GLOBAL SOLUTIONS AND ILL-POSEDNESS FOR THE KAUP SYSTEM AND RELATED BOUSSINESQ SYSTEMS

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Abstract. The two-way propagation of a certain class of long-crested water waves is governed approximately by systems of Boussinesq-type equations. First introduced by Boussinesq in the 1870's, these equations have been put forward in various forms by many authors.

Considered here is a class of such system which include the well known one first introduced by Kaup. The Kaup system is especially interesting since it features an associated inverse scattering formalism, which means that quite detailed aspects of its solutions may be within reach. However, this system and others like it were called into question in earlier work because the initial-value problems for their linearizations around the rest state are ill posed. It is here shown that nonlinearity does not erase this problem. That is to say, the initial-value problem for the Kaup system and others in a certain class of Boussinesq-type systems are ill posed in Sobolev spaces. Indeed, it is shown that arbitrarily small, smooth solutions can blow up in arbitrarily short time in Sobolev-space norms. This norm-inflation result indicates the system is not a good candidate for use in practical problems.

1. Introduction

The purpose of the present essay is to establish the ill-posedness of the initial-value problem for a certain class of Boussinesq systems. Boussinesq systems were originally derived by Boussinesq in the 1870's (see [7], [8]) as approximate models for small amplitude, long wavelength, long-crested water waves propagating over a featureless, horizontal bottom. The original system put forward by Boussinesq was

$$\begin{cases} \partial_t\eta + \partial_x w + \partial_x(w\eta) = 0, \\ \partial_t w + \partial_x\eta + w\partial_x w + \partial_t^2\partial_x\eta = 0, \end{cases} \tag{1}$$

(see Boussinesq [8] or Craik's historical resumés [11] [12]). Here, a standard $x-y-z$ coordinate system has been chosen in which z increases in the direction opposite to which gravity acts and x and y are the horizontal coordinates. As the system is derived for long-crested waves which vary little in the y–direction orthogonal to the primary direction of propagation along the x–axis, a one-space dimensional model is appropriate. Here, x specifies the spatial location in the medium in the primary direction of propagation, t is proportional to elapsed time, $\eta(x,t)$ is the deviation of the free surface from its rest position at the point x at time t and $w(x,t)$ is the depth-averaged horizontal velocity at the station x at time t. Both the independent and dependent variables are scaled so that various physically relevant constants do not appear and the essentials of the partial differential equations are more easily discerned. This system formally allows for propagation of disturbances in both increasing and decreasing directions of x, but it suffers mathematically from having a second-order derivative in time, so requiring more auxiliary data than it is convenient to provide. It has become common to ascribe the alternative system

$$\begin{cases} \partial_t\eta + \partial_x w + \partial_x(w\eta) = 0, \\ \\ \partial_t w + \partial_x\eta + w\partial_x w - \partial_t\partial_x^2 w = 0, \end{cases} \tag{2}$$

as Boussinesq's original system (see Whitham's well known text [19]). The latter system of partial differential equations has a satisfactory well-posedness theory at the hands of Schonbek [18] and Amick [3]. Presumably, one could in fact mount a physcially reasonable theory for the original system (1) by inferring a further initial condition as in the work [10] on the Kruskal variation of the Korteweg-de Vries equation (see [15]).

The regularized system (2) fits into the scheme of *abcd*-systems developed by Bona, Chen and Saut in [4]. These systems take the form

$$\begin{cases} \partial_t\eta + \partial_x w + \partial_x(w\eta) + a\partial_x^3 w - b\partial_x^2\partial_t\eta = 0, \\ \partial_t w + \partial_x\eta + w\partial_x w + c\partial_x^3\eta - d\partial_x^2\partial_t w = 0, \end{cases} \tag{3}$$

where η is as above, the deviation of the free surface from its rest postion and w is the horizontal velocity at a particular height above the bottom. (Since the flow is assumed irrotational and incompressible, the velocity potential is harmonic and hence knowledge of u and η suffices to infer the velocity field everywhere in the flow domain.) While the *abcd*-systems appear to depend upon four parameters, these are not in fact independent. In particular, in the standard scaling for this problem, it must be the case that $a+b+c+d = \frac{1}{3}$ (for more details see [4]).

In [4], a complete analysis was made of which of the *abcd*-systems are linearly well posed. This simply amounts to linearizing the system around the rest state $(\eta, u) = (0,0)$ and solving the resulting linear system using the Fourier transform. In the companion study [5], it was shown that all the systems that are linearly well posed are in fact locally nonlinearly well posed.

The question raised here has to do exactly with the class of *abcd*-systems which are linearly ill-posed. Choosing the constants as $a = \frac{1}{3}$ and $b = c = d = 0$, which is admissible within the detailed formulas for the values of these constants, yields one example that is linearly ill-posed, namely the

Kaup system,

$$\begin{cases} \eta_t + w_x + (w\eta)_x + \frac{1}{3}w_{xxx} = 0, \\ w_t + \eta_x + ww_x = 0. \end{cases} \tag{4}$$

This system was derived by Kaup in [14] as an early example of a coupled pair of equations that admits an inverse-scattering formalism. It has been the object of a number of studies connected with inverse scattering theory.

The theory developed in [6] implies that if the Kaup system has sufficiently smooth solutions that respect the small amplitude, long wavelength presumptions arising in the derivation of Boussinesq systems, then it is indeed a good approximation to the full water wave problem. However, supplementing the linear ill-posedness result in [4], it will be shown here that the initial-value problem with $\eta(x,0) = \eta_0(x)$ and $w(x,0) = w_0(x)$ for the full nonlinear problem is indeed ill-posed. In fact, it transpires that very smooth, small initial data can lead to solutions that lose regularity in a short time. This result, derived for the particular example of the Kaup system, will then be shown to hold true for a generic member of the class of linearly ill-posed *abcd*–sytems.

To effect such results, an idea put forward by Duchon and Robert [13] will come to the fore. Their paper dealt with the existence of small, global solutions to the vortex sheet problem. The vortex sheet problem, like ill-posed Boussinesq equations, has an elliptic flavor, and the Duchon-Robert result can be viewed as solving a boundary-value problem in space-time rather than solving an initial-value problem. This approach will be used here to demonstrate in Section 3 that if suitably chosen, small, initial data for w and η are specified, then global solutions of (4) exist. These solutions are shown to be analytic at positive times, but can be quite rough initially. Making use of the time reversability of the system, it is immediately inferred that small, smooth solutions can lose regularity arbitrarily quickly. This in turn leads to the conclusion propounded in Section 4 that the initial-value problem for the Kaup system is ill-posed in Sobolev spaces.

The first and third authors have previously adapted the method of Duchon and Robert for *abcd*-systems in [17], showing the existence of solutions for temporal boundary value problems (problems in which either Dirichlet or Neumann data is specified at times $t = 0$ and $t = T > 0$). We mention that there is additional work on ill-posedness of the vortex sheet in the literature, for example the early work of Caflisch and Orellana [9] and the more recent foray by S. Wu [20].

2. Preliminaries

As the periodic initial-value problem is our focus, it is natural to consider functions which are periodic on, say, the interval $[0, 1]$. Throughout, all function classes will be presumed to be periodic of period 1 in the spatial variable. Often in what follows, attention will be focussed on functions of the spatial variable which have mean zero, so having a Fourier series representation of the form

$$f(x) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \hat{f}(k) e^{2\pi i k x}. \tag{5}$$

As the functions are real-valued, it must be the case that $\hat{f}(k) = \overline{\hat{f}(-k)}$. For a pseudodifferential operator A, denote its symbol by $\sigma(A) = \sigma(A)(k)$. Thus A acts upon a function f by multiplication of the coefficients of the Fourier expansion of f by the symbol $\sigma(A)(k)$. From (5), it is clear that the symbol of ∂_x is

$$\sigma(\partial_x)(k) = 2\pi i k.$$

The periodic Hilbert transform $\mathscr{H}$ will appear frequently in the analysis to follow. It is defined in terms of its symbol, *viz.*

$$\sigma(\mathscr{H})(k) = -i\mathrm{sgn}(k). \tag{6}$$

When the Hilbert transform acts on mean-zero functions, then (6) implies that

$$\mathscr{H}^2 = -\mathscr{I}, \tag{7}$$

where $\mathscr{I}$ represents the identity operator. Also appearing prominenetly in our analysis is the operator $\Lambda = \mathscr{H}\partial_x$, which has symbol

$$\sigma(\Lambda)(k) = 2\pi |k|.$$

2.1. **Function Spaces.** For any $\rho \in \mathbb{R}$, the space B_ρ is the set of all functions for which the norm

$$\|f\|_{B_\rho} = \sum_{k \in \mathbb{Z}} e^{\rho |k|} |\hat{f}(k)|$$

is finite. This is the periodic analogue of the Gevrey spaces used in [13] (see [17]). It is clear that for any periodic functions f and g and $\rho \geq 0$, the inequality

$$\left| e^{\rho|k|} \widehat{fg}(k) \right| \leq \sum_j \left| e^{\rho|k-j|} \hat{f}(k-j) \right| \left| e^{\rho|j|} \hat{g}(j) \right|,$$

holds, from which is concluded that B_ρ is a Banach algebra with

$$\|fg\|_{B_\rho} \leq \|f\|_{B_\rho} \|g\|_{B_\rho}.$$

Remark 1. *Note that if $f \in B_\rho$ for some $\rho > 0$, then $f \in H^s$ for every $s \in \mathbb{R}$. Furthermore, for any $\rho > 0$, for any $s \in \mathbb{R}$, there exists $C_{\rho,s} > 0$ such that $\|f\|_{H^s} \leq C_{\rho,s} \|f\|_{B_\rho}$ for all $f \in B_\rho$. The best constant for which the last inequality holds depends directly on s for $s \geq 0$ and inversely on $\rho > 0$. Indeed, if $\rho \geq \frac{s}{2}$, then $C_{s,\rho} = 1$ while for $0 < \rho < \frac{s}{2}$,*

$$C_{s,\rho} \leq C_s \left(\frac{1}{\rho^s} \right) \tag{8}$$

where C_s is a constant depending only on the Sobolev index s.

Again following Duchon and Robert, a space-time version of the B_ρ–norm is set forth in which $\rho = \rho(t)$ is a function of time. For $\alpha \geq 0$, define the space $\mathcal{B}_\alpha$ to be the real-valued functions on $[0,1] \times [0,\infty)$ which are continuous in time, and for which the norm

$$\|f\|_{\mathcal{B}_\alpha} = \sum_{k\in\mathbb{Z}} \sup_{t\in[0,\infty)} \Big(e^{\alpha t|k|} |\hat{f}(k,t)| \Big)$$

is finite. Just as for B_ρ, the space $\mathcal{B}_\alpha$ is a Banach algebra with

$$\|fg\|_{\mathcal{B}_\alpha} \leq \|f\|_{\mathcal{B}_\alpha} \|g\|_{\mathcal{B}_\alpha}.$$

For $\alpha \geq 0$ and $j \in \mathbb{N}$, define $\mathcal{B}_\alpha^j$ to be the space of functions f such that $\partial_x^j f \in \mathcal{B}_\alpha$, with norm

$$\|f\|_{\mathcal{B}_\alpha^j} = \|f\|_{\mathcal{B}_\alpha} + \|\partial_x^j f\|_{\mathcal{B}_\alpha}.$$

In the present discussion, interest is especially focussed upon the space $\mathcal{B}_\alpha^1$. Notice that if $f \in \mathcal{B}_\alpha^1$, then $f \in \mathcal{B}_\alpha$ and

$$\|f\|_{\mathcal{B}_\alpha} \leq \|f\|_{\mathcal{B}_\alpha^1}.$$

This inequality implies that

$$\|fg\|_{\mathcal{B}_\alpha^1} \leq 3\|f\|_{\mathcal{B}_\alpha^1} \|g\|_{\mathcal{B}_\alpha^1}$$

on account of Leibnitz' rule $\partial_x(fg) = f\partial_x g + (\partial_x f)g$.

3. Global Solutions

This section is devoted to establishing existence of global solutions of the Kaup system for suitably restricted values of α and small initial data. A word is deserved about the choice of initial data. Henceforth, initial data will be drawn from the closed subspace of mean zero functions, so that their Fourier expansion has the form depicted in (5).

Observe that since the nonlinearities in the Kaup system all appear as derivatives with respect to space and such derivatives always have mean zero, this assumption propagates forward in time for putative solutions of the Kaup system. This point will be important presently.

We begin by rewriting the Kaup system in terms of the variable u, where $u = \mathscr{H}\eta$. In the presence of the mean-zero assumption, in force from now on, it is the case that $\eta = -\mathscr{H}u$. In terms of u and w, the system (4) then becomes

$$(9) \qquad \begin{cases} \partial_t u + \Big(\Lambda - \frac{1}{3}\Lambda^3\Big) w = \partial_x \Big(\mathscr{H}\big(w \mathscr{H}(u)\big)\Big), \\ \partial_t w - \Lambda u = -\frac{1}{2}\partial_x\big(w^2\big); \end{cases}$$

recall that $\Lambda = \mathcal{H}\partial_x$. An invertible Fourier multiplier operator Θ is now introduced to symmetrize the linear part of the system. Since Θ is a Fourier multiplier operator, it will naturally commute with $\mathscr{H}$ and ∂_x. Make the change of dependent variables $v = \Theta u$, so that $u = \Theta^{-1} v$. In terms of v, the system (9) then becomes

$$\partial_t v + \Theta \left(\Lambda - \frac{1}{3}\Lambda^3 \right) w = \partial_x \Theta \mathcal{H} \left(w \mathcal{H} \Theta^{-1}(v) \right),$$

$$\partial_t w - \Lambda \Theta^{-1} v = -\frac{1}{2} \partial_x \left(w^2 \right).$$

It will be convenient to specify Θ so that

$$\Theta \left(\Lambda - \frac{1}{3}\Lambda^3 \right) = -\Lambda \Theta^{-1}.$$

A brief calculation reveals that the latter relation holds when

$$\sigma(\Theta)(k) = \left(\frac{-2\pi|k|}{2\pi|k| - \frac{8\pi^3}{3}|k|^3} \right)^{1/2} = \left(\frac{4\pi^2}{3} k^2 - 1 \right)^{-1/2}. \tag{10}$$

Notice that for all $k \in \mathbb{Z} \setminus \{0\}$, the quantity on the right-hand side of (10) is obviously positive. The related Fourier multiplier operator

$$\mathcal{A} = \Lambda \Theta^{-1} = -\Theta \Big(\Lambda - \frac{1}{3}\Lambda^3 \Big)$$

has the symbol

$$\sigma(\mathcal{A})(k) = (2\pi|k|) \left(\frac{4\pi^2}{3} k^2 - 1 \right)^{1/2}. \tag{11}$$

In terms of the operator $\mathcal{A}$, the Kaup system with mean zero initial data is seen to be equivalent to the symmetric system

$$\begin{cases} \partial_t v - \mathcal{A} w = \partial_x \Big(\Theta \mathscr{H} \big(w \mathscr{H} \Theta^{-1} v \big) \Big), \\ \partial_t w - \mathcal{A} v = -\frac{1}{2} \partial_x \big(w^2 \big). \end{cases} \tag{12}$$

3.1. **Solution Representation.** Adding and subtracting the equations in (12) leads to the system

$$\partial_t (v + w) - \mathcal{A}(v + w) = \partial_x \big(F(v, w) + G(v, w) \big),$$
$$\partial_t (v - w) + \mathcal{A}(v - w) = \partial_x \big(F(v, w) - G(v, w) \big),$$

where

$$F(y, z) = \Theta \mathscr{H} \big(z \mathscr{H} \Theta^{-1} y \big) \quad \text{and} \quad G(y, z) = -\frac{1}{2} z^2. \tag{13}$$

In Fourier transformed variables, the latter system amounts to the coupled pair

$$\partial_t (\hat{v} + \hat{w}) - \sigma(\mathcal{A})(\hat{v} + \hat{w}) = 2\pi i k \left(\widehat{F(v, w)} + \widehat{G(v, w)} \right), \tag{14}$$

$$\partial_t (\hat{v} - \hat{w}) + \sigma(\mathcal{A})(\hat{v} - \hat{w}) = 2\pi i k \left(\widehat{F(v, w)} - \widehat{G(v, w)} \right), \tag{15}$$

where k has been supressed for ease of reading. Duhamel's principle applied to (14) yields

$$(16)\qquad \begin{aligned}(\hat{v}+\hat{w})(\cdot,t) &= e^{\sigma(\mathcal{A})(t-T)}(\hat{v}+\hat{w})(\cdot,T)\\ &\quad -2\pi ik\int_t^T\left(\widehat{F(v,w)}+\widehat{G(v,w)}\right)(\cdot,\tau)e^{\sigma(\mathcal{A})(t-\tau)}d\tau.\end{aligned}$$

Imposing the convenient boundary conditions

$$(17)\qquad \lim_{T\to\infty}\hat{v} = \lim_{T\to\infty}\hat{w} = 0$$

and formally taking the limit of (16) as $T\to+\infty$ leads to

$$(18)\qquad (\hat{v}+\hat{w})(\cdot,t) = -2\pi ik\int_t^\infty\big(\widehat{F(v,w)}+\widehat{G(v,w)}\big)(\cdot,\tau)e^{\sigma(\mathcal{A})(t-\tau)}d\tau.$$

On the other hand, multiplying (15) by the integrating factor $e^{\sigma(\mathcal{A})t}$ followed by an integration in time gives the equation

$$(19)\quad (\hat{v}-\hat{w})(\cdot,t) = e^{-\sigma(\mathcal{A})t}\hat{f}+2\pi ik\int_0^t(\widehat{F(v,w)}-\widehat{G(v,w)})(\cdot,\tau)e^{\sigma(\mathcal{A})(\tau-t)}d\tau,$$

where

$$\hat{f} := (\hat{v}-\hat{w})\Big|_{t=0}.$$

These calculations suggest introducing the operators I^+ and I^-, *viz.*

$$\begin{aligned}I^+h(k,t) &= 2\pi ik\int_0^t e^{\sigma(\mathcal{A})(k)(\tau-t)}\hat{h}(k,\tau)\,d\tau,\\ I^-h(k,t) &= 2\pi ik\int_t^\infty e^{\sigma(\mathcal{A})(k)(t-\tau)}\hat{h}(k,\tau)\,d\tau.\end{aligned}$$

Notice that both I^+ and I^- map into the mean zero subspace. In terms of these operators, equations (18) and (19) may be written in the form

$$\begin{aligned}\hat{v}+\hat{w} &= -I^-(F+G),\\ \hat{v}-\hat{w} &= e^{-\sigma(\mathcal{A})t}\hat{f}+I^+(F-G).\end{aligned}$$

Solving for $\hat{v}$ and $\hat{w}$, there appears the potentially useful formulas

$$(20)\qquad \hat{v} = \frac{1}{2}\left(e^{-\sigma(\mathcal{A})t}\hat{f}+I^+(F-G)-I^-(F+G)\right),$$

$$(21)\qquad \hat{w} = -\frac{1}{2}\left(e^{-\sigma(\mathcal{A})t}\hat{f}+I^+(F-G)+I^-(F+G)\right).$$

It will be helpful to eliminate f in favor of $w(\cdot,0)$. Equation (21), evaluated at $t=0$ implies that

$$\hat{w}(k,0) = -\frac{1}{2}\hat{f}(k)-\frac{1}{2}I_0(k),$$

where

$$\begin{aligned}I_0(k) &= I^-(F+G)(k,0)\\ &= 2\pi ik\int_0^\infty e^{-\sigma(\mathcal{A})(k)\tau}\left(\widehat{F(v,w)}(k,\tau)+\widehat{G(v,w)}(k,\tau)\right)d\tau.\end{aligned}$$

Denoting $w(\cdot,0)$ by w_0, the latter formula becomes

$$\hat{f} = -2\hat{w}_0 - I_0,$$

and the system (20–21) then takes the final form

$$\left\{\begin{array}{l} \hat{v} = -e^{-\sigma(\mathcal{A})t}\hat{w}_0 - \frac{1}{2}e^{-\sigma(\mathcal{A})t}I_0 + \frac{1}{2}I^+(F-G) - \frac{1}{2}I^-(F+G), \\ \\ \hat{w} = e^{-\sigma(\mathcal{A})t}\hat{w}_0 + \frac{1}{2}e^{-\sigma(\mathcal{A})t}I_0 - \frac{1}{2}I^+(F-G) - \frac{1}{2}I^-(F+G). \end{array}\right. \tag{22}$$

Remark that if $(\hat{v},\hat{w})$ satisfies (22), then $\hat{w}(k,0) = \hat{w}_0(k)$ holds automatically. However, notice also that $\hat{v}(k,0) = -\hat{w}(k,0) - I_0(k)$ has not been specified independently in this formulation. Once $\hat{w}_0$ is specified, so is $\hat{v}_0$.

The contraction mapping theorem will be used to prove existence of small global solutions of the system (22). To this end, define a mapping $\mathcal{T}$ via the right-hand side of (22) so that

$$\widehat{\mathcal{T}}(v,w) = \begin{pmatrix} -e^{-\sigma(\mathcal{A})t}\hat{w}_0 - \frac{1}{2}e^{-\sigma(\mathcal{A})t}I_0 + \frac{1}{2}I^+(F-G) - \frac{1}{2}I^-(F+G) \\ e^{-\sigma(\mathcal{A})t}\hat{w}_0 + \frac{1}{2}e^{-\sigma(\mathcal{A})t}I_0 - \frac{1}{2}I^+(F-G) - \frac{1}{2}I^-(F+G) \end{pmatrix}.$$

Of course, it must be kept in mind that the quantities I_0, F and G all depend on (v,w). Notice, however, that they all vanish when $v = w = 0$.

3.2. **Operator Estimates.** In this subsection, the linear operators I^+ and I^- are shown to be bounded on the space $\mathcal{B}^j_\alpha$ for any $j \in \mathbb{N}$, at least for sufficiently small $\alpha > 0$.

Fix $j \in \mathbb{N}$ and let $h \in \mathcal{B}^j_\alpha$. The definition of the norm and the definition of I^+ combine to give

$$\begin{aligned} \|I^+h\|_{\mathcal{B}^j_\alpha} \\ = \sum_{k\in\mathbb{Z}\setminus\{0\}} \sup_{t\in[0,\infty)} \left| (2\pi)^{j+1}k^{j+1}e^{\alpha t|k|} \int_0^t e^{-(t-\tau)\sigma(\mathcal{A})(k)}\hat{h}(k,\tau)\ d\tau \right|. \end{aligned} \tag{23}$$

Rearranging the exponentials, one obtains the estimate

$$\begin{aligned} &\|I^+h\|_{\mathcal{B}^j_\alpha} \\ &\quad\le c \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{j+1} \sup_{t\in[0,\infty)} \Bigg(e^{t\left(\alpha|k|-\sigma(\mathcal{A})(k)\right)} \times \\ &\qquad\qquad \int_0^t e^{\tau\left(\sigma(\mathcal{A})(k)-\alpha|k|\right)} \left| e^{\alpha\tau|k|}\hat{h}(k,\tau)\right| d\tau \Bigg) \\ &\quad\le c \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{j+1} \left(\sup_{t\in[0,\infty)} \left| e^{\alpha t|k|}\hat{h}(k,t)\right| \right) \times \\ &\qquad\qquad \left(\sup_{t\in[0,\infty)} e^{t(\alpha|k|-\sigma(\mathcal{A})(k))} \int_0^t e^{\tau\left(\sigma(\mathcal{A})(k)-\alpha|k|\right)} d\tau \right). \end{aligned}$$

Elementary inequalities employed after evaluating the integral on the right-hand side lead to the more explicit upper bound

$$(24)\quad \|I^+h\|_{\mathcal{B}^j_\alpha} \leq \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{j+1} \left(\sup_{t\in[0,\infty)} \left|e^{\alpha t|k|}\hat{h}(k,t)\right|\right) \sup_{t\in[0,\infty)} \left|\frac{1-e^{t\left(\alpha|k|-\sigma(\mathcal{A})(k)\right)}}{\sigma(\mathcal{A})(k)-\alpha|k|}\right|.$$

Inspection of the symbol of $\mathcal{A}$, which is given above in (11), allows the conclusion that, for sufficiently small α,

$$\sigma(\mathcal{A})(k) - \alpha|k| > 0, \quad \forall k \in \mathbb{Z}\setminus\{0\}.$$

Furthermore, if α is in the range

$$(25)\qquad \alpha \in \left(0, 2\pi\sqrt{\frac{4\pi^2}{3}-1}\right),$$

the quantity

$$(26)\qquad \left|\frac{k}{\sigma(\mathcal{A})(k)-\alpha|k|}\right| = \frac{1}{2\pi\sqrt{\frac{4\pi^2k^2}{3}-1}-\alpha}$$

is bounded by a constant which is independent of k.

Returning to (24), we see that there exists $c_+ > 0$ such that, for all $h \in \mathcal{B}^j_\alpha$,

$$(27)\qquad \|I^+h\|_{\mathcal{B}^j_\alpha} \leq c\sum_{k\in\mathbb{Z}} \sup_{t\in[0,\infty)} \left|k^j e^{\alpha t|k|}\hat{h}(k,t)\right| \leq c_+\|h\|_{\mathcal{B}^j_\alpha}.$$

Attention is now turned to obtaining a bound for I^-. This follows much as did the estimate for I^+. The definition of the norm and of I^- imply that for and $h \in \mathcal{B}^j_\alpha$,

$$\|I^-h\|_{\mathcal{B}^j_\alpha} = \sum_{k\in\mathbb{Z}} \sup_{t\in[0,\infty)} \left|(2\pi)^{j+1}k^{j+1}e^{\alpha t|k|}\int_t^\infty e^{\sigma(\mathcal{A})(k)(t-\tau)}\hat{h}(k,\tau)\ d\tau\right|.$$

As before, straightforward machinations lead quickly to the inequality

$$\begin{aligned}
&\|I^-h\|_{\mathcal{B}^j_\alpha} \\
&\leq c\sum_{k\in\mathbb{Z}} |k|^{j+1} \sup_{t\in[0,\infty)} e^{t(\alpha|k|+\sigma(\mathcal{A})(k))}\int_t^\infty e^{-\tau(\sigma(\mathcal{A})(k)+\alpha|k|)}\left|e^{\alpha\tau|k|}\hat{h}(k,\tau)\right|d\tau \\
&\leq c\sum_{k\in\mathbb{Z}} |k|^{j+1}\left(\sup_{t\in[0,\infty)}\left|e^{\alpha t|k|}\hat{h}(k,t)\right|\right) \\
&\qquad\qquad \times\left(\sup_{t\in[0,\infty)} e^{t(\alpha|k|+\sigma(\mathcal{A})(k))}\int_t^\infty e^{-\tau(\sigma(\mathcal{A})(k)+\alpha|k|)}\ d\tau\right).
\end{aligned}$$

Evaluating the integral simplifies this to

$$\|I^{-}h\|_{\mathcal{B}^j_\alpha} \le c \sum_{k\in\mathbb{Z}\setminus\{0\}} \left(\sup_{t\in[0,\infty)} \left| k^j e^{\alpha t|k|} \hat{h}(k,t) \right| \right) \frac{|k|}{\sigma(\mathcal{A})(k)+\alpha|k|}.$$

Consulting (11) again, it is clear that the quantity $\frac{|k|}{\sigma(\mathcal{A})(k)+\alpha|k|}$ is bounded by a constant which is independent of k. (In this case, the boundedness holds for any $\alpha \ge 0$.) The conclusion is

$$\|I^{-}h\|_{\mathcal{B}^j_\alpha} \le c_{-}\|h\|_{\mathcal{B}^j_\alpha} \tag{28}$$

for some constant c_{-} which only depends upon j and α.

3.3. **Contraction Estimate.** The stage is set to show that the system (22) has a global solution corresponding to sufficiently small data. This is accomplished by applying the contraction mapping theorem to the operator $\mathcal{T}$ defined via the right-hand side of the system (22). The first step is to choose an appropriate Banach space that $\mathcal{T}$ maps to itself. The crux of the matter is then to find a closed subset X, say, of this Banach space that $\mathcal{T}$ maps to itself and on which $\mathcal{T}$ is contractive.

To begin, note that if $w_0 \in B_0$, has mean zero and α satisfies (25), then $e^{-\sigma(\mathcal{A})t}w_0 \in \mathcal{B}^1_\alpha$ and has mean zero for all $t \ge 0$. Indeed, $e^{-\sigma(\mathcal{A})t}$ is a bounded linear operator from B_0 to $\mathcal{B}^1_\alpha$ provided $\alpha > 0$ is small enough. It follows from this remark and the mapping properties exposed in Section 3.2 that $\mathcal{T}$ maps the set of mean zero pairs in $\mathcal{B}^1_\alpha \times \mathcal{B}^1_\alpha$ to itself as long as α satisfies (25). Let X be the closed ball of radius $r > 0$ centered at the origin in $\mathcal{B}^1_\alpha \times \mathcal{B}^1_\alpha$, where r is to be determined. We claim that $\mathcal{T}$ is contractive on X provided r is chosen small enough.

First, local Lipschitz estimates for the nonlinearities F and G are established. Let $(v_1, w_1), (v_2, w_2) \in \mathcal{B}^1_\alpha \times \mathcal{B}^1_\alpha$ be two elements of X. Note that from (10), it is clear that the operator Θ^{-1} acts like one derivative and that Θ smooths by one derivative. Also recall that the Hilbert transform $\mathscr{H}$ is a bounded operator of order zero. Using the definition of F in (13) and the definition of the norm on $\mathcal{B}^1_\alpha$, it is thus inferred that

$$\begin{aligned}\left\|F(v_1,w_1) - F(v_2,w_2)\right\|_{\mathcal{B}^1_\alpha} &= \left\|\partial_x \Theta \mathcal{H}\left(w_1\mathcal{H}\Theta^{-1}v_1 - w_2\mathcal{H}\Theta^{-1}v_2\right)\right\|_{\mathcal{B}_\alpha} \\ &\le c\left\|w_1\mathcal{H}\Theta^{-1}v_1 - w_2\mathcal{H}\Theta^{-1}v_2\right\|_{\mathcal{B}_\alpha}.\end{aligned}$$

The fact that $\mathcal{B}_\alpha$ is an algebra together with the triangle inequality leads to the further estimate

$$\left\|F(v_1,w_1)-F(v_2,w_2)\right\|_{\mathcal{B}^1_\alpha} \le c\|w_1\|_{\mathcal{B}_\alpha}\|\partial_x(v_1-v_2)\|_{\mathcal{B}_\alpha}+\|w_1-w_2\|_{\mathcal{B}_\alpha}\|\partial_x v_2\|_{\mathcal{B}_\alpha}.$$

As (v_1, w_1) and (v_2, w_2) both lie in X, this in turn implies that

$$\left\|F(v_1,w_1) - F(v_2,w_2)\right\|_{\mathcal{B}^1_\alpha} \le rC_F\left\|(v_1-v_2, w_1-w_2)\right\|_{\mathcal{B}^1_\alpha\times\mathcal{B}^1_\alpha} \tag{29}$$

for a suitable constant C_F that is independent of r. The corresponding Lipschitz estimate for G is similar and easier to derive. The final result is

$$\|G(v_1,w_1) - G(v_2,w_2)\|_{\mathcal{B}^1_\alpha} \le rC_G\|w_1 - w_2\|_{\mathcal{B}^1_\alpha} \tag{30}$$

for a possibly different constant C_G which is still independent of r.

It follows immediately from the mapping properties of the linear operators I^+ and I^- derived in Subsection 3.2 that $I^\pm F$ and $I^\pm G$ are both Lipschitz on X with Lipschitz constants bounded by terms of the form Cr where C can be chosen independently of small values of r. The mapping

$$\hat{h} \mapsto 2\pi ik \int_0^\infty e^{-\sigma(\mathcal{A})(k)\tau}\hat{h}(k,\tau)d\tau$$

defined via its Fourier transform is linear and bounded from $\mathcal{B}^1_\alpha$ into mean zero elements of B_0 because α satisfies (25). Composing with $e^{-\sigma(\mathcal{A})t}$ then provides a bounded linear operator from $\mathcal{B}^1_\alpha$ to itself. It thus follows from (29) and (30) again that the contribution from I_0 is Lipschitz on X with a Lipschitz constant of the form Cr with C independent of small values of r.

In light of this discussion, it is evident that as long as r is sufficiently small,

$$\big\|\mathcal{T}(v_1,w_1) - \mathcal{T}(v_2,w_2)\big\|_{\mathcal{B}^1_\alpha\times\mathcal{B}^1_\alpha} \le \lambda\big\|(v_1,w_1) - (v_2,w_2)\big\|_{\mathcal{B}^1_\alpha\times\mathcal{B}^1_\alpha}, \tag{31}$$

for some $\lambda < 1$. Indeed, by choosing r small enough, we can guarantee that the Lipschitz constant λ for $\mathcal{T}$ can be taken to be at most $\frac{1}{2}$, say. Notice that the choice of r does not depend in a direct way upon w_0. It does depend upon the values of the mapping constants $c_\pm$ in (27) and (28) together with the constants C_F and C_G appearing in the local Lipschitz estimates (29) and (30).

Fix a value of $r = r(\alpha)$ such that $\lambda \le \frac{1}{2}$, let $(v,w) \in X$ and focus upon the inequality

$$\begin{aligned}\big\|\mathcal{T}(v,w)\big\|_{\mathcal{B}^1_\alpha\times\mathcal{B}^1_\alpha} &\le \big\|\mathcal{T}(0,0)\|_{\mathcal{B}^1_\alpha\times\mathcal{B}^1_\alpha} + \big\|\mathcal{T}(v,w) - \mathcal{T}(0,0)\big\|_{\mathcal{B}^1_\alpha\times\mathcal{B}^1_\alpha} \\ &\le \big\|\mathcal{T}(0,0)\|_{\mathcal{B}^1_\alpha\times\mathcal{B}^1_\alpha} + \frac{1}{2}\big\|(v,w)\big\|_{\mathcal{B}^1_\alpha\times\mathcal{B}^1_\alpha} \\ &\le \big\|\mathcal{T}(0,0)\big\|_{\mathcal{B}^1_\alpha\times\mathcal{B}^1_\alpha} + \frac{1}{2}r.\end{aligned}$$

This calculation calls attention to the norm of $\mathcal{T}(0,0)$. As already remarked, both F and G vanish at the origin, so

$$\mathcal{T}(0,0) = \begin{pmatrix} -e^{-\sigma(\mathcal{A})t}\hat{w}_0 \\ e^{-\sigma(\mathcal{A})t}\hat{w}_0 \end{pmatrix}.$$

Since $e^{-\sigma(\mathcal{A})t}$ is a bounded linear operator from B_0 to $\mathcal{B}^1_\alpha$, there is an $r_0 > 0$ such that if $\|w_0\|_{B_0} \le r_0$, then $\|\mathcal{T}(0,0)\|_{\mathcal{B}^1_\alpha\times\mathcal{B}^1_\alpha} < \frac{1}{2}r$. In summary, if r is as fixed above and $\|w_0\|_{B_0} \le r_0$, then $\mathcal{T}$ maps X to itself and is contractive there. The value of r_0 only depends upon r and the norm of the operator $e^{-\sigma(\mathcal{A})t}$ considered as a mapping of B_0 to $\mathcal{B}^1_\alpha$. The following theorem results.

Theorem 1. *Let α satisfy (25). There is an $r_0 = r_0(\alpha) > 0$ and a constant $C = C(\alpha)$ such that if $0 < r < r_0$, then for $w_0 \in B_0$ with $\|w_0\|_{B_0} \leq r$, there exists a solution $(v, w) \in \mathcal{B}^1_\alpha \times \mathcal{B}^1_\alpha$ that solves the system* (22) *and*

$$\|(v, w)\|_{\mathcal{B}^1_\alpha \times \mathcal{B}^1_\alpha} \leq Cr. \tag{32}$$

4. Ill-Posedness in Sobolev Spaces

The result of Duchon and Robert [13] on global small solutions for the vortex sheet problem does in fact imply that the vortex sheet problem is ill-posed in Sobolev spaces. In the present section, details are presented of an ill-posedness argument for the Kaup system which carries over *mutatis mutandis* to the vortex sheet problem via the theory developed in [13].

Theorem 2. *The Kaup system is ill-posed in Sobolev spaces. More precisely, for any $s_1, s_2 > 0$, there is a sequence $\{(\eta_0^n, w_0^n)\}_{n\in\mathbb{N}}$ of initial data in $H^{s_1}(\mathbb{T}) \times H^{s_2}(\mathbb{T})$ and positive times $\{t_n\}_{n\in\mathbb{N}}$, both of which tend to zero in their respective norms, such that*

$$\lim_{t\uparrow t_n} \big\|(\eta_n(\cdot, t), w_n(\cdot, t))\big\|_{H^{s_1}(\mathbb{T})\times H^{s_2}(\mathbb{T})} = +\infty.$$

Proof. Choose α in the range specified in (25) and let $r_0 = r_0(\alpha) > 0$ be the value appearing in Theorem 1. Let $w_0 \in B_0$ with $\|w_0\|_{B_0} = 1$ have mean zero and suppose that $w_0 \notin H^s(\mathbb{R})$ for all $s > 0$. Let $\{t_n\}_{n\in\mathbb{N}}$ be any sequence of positive times tending to zero as $n \to \infty$. Define a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ drawn from $(0, r_0)$ with $\varepsilon_n \to 0$ sufficiently rapidly as $n \to \infty$. What 'sufficiently rapidly' means will be made precise presently. Theorem 1 implies there are global solutions (v_n, w_n) of the transformed system (22) corresponding to the initial data $w_0^n = \varepsilon_n w_0$, that lie in $\mathcal{B}^1_\alpha \times \mathcal{B}^1_\alpha$, for each $n = 1, 2, \cdots$. Since $\|w_0^n\|_{B_0} = \varepsilon_n < r_0$ for all $n \in \mathbb{N}$, Theorem 1 further implies that

$$\big\|(v_n, w_n)\big\|_{\mathcal{B}^1_\alpha\times\mathcal{B}^1_\alpha} \leq C\varepsilon_n \tag{33}$$

for a constant C which is independent of n.

Retracing the change of dependent variables introduced in Subsection 3.1, define $\eta_n = \mathcal{H}\Theta^{-1} v_n$. Then, $(\eta_n, w_n) \in \mathcal{B}_\alpha \times \mathcal{B}^1_\alpha$. Moreover, (η_n, w_n) solves the original version (4) of the Kaup system with the mean zero initial data

$$\Big(-\mathcal{H}\Theta^{-1}\big(w_0^n + I_0(v_n, w_n)\big), w_0^n\Big).$$

Observe that w_0^n lies in $H^0(\mathbb{T})$, but not in any smaller L_2–based Sobolev class on account of the assumption about w_0. Also, because of (33),

$$\big\|(\eta_n, w_n)\big\|_{\mathcal{B}_\alpha\times\mathcal{B}^1_\alpha} \leq D\varepsilon_n \tag{34}$$

for another constant D which is also independent of n.

On the other hand, if $f \in \mathcal{B}_\alpha$ has mean zero and if $\bar{t} > 0$, then $f(\cdot,\bar{t}) \in B_{\alpha\bar{t}}$ and

$$\begin{aligned} \|f(\cdot,\bar{t})\|_{B_{\alpha\bar{t}}} &= \sum_{k\in\mathbb{Z}\setminus\{0\}} e^{\alpha\bar{t}|k|}|\hat{f}(k,\bar{t})| \\ &\le \sum_{k\in\mathbb{Z}\setminus\{0\}} \sup_{t\in[0,\infty)} e^{\alpha t|k|}|\hat{f}(k,t)| = \|f\|_{\mathcal{B}_\alpha}. \end{aligned} \tag{35}$$

It thus transpires that $f(\cdot,\bar{t}) \in H^s(\mathbb{T})$ for any $s \in \mathbb{R}$ (see Remark 1).

In light of (35) and (34), the solutions (η_n, w_n) at the time t_n can be estimated in $H^{s_1}(\mathbb{T}) \times H^{s_2}(\mathbb{T})$ as follows:

$$\left\| (\eta_n(\cdot,t_n), w_n(\cdot,t_n)) \right\|_{H^{s_1}(\mathbb{T})\times H^{s_2}(\mathbb{T})} \le C_n\varepsilon_n$$

for $n = 1, 2, \cdots$. The constant C_n depends on n because it depends on t_n. Indeed, according to (8) in Remark 1,

$$C_n \;\le\; E\frac{1}{(t_n)^s}$$

where E is a constant depending on α and s, but not on n. It remains to choose the $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0, r_0)$ so that it tends to zero faster than does the sequence $\{C_n\}_{n\in\mathbb{N}}$ determined by the given sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive times. Once this is done, the inequality

$$\left\| (\eta_n(\cdot,t_n), w_n(\cdot,t_n)) \right\|_{H^{s_1}(\mathbb{T})\times H^{s_2}(\mathbb{T})} \le \gamma_n$$

emerges, where $\gamma_n = C_n\varepsilon_n \to 0$ as $n \to \infty$.

Define new dependent variables

$$\tilde{\eta}_n(x,t) = \eta_n(-x, t_n - t) \quad \text{and} \quad \tilde{w}_n(x,t) = w_n(-x, t_n - t).$$

Since the Kaup system is time reversible and translation invariant, $(\tilde{\eta}_n, \tilde{w}_n)$ is also a solution when $t_n - t \ge 0$. At time $t = 0$,

$$\|(\tilde{\eta}_n(\cdot,0), \tilde{w}_n(\cdot,0))\|_{H^s(\mathbb{T})\times H^s(\mathbb{T})} \le \gamma_n$$

for $n = 1, 2, \cdots$, which can be made as small as desired by choosing n large enough. However, because of the choice of w_0,

$$\lim_{t\uparrow t_n} \|\tilde{w}_n(\cdot,t)\|_{H^s(\mathbb{T})} = +\infty.$$

This completes the proof of the theorem. □

5. Related Systems

While the above arguments were developed specifically for the Kaup system, they can be generalized to a variety of the *abcd*-systems (3). However, it will turn out that the analog of the operator Θ that appeared in the analysis of the Kaup system, and which is of order -1, may have a different order for other values of a, b, c and d. Because of this, other selections of the spaces $\mathcal{B}_\alpha^j$ will be needed.

In [4], the following choices were shown to ensure linear well-posedness of the initial-vaue problems on the real line for the *abcd*-systems:

$$b \geq 0, \quad d \geq 0, \quad a \leq 0, \quad c \leq 0; \tag{36}$$

$$b \geq 0, \qquad d \geq 0, \qquad a = c > 0; \tag{37}$$

$$b = d < 0, \qquad a = c > 0. \tag{38}$$

The authors in [4] referred to the case depicted in (36) as the generic case, for obvious reasons. Local well-posedness for the associated nonlinear initial-value problems on the real line was then established in [5]. Of the three classes just listed, we will not concern ourselves with the non-generic cases (37) or (38). Both of these cases depend upon the rigid condition $a = c$ to cancel out latent singularities in the dispersion. Their use in practice would be problematic because this delicate cancellation could be disrupted in the discrete world of numerical simulation unless special care was taken with the discretization. Moreover, for the class (38), notice that if either $b < 0$ or $d < 0$, there is a problem even with potential local existence. More precisely, suppose that $b < 0$. Then,

$$(I - b\partial_x^2)u_t(x,0) = -\partial_x \left(w_0 + w_0\eta_0 + a\partial_x^2 w_0\right).$$

The right-hand side of the last equation is a more or less arbitrary function $\partial_x W$ which lies in some Sobolev class determined by where one is working. Thus, in the Fourier variables, one is facing

$$(1 + bk^2)\partial_t\hat{\eta}(k,0) = 2\pi i k \hat{W}(k).$$

As $b < 0$, certain values of b would not allow the conculusion that η_t lies in *any* Sobolev space. (Indeed, for the problem posed on the whole real line, this is the case, no matter what the value of $b < 0$.)

Attention is now turned to ill-posedness when condition (36) is negated. This has us concerned with the two cases

$$b \geq 0, \quad d \geq 0, \quad a \leq 0, \quad c > 0 \tag{39}$$

and

$$b \geq 0, \quad d \geq 0, \quad a > 0, \quad c \leq 0. \tag{40}$$

The present argument requires a kind of nonresonance condition on the parameters a and c. Define the set S to be $S = \{(2\pi k)^{-2} : k \in \mathbb{Z} \setminus \{0\}\}$. The nonresonance condition is

$$a \notin S, \quad c \notin S. \tag{41}$$

Clearly, the Kaup system is a special case of (40), which satisfies (41).

The analysis begins as in the Kaup case. Define $u = \mathscr{H}\eta$, which immediately implies (since we continue to use the same function spaces, which includes maintaining the requirement that the data, and hence the solutions, have mean zero) $\eta = -\mathscr{H}u$. With this change of variable, the system (3) becomes

$$\begin{cases} \partial_t u + \Lambda w - \Lambda(w\mathscr{H}u) - a\Lambda^3 w - b\partial_x^2\partial_t u = 0, \\ \partial_t w - \Lambda u + w\partial_x w + c\Lambda^3 u - d\partial_x^2\partial_t w = 0. \end{cases} \tag{42}$$

Introduce the change of dependent variable $v = \Theta u$. As will appear shortly, the Fourier multiplier operator Θ will be invertible, so that $u = \Theta^{-1}v$. Of course, since it is a Fourier multiplier operator, Θ commutes with the Hilbert transform and differentiation. In terms of the new variable v, equation (42) becomes

$$\begin{cases} \partial_t v + \Lambda\Theta w - \Lambda\Theta(w\mathscr{H}\Theta^{-1}v) - a\Lambda^3\Theta w - b\partial_x^2\partial_t v = 0, \\ \partial_t w - \Lambda\Theta^{-1}v + w\partial_x w + c\Lambda^3\Theta^{-1}v - d\partial_x^2\partial_t w = 0. \end{cases} \tag{43}$$

In the case of the Kaup system, $b = d = 0$. This is no longer necessarily the case. It is propitious to invert the operators $(1 - b\partial_x^2)$ and $(1 - d\partial_x^2)$ in (43). At the same time, we take the opportunity to move the nonlinear terms to the right-hand sides of the equations, thereby coming to the system

$$\begin{cases} \partial_t v + (1 - b\partial_x^2)^{-1}(\Lambda - a\Lambda^3)\Theta w = (1 - b\partial_x^2)^{-1}\Lambda\Theta(w\mathscr{H}\Theta^{-1}v), \\ \partial_t w + (1 - d\partial_x^2)^{-1}(-\Lambda + c\Lambda^3)\Theta^{-1}v = -(1 - d\partial_x^2)^{-1}(w\partial_x w). \end{cases} \tag{44}$$

Choose the operator Θ so that

$$(1 - b\partial_x^2)^{-1}(\Lambda - a\Lambda^3)\Theta = (1 - d\partial_x^2)^{-1}(-\Lambda + c\Lambda^3)\Theta^{-1}$$

and notice that a common factor of Λ can be canceled from each side. Calculating in the Fourier transformed variables allows one to solve for the square $\Upsilon(k)$ of the symbol of Θ, *viz.*

$$\big(\sigma(\Theta)(k)\big)^2 = \frac{(1 + b(2\pi k)^2)(-1 + c(2\pi k)^2)}{(1 + d(2\pi k)^2)(1 - a(2\pi k)^2)} = \Upsilon(k). \tag{45}$$

Since (39), (40) and (41) are being assumed, both the numerator and denominator of (45) are nonzero for all $k \in \mathbb{Z} \setminus \{0\}$. Thus if the symbol of Θ is chosen to satisfy (45), then Θ is indeed an invertible Fourier multiplier operator.

Define sets Ω_+ and Ω_- to be

$$\Omega_+ = \{k \in \mathbb{Z} \setminus \{0\} : \Upsilon(k) > 0\} \quad \text{and} \quad \Omega_- = \{k \in \mathbb{Z} \setminus \{0\} : \Upsilon(k) < 0\}.$$

Under either set of assumptions (39) or (40), it is true that if $|k|$ is sufficiently large, then $\Upsilon(k) > 0$, and so Ω_- is a finite (possibly empty) set.

Keeping in mind the required property that Θ maps real-valued functions to real-valued functions, Θ is defined via its symbol by

$$\sigma(\Theta)(k) = \begin{cases} \gamma\sqrt{\Upsilon(k)}, & k \in \Omega_+, \\ i\sqrt{-\Upsilon(k)}, & k \in \Omega_-, \quad k > 0, \\ -i\sqrt{-\Upsilon(k)}, & k \in \Omega_-, \quad k < 0. \end{cases} \tag{46}$$

Here, the positive square root of the positive number under the radical is meant in all cases. The number γ is taken to be 1 if $a \geq 0$, whereas $\gamma = -1$ if $a < 0$. Clearly, this definition leads to a Θ that satisfies (45).

The order of the operator Θ varies depending on the values of the constants a, b, c, and d. This is important as it influences which function space is used in the analysis. More precisely, the associated operator $\mathcal{A}$ turns out to be

$$\mathcal{A} = -(1 - b\partial_x^2)^{-1}(\Lambda - a\Lambda^3)\Theta = (1 - d\partial_x^2)(\Lambda - c\Lambda^3)\Theta^{-1}.$$

The symbol of $\mathcal{A}$ may be calculated from (46). In case $k \in \Omega_+$, the symbol is

$$\sigma(\mathcal{A})(k) = 2\pi|k|\sqrt{\frac{(1 - a(2\pi k)^2)(-1 + c(2\pi k)^2)}{(1 + b(2\pi k)^2)(1 + d(2\pi k)^2)}}. \tag{47}$$

Clearly, the orders of $\mathcal{A}$ and Θ depend strongly on the values of the coefficients a, b, c and d. Table 1 details the cases.

Cases			Order of Θ	Order of $\mathcal{A}$	Name
$a = 0,\ c \neq 0$	$b = 0$	$d = 0$	1	2	(a)
		$d \neq 0$	0	1	(b)
	$b \neq 0$	$d = 0$	2	1	(c)
		$d \neq 0$	1	0	(d)
$a \neq 0,\ c \neq 0$	$b = 0$	$d = 0$	0	3	(e)
		$d \neq 0$	-1	2	(f)
	$b \neq 0$	$d = 0$	1	2	(g)
		$d \neq 0$	0	1	(h)
$a \neq 0,\ c = 0$	$b = 0$	$d = 0$	-1	2	(i)
		$d \neq 0$	-2	1	(j)
	$b \neq 0$	$d = 0$	0	1	(k)
		$d \neq 0$	-1	0	(l)

TABLE 1. The dependence of the orders of the operator Θ and $\mathcal{A}$ on the parameters. For cases (d) and (l), in which the order of $\mathcal{A}$ is zero, no result is available.

The remainder of the paper provides a sketch of the modifications of the ill-posedness argument, successful for the Kaup system, that are telling in the present, more general circumstances. Whether Ω_- is empty or not makes a difference at one step of the proof, so these cases are considered separately.

5.1. **When Ω_- is empty.** In the argument for the Kaup system, the operators I^+ and I^- were shown to be bounded linear operators from $\mathcal{B}^j_\alpha$ to $\mathcal{B}^j_\alpha$. The proof of this statement depended on the fact that the operator $\mathcal{A}$ was of order at least 1. For I^+, for example, our argument for the Kaup system required us to show that there exists an $\alpha > 0$ such that

$$\sup_{k\in\mathbb{Z}\setminus\{0\}} \sup_{t\in[0,\infty)} \left| \frac{|k|\left(1 - e^{t(\alpha|k|-\sigma(\mathcal{A})(k))}\right)}{\sigma(\mathcal{A})(k) - \alpha|k|} \right| < \infty. \tag{48}$$

Obviously, it is important at this stage that Ω_- is empty so that $\sigma(\mathcal{A})(k) \in \mathbb{R}$, for all k. Then, since $\mathcal{A}$ is of order at least one, for α sufficiently small,

$$|1 - e^{t(\alpha|k|-\sigma(\mathcal{A})(k))}| < 1, \quad \forall k \in \mathbb{Z}\setminus\{0\}, \quad \forall t \in [0,\infty).$$

For convenience, let $\alpha^* > 0$ be chosen so that for all $\alpha \in (0, \alpha^*)$, the bound (48) is satisfied for such operators $\mathcal{A}$. Similar considerations apply to the boundedness of I^-.

Another important point in the argument for the Kaup system is that the operators F and G appearing in (13) are maps from $\mathcal{B}^j_\alpha \times \mathcal{B}^j_\alpha$ to $\mathcal{B}^j_\alpha$ where $j = 1$. In the Kaup case, this depended on the fact that Θ was an operator of order -1. The analogue of (13) in the present cases is

$$\begin{cases} F(y,z) = (1 + b\partial_x^2)^{-1}\Theta\mathscr{H}\left(z\mathscr{H}\Theta^{-1}y\right), \\ G(y,z) = -\frac{1}{2}(1 + d\partial_x^2)^{-1}\left(z^2\right). \end{cases} \tag{49}$$

In cases where the order of Θ is less than or equal to 0 while the order of $\mathcal{A}$ is at least equal to one, the proof of the mapping properties of F and G offered in the Kaup case goes through without modification if j is taken to be the negative of the order of Θ. Thus it continues to be true that for any positive α, F and G map $\mathcal{B}^j_\alpha \times \mathcal{B}^j_\alpha$ to $\mathcal{B}^j_\alpha$ for the chosen value of j, with the desired quadratic estimate. This remark applies to cases (b), (e), (f), (h), (i), (j) and (k).

Cases (c) and (g) are similar, but additional benefit is gained from the condition $b \neq 0$. Indeed, because $b \neq 0$ in (c) and (g), it is again true that F maps $\mathcal{B}^j_\alpha$ to itself, and we choose j in these cases to be the order of Θ (that is to say, in case (c) we use the space $\mathcal{B}^2_\alpha$, while in case (g) $\mathcal{B}^1_\alpha$ is a suitable choice).

Continuing to suppose that Ω_- is empty the earlier argument is modified to deal with case (a). A more precise version of the mapping properties of I^+ and I^- are needed. Notice that for the Kaup case use was only made of the fact that I^+ and I^- are bounded from $\mathcal{B}^1_\alpha$ to $\mathcal{B}^1_\alpha$. However, a perusal of the calculations (24)-(27), especially (26), indicate that in fact I^+ and I^- are in fact bounded from $\mathcal{B}^1_\alpha$ to $\mathcal{B}^2_\alpha$. In case (a), the operator $\mathcal{A}$ is of order 2, so the same boundedness property holds. That is, both I^+ and I^- are smoothing operators on the relevant $\mathcal{B}^j_\alpha$ spaces. Inspecting (49), one notes that in case (a), F is bounded from $\mathcal{B}^1_\alpha$ to $\mathcal{B}^0_\alpha$. So, although F loses one derivative in this case, there is an offsetting gain of one derivative from I^+ and I^-. The remainder of the argument for case (a) is then the same as before.

Once these preliminaries are in hand, the contraction mapping argument of Section 3.3 may be used to find existence of global in time, small solutions and then the argument of Section 4 allows the conclusion of ill-posedness of the initial value problem in Sobolev spaces. We pass over the details.

Theorem 3. *Let a, b, c and d satisfy* (39) *or* (40)*, as well as* (41) *and one of the cases (a), (b), (c), (e), (f), (g), (h), (i), (j) or (k), and be such that the set Ω_- is empty. Let α^* be as above, and let $\alpha \in (0, \alpha^*)$ be given. Let j be as described above for the various cases.*

(i) There is an $r_0 > 0$ such that for all $w_0 \in B_0$ with $\|w_0\|_{B_0} < r_0$, there exists a unique solution (η, w) of (3) *lying in $\mathcal{B}^j_\alpha \times \mathcal{B}^j_\alpha$ such that $w(\cdot, 0) = w_0$.*

(ii) Let s_1, s_2 be given. There exist a sequence of initial data $(\eta_0^n, w_0^n) \in H^{s_1} \times H^{s_2}$ and a sequence of times $t_n \in \mathbb{R}$ such that $\|(\eta_0^n, w_0^n)\|_{H^{s_1}\times H^{s_2}} \to 0$ and $t_n \to 0$ as $n \to \infty$, but the corresponding solutions (η^n, w^n) satisfy

$$\lim_{t\uparrow t_n} \|(\eta^n, w^n)\|_{H^{s_1}\times H^{s_2}} = +\infty.$$

Thus, the initial value problem for (3) *is ill-posed in Sobolev spaces for such values of a, b, c, d.*

5.2. **When Ω_- is nonempty.** The estimate for the numerator in (48) is problematic if $\sigma(\mathcal{A})$ is imaginary for some k. Specifically, the term $e^{\alpha t|k|}$ is not controlled by the $\sigma(\mathcal{A})$ term for smaller values of $|k|$, so no matter how small the positive parameter α is chosen, $e^{\alpha t|k|-\sigma(\mathcal{A})(k)}$ will exceed any bound for large values of t. Something different must be entertained in this case.

One solution is to abandon global solutions and focus directly on ill-posedness. This is accomplished by working on a finite time interval rather than all of $\mathbb{R}^+$, but continuing to search for solutions which instantaneously become infinitely smooth whilst starting from nonsmooth initial data. The time reversibilty of the system then comes to our rescue just as before and ill-posedness is concluded. As much of the argument mirrors what has gone before, we content ourselves with an outline.

For any $T > 0$ and $\alpha > 0$ and $j \in \mathbb{N}$, define function spaces $\mathcal{B}^j_{\alpha,T}$ as follows. Consider first the auxiliary function $\beta : [0, T] \to [0, \alpha]$ given by

$$\beta(s) = \begin{cases} 2\alpha s/T, & s \in [0, T/2], \\ 2\alpha - 2\alpha s/T, & s \in [T/2, T]. \end{cases}$$

The norm of $f \in \mathcal{B}^j_{\alpha,T}$ is

$$\|f\|_{\mathcal{B}^j_{\alpha,T}} = \sum_{k\in\mathbb{Z}} \sup_{t\in[0,T]} (1 + |k|^j) e^{\beta(t)|k|} |\hat{f}(k, t)|.$$

These Wiener-algebra based spaces were introduced by the first author in work on mean field games [1], [2]. We remark that these spaces are Banach algebras, just are were the spaces $\mathcal{B}^j_\alpha$.

Finding solutions on intervals of the form $[0, T]$ requires specifying 'final' data as well as initial data. In the previous situation in which the time interval was $[0, \infty)$, we specified $w(\cdot, 0) = w_0$ as initial data and, implicitly, zero data at $t = \infty$. The data for the other dependent variable was then determined (see below (22)).

Thus, the view is that one is solving a boundary-value problem in space-time, so 'half' the boundary data should be specified. For the pair (v, w) on the interval $[0, T]$, specifying half the data means specifying one function at time $t = T$ in addition to continuing to assign $w(\cdot, 0) = w_0$. As explored in some detail in [17], there are several choices that can work, including giving $v(\cdot, T)$, $w(\cdot, T)$, $v_t(\cdot, T)$ or $w_t(\cdot, T)$. For the purposes at hand, take it that $v(\cdot, T) = v_T$ is assigned. This choice has the advantage of a simpler version of the resulting Duhamel formula than obtains for the other options.

Following the same procedure as previously leads to the representation formulas

$$\hat{v}(\cdot, t) = e^{-t\sigma(\mathcal{A})}\hat{f} + e^{(t-T)\sigma(\mathcal{A})}\hat{g} + \frac{1}{2}I^+(F-G)(\cdot, t) - \frac{1}{2}I^-(F+G)(\cdot, t), \tag{50}$$

$$\hat{w}(\cdot, t) = -e^{-t\sigma(\mathcal{A})}\hat{f} + e^{(t-T)\sigma(\mathcal{A})}\hat{g} - \frac{1}{2}I^+(F-G)(\cdot, t) - \frac{1}{2}I^-(F+G)(\cdot, t) \tag{51}$$

where k has been suppressed again for ease of reading. The definition of I^+ is the same as before, but I^- is given as the integral

$$I^- h(k, t) = 2\pi i k \int_t^T e^{\sigma(\mathcal{A})(k)(t-\tau)}\hat{h}(k, \tau)\, d\tau,$$

to reflect the finite temporal domain. For a function $h \in \mathcal{B}^j_{\alpha,T}$, define $I_0 h(k) = I^- h(k, 0)$ and $I_T h(k) = I^+ h(k, T)$. The formulas

$$\begin{cases} \hat{f} = \left(1 + e^{-2T\sigma(\mathcal{A})}\right)^{-1}\Big[-\hat{w}_0 - \frac{1}{2}I_0(F+G) \\ \qquad\qquad + e^{-T\sigma(\mathcal{A})}\left(\hat{v}_T - \frac{1}{2}I_T(F-G)\right)\Big], \\ \hat{g} = \left(1 + e^{-2T\sigma(\mathcal{A})}\right)^{-1}\Big[\hat{v}_T - \frac{1}{2}I_T(F-G)) \\ \qquad\qquad + e^{-T\sigma(\mathcal{A})}(\hat{w}_0 + \frac{1}{2}I_0(F+G))\Big], \end{cases} \tag{52}$$

for $\hat{f}$ and $\hat{g}$ in terms of the boundary data w_0 and v_T are then forthcoming. The wavenumber k has again been suppressed. These formulas then allow one to infer initial data for v and final data for w and full representation formulas for both v and w.

The primary remaining ingredient is the proof that I^+ and I^- are bounded linear operators on the newly-defined function spaces. We present the details only for I^+, those for I^- being entirely similar. Let m denote the order of the operator $\mathcal{A}$. We claim that for sufficiently small positive values of α, the operator I^+ is bounded from $\mathcal{B}^j_{\alpha,T}$ to $\mathcal{B}^{j+m-1}_{\alpha,T}$.

To begin, for $h \in \mathcal{B}^j_{\alpha,T}$, write out the norm of I^+h, and split this into the contributions from Ω_- and Ω_+:

$$\|I^+h\|_{\mathcal{B}^{j+m-1}_{\alpha,T}} = \sum_{k\in\mathbb{Z}} \sup_{t\in[0,T]} (1+|k|^{j+m-1})e^{\beta(t)|k|} \left(2\pi|k| \left|\int_0^t e^{\sigma(\mathcal{A})(k)(\tau-t)}\hat{h}(k,\tau)\ d\tau\right|\right)$$
$$= \sum_{k\in\Omega_-} \sup_{t\in[0,T]} (1+|k|^{j+m-1})e^{\beta(t)|k|} \left(2\pi|k| \left|\int_0^t e^{\sigma(\mathcal{A})(k)(\tau-t)}\hat{h}(k,\tau)\ d\tau\right|\right)$$
$$+ \sum_{k\in\Omega_+} \sup_{t\in[0,T]} (1+|k|^{j+m-1})e^{\beta(t)|k|} \left(2\pi|k| \left|\int_0^t e^{\sigma(\mathcal{A})(k)(\tau-t)}\hat{h}(k,\tau)\ d\tau\right|\right)$$

Recall that the focus is upon when Ω_- is a nonempty. As seen earlier, it is always a finite set and for all $k \in \Omega_-$, the symbol $\sigma(\mathcal{A})$ is purely imaginary. By contrast, Ω_+ is infinite and for all $k \in \Omega_+$, the symbol $\sigma(\mathcal{A})$ is real.

Attention is first given to the contribution to the norm stemming from Ω_-. Take the value k_* to be

$$k_* = \max_{k\in\Omega_-} |k|.$$

Since $0 \le \beta(s) \le \alpha$ for $s \in [0,T]$ the triangle inequality and the fact that $\sigma(\mathcal{A})$ is purely imaginary implies

$$\sum_{k\in\Omega_-} \sup_{t\in[0,T]} (1+|k|^{j+m-1})e^{\beta(t)|k|} \left(2\pi|k| \left|\int_0^t e^{\sigma(\mathcal{A})(k)(\tau-t)}\hat{h}(k,\tau)\ d\tau\right|\right) \tag{53}$$
$$\le 2\pi k_*^m e^{\alpha k_*} \sum_{k\in\Omega_-} \sup_{t\in[0,T]} (1+|k|^j) \int_0^t |\hat{h}(k,\tau)|\ d\tau.$$
$$\le 2\pi k_*^m e^{\alpha k_*} \sum_{k\in\Omega_-} \sup_{t\in[0,T]} (1+|k|^j) \int_0^t \left(\sup_{s\in[0,t]} e^{\beta(s)|k|}|\hat{h}(k,s)|\right)\ d\tau$$
$$\le 2\pi k_*^m e^{\alpha k_*} T \sum_{k\in\mathbb{Z}} \sup_{t\in[0,T]} (1+|k|^j)e^{\beta(t)|k|}|\hat{h}(k,t)| = 2\pi k_*^m e^{\alpha k_*} T \|h\|_{\mathcal{B}^j_{\alpha,T}}.$$

Consider next the contribution from Ω_+. Start with the elementary estimate

$$\sum_{k\in\Omega_+} \sup_{t\in[0,T]} (1+|k|^{j+m-1})e^{\beta(t)|k|} \left(2\pi|k| \left|\int_0^t e^{\sigma(\mathcal{A})(k)(\tau-t)}\hat{h}(k,\tau)\ d\tau\right|\right)$$
$$\le 4\pi \sum_{k\in\Omega_+} \sup_{t\in[0,T]} |k|^{j+m}e^{\beta(t)|k|} \int_0^t e^{\sigma(\mathcal{A})(k)(\tau-t)}|\hat{h}(k,\tau)|\ d\tau.$$

Multiply and divide by $e^{\beta(\tau)|k|}$ and take the supremum of $|e^{\beta(\tau)|k|}\hat{h}(k,\tau)|$. Pulling this through the integral and making additional elementary bounds leads to

$$(54)\quad \sum_{k\in\Omega_+}\sup_{t\in[0,T]}(1+|k|^{j+m-1})e^{\beta(t)|k|}\left(2\pi|k|\left|\int_0^t e^{\sigma(\mathcal{A})(k)(\tau-t)}\hat{h}(k,\tau)\,d\tau\right|\right)$$

$$\le 4\pi\sum_{k\in\Omega_+}\sup_{t\in[0,T]}|k|^{j+m}e^{\beta(t)|k|}\int_0^t e^{\sigma(\mathcal{A})(k)(\tau-t)-\beta(\tau)|k|}|e^{\beta(\tau)|k|}\hat{h}(k,\tau)|\,d\tau$$

$$\le 4\pi\left(\sum_{k\in\mathbb{Z}}(1+|k|^j)\sup_{t\in[0,T]}e^{\beta(t)|k|}|\hat{h}(k,t)|\right)$$

$$\times\left(\sup_{k\in\Omega_+}\sup_{t\in[0,T]}|k|^m e^{\beta(t)|k|-\sigma(\mathcal{A})(k)t}\int_0^t e^{\sigma(\mathcal{A})(k)\tau-\beta(\tau)|k|}\,d\tau\right)$$

$$= 4\pi\|h\|_{\mathcal{B}^j_{\alpha,T}}\left(\sup_{k\in\Omega_+}\sup_{t\in[0,T]}|k|^m e^{\beta(t)|k|-\sigma(\mathcal{A})(k)t}\int_0^t e^{\sigma(\mathcal{A})(k)\tau-\beta(\tau)|k|}\,d\tau\right).$$

To conclude that I^+ is bounded between the given spaces, it suffices to verify that the final quantity in parentheses on the right-hand side of (54) is finite.

For this task, examine separately $t\in[0,T/2]$ and $t\in[T/2,T]$. For $t\in[0,T/2]$, the integration variable $\tau\in[0,T/2]$ so that $\beta(\tau)=2\alpha\tau/T$. In consequence, we have exactly that

$$\int_0^t e^{\sigma(\mathcal{A})(k)\tau-\beta(\tau)|k|}\,d\tau=\int_0^t e^{\sigma(\mathcal{A})(k)\tau-2\alpha\tau|k|/T}\,d\tau=\frac{e^{\sigma(\mathcal{A})(k)t-2\alpha t|k|/T}-1}{\sigma(\mathcal{A})(k)-\frac{2\alpha|k|}{T}}.$$

Now consider the quantity

$$(55)\quad \sup_{k\in\Omega_+}\sup_{t\in[0,T/2]}|k|^m e^{\beta(t)|k|-\sigma(\mathcal{A})(k)t}\int_0^t e^{\sigma(\mathcal{A})(k)\tau-\beta(\tau)|k|}\,d\tau$$

$$=\sup_{k\in\Omega_+}\sup_{t\in[0,T/2]}|k|^m e^{2\alpha t|k|/T-\sigma(\mathcal{A})(k)t}\int_0^t e^{\sigma(\mathcal{A})(k)\tau-\beta(\tau)|k|}\,d\tau$$

$$\le\sup_{k\in\Omega_+}\frac{|k|^m}{\sigma(\mathcal{A})(k)-\frac{2\alpha|k|}{T}}\sup_{t\in[0,T/2]}\left(1-e^{2\alpha t|k|/T-\sigma(\mathcal{A})(k)t}\right).$$

Referring back to equation (47), it is observed that $\sigma(\mathcal{A})$ is positive for all $k\in\Omega_+$. Together with the assumption that the order of $\mathcal{A}$ is $m\ge 1$, it is concluded that there exists $\alpha_*>0$ such that for all $\alpha\in(0,\alpha_*)$ and all $k\in\Omega_+$, $|k|^m/(\sigma(\mathcal{A})(k)-2\alpha|k|/T)>0$. Using again that $\mathcal{A}$ is of order m, it then follows immediately that

$$(56)\quad \sup_{k\in\Omega_+}\sup_{t\in[0,T/2]}|k|^m e^{\beta(t)|k|-\sigma(\mathcal{A})(k)t}\int_0^t e^{\sigma(\mathcal{A})(k)\tau-\beta(\tau)|k|}\,d\tau$$

$$\le\sup_{k\in\Omega_+}\frac{|k|^m}{\sigma(\mathcal{A})(k)-\frac{2\alpha|k|}{T}}<c.$$

Now consider $k \in \Omega_+$ with $t \in [T/2, T]$ and calculate the integral

$$\begin{aligned}\int_0^t & e^{\sigma(\mathcal{A})(k)\tau - \beta(\tau)|k|}\ d\tau \\ &= \int_0^{T/2} e^{\sigma(\mathcal{A})(k)\tau - 2\alpha\tau|k|/T}\ d\tau + \int_{T/2}^t e^{\sigma(\mathcal{A})(k)\tau - 2\alpha|k| + 2\alpha\tau|k|/T}\ d\tau \\ &= \frac{e^{\sigma(\mathcal{A})T/2 - \alpha|k|} - 1}{\sigma(\mathcal{A})(k) - \frac{2\alpha|k|}{T}} + e^{-2\alpha|k|}\left(\frac{e^{\sigma(\mathcal{A})(k)t + 2\alpha t|k|/T} - e^{\sigma(\mathcal{A})(k)T/2 + \alpha|k|}}{\sigma(\mathcal{A})(k) + \frac{2\alpha|k|}{T}}\right)\end{aligned}$$

from the right-hand side of (54). Arguing as above, we find that

$$\sup_{k\in\Omega_+}\sup_{t\in[T/2,T]} |k|^m e^{\beta(t)|k| - \sigma(\mathcal{A})(k)t}\int_0^t e^{\sigma(\mathcal{A})(k)\tau - \beta(\tau)|k|}\ d\tau \leq B_1 + B_2, \tag{57}$$

where

$$B_1 = \sup_{k\in\Omega_+}\sup_{t\in[T/2,T]}\left(|k|^m e^{2\alpha|k| - 2\alpha|k|t/T - \sigma(\mathcal{A})(k)t}\right)\left(\frac{e^{\sigma(\mathcal{A})(k)T/2 - \alpha|k|} - 1}{\sigma(\mathcal{A})(k) - \frac{2\alpha|k|}{T}}\right),$$

and

$$\begin{aligned}B_2 = \sup_{k\in\Omega_+}\sup_{t\in[T/2,T]}&\left(|k|^m e^{-2\alpha|k|t/T - \sigma(\mathcal{A})(k)t}\right)\times \\ &\left(\frac{e^{\sigma(\mathcal{A})(k)t + 2\alpha t|k|/T} - e^{\sigma(\mathcal{A})(k)T/2 + \alpha|k|}}{\sigma(\mathcal{A})(k) + \frac{2\alpha|k|}{T}}\right).\end{aligned}$$

For $\alpha \in (0, \alpha_*)$, it is the case that $\sigma(\mathcal{A})(k) - 2\alpha|k|/T > 0$. Thus the negative term in the numerator of B_1 can be discarded. A small caclulation using the fact that $t \geq T/2$ then reveals that that the remaining exponential has a negative argument. It thus transpires that

$$B_1 \leq \sup_{k\in\Omega_+}\frac{|k|^m}{\sigma(\mathcal{A})(k) - \frac{2\alpha|k|}{T}} < c, \tag{58}$$

where this is actually the same constant c as appears in (56). To bound B_2, first neglect the negative term in the numerator, and then note that for what remains, the exponentials cancel. Thus the estimate

$$B_2 \leq \sup_{k\in\Omega_+}\frac{|k|^m}{\sigma(\mathcal{A})(k) + \frac{2\alpha|k|}{T}} \leq \sup_{k\in\Omega_+}\frac{|k|^m}{\sigma(\mathcal{A})(k) - \frac{2\alpha|k|}{T}} < c \tag{59}$$

obtains, where the constant c is as before.

Combining (53), (54), (56), (57), (58) and (59), it is deduced that if $\alpha \in (0, \alpha_*)$, then I^+ is indeed a bounded linear operator from $\mathcal{B}^j_{\alpha,T}$ to $\mathcal{B}^{j+m-1}_{\alpha,T}$, with operator norm bounded above by $2\pi k_*^m e^{\alpha k_*}T + 8\pi c$.

As already mentioned, the details for I^- are completely analagous, and the conclusion is the same.

With these bounds on I^+ and I^- in hand, the representatation formulas obtaining by using (52) in (50) and (51) provides an operator equation for $(\hat{v}, \hat{w})$ to which the contraction mapping theorem may be applied, just as in Section 3.3. The existence of these solutions and the instantaneous gain of analyticity, then implies ill-posedness of the initial value problem in Sobolev spaces, as in Section 4 above. These ruminations are summarized in the next theorem.

Theorem 4. *Let a, b, c and d be given satisfying* (39), (40), (41), *one of the cases (a), (b), (c), (e), (f), (g), (h), (i), (j), or (k) and are such that the set Ω_- is nonempty. Let α_* be as above and $\alpha \in (0, \alpha_*)$ be given. Let j be as described above.*

(i) For any $T > 0$, there is an $r_0 > 0$ so that if $(w_0, v_T) \in B_0 \times B_0$ satisfies $\|(w_0, v_T)\|_{B_0 \times B_0} < r_0$, then there is a unique $(v, w) \in \mathcal{B}^j_{\alpha,T} \times \mathcal{B}^j_{\alpha,T}$, such that (η, w) satisfy (43), *with $v(\cdot, T) = v_T$ and $w(\cdot, 0) = w_0$.*

(ii) Let s_1, $s_2 > 0$ be given. There exist a sequence of initial data $(v_0^n, w_0^n) \in H^{s_1} \times H^{s_2}$ and a sequence of times $t_n \in \mathbb{R}$ such that $t_n \to 0$ and $\|(v_0^n, w_0^n)\|_{H^{s_1} \times H^{s_2}} \to 0$ as $n \to \infty$ for which the corresponding solutions (v^n, w^n) satisfy

$$\lim_{t \uparrow t_n} \|(v^n, w^n)\|_{H^{s_1} \times H^{s_2}} = +\infty.$$

Thus, the initial value problem for (43) *is ill-posed in Sobolev spaces for such values of a, b, c and d.*

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References

[1] D.M. Ambrose. Small strong solutions for time-dependent mean field games with local coupling. *C. R. Math. Acad. Sci. Paris*, **354**, (2016), 589–594.

[2] D.M. Ambrose. Strong solutions for time-dependent mean field games with non-separable Hamiltonians. *J. Math. Pures Appl.*, (2016), In press.

[3] C. J. Amick. Regularity and uniqueness of solutions to the Boussinesq system of equations. *J. Differential Eq.* **54**, (1984), 231–247.

[4] J.L. Bona, M. Chen, and J.-C. Saut. Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I. Derivation and linear theory. *J. Nonlinear Sci.*, **12**, (2002), 283–318.

[5] J.L. Bona, M. Chen, and J.-C. Saut. Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. II. The nonlinear theory, *Nonlinearity*, **17**, (2004), 925–952.

[6] J.L. Bona, T. Colin and D. Lannes. Long wave approximations for water waves, *Arch. Rattional Mech. Anal.* **178**, (2008), 373–410.

[7] J. Boussinesq. Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond. *J. Math. Pures Appl.* **17**, (1872), 55–108.
[8] J. Boussinesq. Essai sur la théorie des eaux courantes, Mémoires a l'Académie des Sciences de l'Institut de France, tome XXIII, no. 1 (1877) 1–680 and tome XXIII, no. 2, Additions et Éclaircissements 1–64.
[9] R.E. Caflisch and O.F. Orellana. Singular solutions and ill-posedness for the evolution of vortex sheets. *SIAM J. Math. Anal.*, **20**, (1989), 293–307.
[10] H. Chen and J.L. Bona. Comparison of model equations for small-amplitude long waves, *Nonlinear Anal. TMA, Series B* **38**, (1999), 625-647.
[11] A. Craik. The origins or water wave theory. *Annual Rev. Fluid Mech.* **36**, (2004), 1–28.
[12] A. Craik. George Gabriel Stokes on water wave theory. *Annual Rev. Fluid Mech.* **37**, (2005), 23–42.
[13] J. Duchon and R. Robert. Global vortex sheet solutions of Euler equations in the plane. *J. Differential Eq.*, **73**, (1988), 215–224.
[14] D. Kaup. A higher-order water-wave equation and the method for solving it. *Prog. Theoretical Phys.* **54** (1975) 396–408.
[15] M. Kruskal. Nonlinear wave equations, theory and application. In Dynamical Systems Theory and Applications, *Lecture Notes Phys.* **38**, (1975), 310–354.
[16] J.K. Hunter and B. Nachtergaele. Applied Analysis. World Scientific Publishing Company, River Edge, NJ, USA. 2001.
[17] T. Milgrom and D.M. Ambrose. Temporal boundary value problems in interfacial fluid dynamics. *Applicable Analysis*, **92**, (2013), 922-948.
[18] M. E. Schonbek. Existence of solutions for the Boussinesq system of equations. *J. Differential Eq.* **42**, (1981), 325–352.
[19] G. B. Whitham. Linear and nonlinear waves. John Wiley & Sons, New York, 1974.
[20] S. Wu. Mathematical analysis of vortex sheets. *Comm. Pure Appl. Math.*, **59**, (2006), 1065–1206.

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