to a one-parameter family of second-order boundary-value problems, *viz.*

$$\begin{cases} i\lambda\tilde{u}(x,\lambda)+\tilde{u}_{xx}(x,\lambda)=0, \\ \\ \tilde{u}(0,\lambda)=\tilde{h}(\lambda), \qquad \tilde{u}(+\infty,\lambda)=0, \end{cases} \tag{3.6}$$

where $\tilde{u}=\tilde{u}(x,\lambda)$ is the Laplace transform of $u=u(x,t)$ with respect to t and $\operatorname{Re}\lambda>0$ is the dual variable. The solution of (3.6) is given by

$$\tilde{u}(x,\lambda)=e^{r(\lambda)x}\tilde{h}(\lambda)$$

where $r(\lambda)$ is the solution of the quadratic equation

$$i\lambda+r^2=0$$

for which $\operatorname{Re} r < 0$. In consequence, the solution of (3.6) is given formally by

$$u(x,t) = \frac{1}{2\pi i}\int_{-\infty i+\gamma}^{+\infty i+\gamma} e^{\lambda t} e^{r(\lambda)x}\tilde{h}(\lambda)d\lambda$$

for $x,\ t > 0$, where $\gamma > 0$ is fixed. Letting $\gamma \to 0$, one arrives at

$$\begin{aligned}
u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\beta t}e^{r(i\beta)x}\tilde{h}(i\beta)d\beta \qquad \Big(-\beta + r^2 = 0, \qquad \operatorname{Re} r \le 0\Big) \\
&= \frac{1}{2\pi}\int_{-\infty}^{0} e^{i\beta t}e^{i\sqrt{-\beta}x}\tilde{h}(i\beta)d\beta + \frac{1}{2\pi}\int_{0}^{\infty} e^{i\beta t}e^{-\sqrt{\beta}x}\tilde{h}(i\beta)d\beta \\
&= \frac{1}{2\pi}\int_{0}^{\infty} e^{-i\beta t+i\sqrt{\beta}x}\tilde{h}(-i\beta)d\beta + \frac{1}{2\pi}\int_{0}^{\infty} e^{i\beta t-\sqrt{\beta}x}\tilde{h}(i\beta)d\beta \\
&= \frac{1}{\pi}\int_{0}^{\infty} e^{-i\beta^2 t+i\beta x}\beta\tilde{h}(-i\beta^2)d\beta + \frac{1}{\pi}\int_{0}^{\infty} e^{i\beta^2 t-\beta x}\beta\tilde{h}(i\beta^2)d\beta \\
&= I(x,t) + II(x,t).
\end{aligned}$$

For $I(x,t)$, define

$$\nu_1(\beta) = \begin{cases} \frac{1}{\pi}\beta\tilde{h}(-i\beta^2) & \text{for } \beta \ge 0, \\ 0 & \text{for } \beta < 0 \end{cases} \tag{3.7}$$

and

$$\phi_h = \phi_h(x) \tag{3.8}$$

to be the inverse Fourier transform of ν_1, so that the Fourier transform $\hat{\phi}_h$ of ϕ_h is

$$\hat{\phi}_h(\beta) = \nu_1(\beta), \qquad \beta \in \mathbb{R}.$$

Then, $I(x,t)$ can be rewritten as

$$I(x,t) = \int_{-\infty}^{\infty} e^{-i\beta^2 t+i\beta x}\hat{\phi}_h(\beta)d\beta\,,$$

which is exactly the solution formula of the Cauchy problem for the linear Schrödinger equation on $\mathbb{R}$. Similarly, for $II(x,t)$, define

$$\nu_2(\beta) = \begin{cases} \frac{1}{\pi}\beta\tilde{h}(i\beta^2) & \text{for } \beta \ge 0, \\ 0 & \text{for } \beta < 0 \end{cases} \tag{3.9}$$

and

$$\psi_h = \psi_h(x) \tag{3.10}$$

to be the inverse Fourier transform of ν_2, i.e.,

$$\hat{\psi}_h(\beta) = \nu_2(\beta), \qquad \beta \in \mathbb{R}.$$

Thus, $II(x,t)$ can be written as

$$II(x,t) = \int_{-\infty}^{\infty} e^{i\beta^2 t-\beta x}\hat{\psi}_h(\beta)d\beta$$

for $x > 0$.

Proposition 3.3 *The solution of (3.5) may be written as*

$$u(x,t) = [W_b(t)h](x) := [W_{b,1}(t)h](x) + [W_{b,2}(t)h](x)$$

where for $x,\ t > 0$,

$$\left[W_{b,1}(t)h\right](x) = \int_{-\infty}^{\infty} e^{-i\beta^2 t + i\beta x}\hat{\phi}_h(\beta)d\beta,$$

$$\left[W_{b,2}(t)h\right](x) = \int_{-\infty}^{\infty} e^{i\beta^2 t - \beta x}\hat{\psi}_h(\beta)d\beta$$

and ϕ_h, ψ_h *are defined by (3.7)-(3.8) and (3.9)-(3.10), respectively.*

Remark 3.4

(i) It follows from their definitions that for any $s \in \mathbb{R}, \phi_h$ *and* ψ_h *belong to the space* $H^s(\mathbb{R})$ *if and only if* $h \in H_0^{\frac{2s+1}{4}}(\mathbb{R}^+)$.

(ii) The function $v(x,t) = [W_{b,1}(t)h](x)$ *is, in fact, defined for* $x,\ t \in \mathbb{R}$ *and solves the IVP*

$$iv_t + v_{xx} = 0, \quad v(x,0) = \phi_h(x), \qquad x,\ t \in \mathbb{R}$$

for the linear Schrödinger equation posed on $\mathbb{R}$. *As for* $[W_{b,2}(t)h](x)$, *it is defined only for* $x > 0$. *However, it may be extended for* $x < 0$ *by setting*

$$\left[W_{b,2}(t)h\right](x) = \int_{-\infty}^{\infty} e^{i\beta^2 t - \beta|x|}\hat{\psi}_h(\beta)d\beta. \tag{3.11}$$

Note that this extension is not necessarily differentiable at $x = 0$. *Therefore, this small trick is not applicable when* $s > 3/2$.

Next, consider the same linear equation

$$\begin{cases} iu_t + u_{xx} = 0, & x \in \mathbb{R}^+,\ t \in \mathbb{R}^+, \\ u(x,0) = \phi(x), & u(0,t) = 0 \end{cases} \tag{3.12}$$

with zero boundary condition, but non-trivial initial data. By semigroup theory, its solution u may be obtained in the form

$$u(t) = W_0(t)\phi$$

where the spatial variable is suppressed and $W_0(t)$ is the C_0-group in $L^2(\mathbb{R}^+)$ generated by the operator A defined by

$$Av = iv''$$

with domain

$$\mathcal{D}(A) = \{v \in H^2(R^+) \mid v(0) = 0\}.$$

By Duhamel's principle, one may use the semi-group $W_0(t)$ to formally write the solution of the forced linear problem

$$\begin{cases} iv_t + v_{xx} = f, & x \in \mathbb{R}^+,\ t \in \mathbb{R}^+, \\ v(x,0) = 0, & v(0,t) = 0 \end{cases} \tag{3.13}$$

in the form

$$v(x,t) = -i\int_0^t W_0(t-\tau)f(\cdot,\tau)d\tau.$$

Let a function ϕ be defined on the half line $\mathbb{R}^+$ and let ϕ^* be an extension to the whole line $\mathbb{R}$. The mapping $\phi \mapsto \phi^*$ can be organized so that it defines a bounded linear operator B from $H^s(\mathbb{R}^+)$ to $H^s(\mathbb{R})$. Henceforth $\phi^* = B\phi$ will refer to such an extension operator applied to $\phi \in H^s(\mathbb{R}^+)$. Assume that $v = v(x,t) = W_{\mathbb{R}}(t)\phi^*$ is the solution of

$$iv_t + v_{xx} = 0, \quad v(x,0) = \phi^*(x),$$

for $x,\ t \in \mathbb{R}$. If $g(t) = v(0,t)$, then $v_g = v_g(x,t) = W_b(t)g$ is the corresponding solution of the nonhomogeneous boundary-value problem (3.5) with boundary condition $h(t) = g(t)$, for $t \geq 0$. Similarly, the function

$$w \equiv w(x,t) = \int_0^t W_{\mathbb{R}}(t-\tau)f^*(\tau)d\tau$$

with $f^*(x,t) = Bf(x,t)$ solves

$$iw_t + w_{xx} = f^*(x,t), \qquad w(x,0) = 0\,,$$

for $x,\ t \in \mathbb{R}$. If $p(t) = w(0,t)$, then $w_p \equiv w_p(x,t) = W_b(t)p = W_{bdr}(t)p$ is the corresponding solution of the non-homogeneous boundary-value problem (3.5) with boundary condition $h(t) = p(t)$, for $t \geq 0$. The following integral representation thus obtains for solutions of the fully non-homogeneous linear initial-boundary-value problem

$$\begin{cases} iu_t + u_{xx} = f, \quad x,\ t \in \mathbb{R}^+, \\ \\ u(x,0) = \phi(x), \qquad u(0,t) = h(t)\,. \end{cases} \tag{3.14}$$

Proposition 3.5 *The solution u of (3.14) can be expressed as*

$$u(t) = W_{\mathbb{R}}(t)\phi^* + \int_0^t W_{\mathbb{R}}(t-\tau)f^*(\tau)d\tau + W_{bdr}(t)\big(h(t) - g(t) - p(t)\big) \tag{3.15}$$

where

$$\phi^*(x,t) = \big[B\phi\big](x,t), \qquad f^*(x,t) = \big[Bf\big](x,t)$$

and

$$g(t) = W_{\mathbb{R}}(t)\phi^*|_{x=0}\,, \qquad p(t) = \int_0^t W_{\mathbb{R}}(t-\tau)f^*(\tau)d\tau\bigg|_{x=0}.$$

3.2 Linear estimates

As before, for any $q \geq 2$ and $r \geq 2$, the pair (q,r) is called *admissible* if

$$\frac{1}{q} + \frac{1}{2r} = \frac{1}{4}\,. \tag{3.16}$$

For any q with $1 \leq q \leq \infty$, q' will denote the Lebesgue index conjugate to q, which is to say, $\frac{1}{q} + \frac{1}{q'} = 1$.

The following estimates for solutions of the linear Schrödinger equation posed on the whole line $\mathbb{R}$ are well known in the subject and will find use here.

Proposition 3.6 *Let $s \in \mathbb{R}$ and $T > 0$ be given. For any $\phi \in H^s(\mathbb{R})$, let $u = W_{\mathbb{R}}(t)\phi$. Then, there exists a constant C depending only on s such that*

$$\sup_{t\in(0,T)} \|u(\cdot,t)\|_{H^s(\mathbb{R})} \le C\|\phi\|_{H^s(\mathbb{R})},$$

$$\sup_{x\in\mathbb{R}} \|u(x,\cdot)\|_{H^{\frac{2s+1}{4}}(0,T)} \le C\|\phi\|_{H^s(\mathbb{R})}$$

and

$$\|u\|_{L^q(0,T;W^{s,r}(\mathbb{R}))} \le C\|\phi\|_{H^s(\mathbb{R})}\,,$$

for any given admissible pair (q,r).

This proposition is same as Lemma 4.1 in [45].

Proposition 3.7 *Let (q,r) be admissible and $T > 0$ be given. Suppose $f \in L^{q'}(0,T;W^{s,r'}(\mathbb{R}))$ and define*

$$u = \int_0^t W_{\mathbb{R}}(t-\tau)f(\tau)d\tau.$$

(i) For any $s \in \mathbb{R}$, there exists a constant $C > 0$ depending only on s such that

$$\|u\|_{C([0,T];H^s(\mathbb{R}))} + \|u\|_{L^q(0,T;W^{s,r}(\mathbb{R}))} \le C\|f\|_{L^{q'}(0,T;W^{s,r'}(\mathbb{R}))}. \tag{3.17}$$

(ii) For any $s \in (-\frac{3}{2}, \frac{1}{2})$, there exists a constant $C > 0$ depending only on s such that

$$\sup_{x\in\mathbb{R}} \|u(x,\cdot)\|_{H^{\frac{2s+1}{4}}(0,T)} \le C(1+T)^{\frac{1}{2}}\|f\|_{L^{q'}(0,T;W^{s,r'}(\mathbb{R}))}. \tag{3.18}$$

(iii) For any $s \in \mathbb{R}$, there exists a constant $C > 0$ such that

$$\sup_{x\in\mathbb{R}} \|u(x,\cdot)\|_{H^{\frac{2s+1}{4}}(0,T)} \le C\|f\|_{L^1(0,T;H^s(\mathbb{R}))}. \tag{3.19}$$

Proof: The proof of (3.17) can be found in [26]. A proof of (3.18) is provided in [45]. For (3.19), note that

$$\begin{aligned} \sup_{x\in\mathbb{R}} \|u(x,\cdot)\|_{H^{\frac{2s+1}{4}}(0,T)} &\le \int_0^T \sup_{x\in\mathbb{R}} \|W_{\mathbb{R}}(t-\tau)f(\tau)\|_{H_t^{\frac{2s+1}{4}}(0,T)} d\tau \\ &\le C\int_0^T \|f(\cdot,\tau)\|_{H^s(\mathbb{R})} d\tau\,, \end{aligned}$$

thereby completing the analysis. □

Next, consider the boundary integral operator $W_{bdr}(t)$.

Proposition 3.8 *Let $0 \le s \le 1$ and $T > 0$ be given and suppose (q,r) is an admissible pair. There exists a constant $C > 0$ such that*

$$\|W_{bdr}(\cdot)h\|_{L^q(0,T;W^{s,r}(\mathbb{R}))} \le C\|h\|_{H^{\frac{2s+1}{4}}(\mathbb{R}^+)}, \tag{3.20}$$

$$\sup_{0<t<T} \|W_{bdr}(\cdot)h\|_{H^s(\mathbb{R})} \le C\|h\|_{H^{\frac{2s+1}{4}}(\mathbb{R}^+)} \tag{3.21}$$

and

$$\sup_{x\in\mathbb{R}} \|W_{bdr}(\cdot)h\|_{H_t^{\frac{2s+1}{4}}(0,T)} \le C\|h\|_{H^{\frac{2s+1}{4}}(\mathbb{R}^+)}\,, \tag{3.22}$$

for any $h \in H_0^{\frac{2s+1}{4}}(\mathbb{R}^+)$.

Proof: It is sufficient to prove that

$$\|W_{b,2}(\cdot)h\|_{L^q(0,T;W^{s,r}(\mathbb{R}))} \le C\|h\|_{H^{\frac{2s+1}{4}}(\mathbb{R}^+)} \tag{3.23}$$

since

$$\|W_{b,1}(\cdot)h\|_{L^q(0,T;W^{s,r}(\mathbb{R}))} \le C\|h\|_{H^{\frac{2s+1}{4}}(\mathbb{R}^+)} \tag{3.24}$$

can be obtained from the result for the whole real line given in [26] and Remark 3.4. To show (3.23), note that

$$\begin{aligned}
\big[W_{b,2}(t)h\big](x) &= \frac{1}{\pi}\int_0^\infty e^{i\beta^2 t-\beta|x|}\beta\hat{h}(i\beta^2)d\beta \\
&= \frac{1}{\pi}\int_0^\infty e^{i\beta^2 t-\beta|x|}\int_{-\infty}^\infty e^{-iy\beta}\psi_h(y)dyd\beta = \frac{1}{\pi}\int_{-\infty}^\infty \psi_h(y)\int_0^\infty e^{i\beta^2 t-\beta|x|-iy\beta}d\beta dy \\
&:= \int_{-\infty}^\infty \psi_h(y)K_t(x,y)dy
\end{aligned}$$

where

$$K_t(x,y) = \frac{1}{\pi}\int_0^\infty e^{i\beta^2 t-\beta|x|-iy\beta}d\beta.$$

Claim: *There exists a constant $C>0$ independent of t,x,y such that for any $t\neq 0$, $x,\ y\in\mathbb{R}$,*

$$|K_t(x,y)| \le \frac{C}{\sqrt{|t|}}\,. \tag{3.25}$$

Proof of the Claim: Note that although the Van Der Corput lemma (Corollary 1.1 in [58]) can be used to shorten the proof of the claim, we present a self-contained argument in favor of (3.25) here. Our approach is the following:

$$\begin{aligned}
K_t(x,y) &= \frac{1}{\pi}\int_0^\infty e^{i\beta^2 t-\beta|x|-iy\beta}d\beta = \frac{1}{\pi\sqrt{t}}\int_0^\infty e^{i\beta^2-\beta|x|t^{-\frac{1}{2}}-iy\beta t^{-\frac{1}{2}}}d\beta \\
&= \frac{1}{\pi\sqrt{t}}\int_0^\infty e^{i\left(\beta-\frac{y}{2\sqrt{t}}\right)^2-\frac{\beta|x|}{\sqrt{t}}}d\beta e^{-i\frac{y^2}{4t}} = \frac{1}{\pi\sqrt{t}}\int_{-\frac{y}{2\sqrt{t}}}^\infty e^{i\beta^2-\frac{|x|}{\sqrt{t}}\left(\beta+\frac{y}{2\sqrt{t}}\right)}d\beta e^{-i\frac{y^2}{4t}} \\
&= \frac{1}{\pi\sqrt{t}}e^{-\frac{|x|y}{2t}-i\frac{y^2}{4t}}\int_{-\frac{y}{2\sqrt{t}}}^\infty e^{i\beta^2-\frac{|x|}{\sqrt{t}}\beta}d\beta\,.
\end{aligned}$$

If $y\le 0$,

$$|K_t(x,y)| = \frac{1}{\pi\sqrt{t}}e^{-\frac{|x|y}{2t}}\left|\int_{-\frac{y}{2\sqrt{t}}}^\infty e^{i\beta^2-\frac{|x|\beta}{\sqrt{t}}}d\beta\right| = \frac{1}{2\pi\sqrt{t}}e^{-\frac{|x|y}{2t}}\left|\int_{\frac{y^2}{4t}}^\infty \frac{e^{i\beta-\frac{|x|\sqrt{\beta}}{\sqrt{t}}}}{\sqrt{\beta}}d\beta\right|.$$

But $e^{-\frac{|x|\sqrt{\beta}}{t}}/\sqrt{\beta}$ is monotone decreasing as $\beta\to\infty$. Standard results about oscillatory integrals, then imply that

$$|K_t(x,y)| \le \frac{1}{\pi\sqrt{t}}e^{\frac{-|x|y}{2t}}e^{-\frac{|x||y|}{2t}}\left(\frac{|y|}{2\sqrt{t}}\right)^{-1} \le Ct^{-1/2}$$

if $\frac{|y|}{2\sqrt{t}} \geq 1$. For $0 \leq \frac{|y|}{2\sqrt{t}} \leq 1$,

$$\begin{aligned}
|K_t(x,y)| &\leq \frac{1}{2\pi\sqrt{t}} e^{\frac{-|x|y}{2t}} \left(\left| \int_1^\infty e^{i\beta - \frac{|x|}{\sqrt{t}}\sqrt{\beta}} \frac{1}{\sqrt{\beta}} d\beta \right| + \left| \int_{\frac{y^2}{4t}}^1 e^{i\beta - \frac{|x|}{\sqrt{t}}\sqrt{\beta}} \frac{1}{\sqrt{\beta}} d\beta \right| \right) \\
&\leq \frac{1}{2\pi\sqrt{t}} e^{-\frac{|x|y}{2t}} \left(e^{\frac{-|x|}{\sqrt{t}}} + \int_{\frac{y^2}{4t}}^1 \frac{e^{\frac{-|x|}{\sqrt{t}} \cdot \frac{|y|}{2\sqrt{t}}}}{\sqrt{\beta}} d\beta \right) \leq \frac{1}{2\pi\sqrt{t}} \left(1 + \int_{\frac{y^2}{4t}}^1 \frac{d\beta}{\sqrt{\beta}} \right) \leq \frac{C}{\sqrt{t}}.
\end{aligned}$$

Hence, if $y \leq 0$,

$$|K_t(x,y)| \leq \frac{C}{\sqrt{t}}.$$

On the other hand, if $y > 0$,

$$|K_t(x,y)| \leq \frac{1}{\pi\sqrt{t}} e^{-\frac{|x|y}{2t}} \left(\left| \int_{-\frac{y}{2\sqrt{t}}}^0 e^{i\beta^2 - \frac{|x|\beta}{\sqrt{t}}} d\beta \right| + \left| \int_0^\infty e^{i\beta^2 - \frac{|x|\beta}{\sqrt{t}}} d\beta \right| \right) = \frac{1}{\pi\sqrt{t}} e^{-\frac{|x|y}{2t}} (I_1 + I_2),$$

where

$$\begin{aligned}
I_2 &= \left| \int_0^\infty e^{i\beta^2 - \frac{|x|\beta}{\sqrt{t}}} d\beta \right| \leq C, \\
I_1 &= \frac{1}{2} \left| \int_{\frac{y^2}{4t}}^0 \frac{e^{i\beta + \frac{|x|\sqrt{\beta}}{\sqrt{t}}}}{\sqrt{\beta}} d\beta \right| \leq \frac{1}{2} \left| \int_0^{\frac{y^2}{4t}} \frac{\cos\beta e^{\frac{|x|\sqrt{\beta}}{\sqrt{t}}}}{\sqrt{\beta}} d\beta \right| + \frac{1}{2} \left| \int_0^{\frac{y^2}{4t}} \frac{\sin\beta e^{\frac{|x|\sqrt{\beta}}{\sqrt{t}}}}{\sqrt{\beta}} d\beta \right|.
\end{aligned}$$

If $\frac{y^2}{4t} \leq 2\pi$, then $|I_1| \leq Ce^{\frac{|x|y}{2\sqrt{t}}}$. If $\frac{y^2}{4t} > 2\pi$, let $k_0 = \left\lfloor \frac{y^2}{8\pi t} \right\rfloor$ and obtain

$$\begin{aligned}
|I_1| &\leq \frac{1}{2} \left| \sum_{k=0}^{k=k_0-1} \int_{2k\pi}^{2(k+1)\pi} \frac{\cos\beta e^{\frac{|x|\sqrt{\beta}}{\sqrt{t}}}}{\sqrt{\beta}} d\beta \right| + \frac{1}{2} \left| \sum_{k=0}^{k=k_0-1} \int_{2k\pi}^{2(k+1)\pi} \frac{\sin\beta e^{\frac{|x|\sqrt{\beta}}{\sqrt{t}}}}{\sqrt{\beta}} d\beta \right| \\
&\qquad + \frac{1}{2} \left| \int_{2k_0\pi}^{\frac{y^2}{4t}} \frac{\cos\beta e^{\frac{|x|\sqrt{\beta}}{\sqrt{t}}}}{\sqrt{\beta}} d\beta \right| + \frac{1}{2} \left| \int_{2k_0\pi}^{\frac{y^2}{4t}} \frac{\sin\beta e^{\frac{|x|\sqrt{\beta}}{\sqrt{t}}}}{\sqrt{\beta}} d\beta \right| \\
&= II_1 + II_2 + II_3 + II_4.
\end{aligned}$$

It is clear that $|II_3| + |II_4| \leq Ce^{\frac{|x|y}{2\sqrt{t}}}$. The integral II_2 is now analyzed; II_1 can be treated similarly. First, notice that

$$\begin{aligned}
|II_2| &= \frac{1}{2} \left| \sum_{k=0}^{k_0-1} \int_{2k\pi}^{2(k+1)\pi} \frac{\sin\beta}{\sqrt{\beta}} e^{\frac{|x|\sqrt{\beta}}{\sqrt{t}}} d\beta \right| \\
&= \frac{1}{2} \left| \sum_{k=0}^{k_0-1} \int_0^{2\pi} \frac{\sin\beta}{\sqrt{2k\pi + \beta}} e^{\frac{|x|}{\sqrt{t}}\sqrt{2k\pi+\beta}} d\beta \right| \\
&= \frac{1}{2} \left| \sum_{k=0}^{k_0-1} \left(\int_0^\pi \frac{\sin\beta}{\sqrt{2k\pi + \beta}} e^{\frac{|x|}{\sqrt{t}}\sqrt{2k\pi+\beta}} d\beta - \int_0^\pi \frac{\sin\beta}{\sqrt{2k\pi + \pi + \beta}} e^{\frac{|x|}{\sqrt{t}}\sqrt{2k\pi+\pi+\beta}} d\beta \right) \right|.
\end{aligned}$$

Since

$$\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{u}}e^{\frac{|x|\sqrt{u}}{\sqrt{t}}}\right) = \frac{x}{2u\sqrt{t}}e^{\frac{|x|\sqrt{u}}{\sqrt{t}}} - \frac{1}{2\sqrt{u^3}}e^{\frac{|x|\sqrt{u}}{\sqrt{t}}} = \frac{e^{\frac{|x|\sqrt{u}}{\sqrt{t}}}(\frac{|x|\sqrt{u}}{\sqrt{t}}-1)}{2u^{\frac{3}{2}}}$$

$$\begin{cases} >0, & \text{if } \frac{|x|\sqrt{u}}{\sqrt{t}} > 1, \\ <0\,, & \text{if } \frac{|x|\sqrt{u}}{\sqrt{t}} < 1\,, \end{cases}$$

if a_k is defined by

$$a_k = \int_0^{\pi} \frac{\sin\beta}{\sqrt{k\pi+\beta}} e^{\frac{|x|}{\sqrt{t}}\sqrt{k\pi+\beta}} d\beta,$$

then there is an $N \geq 0$ such that a_k is increasing in k if $k > N$ and decreasing if $k \leq N$. In consequence, it transpires that

$$\begin{aligned} |II_2| &\leq \frac{1}{2}\left|\sum_{k=0}^{N} a_k(-1)^k\right| + \frac{1}{2}\left|\sum_{k=N+1}^{2k_0-1}(-1)^k a_k\right| \\ &\leq \frac{1}{2}\left(|a_0|+|a_N|\right) + \frac{1}{2}(|a_N|+|a_{2k_0-1}|) \\ &\leq C\Bigg(\left|\int_0^{\pi}\frac{\sin\beta}{\sqrt{\beta}}e^{\frac{|x|}{\sqrt{t}}\sqrt{\beta}}d\beta\right| + \left|\int_0^{\pi}\frac{\sin\beta}{\sqrt{N\pi+\beta}}e^{\frac{|x|}{\sqrt{t}}\sqrt{N\pi+\beta}}d\beta\right| \\ &\qquad + \left|\int_0^{\pi}\frac{\sin\beta}{\sqrt{(2k_0-1)\pi+\beta}}e^{\frac{|x|}{\sqrt{t}}\sqrt{(2k_0-1)\pi+\beta}}d\beta\right|\Bigg) \\ &\leq Ce^{\frac{|x|}{\sqrt{t}}\sqrt{2k_0\pi}} \leq Ce^{\frac{|x|y}{2t}}. \end{aligned}$$

The integral II_1 has a similar bound, whence

$$|K_t(x,y)| \leq \frac{C}{\pi\sqrt{t}}e^{-\frac{|x|y}{2t}}\left(e^{\frac{|x|y}{2t}}+1\right) \leq \frac{C}{\sqrt{t}}$$

for any $x,\ y \in \mathbb{R}$ and $t > 0$. Similar remarks apply to $K_{-t}(x,y)$ so that for all $t \in \mathbb{R}\backslash\{0\}$, $|K_t(x,y)| \leq \frac{C}{\sqrt{|t|}}$. This completes the proof of the Claim. $\square$

To prove inequality (3.23), let $\mathcal{K}(t)\psi_h = \int_{-\infty}^{+\infty}\psi_h(y)K_t(x,y)dy$. The result of the Claim yields

$$\|\mathcal{K}(t)\psi_h\|_{L^\infty(\mathbb{R})} \leq C|t|^{-\frac{1}{2}}\|\psi_h\|_{L^1(\mathbb{R})}.$$

Also, Proposition 2.2.3 in [26] provides the inequality

$$\|\mathcal{K}(t)\psi_h\|_{L^2(\mathbb{R})} \leq C\|\psi_h\|_{L^2(\mathbb{R})}.$$

The Riesz-Thorin interpolation theorem then implies that

$$\|\mathcal{K}(t)\psi_h\|_{L^p(\mathbb{R})} \leq C|t|^{-(\frac{1}{2}-\frac{1}{p})}\|\psi_h\|_{L^{p'}(\mathbb{R})}\,,$$

where p' is the index conjugate to p as before. From this, there follows the inequality

$$\left\|\int_0^t \mathcal{K}(t-\tau)f(\tau)d\tau\right\|_{L^r(\mathbb{R})} \leq C\int_0^T |t-\tau|^{-(\frac{1}{2}-\frac{1}{r})}\|f(\tau)\|_{L^{r'}(\mathbb{R})}d\tau$$
$$\leq C\int_0^T |t-\tau|^{-\frac{2}{q}}\|f(\tau)\|_{L^{r'}(\mathbb{R})}d\tau,$$

valid for any $f(\cdot,t) = f(t) \in L^{q'}((0,T),L^{r'}(\mathbb{R}))$. The Riesz potential inequalities (see [67], Theorem 1, p. 119) then imply that

$$\left\|\int_0^t \mathcal{K}(t-\tau)f(\tau)d\tau\right\|_{L^q((0,T),L^r(\mathbb{R}))} \leq C\|f\|_{L^{q'}((0,T),L^{r'}(\mathbb{R}))}. \tag{3.26}$$

A similar estimate holds for $\int_0^T \mathcal{K}(t-\tau)f(\tau)d\tau$.

Now, compute the $L^2(\mathbb{R})$–norm of the function

$$\mathcal{K}_1(y) = \int_0^T \int_{-\infty}^{+\infty} \overline{K_t(x,y)}f(x,t)dxdt\,,$$

viz.

$$\|\mathcal{K}_1(y)\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{+\infty}\left(\int_0^T\int_0^T\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overline{K_s(x,y)}f(x,s)K_\sigma(w,y)\overline{f(w,\sigma)}\,dx\,dw\,ds\,d\sigma\right)dy.$$

Note that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} f(x,s)\overline{K_s(x,y)}dx\int_{-\infty}^{\infty}\overline{f(w,\sigma)}K_\sigma(w,y)\,dw\,dy$$
$$= \int_{-\infty}^{\infty} f(x,s)\int_{-\infty}^{\infty}\overline{f(w,\sigma)}\int_{-\infty}^{+\infty}\overline{K_s(x,y)}K_\sigma(w,y)\,dy\,dw\,dx$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,s)\overline{f(w,\sigma)}K_{s,\sigma}(x,w)\,dw\,dx.$$

The inequality (3.25) implies that

$$K_{s,\sigma}(x,w) = \int_{-\infty}^{+\infty}\overline{K_s(x,y)}K_\sigma(w,y)dy$$
$$= \frac{1}{\pi^2}\int_0^\infty\int_{-\infty}^{+\infty}\int_0^\infty e^{-i\tilde{\beta}^2 s-\tilde{\beta}|x|+iy\tilde{\beta}}e^{i\beta^2\sigma-\beta|w|-iy\beta}\,d\beta\,dy\,d\tilde{\beta}$$
$$= \frac{2}{\pi}\int_0^\infty e^{-i\beta^2(s-\sigma)-\beta(|x|+|w|)}d\beta \leq \frac{C}{\sqrt{|s-\sigma|}}$$

for $s \neq \sigma$, where the constant C is independent of $x, w \in \mathbb{R}$. . Rewrite $\|\mathcal{K}_1(y)\|^2_{L^2(\mathbb{R})}$ as

$$\|\mathcal{K}_1(y)\|^2_{L^2(\mathbb{R})} = \int_0^T\int_0^T\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,s)\overline{f(w,\sigma)}K_{s,\sigma}(x,w)\,dw\,dx\,d\sigma\,ds$$

$$= \int_0^T \int_{-\infty}^{+\infty} f(x,s) \int_0^T \int_{-\infty}^{+\infty} \overline{f(w,\sigma)} K_{s,\sigma}(x,w)\, dw\, d\sigma\, dx\, ds\,.$$

Then, using the procedure described for proving (3.26), it is inferred that

$$\left\| \int_0^T \int_{-\infty}^{+\infty} \overline{f(w,\sigma)} K_{s,\sigma}(x,w) dw d\sigma \right\|_{L^q((0,T),L^r(\mathbb{R}))} \le C \|f\|_{L^{q'}((0,T),L^{r'}(\mathbb{R}))}\,,$$

which in turns gives

$$\|\mathcal{K}_1(y)\|^2_{L^2(\mathbb{R})} \le C\|f\|_{L^{q'}((0,T),L^{r'}(\mathbb{R}))} \left(\left\| \int_0^T \int_{-\infty}^{+\infty} \overline{f(w,\sigma)} K_{s,\sigma}(x,w) dw d\sigma \right\|_{L^q((0,T),L^r(\mathbb{R}))} \right)$$

$$\le C\|f\|^2_{L^{q'}((0,T),L^{r'}(\mathbb{R}))}.$$

Finally, consider the integral

$$\int_{-\infty}^{+\infty} \Big(\mathcal{K}(t)\psi_h, \phi(\cdot,t) \Big)_{L^2} dt = \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} K_t(x,y)\psi_h(y) dy \overline{\phi(x,t)} dx \right) dt\,,$$

where $\phi(x,t) \in C_c([0,T], \mathcal{D}(\mathbb{R})), \psi_h \in L^2(\mathbb{R})$. Applying the just obtained estimates yields

$$\begin{aligned} \int_{-\infty}^{+\infty} (\mathcal{K}(t)\psi_h, \phi(\cdot,t))_{L^2(\mathbb{R})} dt &= \int_{-\infty}^{+\infty} \psi_h(y) \overline{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \overline{K_t(x,y)} \phi(x,t) dx dt dy} \\ &\le \|\psi_h\|_{L^2} \left\| \int_0^T \int_{-\infty}^{+\infty} \overline{K_t(x,y)} \phi(x,t) dx dt \right\|_{L^2_y(\mathbb{R})} \\ &\le C\|\psi_h\|_{L^2(\mathbb{R})} \|\phi\|_{L^{q'}((0,T),L^{r'}(\mathbb{R}))}\,. \end{aligned}$$

By duality, $\|\mathcal{K}(t)\psi_h\|_{L^q((0,T),L^r(\mathbb{R}))} \le C\|\psi_h\|_{L^2(\mathbb{R})}$, which gives (3.20) with $s = 0$. Since

$$\partial_x [W_{b,2}(t)h](x) = \frac{1}{\pi} \int_0^\infty e^{i\beta^2 t - \beta|x|}\ sign(x)\ \frac{\beta^2}{b - a\beta} \hat{h}(i\beta^2) d\beta,$$

the same argument suffices to show that (3.20) holds for $s = 1$. When $0 < s < 1$, the relevant estimate follows by interpolation. The inequality (3.21) is a special case of (3.20) and (3.22) is straightforwardly obtained using a classical trace argument and the Fourier transform. □

The following estimates of the temporal regularity of W_{bdr} will also be helpful.

Proposition 3.9 *Let (q, r) be a given admissible pair, $T > 0$ and $s \ge 0$. For any $h \in H^{\frac{1}{4}+s}(\mathbb{R}^+)$, the correspondence $t \mapsto \frac{\partial^s W_{bdr}(t)}{\partial t^s} h$ belongs to the space*

$$L^q(0,T; L^r(\mathbb{R}^+)) \cap C([0,T], L^2(\mathbb{R}^+))$$

and there exists a constant C such that

$$\left\| \frac{\partial^s W_{bdr}(\cdot)}{\partial t^s} h \right\|_{L^q(0,T;L^r(\mathbb{R}^+))} \le C\|h\|_{H^{\frac{1}{4}+s}(\mathbb{R}^+)}\,. \tag{3.27}$$

In particular, for $h \in H^{\frac{1}{4}+s}(\mathbb{R}^+)$,

$$\sup_{0<t<T} \left\| \frac{\partial^s W_{bdr}(\cdot)}{\partial t^s} h \right\|_{L^2(\mathbb{R}^+)} \le C \|h\|_{H^{\frac{1}{4}+s}(\mathbb{R}^+)} \tag{3.28}$$

and

$$\sup_{0<x<\infty} \left\| \frac{\partial^s W_{bdr}(\cdot)}{\partial t^s} h \right\|_{H_t^{\frac{1}{4}}(0,T)} \le C \|h\|_{H^{\frac{1}{4}+s}(\mathbb{R}^+)} . \tag{3.29}$$

Proof: As above, we only have to study $W_{b,2}h$. It is straightforward to calculate that

$$\frac{\partial W_{b,2}(t)h}{\partial t} = \frac{i}{\pi} \int_0^\infty e^{i\beta^2 t - \beta x} \beta^3 \hat{h}(i\beta^2) d\beta = \frac{i}{\pi} \int_0^\infty e^{i\beta^2 t - \beta x} \hat{\psi_1}(\beta) d\beta = i \int_{-\infty}^\infty \psi_1(y) K_t(x,y) dy.$$

It follows immediately that

$$\left\| \frac{\partial W_{b,2}(t)}{\partial t} h \right\|_{L^q(0,T;;L^r(\mathbb{R}^+))} \le C \|\psi_1\|_{L^2(\mathbb{R})} \le C \|h\|_{H^{\frac{1}{4}+1}(\mathbb{R}^+)}.$$

A similar proof holds for all integers $s \ge 0$. The general case then follows by interpolation. Since there are no boundadry conditions involved in the argument, we do not run into trouble when the interpolation index is equal to $\frac{1}{2}$. In particular, the Sobolev space $H^{\frac{1}{2}}(\mathbb{R}^+)$ is the mid-point interpolation space between $L^2(\mathbb{R}^+)$ and $H^1(\mathbb{R}^+)$ in this case. $\Box$

Note that from the equation $iu_t + u_{xx} = 0$, one t-derivative of u is equivalent to two x-derivatives of u. The following proposition holds as a corollary of this observation.

Proposition 3.10 *Let* (q,r) *be a given admissible pair,* $T > 0$ *and* $s \ge 0$. *There exists a constant* $C > 0$ *such that for any* $h \in H^{\frac{1}{4}+s}(\mathbb{R}^+)$, $u = W_{bdr}(t)h$ *satisfies*

$$\|u\|_{L_t^q(0,T;W_x^{s,r}(\mathbb{R}^+))} + \sup_{0<t<T} \|u(\cdot,t)\|_{H^s(\mathbb{R}^+)} + \sup_{0<x<\infty} \|\partial_x^j u(x,\cdot)\|_{H^{\frac{2s+1-2j}{4}}(0,T)} \le C \|h\|_{H^{\frac{2s+1}{4}}(\mathbb{R}^+)}$$

for $j = 0, 1$ and $2s + 1 - 2j \ge 0$.

Finally, we consider the IBVP (3.14). The next proposition follows readily from Propositions 3.6-3.10.

Proposition 3.11 *Let* $T > 0$ *and* $0 \le s < \frac{5}{2}$ *be given. Assume* $f \in L^1(0,T;H^s(\mathbb{R}^+))$, $\phi \in H^s(\mathbb{R}^+)$, $h \in H^{\frac{2s+1}{4}}(0,T)$ *and* $\phi(0) = h(0)$ *if* $\frac{1}{2} < s < \frac{5}{2}$. *Then there exists a constant* $C > 0$ *depending only on* s *such that the solution* u *of the IBVP (3.14) respects the inequality*

$$\begin{aligned} &\sup_{0\le t\le T} \|u(\cdot,t)\|_{H^s(\mathbb{R}^+)} + \sup_{0<x<\infty} \|u(x,\cdot)\|_{H^{\frac{2s+1}{4}}(0,T)} + \|u\|_{L^q(0,T;W^{s,r}(\mathbb{R}^+))} \\ &\quad \le C \left(\|\phi\|_{H^s(\mathbb{R}^+)} + \|h\|_{H^{\frac{2s+1}{4}}(0,T)} + \|f\|_{L^1(0,T;H^s(\mathbb{R}^+))} \right), \end{aligned}$$

where (q,r) *is any admissible pair.*

3.3 Local well-posedness

In this subsection, the local well-posedness of the full nonlinear problem

$$\begin{cases} iu_t + u_{xx} + \lambda u|u|^{p-2} = 0, & x \in \mathbb{R}^+,\ t \in (0,T), \\ u(x,0) = \phi(x), \qquad u(0,t) = h(t)\,, \end{cases} \tag{3.30}$$

is the topic of conversation. Let $\phi^* = B\phi$ be an extension of ϕ from $\mathbb{R}^+$ to $\mathbb{R}$ as before, with

$$\|\phi^*\|_{H^s(\mathbb{R})} \le C_s \|\phi\|_{H^s(\mathbb{R}^+)}.$$

Suppose $0 \le s < 5/2$ and let the operator $\mathcal{W}_{bdr}$ be as introduced in Section 2. Rewrite (3.30) as an integral equation on the domain $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, *viz.*

$$u(t) = W(t)\phi^* + \mathcal{W}_{bdr}(t)\big(h(t) - g_\phi(t)\big) - i\lambda \int_0^t W(t-\tau)|u|^{p-2}u(\tau)d\tau - \mathcal{W}_{bdr}(t) f_u(t) \tag{3.31}$$

where $W(t) = W_{\mathbb{R}}(t)$ and $g_\phi(t), f_u(t)$ are the trace of $W(t)\phi^*$ and $-i\lambda \int_0^t W(t-\tau)|u|^{p-2}u(\tau)d\tau$ at $x = 0$. That is to say,

$$g_\phi(t) = W(t)\tilde{\phi}\Big|_{x=0}, \qquad f_u(t) = -i\lambda \int_0^t W(t-\tau)|u|^{p-2}u(\tau)d\tau\Big|_{x=0}.$$

Proposition 3.12 *Assume*

$$0 \le s < \frac{1}{2} \quad \textit{and} \quad 3 \le p < \frac{6-4s}{1-2s}.$$

Let (γ, ρ) be the admissible pair defined by

$$\rho = \frac{p}{1 + s(p-2)}, \qquad \gamma = \frac{4p}{(p-2)(1-2s)}.$$

For any given $\phi \in H^s(\mathbb{R}^+)$ and $h \in H^{\frac{2s+1}{4}}(0,T)$, there exists a T_{max} with $0 < T_{max} \le T$ such that the integral equation (3.31) admits a unique solution $u \in C([0,T_{\max}); H^s(\mathbb{R}))$ satisfying

$$u \in L^\gamma_{loc}([0,T_{\max}); W^{s,\rho}(\mathbb{R})). \tag{3.32}$$

Moreover, this solution possesses the following additional properties:

(i) The solution $u \in L^q_{loc}([0,T_{\max}); W^{s,r}(\mathbb{R}))$ for every admissible pair (q,r).

(ii) The solution u depends continuously on ϕ and h in the sense that if $\phi_n \to \phi$ in $H^s(\mathbb{R}^+)$ and $h_n \to h$ in $H^{\frac{2s+1}{4}}(\mathbb{R}^+)$, then, for any T with $0 < T < T_{max}$, the corresponding solutions u_n tend to u in $C([0,T]; H^s(\mathbb{R}))$ as $n \to \infty$.

(iii) If $3 \le p < \frac{6-4s}{1-2s}$ and $T_{max} < +\infty$, then

$$\lim_{t \to T_{max}} \|u(\cdot,t)\|_{H^s(\mathbb{R})} = +\infty.$$

Proposition 3.13 *Let $\frac{1}{2} < s < \frac{5}{2}$ and $[s] \le p-2$ be given. For any $\phi \in H^s(\mathbb{R}^+)$ and $h \in H^{\frac{2s+1}{4}}(\mathbb{R}^+)$ satisfying the compatibility condition*

$$\phi(0) = h(0),$$

there exists a $T_{max} > 0$ such that the integral equation (3.31) admits a unique solution

$$u \in C([0, T_{max}); H^s(\mathbb{R}^+)).$$

Moreover, the solution u possesses the following properties:

(i) The solution u belongs to the space $L_x^\infty(\mathbb{R}^+; H_t^{\frac{2s+1}{4}}(\mathbb{R}))$.

(ii) The solution u depends on ϕ and h continuously in the sense that if $\phi_n \to \phi$ in $H^s(\mathbb{R}^+)$ and $h_n \to h$ in $H^{\frac{2s+1}{4}}(\mathbb{R}^+)$, then, for any T with $0 < T < T_{max}$, the corresponding solutions u_n tends to u in $C([0,T]; H^s(\mathbb{R}^+)) \cap L_x^\infty(\mathbb{R}^+; H_t^{\frac{2s+1}{4}}(\mathbb{R}^+))$ as $n \to \infty$.

(iii) If $T_{max} < +\infty$, then

$$\lim_{t \to T_{max}} \|u(\cdot, t)\|_{H^s(\mathbb{R}^+)} = +\infty.$$

The proofs of Propositions 3.12 and 3.13 follow just as does the local existence theory laid out in Holmer [45]. The chain rule and product rule for fractional derivatives and the propositions in the last subsection provide the necessary estimates for applying the contraction mapping theorem to the right-hand side of (3.31). The details are omitted.

Remark 3.14 *Proposition 3.13 also holds for $s \in (\frac{5}{2}, \frac{9}{2})$ if the following compatibility conditions are satisfied;*

$$\phi(0) = h(0)\,, \quad h_t(0) = i\phi_{xx}(0) + i\lambda|\phi(0)|^{p-2}\phi(0)\,.$$

The only difference from the proof of the local existence in Holmer [45] is to use the function space

$$C((0,T); H_x^s(\mathbb{R}^+)) \cap C(\mathbb{R}_x^+, H^{\frac{2s+1}{4}}(0,T)) \cap C_t^1((0,T); H_x^{s-2}(\mathbb{R}^+))\,.$$

Note again that one t-derivative of u corresponds to two x-derivatives of u.

Remark 3.15 *In case of $s = 1$ or $s = 2$, the assumption $\lfloor s \rfloor \le p - 2 < +\infty$ is not needed. The result of Proposition 3.13 holds for any p with $p > 2$ in these situations.*

Remark 3.16 *According to Proposition 3.12, for $\phi \in H^s(\mathbb{R}^+)$, $h \in H_{loc}^{\frac{2s+1}{4}}(\mathbb{R}^+)$ with $0 \le s < \frac{1}{2}$, there exists a T_{max} depending only on s such that the corresponding solution $u \in C([0, T_{max}); H^s(\mathbb{R}))$ blows up at T_{max}, i.e.,*

$$\lim_{t \to T_{max}} \|u(\cdot, t)\|_{H^s(\mathbb{R})} = +\infty$$

if $T_{max} < \infty$. However, if (ϕ, h) also belongs to the space $H^2(\mathbb{R}^+) \times H_{loc}^{\frac{5}{4}}(\mathbb{R}^+)$, then by Proposition 3.13, there exists a $T_{max}^* > 0$ such that $u \in C([0, T_{max}^*); H^2(\mathbb{R}))$ and

$$\lim_{t \to T_{max}} \|u(\cdot, t)\|_{H^2(\mathbb{R})} = +\infty$$

if $T_{max}^* < \infty$. It is obviously the case that $T_{max}^* \le T_{max}$. Is it true that $T_{max}^* = T_{max}$? This is a well-known regularity issue (see [26]). For the pure Cauchy problem (3.2), the answer is positive. The same proof can be applied to the IBVP considered here to show that $T_{max}^* = T_{max}$.

A solution of the integral equation (3.31) on $\mathbb{R}$ as given in Propositions 3.12 and 3.13, when restricted to $\mathbb{R}^+$, is a distributional solution of the IBVP (3.30) with strong traces. However, as the IBVP (3.30) can be converted to other integral equations similar to (3.31) on $\mathbb{R}$, whose solutions, when restricted to $\mathbb{R}^+$, yield distributional solutions to the IBVP (3.30), the following question arises naturally.

Are solutions of the various integral equations on $\mathbb{R}$ equal to each other when restricted to $\mathbb{R}^+$? In other words, Propositions 3.12 and 3.13 lead to the existence of distributional solutions with strong traces for the IBVP (3.30). As for its uniqueness, in the case of $s > \frac{1}{2}$, since the space $H^s(\mathbb{R}^+)$ is continuously imbedded into the space $L^\infty(\mathbb{R}^+)$, it is straightforward to ascertain that the IBVP (3.30) admits at most one distributional solution with strong traces in the space $C([0,T];H^s(\mathbb{R}^+))$. The following well-posedness theory for the IBVP (3.30) results as a corollary of Proposition 3.13.

Corollary 3.17 *Let $\frac{1}{2} < s < \frac{5}{2}$ and $3 \le p < +\infty$ be given. For any $\phi \in H^s(\mathbb{R}^+)$ and $h \in H^{\frac{2s+1}{4}}(\mathbb{R}^+)$ satisfying the compatibility condition*

$$\phi(0) = h(0),$$

there exists a $T_{max} > 0$ such that (3.31) admits a unique solution $u \in C([0,T_{max});H^s(\mathbb{R}^+))$. Additionally, the solution $u \in L^\infty_x(\mathbb{R};H_t^{\frac{2s+1}{4}}(\mathbb{R}^+))$ and if $T_{max} < +\infty$, then

$$\lim_{t\to T_{max}} \|u(\cdot,t)\|_{H^s(\mathbb{R}^+)} = +\infty.$$

Moreover, the solution u depends continuously on ϕ and h in the sense that if $\phi_n \to \phi$ in $H^s(\mathbb{R}^+)$ and $h_n \to h$ in $H^{\frac{2s+1}{4}}(\mathbb{R}^+)$, then, for any $0 < T < T_{max}$, the corresponding solutions u_n tend to u in $C([0,T];H^s(\mathbb{R}^+))$ as $n \to \infty$.

The uniqueness of the IBVP (3.30) in the space $C([0,T];H^s(\mathbb{R}^+))$ remains open in case $0 \le s < \frac{1}{2}$. To resolve this issue, we first show that the solution given in Proposition 3.12, when restricted on $\mathbb{R}^+$ is a mild solution of the IBVP (3.30).

Proposition 3.18 *Let $0 \le s < \frac{1}{2}$ be given and assume that $3 \le p < \frac{6-4s}{1-2s}$. For any given $\phi \in H^s(\mathbb{R}^+)$ and $h \in H_{loc}^{\frac{2s+1}{4}}(\mathbb{R}^+)$, there exists a $T_{max} > 0$ such that the IBVP (3.30) admits a mild solution $u \in C([0,T_{\max});H^s(\mathbb{R}))$.*

Proof: It suffices to show that for $0 \le s < 1/2$ the solution $u \in C([0,T_{max});H^s(\mathbb{R}))$ of (3.31) given by Proposition 3.12, when restricted to $\mathbb{R}^+$, is a mild solution of the IBVP (3.30). To this end, let $(\phi_n,h_n) \in H^2(\mathbb{R}^+) \times H_{loc}^{\frac{5}{4}}(\mathbb{R}^+)$ with $\phi_n(0) = h_n(0)$ and

$$\lim_{n\to\infty} \|(\phi_n,h_n) - (\phi,h)\|_{H^s(\mathbb{R}^+)\times H_{loc}^{\frac{2s+1}{4}}(\mathbb{R}^+)} = 0.$$

Then by Proposition 3.12, there exists $u_n \in C([0,T_{max});H^2(\mathbb{R}^+))$ solving the integral equation (3.31) with (ϕ,h) replaced by (ϕ_n,h_n). Moreover, u_n tends to u in the space $C([0,T];H^s(\mathbb{R}^+))$ as $n \to \infty$ for any $T < T_{max}$. According to Remarks 3.15 and 3.16, when restricted to $\mathbb{R}^+$, u_n lies in $C([0,T_{max});H^2(\mathbb{R}^+))$ and it solves the IBVP (3.30). In particular, u_n tends to u in the space $C([0,T];H^s(\mathbb{R}^+))$ as $n \to \infty$ for any $T < T_{max}$. Thus, the solution u of (3.31) when restricted to $\mathbb{R}^+$ is a mild solution of the IBVP (3.30). $\Box$

Next, we show that the IBVP (3.30) admits at most one mild solution.

Proposition 3.19 *Assume that s and p are such that*

$$0 \le s < \frac{1}{2} \quad \text{and} \quad 3 \le p < \frac{6-4s}{1-2s}.$$

For any $\phi \in H^s(\mathbb{R}^+)$, $h \in H_{loc}^{\frac{2s+1}{4}}(\mathbb{R}^+)$, the IBVP (3.30) admits at most one mild solution.

Proof: Suppose that for a given $\phi \in H^s(\mathbb{R}^+)$ and $h \in H_{loc}^{\frac{2s+1}{4}}(\mathbb{R}^+)$, the IBVP (3.30) admits two mild solutions u and v which lie in in the space $C([0,T']; H^s(\mathbb{R}^+))$ for some $T' > 0$. By definition, there exist two sequences $\{u_n\}$ and $\{v_n\}$ in the space $C([0,T']; H^2(\mathbb{R}^+))$ such that both u_n and v_n solve the equation in (3.30) for $n = 1, 2, \cdots$, and if

$$u_n(x,0) = \phi_n(x),\ v_n(x,0) = \psi_n(x),\ u_n(0,t) = h_n(t),\ v_n(0,t) = g_n(t),$$

then as $n \to \infty$,

$$u_n \to u, \quad v_n \to v \quad \text{in} \quad C([0,T']; H^s(\mathbb{R}^+)), \quad \phi_n \to \phi, \quad \psi_n \to \phi \quad \text{in} \quad H^s(\mathbb{R}^+)$$

and

$$g_n \to h, \quad h_n \to h \quad \text{in} \quad H^{\frac{2s+1}{4}}(0,T).$$

Let u_n^*, v_n^* and w be the solutions of the integral equation (3.31) corresponding to (ϕ_n, h_n), (ψ_n, g_n) and (ϕ, h), respectively, given by Proposition 3.12 (restricted to $\mathbb{R}^+$). It follows that u_n^*, v_n^* and w lie in $C([0,T; H^s(\mathbb{R}^+)) \cap L^q(0,T; W^{s,r}(\mathbb{R}^+))$ for some $T > 0$. Then, by Proposition 3.13 and Remarks 3.15 and 3.16, u_n^* and v_n^* are in $C([0,T]; H^2(\mathbb{R}^+))$. Note that the time interval over which u_n^* and v_n^* exist in the space $H^2(\mathbb{R}^+)$ is $(0,T)$ for any n, as guaranteed by Remark 3.16. By the uniqueness result in Corollary 3.17, it must be the case that

$$u_n \equiv u_n^*, \qquad v_n \equiv v_n^*,\ n = 1, 2, \cdots.$$

Since (ϕ_n, h_n) and (ψ_n, g_n) are both convergent to (ϕ, h) in $H^s(\mathbb{R}^+) \times H^{\frac{2s+1}{4}}(0,T)$, it follows from Proposition 3.12 that both u_n and v_n converge to w in $C([0,T]; H^s(\mathbb{R}^+))$. Consequently, $u \equiv v$. The proof is complete. □

The last result of the section summarizes the previous ruminations.

Theorem 3.20 *Assume either*

$$3 \le p < \frac{6-4s}{1-2s}, \quad 0 \le s < \frac{1}{2}, \quad s = 1, 2$$

or

$$\frac{1}{2} < s < \frac{5}{2}, \qquad \lfloor s \rfloor < p - 2 < \infty.$$

For any $\phi \in H^s(\mathbb{R}^+)$ and $H_{loc}^{\frac{2s+1}{4}}(\mathbb{R}^+)$ satisfying $\phi(0) = h(0)$ if $s > \frac{1}{2}$, there exists a $T_{\max} > 0$ such that the IBVP (3.30) admits a unique mild solution $u \in C([0, T_{max}); H^s(\mathbb{R}^+))$. Moreover, the solution u has the following properties:

(i) The solution $u \in L_x^\infty(\mathbb{R}^+; H_{loc}^{\frac{2s+1}{4}}(\mathbb{R}^+))$.

(ii) The solution u depends on ϕ and h continuously in the sense that if $\phi_n \to \phi$ in $H^s(\mathbb{R}^+)$ and $h_n \to h$ in $H^{\frac{2s+1}{4}-loc}(\mathbb{R}^+)$, then, for any T with $0 < T < T_{max}$, the corresponding solutions u_n tend to u in $C([0,T]; H^s(\mathbb{R}^+))$ as $n \to \infty$.

4 The Schrödinger equation posed on a finite interval

In this section, consideration is given to the well-posedness in $H^s(0,L)$ of the IBVP

$$\begin{cases} iu_t + u_{xx} + \lambda |u|^{p-2}u = 0, & x \in (0,L), \ t \in \mathbb{R}, \\ u(x,0) = \phi(x), \qquad u(0,t) = h_1(t), & u(L,t) = h_2(t), \end{cases} \tag{4.1}$$

for the NLS equation posed on a finite interval $(0,L)$. Without loss of generality, take $L = 1$.

First, the homogeneous boundary-value problem

$$\begin{cases} iu_t + u_{xx} + \lambda u|u|^{p-2} = 0, & x \in (0,1), \ t \in \mathbb{R}, \\ u(x,0) = \psi(x), \qquad u(0,t) = 0, & u(1,t) = 0, \end{cases} \tag{4.2}$$

is discussed. The well-posedness of (4.2) in $H^s(0,1)$ can be reduced to a special case of the IVP

$$\begin{cases} iu_t + u_{xx} + \lambda u|u|^{p-2} = 0, & -1 < x < 1, \ t \in \mathbb{R}, \\ u(x,0) = \psi(x), \qquad u(-1,t) = u(1,t), & u_x(-1,t) = u_x(1,t), \end{cases} \tag{4.3}$$

of the NLS equation posed on the interval $(-1,1)$ with periodic boundary conditions. Observe that solutions of the IVP (4.3) are even (odd) in x if ψ is even (odd). On the other hand, if u is an odd function with respect to x and solves the IVP (4.3), then its restriction to the interval $(0,1)$ solves the IBVP (4.2) since the boundary conditions $u(0,t) = u(1,t) = 0$ are automatically satisfied. Thus, the following well-posedness result follows immediately from the known results for (4.3).

Theorem 4.1 *Assume that $3 \le p < \infty$ if $\lambda < 0$ and $3 \le p < 6$ if $\lambda > 0$. Then, for any $s \in [0, \frac{5}{2})$ (s not equal to $\frac{1}{2}$ or $\frac{3}{2}$, see (1.5)), the IBVP (4.2) is unconditionally locally well-posed in $H^s(0,1)$ under the conditions that $\lfloor s \rfloor < p - 2$ if p is not an integer and $\psi(0) = \psi(1) = 0$ if $\frac{1}{2} < s < \frac{5}{2}$.*

Now, consider (4.1) with nonhomogeneous boundary data. This is analyzed in several stages.

4.1 Linear problem

First, consider the IBVP

$$\begin{cases} iu_t + u_{xx} = 0, & x \in (0,1), \ t \in \mathbb{R}, \\ u(x,0) = \phi(x), & u(0,t) = u(1,t) = 0, \end{cases} \tag{4.4}$$

for the linear Schrödinger equation posed on the finite interval $(0,1)$. According to standard semigroup theory, for any $\phi \in L^2(0,1)$, the IBVP admits a unique solution $u \in C(\mathbb{R}^+; L^2(0,1))$ given by

$$u(t) = W_0(t)\phi$$

where $W_0(t)$ is the C_0-group in $L^2(0,1)$ generated by the operator $Av = iv''$ with domain $\mathcal{D}(A) = H^2(0,1) \cap H_0^1(0,1)$. Moreover, the solution of the following nonhomogeneous problem

$$\begin{cases} iu_t + u_{xx} = f, \qquad x \in (0,1), \;\; t \in \mathbb{R}, \\ \\ u(x,0) = 0, \qquad u(0,t) = u(1,t) = 0 \end{cases} \tag{4.5}$$

can be expressed, via Duhamel's principle, as

$$u(t) = -i \int_0^t W_0(t-\tau) f(\cdot,\tau) d\tau.$$

Proposition 4.2 *Let $0 \le s \le 2$ and $T > 0$ be given. Let*

$$u(t) = W_0(t)\phi, \quad v(t) = \int_0^t W_0(t-\tau) f(\cdot,\tau) dt\tau$$

and

$$w(t) = \int_0^t W_0(t-\tau) g(\cdot,\tau) d\tau,$$

with $\phi \in H^s(0,1)$, $f \in L^1(0,T;H^s(0,1))$ and $g \in W^{\frac{s}{2},1}(0,T;L^2(0,1))$ satisfying

$$\phi(0) = \phi(1) = 0, \quad f(0,t) = f(1,t) \equiv 0$$

if $s > \frac{1}{2}$. Then, $u,\ v,\ w \in C([0,T];H^s(0,1))$ and

$$\|u\|_{C([0,T];H^s(0,1))} \le C_{T,s} \|\phi\|_{H^s(0,1)},$$

$$\|v\|_{C([0,T];H^s(0,1))} \le C_{T,s} \|f\|_{L^1(0,T;H^s(0,1))}$$

and

$$\|w\|_{C([0,T];H^s(0,1))} \le C_{T,s} \|g\|_{W^{\frac{s}{2},1}(0,T;L^2(0,1))},$$

where the constant $C_{T,s}$ depends only on s and T.

Proof: The cases $s = 0$ and $s = 2$ follow from standard semigroup theory. When $0 < s < 2$, these inequalities are follow from standard interpolation theory. □

In terms of Fourier sine series, the solution u is given explicitly by

$$u(x,t) = \left[W_0(t)\phi\right](x) = \sum_{n=1}^{+\infty} c_n e^{-i(n\pi)^2 t} \sin(n\pi x) \quad \text{where} \quad c_n = 2\int_0^1 \phi(x)\sin(n\pi x) dx.$$

This can be written in the complex form

$$u(x,t) = \sum_{n=-\infty}^{\infty} \tilde{c}_n e^{-i(n\pi)^2 t + in\pi x}$$

where

$$\tilde{c}_n = \begin{cases} c_n & \text{if } \; n \ge 1, \\ 0 & \text{if } \; n = 0, \\ -c_n & \text{if } \; n \le -1. \end{cases}$$

In this form, it is clear that u may be viewed as the solution $u(x,t)$ of the Cauchy problem

$$\begin{cases} iu_t + u_{xx} = 0, \ u(x,0) = \phi^*(x), \ x \in (-1,1), \\ u(-1,t) = u(1,t), \ u_x(-1,t) = u_x(1,t)\,, \end{cases} \tag{4.6}$$

where ϕ^* is the odd extension of ϕ from $(0,1)$ to $(-1,1)$. On the other hand, if u is a solution of (4.6) and is also an odd function, then its restriction to $(0,1)$ solves (4.5). Thus

$$\big[W_{\mathbb{T}}(t)\phi^*\big](x) = \big[W_0(t)\phi\big](x), \quad x \in (0,1).$$

Here, $W_{\mathbb{T}}(t)$ is the C_0-group in $L^2(\mathbb{T})$ generated by the operator $A_{\mathbb{T}}$ in $L^2(\mathbb{T})$ with domain $\mathcal{D}(A_{\mathbb{T}}) = H^2(\mathbb{T})$. Consequently, the following proposition follows from the theory developed in [16].

Proposition 4.3 *Let $0 \le s < \frac{1}{2}$ and $T > 0$ be given and let $\Omega_T = (0,1)\times(0,T)$. For any $\phi \in H^s(0,1)$, $u = W_0(t)\phi \in L^4(\Omega_T) \cap C([0,T];H^s(0,1))$ has*

$$\|u\|_{L^4(\Omega_T)\cap C([0,T];H^s(0,1))} \le C\|\phi\|_{H^s(0,1)}\,,$$

where $C > 0$ depends only on s and T.

Next is discussed the IBVP of the associated linear problem with nonhomogeneous Dirichlet boundary data, namely,

$$\begin{cases} iu_t + u_{xx} = 0, \qquad x \in (0,1), \ t \in \mathbb{R}, \\ u(x,0) = 0, \qquad u(0,t) = h_1(t), \quad u(1,t) = h_2(t), \end{cases} \tag{4.7}$$

with the compatibility conditions $h_1(0) = h_2(0) = 0$ if necessary.

Proposition 4.4 *The solution of (4.7) can be expressed as*

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{+\infty} 2in\pi e^{-i(n\pi)^2 t}\int_0^t e^{i(n\pi)^2\tau}\Big(h_1(\tau) - (-1)^n h_2(\tau)\Big)d\tau \sin n\pi x \\ &= W_h h_1 + (W_h h_2)\Big|_{x\to 1-x}. \end{aligned} \tag{4.8}$$

Proof: Consider first the special case where $h_2 \equiv 0$ and $h_1(0) = 0$. Define v by

$$u(x,t) = v(x,t) + (1-x)h_1(t).$$

Then $v(x,t)$ solves

$$\begin{cases} iv_t + v_{xx} = -i(1-x)h_1'(t), \qquad x \in (0,1), \ t \in \mathbb{R}, \\ v(x,0) = 0, \qquad v(0,t) = 0, \quad v(1,t) = 0\,, \end{cases}$$

if u solves (4.7). As above, write $v(x,t)$ as

$$v(x,t) = \sum_{k=1}^{\infty} \alpha_k(t) \sin k\pi x.$$

Then, for $k = 1, 2, \cdots$,

$$\frac{d}{dt}\alpha_k(t) + i(k\pi)^2\alpha_k(t) = \beta_k h_1'(t), \qquad \alpha_k(0) = 0,$$

where

$$\beta_k = -2\int_0^1 (1-x)\sin k\pi x dx = -\frac{2}{k\pi}.$$

It follows that

$$\alpha_k(t) = \beta_k \int_0^t e^{-i(k\pi)^2(t-\tau)} h_1'(\tau)d\tau = \beta_k h_1(t) - i\beta_k(\pi k)^2 \int_0^t e^{-i(k\pi)^2(t-\tau)} h_1(\tau)d\tau\,.$$

Substituting the latter into the original Fourier series representation yields

$$v(x,t) = -(1-x)h_1(t) - \sum_{k=1}^{\infty} i\beta_k(\pi k)^2 \int_0^t e^{-i(k\pi)^2(t-\tau)} h_1(\tau)d\tau\,,$$

which in turn implies that

$$u(x,t) = \sum_{k=1}^{\infty} 2i\pi k \int_0^t e^{-i(k\pi)^2(t-\tau)} h_1(\tau)d\tau \sin k\pi x.$$

Next, consider the case of $h_1 \equiv 0$ and $h_2(0) = 0$. If we let $x' = 1 - x$, this situation can be reduced to the case just studied. Thus, if $h_1 \equiv 0$ and $h_2(0) = 0$,

$$u(x,t) = \sum_{k=1}^{\infty} (-1)^{k+1} 2i\pi k \int_0^t e^{-i(k\pi)^2(t-\tau)} h_2(\tau)d\tau \sin k\pi x.$$

The full representation (4.8) now follows. □

Remark 4.5 *One may view the solution u in (4.8) of (4.7) as being written in the form*

$$u(x,t) = \int_0^t W_0(t-\tau)q(\cdot,\tau)d\tau \tag{4.9}$$

where

$$q(x,t) = \Big(h_1(t) - (-1)^n h_2(t)\Big)\sum_{n=1}^{\infty} 2in\pi \sin n\pi x\,.$$

Of course, q belongs to the space $H^s\left(0,T; H^{-(3/2)-\epsilon}(0,1)\right)$ for any $\epsilon > 0$ if $h_1, h_2 \in H^s(0,T)$. By semigroup theory, if $h_1,\ h_2 \in W^{1,1}(0,T)$, then $u \in C\left([0,T]; H^{(1/2)-\epsilon}(0,1)\right)$.

Attention is now turned to the boundary integral

$$\begin{aligned} u_h = W_h h \quad &= \sum_{n=1}^{\infty} 2in\pi e^{-i(n\pi)^2 t} \int_0^t e^{i(n\pi)^2\tau} h(\tau)d\tau \sin n\pi x \\ &= \sum_{n=-\infty}^{\infty} n\pi e^{-i(n\pi)^2 t} \int_0^t e^{i(n\pi)^2\tau} h(\tau)d\tau e^{in\pi x}. \end{aligned} \tag{4.10}$$

In the following, we will use the Lions-Magenes space $H_{00}^{1/2}(0,T)$ [59], which is the interpolation space $[H_0^1(0,T), L^2(0,T)]_\theta$ with $\theta = 1/2$.

Proposition 4.6 *For a given $T>0$, let $\Omega_T=(0,1)\times(0,T)$. If $h\in H_{00}^{\frac{1}{2}}(0,T)$, then*

$$u_h=W_h(\cdot)h\in L^4(\Omega_T)\cap C\big([0,T];L^2(0,1)\big)$$

and there is a constant C_T depending only on T such that

$$\|u_h\|_{L^4(\Omega_T)}\le C_T\|h\|_{H_{00}^{\frac{1}{2}}(0,T)} \tag{4.11}$$

and

$$\sup_{0\le t\le T}\|u_h(\cdot,t)\|_{L^2(0,1)}\le C_T\|h\|_{H_{00}^{\frac{1}{2}}(0,T)}. \tag{4.12}$$

Proof: These results follow from analysis provided in Bourgain's paper [16]. In more detail, let $h(\tau)=\int_{-\infty}^{\infty}e^{-\pi^2 i\lambda\tau}\hat{h}(\lambda)d\lambda$. Write u_h as follows:

$$\begin{aligned}
u_h &= \sum_{n=-\infty}^{\infty}e^{-i(n\pi)^2t}e^{in\pi x}n\pi\int_{-\infty}^{\infty}\hat{h}(\lambda)\int_0^t e^{i(n\pi)^2\tau-\pi^2 i\lambda\tau}d\tau d\lambda\\
&= \sum_{n=-\infty}^{\infty}e^{-i(n\pi)^2t}e^{in\pi x}n\pi\int_{-\infty}^{\infty}\hat{h}(\lambda)\frac{e^{i(n\pi)^2t-\pi^2 i\lambda t}-1}{(n^2-\lambda)\pi^2 i}d\lambda\\
&= \sum_{n=-\infty}^{\infty}e^{-i(n\pi)^2t}e^{in\pi x}n\pi\left(\int_{-\infty}^{0}+\int_0^{\infty}\right)\hat{h}(\lambda)\frac{e^{i(n\pi)^2t-\pi^2 i\lambda t}-1}{(n^2-\lambda)\pi^2 i}d\lambda\\
&= I^-(x,t)+I^+(x,t).
\end{aligned}$$

Note that $I^+(x,t)$ also takes the form

$$I^+(x,t)=\sum_{n=1}^{\infty}2e^{-i(n\pi)^2t}n\pi\sin n\pi x\int_0^{\infty}\hat{h}(\lambda)\frac{e^{i(n\pi)^2t-\pi^2 i\lambda t}-1}{(n^2-\lambda)\pi^2}d\lambda\,.$$

The quantity $I^+(x,t)$ is studied first. Write

$$\begin{aligned}
I^+(x,t) &= \sum_{n=-\infty}^{\infty}e^{-i(n\pi)^2t}e^{in\pi x}n\pi\int_0^{\infty}\hat{h}(\lambda)\psi(n^2-\lambda)\sum_{k=1}^{\infty}\frac{\big((n^2-\lambda)t\pi^2 i\big)^k}{k!(n^2-\lambda)\pi^2 i}d\lambda\\
&\qquad+\sum_{n=-\infty}^{\infty}e^{in\pi x}n\pi\int_0^{\infty}\hat{h}(\lambda)\Big(1-\psi(n^2-\lambda)\Big)\frac{e^{-\lambda\pi^2 it}}{(n^2-\lambda)\pi^2 i}d\lambda\\
&\qquad-\sum_{n=-\infty}^{\infty}e^{-i(n\pi)^2t}e^{in\pi x}n\pi\int_0^{\infty}\hat{h}(\lambda)\Big(1-\psi(n^2-\lambda)\Big)\frac{1}{(n^2-\lambda)\pi^2 i}d\lambda\\
&= I_1^++I_2^++I_3^+,
\end{aligned}$$

where ψ is a suitable C^∞ cut-off function (see [16]). For I_1^+, consider the individual summand

$$I_{1,k}^+=\sum_{n=-\infty}^{\infty}e^{-i(n\pi)^2t}e^{in\pi x}n\pi\int_0^{\infty}\hat{h}(\lambda)\psi(n^2-\lambda)((n^2-\lambda)^k d\lambda\,,$$

for $k = 1, 2, \cdots$. By Proposition 2.1 in [16],

$$
\begin{aligned}
\left\| I_{1,k}^{+} \right\|^2_{L^4(\Omega_T)\cap L^\infty(0,T;L^2(0,1))} &\leq C \left(\sum_{n=-\infty}^{\infty} n^2 \left| \int_0^\infty \hat{h}(\lambda)\psi(n^2-\lambda)\Big(n^2-\lambda\Big)^k d\lambda \right|^2 \right) \\
&\leq CB^k \left(\sum_{n=-\infty}^{\infty} n^2 \left| \int_{|\lambda-n^2|\leq B} \hat{h}(\lambda) d\lambda \right|^2 \right) \\
&\leq CB^{k+1} \left(\sum_{n=-\infty}^{\infty} n^2 \left| \int_{|\lambda-n^2|\leq B} |\hat{h}(\lambda)|^2 d\lambda \right| \right) \\
&\leq CB^{k+1} \left(\sum_{n=-\infty}^{\infty} \left| \int_{|\lambda-n^2|\leq B} |\lambda||\hat{h}(\lambda)|^2 d\lambda \right| \right) \\
&\leq CB^{k+1} \int_0^\infty |\lambda||\hat{h}(\lambda)|^2 d\lambda \leq CB^{k+1} \|h\|^2_{H^{\frac{1}{2}}(\mathbb{R})}.
\end{aligned}
$$

Bounds on I_1^+ follow. Rewrite I_2^+ as

$$
\begin{aligned}
I_2^+(x,t) &= \sum_{n=0}^{\infty} 2\sin n\pi x \int_0^\infty \hat{h}(\lambda) n\pi \Big(1-\psi(n^2-\lambda)\Big) \frac{e^{-\lambda\pi^2 it}}{(n^2-\lambda)\pi^2} d\lambda \\
&= \sum_{n=1}^{\infty} \frac{1}{\pi} \int_0^\infty \hat{h}(\lambda) e^{-\lambda\pi^2 it} \Big(1-\psi(n^2-\lambda)\Big) \left(\frac{1}{n-\sqrt{\lambda}} + \frac{1}{n+\sqrt{\lambda}} \right) \sin n\pi x \, d\lambda \\
&= \sum_{n=1}^{\infty} \frac{1}{\pi} \int_0^\infty 2\mu \hat{h}(\mu^2) e^{-\mu^2\pi^2 it} \Big(1-\psi(n^2-\mu^2)\Big) \left(\frac{1}{n-\mu} + \frac{1}{n+\mu} \right) \sin n\pi x \, d\mu.
\end{aligned}
$$

Applying Lemma A-1 in the Appendix leads to

$$
\begin{aligned}
\sup_{0\leq t\leq T} \|I_2^+(\cdot,t)\|_{L^2(0,1)} &\leq C \sum_{n=1}^{\infty} \left| \int_0^\infty \mu \hat{h}(\mu^2) \Big(1-\psi(n^2-\mu^2)\Big) \left(\frac{1}{n-\mu} + \frac{1}{n+\mu} \right) d\mu \right|^2 \\
&\leq C \left\| (1+|\mu|)^{3/2} \, \hat{h}(\mu^2) \right\|_{L^2(\mathbb{R})} \leq C\|h\|^2_{H^{\frac{1}{2}}(\mathbb{R}^+)}.
\end{aligned}
$$

To estimate the $L^4(\Omega_T)$-norm, rewrite $I_2^+(x,t)$ as

$$
\begin{aligned}
I_2^+(x,t) &= \sum_{n=-\infty}^{\infty} e^{in\pi x} n\pi \left(\int_0^{\frac{n^2}{2}} + \int_{\frac{n^2}{2}}^\infty \right) \hat{h}(\lambda) \Big(1-\psi(n^2-\lambda)\Big) \frac{e^{-\lambda\pi^2 it}}{(n^2-\lambda)\pi^2 i} d\lambda \\
&:= I_{2,1}^+ + I_{2,2}^+ .
\end{aligned}
$$

Proposition 2.6 in [16] implies

$$
\|I_{2,2}^+\|^2_{L^4(\Omega_T)} \leq C \left(\sum_{n=-\infty}^{\infty} \int_0^\infty \frac{n^2\pi^2 |\hat{h}(\lambda)|^2}{\big(|\lambda-n^2|+1\big)^2} \Big(|\lambda-n^2|+1\Big)^{\frac{3}{4}} \chi_{[\frac{n^2}{2},\infty)}(\lambda) \Big(1-\psi(n^2-\lambda)\Big)^2 d\lambda \right)
$$

$$\leq C\int_0^\infty |\lambda||\hat{h}(\lambda)|^2\sum_{n=-\infty}^{\infty}\frac{1}{(1+|\lambda-n^2|)^{\frac{5}{4}}}d\lambda \leq C\int_0^\infty |\lambda||\hat{h}(\lambda)|^2 d\lambda \leq C\|h\|^2_{H^{\frac{1}{2}}(R)}.$$

Rewrite $I^+_{2,1}$ as

$$\begin{aligned}
\left|I^+_{2,1}\right| &= \left|2\int_0^\infty\left(\sum_{n=1}^{\infty}\chi_{[0,\frac{n^2}{2}]}(\lambda)(1-\psi(n^2-\lambda))\frac{n\pi\sin n\pi x}{(n^2-\lambda)\pi^2}\right)e^{-\lambda\pi^2 it}\hat{h}(\lambda)\,d\lambda\right| \\
&= \left|\frac{1}{\pi}\int_0^\infty e^{-\lambda\pi^2 it}\hat{h}(\lambda)\left(\sum_{n=[\sqrt{2\lambda}]}^{\infty}\sin n\pi x\chi_{[0,\frac{n^2}{2}]}(\lambda)\left(\frac{1}{n-\sqrt{\lambda}}+\frac{1}{n+\sqrt{\lambda}}\right)\right)d\lambda\right| \\
&\leq \frac{1}{\pi}\int_0^\infty|\hat{h}(\lambda)|\left|\sum_{n=[\sqrt{2\lambda}]}^{\infty}\left(\frac{1}{n-\sqrt{\lambda}}+\frac{1}{n+\sqrt{\lambda}}\right)\sin n\pi x\right|d\lambda\,.
\end{aligned}$$

To estimate the last sum, let

$$S_k=\sum_{n=1}^{k}\sin n\pi x=\frac{\sin((k+1)\pi x/2)\sin(k\pi x/2)}{\sin(\pi x/2)}\qquad (S_0=0)\,.$$

For any $\alpha\in[0,1]$ and $0<x\leq 1$, $|S_k|\leq Ck^\alpha/|x|^{1-\alpha}$. Consequently,

$$\begin{aligned}
\sum_{n=\lfloor\sqrt{2\lambda}\rfloor}^{k}\frac{1}{n-\sqrt{\lambda}}(S_n-S_{n-1}) &= \sum_{n=\lfloor\sqrt{2\lambda}\rfloor}^{k}\frac{1}{n-\sqrt{\lambda}}S_n-\sum_{n=\lfloor\sqrt{2\lambda}\rfloor}^{k}\frac{1}{n-\sqrt{\lambda}}S_{n-1} \\
&= \sum_{n=\lfloor\sqrt{2\lambda}\rfloor}^{k-1}\left(\frac{1}{n-\sqrt{\lambda}}-\frac{1}{n+1-\sqrt{\lambda}}\right)S_n+\frac{1}{k-\sqrt{\lambda}}S_k-\frac{1}{\lfloor\sqrt{2\lambda}\rfloor-\sqrt{\lambda}}S_{\lfloor\sqrt{2\lambda}\rfloor-1}\,.
\end{aligned}$$

Choose $3/4<\alpha<1$ and let $k\to\infty$ to come to the inequality

$$\begin{aligned}
\left|\sum_{n=\lfloor\sqrt{2\lambda}\rfloor}^{\infty}\frac{1}{n-\sqrt{\lambda}}\sin n\pi x\right| &\leq C|x|^{\alpha-1}\left(\left(\sum_{n=\lfloor\sqrt{2\lambda}\rfloor}^{\infty}\frac{n^\alpha}{(n-\sqrt{\lambda})^2}\right)+\frac{\lambda^{\alpha/2}}{\sqrt{\lambda}+1}\right) \\
&\leq C|x|^{\alpha-1}\left(\frac{\lambda^{\alpha/2}}{\sqrt{\lambda}+1}+\sum_{n=1}^{\infty}\frac{1}{(n+\sqrt{\lambda})^{2-\alpha}}\right) \\
&\leq \frac{C}{|x|^{1-\alpha}(1+\sqrt{\lambda})^{1-\alpha}}\,.
\end{aligned}$$

Using a similar argument for the other term gives

$$\begin{aligned}
\left|I^+_{2,1}(x,t)\right| &\leq C|x|^{\alpha-1}\int_0^\infty\frac{|\hat{h}(\lambda)|}{(1+\sqrt{\lambda})^{1-\alpha}}d\lambda \\
&\leq C|x|^{\alpha-1}\int_0^\infty(1+|\lambda|)^{\tilde{\alpha}}|\hat{h}(\lambda)|\frac{d\lambda}{(1+\sqrt{\lambda})^{1-\alpha}(1+|\lambda|)^{\tilde{\alpha}}}\qquad \left(1/2\geq\tilde{\alpha}>\alpha/2\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad C|x|^{\alpha-1}\left(\int_0^\infty (1+|\lambda|)^{2\tilde{\alpha}}|\hat{h}(\lambda)|^2 d\lambda\right)^{\frac{1}{2}}\left(\int_0^\infty \frac{d\lambda}{(1+\sqrt{\lambda})^{2-2\alpha}(1+|\lambda|)^{2\tilde{\alpha}}}\right)^{\frac{1}{2}}\\
&\leq \quad C|x|^{\alpha-1}\|h\|_{H^{\tilde{\alpha}}(\mathbb{R}^+)}.
\end{aligned}
$$

Combining the foregoing result leads to the desired bound,

$$
\|I_2^+\|_{L^4}^2 \leq C\|h\|^2_{H^{\frac{1}{2}}(\mathbb{R}^+)}.
$$

To study $I_3^+(x,t)$, use again Proposition 2.1 in [16] to write

$$
\begin{aligned}
\|I_3^+\|_{L^4(\Omega_T)\cap L^\infty(0,T;L^2(0,1))} &\leq C\left(\sum_{n=1}^\infty n^2\left|\int_0^\infty \hat{h}(\lambda)\frac{1-\psi(n^2-\lambda)}{\lambda-n^2}d\lambda\right|^2\right)\\
&= C\sum_{n=1}^\infty\left|\int_0^\infty \hat{h}(\lambda)\left(\frac{1}{\sqrt{\lambda}-n}-\frac{1}{\sqrt{\lambda}+n}\right)(1-\psi(n^2-\lambda))d\lambda\right|^2\Bigg)\\
&\leq C\left(\sum_{n=1}^\infty\left|\int_0^\infty \mu\hat{h}(\mu^2)\frac{1}{\mu-n}(1-\psi(n^2-\mu^2))d\mu\right|^2\right.\\
&+ \left.\sum_{n=1}^\infty\left|\int_0^\infty \mu\hat{h}(\mu^2)\frac{1}{\mu+n}(1-\psi(n^2-\mu^2))d\mu\right|^2\right)\\
&\leq C\|\mu\hat{h}(\mu^2)\|^2_{L^2}+\int_0^\infty\left(\int_0^\infty \frac{|\mu\hat{h}(\mu^2)|}{\mu+y}d\mu\right)^2 dy\\
&\leq C\|\mu\hat{h}(\mu^2)\|^2_{L^2}\leq C\|h\|^2_{H^{\frac{1}{4}}(\mathbb{R}^+)}.
\end{aligned}
$$

In summary, it appears that

$$
\|I^+\|_{L^4}^2 \leq C\|h\|^2_{H^{\frac{1}{2}}(\mathbb{R}^+)}.
$$

Now consider $I^-(x,t)$ and express it in the form

$$
\begin{aligned}
I^-(x,t) &= \sum_{n=-\infty}^\infty e^{-i(n\pi)^2t}e^{in\pi x}n\pi\int_0^\infty \frac{e^{i(n\pi)^2t+i\lambda\pi^2t}-1}{(\lambda+n^2)\pi^2 i}\hat{h}(-\lambda)d\lambda\\
&= \sum_{n=-\infty}^\infty e^{in\pi x}n\pi\int_0^\infty \frac{e^{i\lambda\pi^2t}-e^{-i(n\pi)^2t}}{(\lambda+n^2)\pi^2 i}\hat{h}(-\lambda)d\lambda\\
&:= I_1^- - I_2^-.
\end{aligned}
$$

For I_2^-, Proposition 2.1 of [16] implies

$$
\|I_2^-\|^2_{L^4(\Omega_T)}+\|I_2^-\|^2_{L^\infty(0,T;L^2(0,1))}\leq C\sum_{n=-\infty}^\infty n^2\pi^2\left|\int_0^\infty \frac{\hat{h}(-\lambda)}{\lambda+n^2}d\lambda\right|^2
$$

$$
\leq \quad C\sum_{n=1}^\infty\left|\int_0^\infty \frac{\hat{h}(-\lambda)n}{\lambda+n^2}d\lambda\right|^2\leq C\int_1^\infty\left|\int_0^\infty \frac{\hat{h}(-\lambda)y}{\lambda+y^2}d\lambda\right|^2 dy
$$

$$\leq \quad C\int_1^\infty \left(\int_0^\infty (1+|\lambda|^{\frac{1}{2}})^2|\hat{h}(-\lambda)|^2\,d\lambda \int_0^\infty \frac{y^2}{(y^2+\lambda)^2(1+|\lambda|^{\frac{1}{2}})^2}\,d\lambda\right)dy \leq C\|h\|^2_{H^{\frac{1}{2}}(\mathbb{R}^+)}.$$

The formula

$$\sum_{n=1}^\infty \frac{n\sin nx}{n^2+a^2} = \frac{\pi}{2}\frac{\sinh a(\pi-x)}{\sinh a\pi}, \qquad \text{for} \quad 0<x<2\pi,$$

which holds for all a, allows us to write

$$\begin{aligned} I_1^- &= \sum_{n=1}^\infty 2n\pi\sin n\pi x\int_0^\infty \hat{h}(-\lambda)e^{i\lambda\pi^2 t}\frac{1}{(n^2+\lambda)\pi^2}d\lambda \\ &= \int_0^\infty \frac{2\hat{h}(-\lambda)e^{i\lambda\pi^2 t}}{\pi}\sum_{n=1}^\infty \frac{n\sin\pi nx}{n^2+\lambda}d\lambda = \int_0^\infty \hat{h}(-\lambda)e^{i\lambda\pi^2 t}\frac{\sinh\sqrt{\lambda}(\pi-x)}{\sinh\sqrt{\lambda}\pi}d\lambda\,. \end{aligned}$$

Consequently, it is seen that

$$\begin{aligned} \left|I_1^-(x,t)\right| &\leq C\int_0^\infty |\hat{h}(-\lambda)|e^{-\sqrt{\lambda}\pi x}d\lambda \\ &\leq \left|\int_0^\infty |\hat{h}(-\lambda)|^2(1+|\lambda|)d\lambda\right|^{\frac{1}{2}}\left|\int_0^\infty \frac{e^{-2\sqrt{\lambda}\pi x}}{1+|\lambda|}d\lambda\right|^{\frac{1}{2}}, \end{aligned}$$

which implies

$$\sup_{0\leq t\leq T}\int_0^1 \left|I_1^-(x,t)\right|^2 dx \leq C\int_0^1\left(\int_0^\infty |\hat{h}(-\lambda)|^2(1+\lambda)\,d\lambda\int_0^\infty \frac{e^{-2\sqrt{\lambda}\pi x}}{1+\lambda}\,d\lambda\right)dx$$

$$\leq C\|h\|^2_{H^{\frac{1}{2}}(\mathbb{R}^+)}\int_0^\infty\int_0^1 \frac{e^{-2\sqrt{\lambda}\pi x}}{(1+\lambda)}dxd\lambda \leq C\|h\|^2_{H^{\frac{1}{2}}(\mathbb{R}^+)}$$

and

$$\begin{aligned} \int_0^T\int_0^1 |I_1^-(x,t)|^4dxdt &\leq C\int_0^1\left|\int_0^\infty |\hat{h}(-\lambda)|e^{-\sqrt{\lambda}\pi x}d\lambda\right|^4dx \\ &\leq C\int_0^1\left|\int_0^\infty |\hat{h}(-\lambda)|^2(1+|\lambda|)d\lambda\right|^2\left|\int_0^\infty \frac{e^{-2\sqrt{\lambda}\pi x}}{1+|\lambda|}d\lambda\right|^2dx \\ &= C\|h\|^4_{H^{\frac{1}{2}}(\mathbb{R}^+)}\left(\int_0^\infty\left(\int_0^1 \frac{e^{-4\sqrt{\lambda}\pi x}}{(1+|\lambda|)^2}dx\right)^{\frac{1}{2}}d\lambda\right)^2 \\ &\leq C\|h\|^4_{H^{\frac{1}{2}}(\mathbb{R}^+)}\left(\int_0^\infty \frac{1}{1+|\lambda|}\frac{d\lambda}{\lambda^{\frac{1}{4}}}\right)^2 \leq C\|h\|^4_{H^{\frac{1}{2}}(\mathbb{R}^+)}\,. \end{aligned}$$

Hence, we arrive at $\|u_h\|_{L^4(\Omega_T)\cap L^\infty(0,T;L^2(0,1))} \leq C\|h\|_{H^{\frac{1}{2}}(\mathbb{R}^+)}$ and the proof is complete. □

If the regularity of $h(t)$ is higher, $W_h(t)h$ is smoother.

Proposition 4.7 *Let $s \geq 0$ be given. For any $h \in H_0^{\frac{1+s}{2}}(0,T)$ (here for s an even integer, h should be in $H_{00}^{\frac{1+s}{2}}(0,T)$), let $u = W_h h$. Then, $\partial_x^s u$ belongs to $L^4((0,1)\times[0,T])\cap C([0,T];L^2(0,1))$ and satisfies*

$$\|\partial_x^s u\|_{L^4(\Omega_T)} \leq C\|h\|_{H^{\frac{1+s}{2}}(0,T)}$$

and

$$\sup_{0\leq t\leq T} \| \ \partial_x^s u\|_{L^2((0,1)} \leq C\|h\|_{H^{\frac{1+s}{2}}(0,T)}$$

where $C > 0$ is a constant independent of h.

Proof: We only need to prove it for $s = 2$. The cases where $s \in (0,2)$ can then be obtained by interpolation, where we note that $H_0^s(0,T)$ is an interpolation space for $s \neq$ integer $+1/2$ while for $s =$ integer $+1/2$, the corresponding interpolation space is the Lions-Magenes space $H_{00}^s(0,T)$ [59]. The proof for $s > 2$ is same as for $s = 2$.

Notice that the t-derivative of $W_h(\cdot)h$ satisfies the system (4.7) with boundary condition $h'(t)$ and zero initial condition. Hence, by Proposition 4.6, there obtains

$$\left\|\frac{\partial W_h(\cdot)h}{\partial t}\right\|_{L^4((0,1)\times[0,T])} \leq C\|h'(t)\|_{H^{\frac{1}{2}}(0,T)} \leq C\|h(t)\|_{H^{\frac{3}{2}}(0,T)}.$$

But, bounds on one t-derivative of $W_h(t)h$ give bounds on two x-derivatives of $W_h(t)h$. Thus, the case for $s = 2$ is established. □

Remark 4.8 *Notice that*

$$\|W_0(t)\phi\|_{L^4((0,1)\times(0,T))} \leq C\|\phi\|_{L^2(0,1)}$$

for any $\phi \in L^2(0,1)$ and, in addition, for the linear Schrödinger equation posed on the half-line,

$$\|W_b(\cdot)h\|_{L^q(\mathbb{R}^+;L^r(\mathbb{R}^+))} \leq C\|h\|_{H^{\frac{1}{4}}(\mathbb{R}^+)}$$

for any $h \in H^{\frac{1}{4}}(\mathbb{R}^+)$, where (q,r) is an admissible pair satisfying $\frac{1}{q} + \frac{1}{2r} = \frac{1}{4}$. One thus wonders whether the estimate (4.11) or (4.12) can be improved. Example A-2 in the Appendix shows that if $\|W_b(\cdot)h\|_{L^2([0,1]\times[0,T])} \leq C\|h\|_{H^s([0,T])}$ for all $h(t) \in H^s([0,T])$, then it must be the case that $s \geq \frac{1}{2}$. Thus, the estimates in (4.11) and (4.12) are optimal.

4.2 The nonlinear problem

In this subsection, the full nonlinear IBVP

$$\begin{cases} iu_t + u_{xx} + \lambda u|u|^{p-2} = 0, \qquad x \in (0,1),\ t \in \mathbb{R}^+, \\ \\ u(x,0) = \phi(x), \qquad u(0,t) = h_1(t), \quad u(1,t) = h_2(t) \end{cases} \tag{4.13}$$

with $\phi \in H^s(0,1)$ and $h_1,\ h_2 \in H_{loc}^{\frac{s+1}{2}}(\mathbb{R}^+)$ is studied. A local well-posedness theorem is formulated and proved.

Theorem 4.9 *Let $3 \le p < \infty$, $\frac{1}{2} < s < \frac{5}{2}$ and $\lfloor s \rfloor < p-2$, $T > 0$ and $r > 0$ be given. There exists a $T^* > 0$ such that if $(\phi, h_1, h_2) \in \mathcal{X}_{s,T} := H^s(0,1) \times H^{\frac{s+1}{2}}(0,T) \times H^{\frac{s+1}{2}}(0,T)$ satisfies $h_1(0) = \phi(0), h_2(0) = \phi(1)$ and $\|(\phi, h_1, h_2)\|_{\mathcal{X}_{s,T}} \le r$, the IBVP (4.13) admits a unique solution $u \in C([0,T^*]; H^s(0,1))$. Moreover, the solution u depends on (ϕ, h_1, h_2) continuously in the corresponding spaces.*

Proof: We only consider the cases where $\frac{1}{2} < s \le 2$. In addition, without loss of generality, we assume that $\phi(0) = h_1(0) = 0$ and $\phi(1) = h_2(0) = 0$. For if not, we can homogenize the boundary conditions by writing $u = v + h_1(0)(1-x) + h_2(0)x = v + \gamma(x)$. Then v satisfies homogeneous compatibility conditions and the equation

$$iv_t + v_{xx} + \lambda |v+\gamma|^{p-2}(v+\gamma) = 0.$$

As γ is smooth and the direct estimates made of the nonlinear term, *e.g.* (4.14), are very simple, theory for either u or v follows exactly the same lines.

For $s > \frac{1}{2}$, $H^s(0,1)$ is a Banach algebra. It follows that there is a constant $C = C_s$ such that

$$\big\| v|v|^{p-2} \big\|_{H^s(0,1)} \le C \big\| v \big\|^{p-1}_{H^s(0,1)}, \tag{4.14}$$

when $s = 1, 2$. Indeed, for any s with $\lfloor s \rfloor < p-2$, the chain rule for fractional derivatives implies the same result.

For any θ with $0 < \theta \le T$ and $v \in C([0,\theta]; H^s(0,1))$, Propositions 4.2 and 4.7 imply that the linear IBVP

$$\begin{cases} iu_t + u_{xx} + \lambda v|v|^{p-2} = 0, & x \in (0,1),\ t \in \mathbb{R}, \\ u(x,0) = \phi(x), \quad u(0,t) = h_1(t), & u(1,t) = h_2(t)\,, \end{cases} \tag{4.15}$$

admits a unique solution $u \in C([0,\theta]; H^s(0,1))$. Moreover, there exists a constant $C > 0$ independent of θ such that

$$\|u\|_{C([0,\theta];H^s(0,1))} \le C\|(\phi, h_1, h_2)\|_{\mathcal{X}_{s,T}} + C\theta \|v\|^{p-1}_{C([0,\theta];H^s(0,1))}.$$

Thus, for any given $(\phi, h_1, h_2) \in \mathcal{X}_{s,T}$, the IBVP (4.15) defines a nonlinear map Γ from $Y_{s,\theta} := \{w \in C([0,\theta]; H^s(0,1))\}$ to $Y_{s,\theta}$. A well understood argument, similar to the contraction mapping argument in Section 7 of [45] using the chain rule, now reveals that if $\theta > 0$ is chosen small enough, there exists an $M > 0$ such that

$$\|\Gamma(v_0)\|_{C([0,\theta];H^s(0,1))} \le M$$

and

$$\|\Gamma(v_1) - \Gamma(v_2)\|_{C([0,\theta];H^s(0,1))} \le \frac{1}{2}\|v_1 - v_2\|_{C([0,\theta];H^s(0,1))}$$

for any $v_0,\ v_1,\ v_2 \in C([0,\theta]; H^s(0,1))$ with

$$\|v_j\|_{C([0,\theta];H^s(0,1))} \le M, \qquad j = 0,1,2.$$

Hence, the map Γ is a contraction whose unique fixed point is the desired solution u of (4.13). The proof is complete. □

Next, we aim to show the well-posedness of the IBVP (4.13) in $H^s(0,1)$ for $0 \leq s < \frac{1}{2}$. To this end, consider the integral equation

$$u(\cdot,t) = W_0(t)\phi + W_h h_1 + (W_h h_2)\Big|_{x\to 1-x} + i\lambda \int_0^t W_0(t-\tau)\Big(u(\cdot,\tau)|u(\cdot,\tau)|^{p-2}\Big)d\tau\,, \qquad (4.16)$$

associated with the IBVP (4.13).

Proposition 4.10 *Let $0 \leq s < \frac{1}{2}$ and $T > 0$. Suppose $r > 0$ to be given and $3 \leq p \leq 4$. There exists a $T^* = T^*(r) > 0$ such that for any*

$$(\phi, h_1, h_2) \in \mathcal{X}_{s,T}$$

with $\|(\phi, h_1, h_2)\|_{\mathcal{X}_{s,T}} \leq r$, (4.16) admits a unique solution

$$u \in \mathcal{Y}_{s,T^*} := L^4((0,1)\times(0,T^*)) \cap C([0,T^*]; H^s(0,1))$$

which depends continuously on (ϕ, h_1, h_2) in the corresponding spaces.

Proof: Solving (4.16) can be viewed as a problem of finding a fixed point of a nonlinear operator. Consequently, the proposition follows using the argument that appears already in [16] along with our Proposition 4.6 for the boundary integrals. □

The solution u of (4.16) given by Proposition 4.10 is a mild solution of the IBVP (4.13). By the same arguments as put forward already in the proofs of Propositions 3.18 and 3.19, it is deduced that the IBVP (4.13) admits at most one mild solution, thereby settling the validity of the following theorem.

Theorem 4.11 *Under the conditions in Proposition 4.10, the IBVP (4.13) is unconditionally locally well-posed in $H^s(0,1)$ for $0 \leq s < \frac{1}{2}$.*

5 Global Well-Posedness

In this section, consideration is given to the issue of global well-posedness for both the problems,

$$\begin{cases} iu_t + u_{xx} + \lambda u|u|^{p-2} = 0, & x \in \mathbb{R}^+,\ t \in \mathbb{R}, \\ u(x,0) = \phi(x), \qquad u(0,t) = h(t) \end{cases} \qquad (5.1)$$

and

$$\begin{cases} iu_t + u_{xx} + \lambda u|u|^{p-2} = 0, & x \in (0,1),\ t \in \mathbb{R}, \\ u(x,0) = \phi(x), \qquad u(0,t) = h_1(t), \quad u(1,t) = h_2(t) \end{cases} \qquad (5.2)$$

in $H^s(\mathbb{R}^+)$ and $H^s(0,1)$, respectively. Since the local well-posedness of both problems has been established, global well-posedness will follow from suitable *a-priori* estimates.

First, recall that if $u(x,t)$ is a smooth solution of the NLS equation

$$iu_t + u_{xx} + \lambda u|u|^{p-2} = 0,$$

then the following identities

$$\frac{\partial}{\partial t}(|u|^2) = -2\,\mathrm{Im}(u_x(x,t)\bar{u}(x,t))_x\,, \tag{5.3}$$

$$\frac{\partial}{\partial t}\left(|u_x|^2 - \frac{2\lambda}{p}|u|^p\right) = 2\,\mathrm{Re}\left(u_x(x,t)\bar{u}_t(x,t)\right)_x\,, \tag{5.4}$$

$$\left(|u_x(x,t)|^2 + \frac{2\lambda}{p}|u(x,t)|^p\right)_x = -i\left(\frac{\partial}{\partial t}(u\bar{u}_x) - (u(x,t)\bar{u}_t(x,t))_x\right) \tag{5.5}$$

were obtained in [30]. Multiply both sides of (5.5) by a smooth, time-independent function $\eta(x)$ and write

$$u\bar{u}_t = u(-i\bar{u}_{xx} - i\lambda\bar{u}|u|^{p-2}) = -i((u\bar{u}_x)_x - u_x\bar{u}_x + \lambda|u|^p)$$

to derive the formula

$$\begin{aligned}\eta(x)\left(|u_x(x,t)|^2 + \frac{2\lambda}{p}|u(x,t)|^p\right)_x &= -i\left(\frac{\partial}{\partial t}(\eta(x)u\bar{u}_x)\right) + i(\eta u\bar{u}_t)_x \\ &\quad - (\eta_x u\bar{u}_x)_x + \eta_{xx}(u\bar{u}_x) + \eta_x|u_x|^2 - \lambda\eta_x|u|^p\,. \end{aligned} \tag{5.6}$$

By choosing appropriate functions $\eta(x)$, one can obtain various pointwise estimates of $u(x,t)$. In particular, for any given interval $[a,b]$, choose $\eta(x)\in C^\infty(\mathbb{R})$ such that $\eta = 1$ for $x \le a$ and $\eta = 0$ for $x \ge b$ with $|\eta(x)| \le 1$ for all x. Integrating (5.6) from a to b with respect to x and integrating by parts yields

$$\begin{aligned}&-|u_x(a,t)|^2 - \frac{2\lambda}{p}|u(a,t)|^p - \int_a^b \eta_x(x)\left(|u_x(x,t)|^2 + \frac{2\lambda}{p}|u(x,t)|^p\right)dx \\ &= -i\frac{\partial}{\partial t}\left(\int_a^b \eta(x)u\bar{u}_x dx\right) - i\Big(\eta(a)u(a,t)\bar{u}_t(a,t)\Big) + \int_a^b \Big(\eta_{xx}(u\bar{u}_x) + \eta_x|u_x|^2 - \lambda\eta_x|u|^p\Big)dx\,.\end{aligned}$$

If $v = u_t$, then v satisfies the equation

$$iv_t + v_{xx} + (\lambda p/2)|u|^{p-2}v + (\lambda(p-2)/2)|u|^{p-4}u^2\bar{v} = 0\,, \tag{5.7}$$

which is linear in terms of v. Similar identities as (5.3)-(5.6) hold for v;

$$\frac{\partial}{\partial t}(|v|^2) = -2\,\mathrm{Im}\big(v_x(x,t)\bar{v}(x,t)\big)_x - \lambda(p-2)|u|^{p-4}\,\mathrm{Im}\big(u^2\bar{v}^2\big)\,, \tag{5.8}$$

$$\frac{\partial}{\partial t}|v_x|^2 = 2\,\mathrm{Re}\left(v_x(x,t)\bar{v}_t(x,t)\right)_x + \mathrm{Im}\big(\lambda p|u|^{p-2}v\bar{v}_{xx}\big) + \mathrm{Im}\big(\lambda(p-2)|u|^{p-4}u^2\bar{v}\bar{v}_{xx}\big)\,, \tag{5.9}$$

$$\begin{aligned}\big(|v_x(x,t)|^2\big)_x &= -i\left(\frac{\partial}{\partial t}(v\bar{v}_x) - (v(x,t)\bar{v}_t(x,t))_x\right) \\ &\quad - \lambda(p/2)|u|^{p-2}(|v|^2)_x - \lambda(p-2)|u|^{p-4}\,\mathrm{Re}\big(u^2\bar{v}\bar{v}_x\big)\,.\end{aligned} \tag{5.10}$$

These identities will play a role in our study of global well-posedness. The quarter-plane IBVP (5.1) will be considered next while the IBVP (5.2) will be dealt with in Subsection 5.2.

5.1 Global well-posedness on $\mathbb{R}^+$

Proposition 5.1 *Assume that $p \geq 2$ if $\lambda < 0$ and $2 \leq p \leq 4$ if $\lambda > 0$. Let $T > 0$ be given. Then there exists a nondecreasing, continuous function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ with $\alpha(0) = 0$ such that any smooth solution u of (5.1) satisfies*

$$\sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{H^1(R^+)} \leq \alpha\Big(\|\phi\|_{H^1(R^+)} + \|h\|_{H^1(0,T)}\Big). \tag{5.11}$$

Here α also depends upon T and other constants and is bounded for any $T > 0$.

Remark 5.2 *The calculations to follow can easily be justified for solutions that are in $H^2(\mathbb{R}^+)$ in space with boundary traces that are continuous functions of time. Note that this result does not depend upon how the solution is obtained, but simply asserts a priori information that it must obey.*

Proof: First, integrate (5.6) with $\eta = 1$ from 0 to t to obtain

$$\begin{aligned}
\int_0^t |u_x(0,s)|^2 ds =& i\left(\int_0^\infty u\bar{u}_x dx\right)\Big|_0^t - \frac{2\lambda}{p}\int_0^t |u(0,s)|^p ds + i\int_0^t u(0,s)\bar{u}_t(0,s) ds \\
=& i\left(\int_0^\infty u(x,t)\bar{u}_x(x,t)dx\right) - i\left(\int_0^\infty u(x,0)\bar{u}_x(x,0)dx\right) + C_1(t) \\
=& i\left(\int_0^\infty u(x,t)\bar{u}_x(x,t)dx\right) + c_1 + C_1(t) \\
\leq& \left(\int_0^\infty |u(x,t)|^2 dx\right)^{1/2}\left(\int_0^\infty |u_x(x,t)|^2 dx\right)^{1/2} + c_1 + C_1(t)\,,
\end{aligned} \tag{5.12}$$

where c_1 is dependent on the initial data and $C_1(t)$ is dependent on the boundary data with $C_1(0) = 0$. It follows from (5.3) that

$$\begin{aligned}
\int_0^\infty |u(x,t)|^2 dx =& \int_0^\infty |u(x,0)|^2 dx + 2\,\mathrm{Im}\int_0^t (u_x(0,s)\bar{u}(0,s)) ds \\
\leq& c_1 + 2\left(\int_0^t |u_x(0,s)|^2 ds \int_0^t |u(0,s)|^2 ds\right)^{1/2} \\
=& c_1 + 2C_1(t)\left(\int_0^t |u_x(0,s)|^2 ds\right)^{1/2} \\
\leq& c_1 + 2C_1(t)\left(\left(\int_0^\infty |u(x,t)|^2 dx\right)^{1/2}\left(\int_0^\infty |u_x(x,t)|^2 dx\right)^{1/2} + c_1 + C_1(t)\right)^{1/2} \\
\leq& c_1 + 2C_1(t)\left(\left(\int_0^\infty |u(x,t)|^2 dx\right)^{1/4}\left(\int_0^\infty |u_x(x,t)|^2 dx\right)^{1/4} + (c_1 + C_1(t))^{1/2}\right) \\
\leq& c_1 + 2C_1(t)(c_1 + C_1(t))^{1/2} + \frac{1}{4}\int_0^\infty |u(x,t)|^2 dx \\
&+ \frac{3}{4}\left(2C_1(t)\left(\int_0^\infty |u_x(x,t)|^2 dx\right)^{1/4}\right)^{4/3}.
\end{aligned}$$

A direct consequence is the inequality

$$\int_0^\infty |u(x,t)|^2 dx \le \frac{4}{3}\big(c_1 + 2C_1(t)\big(c_1 + C_1(t)\big)^{1/2}\big) + (2C_1(t))^{4/3}\left(\int_0^\infty |u_x(x,t)|^2 dx\right)^{1/3}$$
$$= D_1 + (2C_1(t))^{4/3}\left(\int_0^\infty |u_x(x,t)|^2 dx\right)^{1/3} \tag{5.13}$$

where D_1 is a constant depending on both the initial and boundary data.

If $\lambda < 0$, integrating both sides of (5.4) in x over $\mathbb{R}^+$ and t over $[0,t]$ yields

$$\int_0^\infty |u_x(x,t)|^2 dx = \frac{2\lambda}{p}\int_0^\infty |u(x,t)|^p dx + \int_0^\infty \left(|u_x(x,0)|^2 - \frac{2\lambda}{p}|u(x,0)|^p\right) dx$$
$$- 2\,\mathrm{Re}\int_0^t u_x(0,s)\bar{u}_s(0,s) ds\,.$$

The right-hand side of this equation may be bounded thusly (note that the first term is negative and the second term only depends on initial data):

$$\begin{aligned}
\text{rhs} &\le c_1 + \int_0^t |u_x(0,s)|^2 ds + \int_0^t |u_s(0,s)|^2 ds = D_1 + \int_0^t |u_x(0,s)|^2 ds \\
&\le D_1 + \left(\int_0^\infty |u(x,t)|^2 dx\right)^{1/2}\left(\int_0^\infty |u_x(x,t)|^2 dx\right)^{1/2} + c_1 + C_1(t) \\
&\le D_1 + c_1 + C_1(t) + \left(D_1 + \big(2C_1(t)\big)^{4/3}\left(\int_0^\infty |u_x(x,t)|^2 dx\right)^{1/3}\right)^{1/2}\left(\int_0^\infty |u_x(x,t)|^2 dx\right)^{1/2} \\
&\le D_1 + c_1 + C_1(t) + \left(D_1^{1/2} + \big(2C_1(t)\big)^{2/3}\left(\int_0^\infty |u_x(x,t)|^2 dx\right)^{1/6}\right)\left(\int_0^\infty |u_x(x,t)|^2 dx\right)^{1/2} \\
&\le D_1 + c_1 + C_1(t) + D_1^{1/2}\|u_x(\cdot,t)\|_{L^2} + \big(2C_1(t)\big)^{2/3}\|u_x(\cdot,t)\|_{L^2}^{4/3},
\end{aligned}$$

where c_1 again depends only on the initial data and D_1 depends on initial and boundary data. Here, $2ab \le a^2 + b^2$ is used for the first inequality, (5.12) is applied for the second inequality, the third inequality is from (5.13), and the fourth inequality uses the inequality $(a+b)^{1/2} \le a^{1/2} + b^{1/2}$. Hence, over any finite time interval, $\|u_x(\cdot,t)\|_{L^2(\mathbb{R}^+)}$ is uniformly bounded. Appealing to (5.13) again reveals that $\|u(\cdot,t)\|_{L^2(\mathbb{R}^+)}$ is also bounded for any bounded time interval.

If $\lambda > 0$, equation (5.4) implies

$$\begin{aligned}
\int_0^\infty |u_x(x,t)|^2 dx &= \frac{2\lambda}{p}\int_0^\infty |u(x,t)|^p dx + \int_0^\infty \left(|u_x(x,0)|^2 - \frac{2\lambda}{p}|u(x,0)|^p\right) dx \\
&\qquad - 2\,\mathrm{Re}\int_0^t u_x(0,s)\bar{u}_s(0,s) ds \\
&\le \frac{4\lambda}{p}\Big(\|u(\cdot,t)\|_{L^2}\|u_x(\cdot,t)\|_{L^2}\Big)^{(p-2)/2}\int_0^\infty |u(x,t)|^2 dx + c_2 + \int_0^t |u_x(0,s)|^2 ds + \int_0^t |u_s(0,s)|^2 ds \\
&= \frac{4\lambda}{p}\Big(\|u(\cdot,t)\|_{L^2}\|u_x(\cdot,t)\|_{L^2}\Big)^{(p-2)/2}\int_0^\infty |u(x,t)|^2 dx + \tilde{D} + \int_0^t |u_x(0,s)|^2 ds \\
&\le \tilde{D} + \frac{4\lambda}{p}\|u_x(\cdot,t)\|_{L^2}^{(p-2)/2}\left(D_1 + \big(2C_1(t)\big)^{4/3}\left(\int_0^\infty |u_x(x,t)|^2 dx\right)^{1/3}\right)^{(p+2)/4}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\int_0^\infty |u(x,t)|^2dx\right)^{1/2}\left(\int_0^\infty |u_x(x,t)|^2dx\right)^{1/2}+c_1+C_1(t)\\
\le\,& \tilde{D}+c_1+C_1(t)+\frac{4\lambda}{p}\|u_x(\cdot,t)\|_{L^2}^{(p-2)/2}\left(D_1+\big(2C_1(t)\big)^{4/3}\left(\int_0^\infty |u_x(x,t)|^2dx\right)^{1/3}\right)^{(p+2)/4}\\
&+\left(D_1+\big(2C_1(t)\big)^{4/3}\left(\int_0^\infty |u_x(x,t)|^2dx\right)^{1/3}\right)^{1/2}\left(\int_0^\infty |u_x(x,t)|^2dx\right)^{1/2}\\
\le\,& \tilde{D}+c_1+C_1(t)+\frac{2^{(p+6)/4}\lambda}{p}\|u_x(\cdot,t)\|_{L^2}^{(p-2)/2}\Bigg(D_1^{(p+2)/4}\\
&+(2C_1(t))^{(p+2)/3}\left(\int_0^\infty |u_x(x,t)|^2dx\right)^{(p+2)/12}\Bigg)\\
&+\left(D_1^{1/2}+\big(2C_1(t)\big)^{2/3}\left(\int_0^\infty |u_x(x,t)|^2dx\right)^{1/6}\right)\left(\int_0^\infty |u_x(x,t)|^2dx\right)^{1/2}\\
=\,& \tilde{D}+c_1+C_1(t)+\frac{2^{(p+6)/4}\lambda\,(2C_1(t))^{(p+2)/3}}{p}\|u_x(\cdot,t)\|_{L^2}^{2(p-1)/3}\\
&+\frac{2^{(p+6)/4}\lambda D_1^{(p+2)/4}}{p}\|u_x(\cdot,t)\|_{L^2}^{(p-2)/2}+D_1^{1/2}\|u_x(\cdot,t)\|_{L^2}+(2C_1(t))^{2/3}\,\|u_x(\cdot,t)\|_{L^2}^{4/3}\,,
\end{aligned}
$$

where the first inequality is derived from the fact that $H^1(\mathbb{R})$ is embedded in $L^\infty(\mathbb{R})$. The second and third steps in the last chain of inequalities follow from (5.12) and (5.13) whilst the last step is a consequence of the elementary fact that if $a,b\ge 0$, then $(a+b)^m\le 2^{m-1}(a^m+b^m)$ when $m\ge 1$. When $p<4$, $2(p-1)/3<2$, so, for any finite time interval, $\|u_x(\cdot,t)\|_{L^2}$ is uniformly bounded. It follows again from (5.13) that $\|u(\cdot,t)\|_{L^2}$ is likewise bounded on bounded time intervals.

Suppose $p=4$ and let $\delta>0$ be given, to be specified presently. Then it follows that

$$
\begin{aligned}
&\left(1-2^{5/2}\lambda C_0^2(t)\right)\|u_x(\cdot,t)\|_{L^2}^2\le c_2+c_1+C_1(t)+2^{1/2}\lambda D_0^{3/2}\|u_x(\cdot,t)\|_{L^2}\\
&\qquad+D_0^{1/2}\|u_x(\cdot,t)\|_{L^2}+(2C_0(t))^{2/3}\,\|u_x(\cdot,t)\|_{L^2}^{4/3}\\
&=D_1+D_2\|u_x(\cdot,t)\|_{L^2}+D_3\|u_x(\cdot,t)\|_{L^2}^{4/3}\\
&\le D_1+\frac{1}{8}\|u_x(\cdot,t)\|_{L^2}^2+2D_2^2+\frac{1}{3}\left(\frac{D_3}{\delta}\right)^3+\frac{2}{3}\delta^{3/2}\|u_x(\cdot,t)\|_{L^2}^2\,.
\end{aligned}
$$

Determine δ by demanding $\frac{2}{3}\delta^{3/2}=\frac{1}{8}$ so that

$$
\left(\frac{3}{4}-2^{5/2}\lambda C_0^2(t)\right)\|u_x(\cdot,t)\|_{L^2}^2\le D_1+2D_2^2+\frac{1}{3}\left(\frac{D_3}{\delta}\right)^3,
$$

where the right-hand side only depends on the initial and boundary data. Since $C_0^2(t)=\int_0^t|u(0,s)|^2ds$, choose t_1 small so that $2^{5/2}\lambda C_0^2(t_1)\le 1/4$. With such a choice, if $0<t\le t_1$, then

$$
\|u_x(\cdot,t)\|_{L^2}^2\le 2D_1+4D_2^2+\frac{2}{3}\left(\frac{D_3}{\delta}\right)^3.
$$

Use the solution at $t=t_1$ as the initial data and apply the same argument to extend the solution to $t_2>t_1$. Since $s\ge 1$ here, the boundary values lie at least in $H^1_{loc}(\mathbb{R}^+)$. Hence, given any

$T > 0$, there are positive values $\mu = \mu(T)$, say, such that $\int_t^{t+\mu} |u(0,s)|^2 ds$ can be made uniformly small for all $t \in [0,T]$. Hence, the argument just presented can be iterated at least out to time T. As T was arbitrary, the proof is complete. $\square$

Theorem 5.3 *Let $1 \le s < \frac{5}{2}$ be given and assume that*

$$p \ge 2 \quad \text{if} \quad \lambda < 0 \quad \text{or} \quad 2 \le p \le 4 \quad \text{if} \quad \lambda > 0.$$

Then, the IBVP (5.1) is globally well-posed in $H^s(\mathbb{R}^+)$ for $\phi \in H^s(\mathbb{R}^+)$ with $h \in H_{loc}^{\frac{s+3}{4}}(\mathbb{R}^+)$ if $1 \le s \le 2$ and $h \in H_{loc}^{\frac{2s+1}{4}}(\mathbb{R}^+)$ if $2 \le s < \frac{5}{2}$.

Proof: In (5.1), assume that $\phi(x) \in H^2(R^+)$ and $h(t) \in H^{\frac{5}{4}}(0,T)$ satisfy the compatibility condition $\phi(0) = h(0)$. Proposition 5.1 implies the global existence of the solution u which lies in $C([0,T], H^1(R^+))$, for any $T > 0$. Let $T > 0$ be fixed, but arbitrary. To prove the existence in $C([0,T], H^2(R^+))$, take the derivative of (5.1) with respect to t to obtain (5.7) where $v = u_t$. The initial and boundary conditions for v are

$$v(x,0) = i(\phi_{xx} + \lambda\phi|\phi|^{p-2}) = \phi_1(x), \quad v(0,t) = h'(t) = h_1(t)\,.$$

Note that (5.7) is linear in terms of v. Let $v = w + z$ be such that z satisfies

$$iz_t + z_{xx} = 0, \quad z(x,0) = 0, \quad z(0,t) = h_1(t)$$

and w solves

$$\begin{aligned} iw_t + w_{xx} &+ (\lambda p/2)|u|^{p-2}(w+z) + (\lambda(p-2)/2)|u|^{p-4}u^2\overline{(w+z)} = 0\,, \\ w(x,0) &= \phi_1(x), \qquad w(0,t) = 0\,. \end{aligned}$$

From (3.21), for $s = 0$ and any $T > 0$,

$$\sup_{0<t<T} \|z\|_{L^2(\mathbb{R})} = \sup_{0<t<T} \|W_{bdr}(\cdot)h_1\|_{L^2(\mathbb{R})} \le C(T)\|h_1\|_{H^{\frac{1}{4}}(\mathbb{R}^+)}\,.$$

A similar identity as appears in (5.8) applied to w gives

$$\frac{\partial}{\partial t}(|w|^2) = -2\,\mathrm{Im}\big(w_x(x,t)\bar{w}(x,t)\big)_x - \lambda p|u|^{p-2}\,\mathrm{Im}(z\bar{w}) - \lambda(p-2)|u|^{p-4}\,\mathrm{Im}\big(u^2\overline{(w+z)}\bar{w}\big)\,.$$

Integrating this over the half line yields

$$\begin{aligned} \frac{d}{dt}\int_0^\infty |w|^2 dx \le& |\lambda|p\|u\|_{H^1(\mathbb{R}^+)}^{p-2}\|z\|_{L^2(\mathbb{R}^+)}\|w\|_{L^2(\mathbb{R}^+)} \\ &+ |\lambda|(p-2)\|u\|_{H^1(\mathbb{R}^+)}^{p-2}\big(\|w\|_{L^2(\mathbb{R}^+)}^2 + \|z\|_{L^2(\mathbb{R}^+)}\|w\|_{L^2(\mathbb{R}^+)}\big), \end{aligned}$$

or, what is the same,

$$\begin{aligned} \frac{d}{dt}\|w\|_{L^2(\mathbb{R}^+)} \le& (1/2)|\lambda|p\|u\|_{H^1(\mathbb{R}^+)}^{p-2}\|z\|_{L^2(\mathbb{R}^+)} \\ &+ (1/2)|\lambda|(p-2)\|u\|_{H^1(\mathbb{R}^+)}^{p-2}\big(\|w\|_{L^2(\mathbb{R}^+)} + \|z\|_{L^2(\mathbb{R}^+)}\big). \end{aligned}$$

This in turn implies by way of Gronwall's Lemma that $\|w(\cdot,t)\|_{C([0,T];L^2(\mathbb{R}^+)}$ is bounded. The inequality (5.11) in 5.1 implies that $\|u\|_{C([0,T];H^1(\mathbb{R}^+))}$ is bounded by $\alpha_0\left(\|\phi\|_{H^1(R^+)}+\|h\|_{H^1(0,T)}\right)$ for some function α_0. By combining the foregoing inequalities, there obtains

$$\sup_{0\le t\le T}\|v(\cdot,t)\|_{L^2(R^+)}\le\alpha_0\left(\|\phi\|_{H^1(R^+)}+\|h\|_{H^1(0,T)}\right)\left(\|\phi_1\|_{L^2(R^+)}+\|h_1\|_{H^{\frac14}(0,T)}\right),$$

where $\alpha_0:\mathbb{R}^+\to\mathbb{R}^+$ with $\alpha_0(0)=0$ is a nondecreasing, continuous function which may depend upon T as well. Thus, (5.1) implies

$$\sup_{0\le t\le T}\|u_{xx}(\cdot,t)\|_{L^2(R^+)}\le\alpha_0\left(\|\phi\|_{H^1(R^+)}+\|h\|_{H^1(0,T)}\right)\left(\|\phi\|_{H^2(R^+)}+\|h\|_{H^{\frac54}(0,T)}\right),$$

or

$$\sup_{0\le t\le T}\|u(\cdot,t)\|_{H^2(R^+)}\le\alpha_0\left(\|\phi\|_{H^1(R^+)}+\|h\|_{H^1(0,T)}\right)\left(\|\phi\|_{H^2(R^+)}+\|h\|_{H^{\frac54}(0,T)}\right).$$

By the local existence theory presented in Section 4 subject to the compatibility condition $\phi(0)=h(0)$ (see Proposition 3.13 and Remark 3.14), nonlinear interpolation theory applied for s in the range $1<s<2$ yields the desired result for this range of s [4] (for details, see [13] in the context of the Korteweg-de Vries equation). As $T>0$ was arbitrary, this in turn implies that the theorem holds for $1\le s\le 2$.

Now suppose that $2\le s\le 4$. First, assume $\phi(x)\in H^4(R^+)$ and $h\in H^{\frac94}(0,T)$. Take the derivative of (5.7) with respect to t and let $v_t=v_1=u_{tt}$. Then, the equation for v_1 is linear in v_1 with nonhomogeneous terms that are globally defined. The initial and boundary conditions for v_1 are

$$\begin{aligned}&v_1(x,0)=i\phi_1''+i(\lambda p/2)|\phi|^{p-2}\phi_1+i(\lambda(p-2)/2)|\phi|^{p-4}\phi^2\bar{\phi}_1=\phi_2(x)\in L^2(R^+),\\&v_1(0,t)=h''(t)=h_1'(t)=h_2(t)\in H^{\frac14}(0,T).\end{aligned}$$

A similar argument as that applied to $v(x,t)=u_t(x,t)$ shows that $\sup_{0<t<T}\|v_1(\cdot,t)\|_{L^2(\mathbb{R}^+)}$ is bounded for any $T>0$. Therefore, $\sup_{0<t<T}\|v(\cdot,t)\|_{H^2(\mathbb{R}^+)}$ is bounded. Now, consider u in (5.7) as a fixed function in $C([0,T];H^1(\mathbb{R}^+))$ and $\phi_1(x),h_1(t)$ as functions unrelated to $\phi(x),h(x)$. Then, by the above argument, if $\phi_1(x)\in L^2(\mathbb{R}^+),h_1(t)\in H^{\frac14}(\mathbb{R}^+)$, then $v(x,t)\in C([0,T];L^2(\mathbb{R}^+))$, while if $\phi_1(x)\in H^2(\mathbb{R}^+),h_1(t)\in H^{\frac54}(\mathbb{R}^+)$, then $v(x,t)\in C([0,T];H^2(\mathbb{R}^+))$. This uses only the simple compatibility condition $\phi_1(0)=h_1(0)$. The usual nonlinear interpolation theory applied to v with s in the range $0<s<2$ gives the desired result for v with $0\le s\le 2$. This immediately implies the advertised result for u with $2\le s\le 4$. If p is an even integer or p is large, this argument can be continued for higher values of s (see a similar and detailed argument for the KdV equation in a quarter plane [8]). $\square$

[4] Here, the following interpolation result has been used. Its proof is presented in Appendix 2. Let

$$X:=\left\{(\phi,h)\in H^1(\mathbb{R}^+)\times H^{\frac34}(\mathbb{R}^+);\ \phi(0=h(0)\right\},\qquad Y:=\left\{(\phi,h)\in H^2(\mathbb{R}^+)\times H^{\frac54}(\mathbb{R}^+);\quad \phi(0=h(0)\right\}.$$

Then, for any θ with $0\le\theta\le1$,

$$[X,Y]_\theta=\left\{(\phi,h)\in H^{1+\theta}(\mathbb{R}^+)\times H^{\frac{2\theta+3}{4}}(0,T);\ \phi(0)=h(0)\right\}$$

5.2 Global well-posedness on $(0,1)$

Proposition 5.4 *Assume that $p \geq 2$ if $\lambda < 0$ and $2 \leq p \leq \frac{10}{3}$ if $\lambda > 0$. Let $T > 0$ be given. Then there exists a nondecreasing continuous function $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\beta(0) = 0$ such that any smooth solution u of (5.2) satisfies*

$$\sup_{0\leq t\leq T} \|u(\cdot,t)\|_{H^1(0,1)} \leq \beta\Big(\|\phi\|_{H^1(0,1)} + \|h_1\|_{H^1(0,T)} + \|h_2\|_{H^1(0,T)}\Big).$$

Proof: Let $\eta(x) = x - (1/2)$ in (5.6) and integrate with respect to x from 0 to 1 to obtain

$$\begin{aligned}
&\frac{1}{2}\left(|u_x(1,t)|^2 + \frac{2\lambda}{p}|u(1,t)|^p + |u_x(0,t)|^2 + \frac{2\lambda}{p}|u(0,t)|^p\right) \\
&= -i\int_0^1 \left(\frac{d}{dt}((x-(1/2))u\bar{u}_x)\right) dx + i(1/2)(u(1,t)\bar{u}_t(1,t) + u(0,t)\bar{u}_t(0,t)) \\
&- (u(1,t)\bar{u}_x(1,t) - u(0,t)\bar{u}_x(0,t)) + \int_0^1 \left(2|u_x(x,t)|^2 - \lambda(1-(2/p))|u(x,t)|^p\right) dx\,. \qquad (5.14)
\end{aligned}$$

In the following, we again use D as a constant dependent on the initial and boundary data, c as a constant only dependent on the initial data and $C(t)$ as a constant only dependent on the boundary data, while C is just a fixed constant, independent of the initial and boundary data. Integrate (5.14) with respect to t from 0 to t to derive

$$\begin{aligned}
&\int_0^t \left(|u_x(1,s)|^2 + |u_x(0,s)|^2\right) ds = D_0 + 2\int_0^t\int_0^1 \left(2|u_x(x,s)|^2 - \lambda(1-(2/p))|u(x,s)|^p\right) dxds \\
&\quad - 2i\int_0^1 ((x-(1/2))u(x,t)\bar{u}_x(x,t))dx - 2\int_0^t \Big(u(1,s)\bar{u}_x(1,s) - u(0,s)\bar{u}_x(0,s)\Big)ds\,. \qquad (5.15)
\end{aligned}$$

Consider the cases $\lambda > 0$ and $\lambda < 0$ separately.

(a) $\lambda < 0$

For this case, (5.15) gives

$$\begin{aligned}
&\int_0^t \left(|u_x(1,s)|^2 + |u_x(0,s)|^2\right) ds \leq D_0 + C\int_0^t\int_0^1 \left(|u_x(x,s)|^2 + |u(x,s)|^p\right) dxds \\
&\quad + \int_0^1 |u(x,t)\bar{u}_x(x,t))|\, dx + 2\int_0^t \Big|u(1,s)\bar{u}_x(1,s) - u(0,s)\bar{u}_x(0,s)\Big|ds \\
&\leq D_1 + C\int_0^t\int_0^1 \left(|u_x(x,s)|^2 + |u(x,s)|^p\right) dxds \\
&\quad + \left(\int_0^1 |u(x,t)|^2 dx\right)^{1/2}\left(\int_0^1 |u_x(x,t)|^2 dx\right)^{1/2} + \frac{1}{2}\int_0^t \Big(|u_x(1,s)|^2 + |u_x(0,s)|^2\Big)ds\,,
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_0^t \left(|u_x(1,s)|^2 + |u_x(0,s)|^2\right) ds \leq 2D_1 + 2C\int_0^t\int_0^1 \left(|u_x(x,s)|^2 + |u(x,s)|^p\right) dxds \\
&\quad + 2\left(\int_0^1 |u(x,t)|^2 dx\right)^{1/2}\left(\int_0^1 |u_x(x,t)|^2 dx\right)^{1/2}.
\end{aligned}$$

We use techniques that are by now familiar to obtain from (5.3) that

$$
\begin{aligned}
&\int_0^1 |u(x,t)|^2dx = \int_0^1 |u(x,0)|^2dx - 2\,\mathrm{Im}\int_0^t (u_x(1,s)\bar{u}(1,s) - u_x(0,s)\bar{u}(0,s))ds \\
&\leq \int_0^1 |u(x,0)|^2dx + 2\left(\int_0^t \left(|u(1,s)|^2 + |u(0,s)|^2\right) ds\right)^{1/2} \left(\int_0^t \left(|u_x(1,s)|^2 + |u_x(0,s)|^2\right) ds\right)^{1/2} \\
&\leq c_0 + 2C_0(t)\Bigg(2D_1 + 2C\int_0^t\int_0^1 \left(|u_x(x,s)|^2 + |u(x,s)|^p\right) dxds \\
&\qquad + 2\left(\int_0^1 |u(x,t)|^2dx\right)^{1/2}\left(\int_0^1 |u_x(x,t)|^2dx\right)^{1/2}\Bigg)^{1/2} \\
&\leq c_0 + 2\sqrt{2}C_0(t)\Bigg(D_1^{1/2} + \left(C\int_0^t\int_0^1 \left(|u_x(x,s)|^2 + |u(x,s)|^p\right) dxds\right)^{1/2} \\
&\qquad + \left(\int_0^1 |u(x,t)|^2dx\right)^{1/4}\left(\int_0^1 |u_x(x,t)|^2dx\right)^{1/4}\Bigg) \\
&\leq D_2 + (1/4)\int_0^1 |u(x,t)|^2dx + (3/4)\left(2\sqrt{2}C_0(t)\left(\int_0^1 |u_x(x,t)|^2dx\right)^{1/4}\right)^{4/3} \\
&\qquad + 2\sqrt{2}C_0(t)\left(C\int_0^t\int_0^1 \left(|u_x(x,s)|^2 + |u(x,s)|^p\right) dxds\right)^{1/2},
\end{aligned}
$$

or

$$
\begin{aligned}
\int_0^1 |u(x,t)|^2dx \leq D_3 + 4C_0^{4/3}(t)\left(\int_0^1 |u_x(x,t)|^2dx\right)^{1/3}& \\
+ 4C_0(t)\left(C\int_0^t\int_0^1 \left(|u_x(x,s)|^2 + |u(x,s)|^p\right) dxds\right)^{1/2}&.
\end{aligned}
$$

To obtain an estimate for u_x for the case $\lambda < 0$, integrate (5.4) with respect to x and t to reach

$$
\begin{aligned}
&\int_0^1 \left(|u_x(x,t)|^2 + \frac{2|\lambda|}{p}|u(x,t)|^p\right) dx \leq \int_0^1 \left(|u_x(x,0)|^2 + \frac{2|\lambda|}{p}|u(x,0)|^p\right) dx \\
&\qquad + 2\int_0^t \Big|u_x(1,s)\bar{u}_s(1,s) - u_x(0,s)\bar{u}_s(0,s)\Big|ds \\
&\leq D_4 + \int_0^t |u_x(1,s)|^2ds + \int_0^t |u_x(0,s)|^2ds \\
&\leq D_4 + 2D_1 + 2C\int_0^t\int_0^1 \left(|u_x(x,s)|^2 + |u(x,s)|^p\right) dxds \\
&\qquad + 2\left(\int_0^1 |u(x,t)|^2dx\right)^{1/2}\left(\int_0^1 |u_x(x,t)|^2dx\right)^{1/2} \\
&\leq D_4 + 2D_1 + 2C\int_0^t\int_0^1 \left(|u_x(x,s)|^2 + |u(x,s)|^p\right) dxds \\
&\qquad + 2\Bigg(D_3 + 4C_0^{4/3}(t)\left(\int_0^1 |u_x(x,t)|^2dx\right)^{1/3}
\end{aligned}
$$

$$
\begin{aligned}
&+ 4C_0(t) \left(C \int_0^t \int_0^1 \left(|u_x(x,s)|^2 + |u(x,s)|^p\right) dxds \right)^{1/2} \Bigg)^{1/2} \left(\int_0^1 |u_x(x,t)|^2 dx \right)^{1/2} \\
\le\ & D_5 + 2C \int_0^t \int_0^1 \left(|u_x(x,s)|^2 + |u(x,s)|^p\right) dxds \\
&+ 2D_3^{1/2} \left(\int_0^1 |u_x(x,t)|^2 dx \right)^{1/2} + 4C_0^{2/3}(t) \left(\int_0^1 |u_x(x,t)|^2 dx \right)^{2/3} \\
&+ 2(C_0(t))^{1/2} \left(C \int_0^t \int_0^1 \left(|u_x(x,s)|^2 + |u(x,s)|^p\right) dxds \right)^{1/4} \left(\int_0^1 |u_x(x,t)|^2 dx \right)^{1/2} \\
\le\ & D_6 + \frac{1}{2} \int_0^1 |u_x(x,t)|^2 dx + 3C \int_0^t \int_0^1 \left(|u_x(x,s)|^2 + |u(x,s)|^p\right) dxds \,,
\end{aligned}
$$

where use has been made of Young's inequality. It follows that

$$
\int_0^1 \Big(|u_x(x,t)|^2 + |u(x,t)|^p\Big) dx \le D_7 + C \int_0^t \int_0^1 \Big(|u_x(x,s)|^2 + |u(x,s)|^p\Big) dxds \,.
$$

for suitable constants. Gronwall's lemma then provides a global bound on the solution u in $H^1(0,1)$.

(b) $\lambda > 0$

From (5.15) with $p \ge 2$, it happens that

$$
\begin{aligned}
&\int_0^t \left(|u_x(1,s)|^2 + |u_x(0,s)|^2\right) ds + \int_0^t \int_0^1 \lambda\big(1 - (2/p)\big)|u(x,s)|^p dxds \\
&= D_0 + 4 \int_0^t \int_0^1 |u_x(x,s)|^2 dxds - 2i \int_0^1 \big(x - (1/2)\big) u(x,t) \bar{u}_x(x,t) dx \\
&\qquad - 2 \int_0^t \Big(u(1,s)\bar{u}_x(1,s) - u(0,s)\bar{u}_x(0,s)\Big) ds \\
&\le D_1 + C \int_0^t \int_0^1 |u_x(x,s)|^2 dxds + \left(\int_0^1 |u(x,t)|^2 dx \right)^{1/2} \left(\int_0^1 |u_x(x,t)|^2 dx \right)^{1/2} \\
&\qquad + \frac{1}{2} \int_0^t \Big(|u_x(1,s)|^2 + |u_x(0,s)|^2\Big) ds \,,
\end{aligned}
$$

which implies

$$
\begin{aligned}
\int_0^t \left(|u_x(1,s)|^2 + |u_x(0,s)|^2\right) ds &\le 2D_1 + 2C \int_0^t \int_0^1 |u_x(x,s)|^2 dxds \\
&\quad + 2 \left(\int_0^1 |u(x,t)|^2 dx \right)^{1/2} \left(\int_0^1 |u_x(x,t)|^2 dx \right)^{1/2} .
\end{aligned}
$$

By the same argument as in the case with $\lambda < 0$, it is seen that

$$
\int_0^1 |u(x,t)|^2 dx \le D_3 + 4C_0^{4/3}(t) \left(\int_0^1 |u_x(x,t)|^2 dx \right)^{1/3} + 4C_0(t) \left(C \int_0^t \int_0^1 |u_x(x,s)|^2 dxds \right)^{1/2} .
$$

The estimate for $\|u_x\|_{H^1(0,1)}$ can be obtained from (5.4) as follows:

$$
\int_0^1 |u_x(x,t)|^2 dx = \frac{2\lambda}{p} \int_0^1 |u(x,t)|^p dx + \int_0^1 \left(|u_x(x,0)|^2 - \frac{2\lambda}{p} |u(x,0)|^p \right) dx
$$

$$
\begin{aligned}
&\quad + 2\,\mathrm{Re}\int_0^t \left(u_x(1,s)\bar{u}_s(1,s) - u_x(0,s)\bar{u}_s(0,s)\right) ds \\
&\le \frac{4\lambda}{p}\Big(\|u(\cdot,t)\|_{L^2}\|u_x(\cdot,t)\|_{L^2}\Big)^{(p-2)/2}\int_0^1 |u(x,t)|^2dx + D_4 + \int_0^t |u_x(1,s)|^2ds + \int_0^t |u_x(0,s)|^2ds \\
&\le D_4 + \frac{4\lambda}{p}\|u_x(\cdot,t)\|_{L^2}^{(p-2)/2}\Bigg[D_3 + 4C_0^{4/3}(t)\left(\int_0^1 |u_x(x,t)|^2dx\right)^{1/3} \\
&\quad + 4C_0(t)\left(C\int_0^t\int_0^1 |u_x(x,s)|^2dxds\right)^{1/2}\Bigg]^{(p+2)/4} + 2D_1 + 2C\int_0^t\int_0^1 |u_x(x,s)|^2dxds \\
&\quad + 2\left(\int_0^1 |u(x,t)|^2dx\right)^{1/2}\left(\int_0^1 |u_x(x,t)|^2dx\right)^{1/2} \\
&\le D_4 + \frac{4\lambda}{p}\|u_x(\cdot,t)\|_{L^2}^{(p-2)/2}\Bigg[D_3 + 4C_0^{4/3}(t)\left(\int_0^1 |u_x(x,t)|^2dx\right)^{1/3} \\
&\quad + 4C_0(t)\left(C\int_0^t\int_0^1 |u_x(x,s)|^2dxds\right)^{1/2}\Bigg]^{(p+2)/4} \\
&\quad + 2D_1 + 2C\int_0^t\int_0^1 |u_x(x,s)|^2dxds + 2\Bigg[D_3 + 4C_0^{4/3}(t)\left(\int_0^1 |u_x(x,t)|^2dx\right)^{1/3} \\
&\quad + 4C_0(t)\left(C\int_0^t\int_0^1 |u_x(x,s)|^2dxds\right)^{1/2}\Bigg]^{1/2}\left(\int_0^1 |u_x(x,t)|^2dx\right)^{1/2} \\
&\le D_4 + C\|u_x(\cdot,t)\|_{L^2}^{(p-2)/2}\Bigg[D_3^{(p+2)/4} + \left(4C_0^{4/3}(t)\left(\int_0^1 |u_x(x,t)|^2dx\right)^{1/3}\right)^{(p+2)/4} \\
&\quad + \left(4C_0(t)\left(C\int_0^t\int_0^1 |u_x(x,s)|^2dxds\right)^{1/2}\right)^{(p+2)/4}\Bigg] \\
&\quad + 2D_1 + 2C\int_0^t\int_0^1 |u_x(x,s)|^2dxds + 2\Bigg[D_3^{1/2} + 2C_0^{2/3}(t)\left(\int_0^1 |u_x(x,t)|^2dx\right)^{1/6} \\
&\quad + 2C_0^{1/2}(t)\left(C\int_0^t\int_0^1 |u_x(x,s)|^2dxds\right)^{1/4}\Bigg]\left(\int_0^1 |u_x(x,t)|^2dx\right)^{1/2} \\
&\le D_5 + D_6\|u_x(\cdot,t)\|_{L^2}^{(p-2)/2} + CC_0^{(p+2)/3}(t)\|u_x(\cdot,t)\|_{L^2}^{2(p-1)/3} \\
&\quad + C_1(t)\|u_x(\cdot,t)\|_{L^2}^{(p-2)/2}\left(\int_0^t\int_0^1 |u_x(x,s)|^2dxds\right)^{(p+2)/8} + 2C\int_0^t\int_0^1 |u_x(x,s)|^2dxds \\
&\quad + D_7\|u_x(\cdot,t)\|_{L^2} + C_2(t)\|u_x(\cdot,t)\|_{L^2}^{4/3} + C_3(t)\|u_x(\cdot,t)\|_{L^2}\left(\int_0^t\int_0^1 |u_x(x,s)|^2dxds\right)^{1/4}.
\end{aligned}
\tag{5.16}
$$

For $2 \le p \le \frac{10}{3}$, Young's inequality $ab \le (1/m)a^m + (1/n)b^n$ with $m^{-1} + n^{-1} = 1$ leads from (5.16) to

$$\int_0^1 |u_x(x,t)|^2dx \le D_8 + D_9\int_0^t\int_0^1 |u_x(x,s)|^2dxds$$

which, by Gronwall's lemma, gives a uniform bound for $\int_0^1 |u_x(x,t)|^2dx$ on the interval $0 \le t \le T$. As the time T is arbitrary, the proof is complete. □

Using the same argument as that for proving Theorem 5.3 leads to the following global well-posedness result.

Theorem 5.5 *Assume that*

$$p \geq 3 \quad \textit{if} \quad \lambda < 0 \quad \textit{or} \quad 3 \leq p \leq \tfrac{10}{3} \quad \textit{if} \quad \lambda > 0$$

and let $1 \leq s < 5/2$ *be given. Then the IBVP (5.2) is globally well-posed in* $H^s(0,1)$ *with* $\phi \in H^s(0,1)$ *and* $h_1,\ h_2 \in H_{loc}^{\frac{s+1}{2}}(\mathbb{R}^+)$ *subject to the compatibility conditions on* ϕ, h_1 *and* h_2.

6 Appendices

6.1 Appendix 1

The following Lemma is used to obtain an estimate in the proof of Proposition 4.6. **Lemma A-1:** Let ψ be an even, non-negative, C^∞ cut-off function with $\mathrm{supp}(\psi) \subset [-1,1]$ and with $\psi(x) \equiv 1$ for $|x| \leq \frac{1}{2}$. Suppose also that ψ is strictly decreasing on $[\frac{1}{2}, 1]$. There exists a constant $C > 0$ such that for any $g \in H_{00}^{\frac{1}{2}}(\mathbb{R}^+)$,

$$\sum_{n=1}^{\infty} \left| \int_0^\infty f(\mu) \frac{1}{\mu - n} \Big(1 - \psi(n^2 - \mu^2)\Big) d\mu \right|^2 \leq C \int_0^\infty (\mu+1)|f(\mu)|^2 d\mu\,,$$

where f is the Fourier transform of the extension by zero of g to all of $\mathbb{R}$.

Proof: Write

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left| \int_0^\infty f(\mu) \frac{1}{\mu - n} \big(1 - \psi(n^2 - \mu^2)\big) d\mu \right|^2 \\
&\quad = \sum_{n=1}^{\infty} \left| \left(\int_0^{n-1} + \int_{n-1}^{n+1} + \int_{n+1}^{\infty} \right) f(\mu) \frac{1}{\mu - n} \big(1 - \psi(n^2 - \mu^2)\big) d\mu \right|^2 \\
&\quad \leq I_1 + I_2 + I_3\,.
\end{aligned}$$

Since the estimates for I_1 and I_3 are similar, we only study I_3. Let $\alpha, \beta > 0$ so the Cauchy-Schwartz inequality implies that

$$I_3 = \sum_{n=1}^{\infty} \left| \int_{n+1}^\infty f(\mu) \frac{1}{\mu - n} \big(1 - \psi(n^2 - \mu^2)\big) d\mu \right|^2 \leq \sum_{n=1}^{\infty} \int_{n+1}^\infty \frac{|f(\mu)|^2 \mu^{2\alpha}}{|\mu - n|^{2-2\beta}} d\mu \int_{n+1}^\infty \frac{1}{|\mu - n|^{2\beta} \mu^{2\alpha}} d\mu\,.$$

If $2\alpha + 2\beta > 1$, then

$$\begin{aligned}
\left| \int_{n+1}^\infty \frac{1}{|\mu - n|^{2\beta} \mu^{2\alpha}} d\mu \right| &= \left| \left(\int_{n+1}^{3n} + \int_{3n}^\infty \right) \frac{1}{|\mu - n|^{2\beta} \mu^{2\alpha}} d\mu \right| \\
&\leq C \left(\int_{n+1}^{3n} \frac{1}{|\mu - n|^{2\beta} n^{2\alpha}} d\mu + \int_{3n}^\infty \frac{1}{\mu^{2\alpha+2\beta}} d\mu \right) \leq C n^{1-2\alpha-2\beta} \leq C,
\end{aligned}$$

where C is independent of n. It thus transpires that if $2-2\beta>1$, then

$$
\begin{aligned}
I_3 &\le C\sum_{n=1}^{\infty}\int_{n+1}^{\infty}\frac{|f(\mu)|^2\mu^{2\alpha}}{(|\mu-n|+1)^{2-2\beta}}d\mu \le C\int_0^{\infty}|f(\mu)|^2\mu^{2\alpha}\sum_{n=1}^{\infty}\frac{1}{(|\mu-n|+1)^{2-2\beta}}d\mu\\
&\le C\int_0^{\infty}|f(\mu)|^2\mu^{2\alpha}d\mu.
\end{aligned}
$$

Choosing $\alpha=\frac{1}{4}$ and, say, $\beta=\frac{3}{8}$ yields the advertised bound.

To study I_2, note that in the integrals, the integrand vanishes unless $\mu\ge\sqrt{n^2+1/2}$ or $0\le\mu\le\sqrt{n^2-1/2}$. Consequently, it must be the case that

$$
\begin{aligned}
I_2 &\le \sum_{n=1}^{\infty}\left(\left(\int_{\sqrt{n^2+1/2}}^{n+1}+\int_{n-1}^{\sqrt{n^2-1/2}}\right)\frac{|f(\mu)|}{|\mu-n|}d\mu\right)^2\\
&\le \sum_{n=1}^{\infty}\Bigg(\int_{\sqrt{n^2+1/2}}^{n+1}|f(\mu)|^2d\mu\int_{\sqrt{n^2+1/2}}^{n+1}|\mu-n|^{-2}d\mu\\
&\qquad\qquad+\int_{n-1}^{\sqrt{n^2-1/2}}|f(\mu)|^2d\mu\int_{n-1}^{\sqrt{n^2-1/2}}|\mu-n|^{-2}d\mu\Bigg)\\
&\le C\sum_{n=1}^{\infty}\left(\int_n^{n+1}n|f(\mu)|^2d\mu+\int_{n-1}^{n}n|f(\mu)|^2d\mu\right)\\
&\le C\sum_{n=1}^{\infty}\left(\int_n^{n+1}\mu|f(\mu)|^2d\mu+\int_{n-1}^{n}(\mu+1)|f(\mu)|^2d\mu\right)\\
&\le C\int_0^{\infty}(\mu+1)|f(\mu)|^2d\mu\,.
\end{aligned}
$$

The lemma is proved. □

The following example shows the optimality of the assumption $h\in H^{1/2}(0,T)$ in (4.11) and (4.12). This result then implies that the assumptions on (h_1,h_2) in Theorem 1.2 are optimal.
Example A-2: Notice that if (4.11) or (4.12) holds, then

$$
\|u_h\|_{L^2(\Omega_T)}=\|u_h\|_{L^2((0,1)\times(0,T))}\le C_T\|h\|_{H^{\frac{1}{2}}(0,T)}\,,
$$

where we recall for the reader's convenience that

$$
\begin{aligned}
u_h &= \sum_{n=1}^{\infty}2in\pi e^{-i(n\pi)^2t}\int_0^t e^{i(n\pi)^2\tau}h(\tau)d\tau\sin n\pi x\\
&= \sum_{n=-\infty}^{\infty}n\pi e^{-i(n\pi)^2t+in\pi x}\int_0^t e^{i(n\pi)^2\tau}h(\tau)d\tau
\end{aligned}
$$

(see (4.10)). Assume that $h(t)$ has the Fourier series expansion

$$
h(t)=\sum_{k=-\infty}^{\infty}e^{-\pi^2ikt}a_k \quad \text{with} \quad a_k=\int_0^{\frac{2\pi}{\pi^2}}e^{\pi^2ikt}h(t)dt\ .
$$

It follows that

$$u_h = \sum_{n=-\infty}^{\infty} n\pi e^{-i(n\pi)^2 t + in\pi x} \int_0^t \sum_k e^{i(n^2-k)\pi^2\tau} a_k d\tau$$

$$= \left(\sum_{n=-\infty}^{\infty} n\pi e^{-i(n\pi)^2 t + in\pi x} \sum_{k \neq n^2} \frac{e^{i(n^2-k)\pi^2 t} - 1}{n^2 - k} a_k \right) + \left(\sum_{n=-\infty}^{\infty} n\pi e^{-i(n\pi)^2 t + in\pi x} t a_{n^2} \right)$$

$$= \left(\sum_{n=-\infty}^{\infty} n\pi e^{in\pi x} \sum_{k \neq n^2} \frac{e^{-ki\pi^2 t} - e^{-in^2\pi^2 t}}{n^2 - k} a_k \right) + \left(\sum_{n=-\infty}^{\infty} n\pi e^{-i(n\pi)^2 t + in\pi x} t a_{n^2} \right)$$

$$= \left(\sum_{n=-\infty}^{\infty} n\pi e^{in\pi x} \sum_{k \neq n^2} \frac{e^{-ki\pi^2 t}}{n^2 - k} a_k \right) + \left(\sum_{n=-\infty}^{\infty} n\pi e^{-i(n\pi)^2 t + in\pi x} \left(t a_{n^2} - \sum_{k \neq n^2} \frac{a_k}{n^2 - k} \right) \right).$$

Choose $h(t)$ so that

$$a_{n^2} = \int_0^{\frac{2}{\pi}} e^{\pi^2 i n^2 t} h(t) dt = 0\,, \quad n \in \mathbb{Z}\,.$$

Then, the last formula condenses to

$$u_h = \sum_{n=-\infty}^{\infty} n\pi e^{in\pi x} \sum_{k \neq n^2} \frac{e^{-ki\pi^2 t}}{n^2 - k} a_k + \sum_{n=-\infty}^{\infty} n\pi e^{-i(n\pi)^2 t + in\pi x} \left(\sum_{k \neq n^2} \frac{a_k}{k - n^2} \right).$$

As $k \neq n^2$, the exponentials $e^{in\pi x - ik\pi^2 t}$ and $e^{i(n\pi x - (n\pi)^2)t}$ are orthogonal, whence

$$\begin{aligned} \|u_h\|^2_{L^2\left((0,1)\times\left(0,\frac{2}{\pi}\right)\right)} &= \sum_{n=-\infty}^{\infty} \sum_{k \neq n^2} n^2\pi^2 \frac{a_k^2}{(n^2-k)^2} + \sum_{n=-\infty}^{\infty} (n\pi)^2 \left(\sum_{k \neq n^2} \frac{a_k}{k - n^2} \right)^2 \\ &\geq \sum_{n=-\infty}^{\infty} n^2\pi^2 a_{n^2+1}^2, \end{aligned}$$

the latter inequality obtained by only considering the terms where $k = n^2 + 1$.

If there were a constant C such that for all $h \in H^\alpha\left(0, \frac{2}{\pi}\right)$, $\|u_h\|^2_{L^2\left((0,1)\times\left(0,\frac{2}{\pi}\right)\right)} \leq C\|h\|^2_{H^\alpha\left(0,\frac{2}{\pi}\right)}$, then it would follow that $\alpha \geq \frac{1}{2}$. Suppose instead that there is a constant C such that

$$\|u_h\|^2_{L^2\left((0,1)\times\left(0,\frac{2}{\pi}\right)\right)} \leq C\|h\|^2_{H^\alpha\left(0,\frac{2}{\pi}\right)} \qquad \text{for some } \alpha \text{ with } 0 < \alpha < \tfrac{1}{2}\,.$$

Define the function h by its Fourier series, *viz.*

$$h(t) = \sum_{n \neq 0} \frac{1}{|n|^\beta} e^{-\pi^2 i(n^2+1)t}\,.$$

For h to lie in $H^\alpha\left(0, \frac{2}{\pi}\right)$, we need

$$\sum_{n\neq 0} \left| \frac{(n^2+1)^\alpha}{|n|^\beta} \right|^2 < +\infty\,,$$

or $2\beta - 4\alpha > 1$ which implies that $\beta > 2\alpha + \frac{1}{2}$. But, for this h,

$$\|u_h\|^2_{L^2\left((0,1)\times\left(0,\frac{2}{\pi}\right)\right)} \geq \sum_{n=-\infty}^{\infty} n^2\pi^2 \frac{1}{|n|^{2\beta}} = \sum_{n=-\infty,\ n\neq 0}^{\infty} \frac{\pi^2}{|n|^{2\beta-2}}\,.$$

Since $\alpha < \frac{1}{2}$, β can be chosen so that $2\alpha + \frac{1}{2} < \beta < \frac{3}{2}$. For such a value of β, it is clear that

$$\sum_{n=-\infty,\ n\neq 0}^{\infty} \frac{1}{|n|^{2\beta-2}} = +\infty.$$

The partial sums

$$h_k(t) = \sum_{n\neq 0}^{|n|=k} \frac{1}{|n|^\beta} e^{-\pi^2 i(n^2+1)t}$$

lie in C^∞ and therefore, according to our hypothesis,

$$\|u_{h_k}\|^2_{L^2\left((0,1)\times\left(0,\frac{2}{\pi}\right)\right)} \leq C\|h_k\|^2_{H^\alpha}\,.$$

But, as $k \to \infty$, the right side of the last inequality is bounded while the left side tends to ∞, which is a contradiction. Hence, we must have $\alpha \geq \frac{1}{2}$.

6.2 Appendix 2

Let

$$X = \left\{(\phi, h) \in H^1(\mathbb{R}^+) \times H^{\frac{3}{4}}(\mathbb{R}^+);\quad \phi(0) = h(0)\right\},$$
$$Y = \left\{(\phi, h) \in H^2(\mathbb{R}^+) \times H^{\frac{5}{4}}(\mathbb{R}^+);\quad \phi(0) = h(0)\right\},$$

and

$$X^* = H^1(\mathbb{R}^+) \times H^{\frac{3}{4}}(\mathbb{R}^+), \quad Y^* = H^2(\mathbb{R}^+) \times H^{\frac{5}{4}}(\mathbb{R}^+).$$

While it is well-known (cf. [59]) that for any θ with $0 \leq \theta \leq 1$,

$$[X^*, Y^*]_\theta = H^{1+\theta}(\mathbb{R}^+) \times H^{\frac{2\theta+3}{4}}(\mathbb{R}^+),$$

however, as pointed out by an anonymous referee, it seems that no rigorous proof can be found in literature for the interpolation result

$$[X, Y]_\theta = \left\{(\phi, h) \in H^{1+\theta}(\mathbb{R}^+) \times H^{\frac{2\theta+3}{4}}(\mathbb{R}^+);\quad \phi(0) = h(0)\right\} \tag{6.1}$$

used in our analysis. It is mentioned (in a much more general setting) as “most likely” true in the book of Lions-Magenes [60] (Chapter 4, Section 14, remark after Theorem 14.1). The following short proof of (6.1) was suggested by the referee.

First, it is claimed that there exists a bounded linear "lifting" operator L from the space

$$Z_s := \Big\{(\phi, h) \in H^{1+s}(\mathbb{R}^+) \times H^{\frac{2s+3}{4}}(\mathbb{R}^+); \quad \phi(0) = h(0)\Big\}$$

to $H^{s+2,\frac{s+2}{2}}(\mathbb{R}^+ \times \mathbb{R}^+) := L^2_t(\mathbb{R}^+; H^{s+2}_x(\mathbb{R}^+) \cap H_t^{\frac{s+2}{2}}(\mathbb{R}^+; L^2_x(\mathbb{R}^+))$ for $0 \le s \le 1$ such that $w = L(\phi, h) \in H^{s+2,\frac{s+2}{2}}(\mathbb{R}^+ \times \mathbb{R}^+)$ for any $(\phi, h) \in Z_s$ and

$$w(x, 0) = \phi(x), \qquad w(0, t) = h(t).$$

Then

$$\mathbb{T} \circ L = I$$

where I denotes the identity operator and $\mathbb{T}$ is the trace operator defined by

$$\mathbb{T} : H^{s+2,\frac{s+2}{2}}(\mathbb{R}^+ \times \mathbb{R}^+) \to Z_s, \quad \mathbb{T}w = (w(x, 0), w(0, t)).$$

One has ([60] Proposition 2.1, Chapter 4)

$$\Big[H^{2,1}(\mathbb{R}^+ \times \mathbb{R}^+), H^{3,\frac{3}{2}}(\mathbb{R}^+ \times \mathbb{R}^+)\Big]_\theta = H^{2+\theta,1+\frac{\theta}{2}}(\mathbb{R}^+ \times \mathbb{R}^+)$$

for $0 \le \theta \le 1$. Consequently, $[X, Y]_\theta$ can be identified with $\mathbb{T}\left(H^{2+\theta,1+\frac{\theta}{2}}(\mathbb{R}^+ \times \mathbb{R}^+)\right)$ which is exactly Z_θ.

It remains to prove the existence of the lifting operator

$$L : Z_s \to H^{s+2,1+\frac{s}{2}}(\mathbb{R}^+ \times \mathbb{R}^+)$$

for $0 \le s \le 1$. To this end, consider the following IBVP

$$\begin{cases} u_t = u_{xx}, \quad x \in \mathbb{R}^+,\ t \in \mathbb{R}^+, \\ u(x, 0) = \phi(x), \quad u(0, t) = h(t), \quad x \in \mathbb{R}^+,\ t \in \mathbb{R}^+, \end{cases} \tag{6.2}$$

for the heat equation, where $\phi \in H^{s+1}(\mathbb{R}^+), h(t) \in H^{\frac{2s+3}{4}}(\mathbb{R}^+)$ with $0 \le s \le 1$ and $\phi(0) = h(0)$. The existence of the solution $u(x, t) \in H^{s+2,1+\frac{s}{2}}(\mathbb{R}^+ \times \mathbb{R}^+)$ for (6.2) is established in Theorems 6.1 and 6.2 in Chapter 4 of [60]. Therefore, given s with $0 \le s \le 1$ and $(\phi, h) \in Z_s$, we may define the lifting operator L by

$$L(\phi, h) := u,$$

where u is the solution of the IBVP (6.2).

7 Acknowledgment

JLB and SMS were partially supported by the US National Science Foundation. BYZ was partially supported by a grant from the Simons Foundation (201615) and NSF of China (11231007, 11571244). JLB also thanks the Université de Paris Nord and the Ulsan National Institute of Science and Technology for hospitality and very good working conditions during parts of the writing phase of this project. We all tender heartfelt thanks to anonymous referees for careful readings of the script and many helpful comments, corrections and suggestions.

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