Math 160 , Finite Mathematics for Business

Section 5.7 - Discussion Notes
Brian Powers - TA - Fall 2011

Review problems:

1) You have to arrange 2 novels, 6 biographies and 4 how-to books on a book shelf. How many ways can they be arranged if the books from each category need to stay together?

This is determined using the multiplication principal. The choices you need to make are:

a) the order of the 3 categories,
b) the order of the novels
c) the order of the biographies and
d) the order of the how-to books.
These can be arranged in P(3,3), P(2,2), P(6,6) and P(4,4) ways respectively.
Recall that a permulation of n out of n things is simply calculated as n!. So the solution is:
$3! \cdot 2! \cdot 6! \cdot 4! = 207,360$

2) In how many ways can you choose 12 jurors and 2 alternates from a pool of 20 people?

The choices you need to make are:

a) choose 12 jurors from 20 people, and
b) choose 2 alternates from the remaining 8.
This is calculated using the multiplication principle and the combination formula. The solution is:
$C(20, 12) \cdot C(8, 2) = \binom{20}{12} \cdot \binom{8}{2} = 3,527,160$
Note that you can do this in the opposite order - first choose the 2 alternates and then the 12 jurors. The calculation will give you the same result:
$C(20, 2) \cdot C(18, 12) = \binom{20}{2} \cdot \binom{18}{12} = 3,527,160$

3) In how many ways can you appoint a president, a vice president and a 3 person council of advisors from a total of 45 people?

The choices you need to make are:

a) Who is president
b) Who is vice president, and
c) Who 3 will be on the council.
There are 45 choices for president. Once president is picked there are 44 choices left for vice president. After the VP has been chosen there are 43 remaining of which you choose 3 for the council. Using the multiplication principal, the solution is:
$45 \cdot 44 \cdot \binom{43}{3} = 24,435,180$

Note that if we reverse the order and appoint the council first the calculation will be as follows (though the answer remains unchanged):
$\binom{45}{3} \cdot 42 \cdot 41 = 24,435,180$

Binomial Coefficients

We have been writing the notation for C(n,r) as $\binom{n}{r}$ and the formula is as follows

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

The expression $\binom{n}{r}$ is known as the binomial coefficient and arises when we expand $(x+y)^n$.
$(x+y)^2 = x^2 + 2xy + y^2$
$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
And so on...
The coefficients in front of each variable term follows a pattern:
$(x+y)^2 = \binom{2}{0}x^2 + \binom{2}{1}xy + \binom{2}{2}y^2$
$(x+y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$
$(x+y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$

This is generalized in the Binomial Theorm which states:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-2}x^2y^{n-2} + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

It is important to note that the pattern of binomial coefficients is symmetric. This is to say:

$$\binom{n}{r} = \binom{n}{n-r}$$

For example, $\binom{9}{3} = \binom{9}{6}$.

Problems

1) Calculate by hand $\binom{25}{24}$

Rather than writing out the entire formula for $\binom{25}{24}$, we can use the fact that $\binom{25}{24} = \binom{25}{1}$ to simplify things. Thus the expanded formula looks like this:
$\binom{25}{24} = \binom{25}{1} = \frac{25!}{24!\cdot 1!} = \frac{25\cdot 24!}{24!\cdot 1!} = \frac{25}{1} = 25$

2) Calculate by hand $\binom{18}{16}$

Again, rather than writing out $\binom{18}{16}$, we can calculate $\binom{18}{2}$:
$\binom{18}{16} = \binom{16}{2} = \frac{18!}{16!\cdot 2!} = \frac{18\cdot 17\cdot 16!}{16!\cdot 2!} = \frac{18\cdot 17}{2} = 153$

3) What are the first three terms of $(x+y)^{10}$?

This is a direct application of the Binomial Theorm. If we use n=10 in the Binomial Theorm, the first three terms are:
$\binom{10}{0}x^{10} + \binom{10}{1}x^9y + \binom{10}{2}x^8y^2$
Which simplifies to $x^{10} + 10x^9y + 45x^8y^2$

4) How many subsets are there from a set of 7 elements?

You can think about this problem in a couple of ways. If you were to sum up all of the possibilities, you would count:

 # Subsets of 0 elements + # subsets of 1 element + $\cdots$ + # subsets of 7 elements

Which is equal to:

$$\binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7}$$

i.e. there are $\binom{7}{0}$ subsets with 0 elements, $\binom{7}{1}$ subsets with 1 element, and so forth. This is

the sum of binomial coefficients. Recall that

$$(x+y)^7 = \binom{7}{0}x^7 + \binom{7}{1}x^6y + \binom{7}{2}x^5y^2 + \binom{7}{3}x^4y^3 + \binom{7}{4}x^3y^4 + \binom{7}{5}x^2y^5 + \binom{7}{6}xy^6 + \binom{7}{7}y^7$$

If x and y both = 1, then all of the xy variable products will be 1 and we will get our sum. On the left side of the equation, when x and y are 1 we get $(1+1)^7$ **which is** $2^7 = 128$**. So there are 128 subsets from a set with 7 elements.**

Another way to think about this: You can index the elements in the set of 7 with numbers 1-7. In a particular subset, each of these elements is either present or not present - there are 2 choices for each element. By the multiplication principal if you have a sequence of 7 choices and each one can be made in 2 ways, you have $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7 = 128$ **possible ways to create a subset.**

5) A cable company offers 20 basic channels and 5 premium channels, each one for an additional cost. How many possible ways can you order cable channels from this company?

In this problem the fact that 20 basic channels are offered is irrelevant, because these 20 channels are part of your package no matter what - the question is asking how many ways can you pick and choose from the 5 premium channels. Every combination of premium channels is a subset of the premium channels (for example, you could order premium channels {1,2,5}, or {2,4}, none of them, all of them, or anywhere in between. So simply put, this problem is asking "how many subsets are there from a set of 5 elements". From the previous example the answer is found to be $2^5 = 32$ **possible ways.**

6) In how many ways can you select at least 1 book from 8 books?

If you were asked "In how many ways can you select any number of books, including zero, from 8 books?" then this would be another simple subset question (how many subsets are there from a set of 8 elements). But we are not counting the subset with 0 books. So the solution is:
#selections of at least 1 book =
#selections of any number of books - #selections of 0 books =
$2^8 - \binom{8}{0} = 2^8 - 1 = 256 - 1 = 255$.

7) An ice cream parlor offers 4 flavors of ice cream, 3 different sauces and 2 types of nuts. In how many ways can you make a sundae of a single flavor of ice cream and at least 1 topping?

There is no distinction between nuts and sauces when it comes to toppings, so don't assume you can't have all 3 sauces and both types of nuts - There are simply 5 toppings to choice from. This problem can thus be broken down into two choices:
a) Which flavor of ice cream, and
b) Which of the 5 toppings (at least 1 of them)
There are 4 flavors of ice cream and from the previous example, we can calculate that there are $2^5 - 1$ **ways of choosing at least 1 topping from 5 choices. Therefore there are:**
$4 \cdot (2^5 - 1) = 4 \cdot (32 - 1) = 4 \cdot 31 = 124$ **possible sundaes.**