Math 160 Discussion Notes
Brian Powers – TA – Fall 2011

2.1 & 2.2 Solving Systems of Linear Equations

A **linear equation** is an equation of the form $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$ where $a_1, a_2, \ldots, a_n$, and b are real numbers and $x_1, x_2, \ldots, x_n$ are variables. For example, 3x+5y-2z=5

A **system of linear equations** are a number of equations that all exist together. A system of m equations of n variables looks like this:

$$\begin{cases} a_{1,1} x_1 + a_{1,2} x_2 + \cdots + a_{1,n} x_n = b_1 \\ a_{2,1} x_1 + a_{2,2} x_2 + \cdots + a_{2,n} x_n = b_2 \\ \vdots \\ a_{m,1} x_1 + a_{m,2} x_2 + \cdots + a_{m,n} x_n = b_m \end{cases}$$

For example,

$$\begin{cases} x \quad -3y \ +4z = 1 \\ 4x \ -10y + 10z = 4 \\ -3x + 9y \ -5z = -6 \end{cases}$$

A **solution to the system** would be values x_1, x_2, … , x_n that simultaneously satisfy all m equations. If the system that has no solution is called **inconsistent**. Otherwise the system is **consistent**. A consistent system either has one solution or infinite solutions.

We have some basic methods for solving these systems. We can:

- Use substitution: solve one equation in terms of one of the variables, then substitute this into the other equations. Repeat this until you solve for one variable, then use back-substitution to get the values of the other variables.
- Add or Subtract two equations

We will use a more sophisticated method called Gauss-Jordan Elimination.

Elementary Row Operations

There are three operations we can do to a system that does not change the solution:

1. Swap the position of two equations in the system
2. Multiply both sides of an equation by a non-zero constant number
3. Add a non-zero multiple of one equation to another equation in the system.

We will use these three operations, but rather tan operate on the equations, we will simplify the notation by using matrices.

A Matrix is a rectangular grid of numbers. We refer to the dimensions of a matrix as the number of rows x the number of columns. Thus a 2x3 matrix has 2 rows and 3 columns. We typically use capital letters to refer to a matrix, and lower case letters to refer to the entries of the matrix, or the numbers in the grid. The entries are addressed with subscripts, giving first the row, then column. In general, we have the nxm matrix A:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{bmatrix}$$

We can use matrix notation to solve a system of equations.

To represent the system $\begin{cases} x & -3y & +4z = 1 \\ 4x & -10y + 10z = 4 \\ -3x + 9y & -5z = -6 \end{cases}$ as a matrix, we write $\left[\begin{array}{ccc|c} 1 & -3 & 4 & 1 \\ 4 & -10 & 10 & 4 \\ -3 & 9 & -5 & -6 \end{array}\right]$ although the vertical line can be omitted (it is only added for clarity).

The three equation operations become three row operations:

1. Swap the positions of two rows of the matrix
2. Multiply a row by a non-zero constant
3. Add a non-zero multiple of one row to another row

The goal is to use row operations to arrive at a matrix of the form $\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$ (which is called reduced row eschelon form) which translates to the solution x=a, y=b, z=c. We do this by using the Gauss-Jordan method.

Pivot

To **pivot** on non-zero entry of the matrix, we first divide the row by the value of this entry, we then add a multiple of this row to each other row of the matrix to get a zero in the column above and below this pivoted entry.

ex) Pivot the entry $e_{2,4}$ in the matrix $\boldsymbol{E = \begin{bmatrix} 7 & -9 & 12 & 6 & 5 \\ -6 & 12 & 9 & -3 & 3 \\ 2 & 5 & 8 & -11 & 6 \\ -4 & -2 & 8 & 12 & -10 \end{bmatrix}}$

$e_{2,4}$ is circled here $\begin{bmatrix} 7 & -9 & 12 & 6 & 5 \\ -6 & 12 & 9 & (-3) & 3 \\ 2 & 5 & 8 & -11 & 6 \\ -4 & -2 & 8 & 12 & -10 \end{bmatrix}$. The first step is to divide row 2 by -3

$$\begin{array}{c} \frac{-1}{3}R_2 \\ \rightarrow \end{array} \begin{bmatrix} 7 & -9 & 12 & 6 & 5 \\ 2 & -4 & -3 & (1) & -1 \\ 2 & 5 & 8 & -11 & 6 \\ -4 & -2 & 8 & 12 & -10 \end{bmatrix}.$$

Next we subtract $6R_2$ from R_1, add $11R_2$ to R_3, and subtract 12R2 from R4:

$$\begin{array}{c} R_1 - 6R_2 \\ R_3 + 11R_2 \\ R_4 - 12R_2 \\ \rightarrow \end{array} \begin{bmatrix} 7-12 & -9+24 & 12+18 & 6-6 & 5+6 \\ 2 & -4 & -3 & (1) & -1 \\ 2+22 & 5-44 & 8-33 & -11+11 & 6-11 \\ -4-24 & -2+48 & 8+36 & 12-12 & -10+12 \end{bmatrix} = \begin{bmatrix} -5 & 13 & 30 & 0 & 11 \\ 2 & -4 & -3 & (1) & -1 \\ 24 & -39 & -25 & 0 & -5 \\ -28 & 46 & 44 & 0 & 2 \end{bmatrix}$$

The Gauss-Jordan method for solving a sytems is done like this:

1) if the leading non-zero entry of the first row is a zero, swap this row with a row below it that has a non-zero leading entry
2) Pivot The leading non-zero entry of row 1
3) now consider the next column and row 2... continue in this manner.

ex) Solve the system $\boldsymbol{\begin{cases} x & -3y & +4z = 1 \\ 4x & -10y + 10z = 4 \\ -3x + 9y & -5z = -6 \end{cases}}$

In matrix form, we have $\begin{bmatrix} 1 & -3 & 4 & 1 \\ 4 & -10 & 10 & 4 \\ -3 & 9 & -5 & -6 \end{bmatrix}$

We pivot entry 1,1 $\begin{matrix} R_2-4R_1 \\ R_3+3R_1 \\ \to \end{matrix}\begin{bmatrix} 1 & -3 & 4 & 1 \\ 4-4 & -10+12 & 10-16 & 4-4 \\ -3+3 & 9-9 & -5+12 & -6+3 \end{bmatrix}=\begin{bmatrix} 1 & -3 & 4 & 1 \\ 0 & 2 & -6 & 0 \\ 0 & 0 & 7 & -3 \end{bmatrix}$

We next pivot entry 2,2 $\begin{matrix} R_2/2 \\ \to \end{matrix}\begin{bmatrix} 1 & -3 & 4 & 1 \\ 0 & 2/2 & -6/2 & 0 \\ 0 & 0 & 7 & -3 \end{bmatrix}=\begin{bmatrix} 1 & -3 & 4 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 7 & -3 \end{bmatrix}$

$\begin{matrix} R_1+3R_2 \\ \to \end{matrix}\begin{bmatrix} 1 & -3+3 & 4-9 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 7 & -3 \end{bmatrix}=\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 7 & -3 \end{bmatrix}$ And next pivot entry 3,3

$\begin{matrix} R_3/7 \\ \to \end{matrix}\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3/7 \end{bmatrix}$ $\begin{matrix} R_1+5R_3 \\ R_2+3R_3 \\ \to \end{matrix}\begin{bmatrix} 1 & 0 & -5+5 & 1-15/7 \\ 0 & 1 & -3+3 & 0-9/7 \\ 0 & 0 & 1 & -3/7 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & -8/7 \\ 0 & 1 & 0 & -9/7 \\ 0 & 0 & 1 & -3/7 \end{bmatrix}$

Which gives the solution x=-8/7, y=-9/7, z=-3/7

When you put your equation in reduced row eschelon form, you will have one of three results.

1 Solution

ex) $\begin{bmatrix} 1 & 0 & 0 & -8/7 \\ 0 & 1 & 0 & -9/7 \\ 0 & 0 & 1 & -3/7 \end{bmatrix}$ as in the above example.

No solutions

when you have a row where all entries except for the right entry are zero

ex) $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 87 \end{bmatrix}$ represents a system with no solution, because this translates to: x=5, y=9, 0=87

Infinite Solutions

ex) $\begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 2 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ represents x+3z=5, y+2z=9. The variable “z” is known as a free variable, which can take any value. We express x and y in terms of z:

x=5-3z, y=9-2z, z=any real number