HW Solutions

MATH 170, CALC. FOR LIFE SCIENCES, SPR 2016
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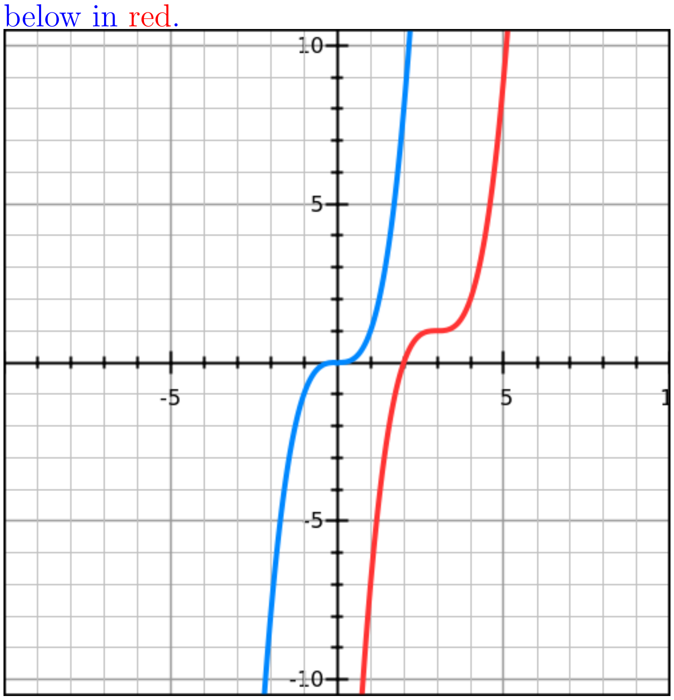
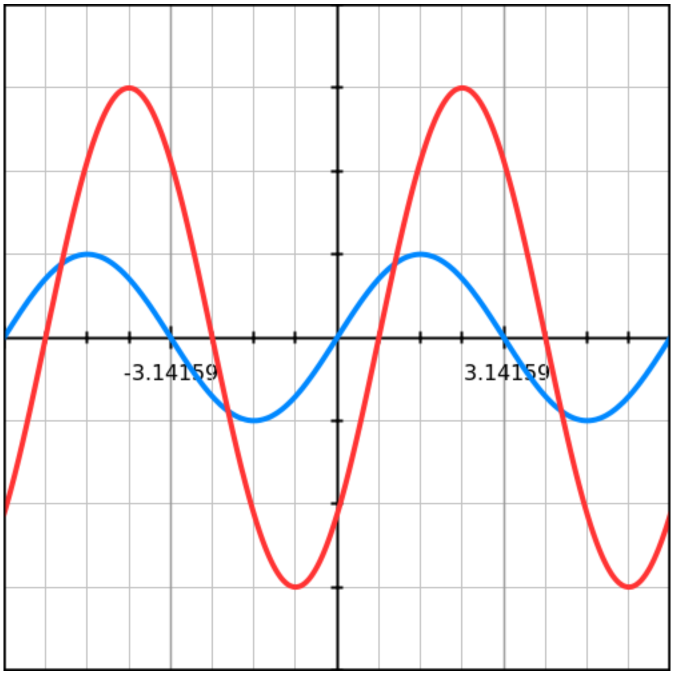
1 Due Jan 20

1. Compute the following trigonometric numbers:
   $\sin(5\pi/6) = \frac{1}{2}$
   $\cos(2\pi/3) = -\frac{1}{2}$
   $\sin(3\pi/2) = -1$
   $\tan(7\pi/6) = \frac{\sqrt{3}}{3}$
   $\sec(5\pi/4) = -\sqrt{2}$

2. Sketch the graph of $f(x) = x^3$ and use it to sketch the graph of $g(x) = f(x-3)+1$.
   Translating a function means shifting it left, right, up, down and scaling it. To see the effect of the translation, we start with the x and work our way out. From $f(x)$ to $f(x-3)$ we shift RIGHT by 3 units. From $f(x-3)$ to $f(x-3)+1$ we shift UP 1 unit. So $g(x) = f(x-3)+1 = (x-3)^3+1$ is the new graph, shown below in red.
   

3. Sketch the graph of $y = \sin x$ and $y = \sin(x - \pi/4)$
   The translation from $\sin(x)$ to $\sin(x-\pi/4)$ shifts the graph RIGHT by $\pi/4$ units. Then the change from $\sin(x-\pi/4)$ to $3\sin(x-\pi/4)$ scales vertically by a factor of 3. $\sin(x)$ is in blue while $3\sin(x-\pi/4)$ is in red.



4. Find the inverse of the function $f(x) = \dfrac{3x+1}{x-3}$. Also determine the domain of f and that of f^{-1}.
To find the inverse, we follow these steps:
1) Put y in place of the xs, and put x in place of $f(x)$
2) Using algebra solve for y (if possible)
3) What you have now is the inverse function.
So the steps are:
1) $x = \dfrac{3y+1}{y-3}$
2) Using algebra:

$$x = \frac{3y+1}{y-3} \Rightarrow (y-3)x = 3y+1 \Rightarrow xy - 3x = 3y+1 \Rightarrow xy - 3y = 3x+1$$

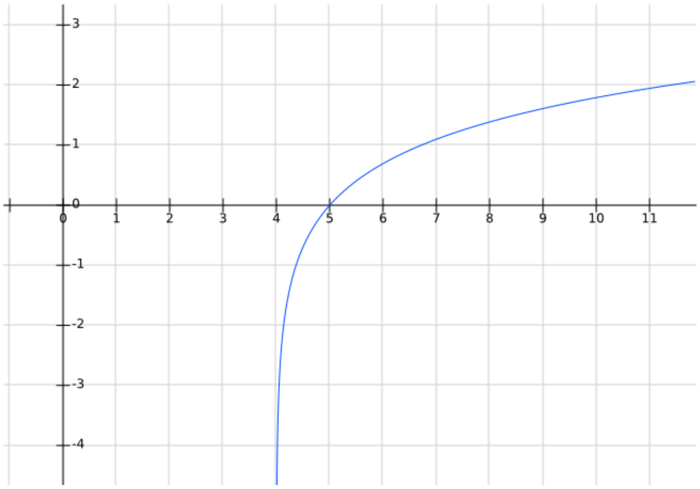
$$\Rightarrow y(x-3) = 3x+1 \Rightarrow y = \frac{3x+1}{x-3}$$

So the inverse function is

$$f^{-1}(x) = \frac{3x+1}{x-3}$$

All values of x are ok in $f(x)$ except $x = 3$, which will cause division by zero. So The domain of f can be written as "All real numbers except 3", or using set notation $\mathbb{R}\backslash\{3\}$, or using interval notation $(-\infty, 3) \cup (3, \infty)$. Since $f(x) = f^{-1}$ (coincidentally), it of course has the same domain.

5. Sketch the graph of $f(x) = \ln(x-4)$.
The graph of $\ln(x)$ passes through $(1, 0)$ and has a vertical asymptote at $x = 0$. $\ln(x-4)$ is shifted 4 units to the RIGHT, so it passes through the point $(5, 0)$ and has the vertical asymptote at $x = 4$.



2 Due Jan 22

1. A population triples every two days. Suppose that the initial count N_0 is 5 and that the unit of time is 4 days. Give the formula for the sequence N_t that gives the population count at t units of time.
   This is confusing because the unit of time is 4 days. So N_1 is the population on day 4. Since the population triples every 2 days, in 4 days it will have tripled twice, so it will be 9 times as big. So
   $$N_1 = N_0 \cdot 9$$
   When $t = 2$, that is after 8 days, so the population will be increased by another factor of 9:
   $$N_2 = N_1 \cdot 9 = (N_0 \cdot 9) \cdot 9 = N_0 \cdot 9^2$$
   The pattern we see is
   $$N_t = N_0 \cdot 9^t$$
   Since $N_0 = 5$, this is
   $$N_t = 5 \cdot 9^t$$
   This can also be written
   $$N_t = 5 \cdot 3^{2t}$$

2. The Fibonacci sequence is given recursively by $a_1 = a_2 = 1$ and $a_{n+2} = a_n + a_{n+1}$. Give the first ten terms of the Fibonacci sequence.
   $a_1 = 1$
   $a_2 = 1$
   $a_3 = a_1 + a_2 = 1 + 1 = 2$
   $a_4 = a_2 + a_3 = 1 + 2 = 3$
   $a_5 = a_3 + a_4 = 2 + 3 = 5$
   $a_6 = a_4 + a_5 = 3 + 5 = 8$
   $a_7 = a_5 + a_6 = 5 + 8 = 13$
   $a_8 = a_6 + a_7 = 8 + 13 = 21$
   $a_9 = a_7 + a_8 = 13 + 21 = 34$

$a_{10} = a_8 + a_9 = 21 + 34 = 55$

3. Use the definition of the limit in order to justify that

$$\lim_{n\to+\infty} \frac{2n+3}{3n+2} = \frac{2}{3}$$

Let $\epsilon > 0$. Suppose $\left|\frac{2n+3}{3n+2} - \frac{2}{3}\right| < \epsilon$. This is equivalent to
$\left|\frac{3(2n+3)}{3(3n+2)} - \frac{2(3n+2)}{3(3n+2)}\right| < \epsilon$
$\Rightarrow \left|\frac{6n+9}{9n+6} - \frac{6n+4}{9n+6}\right| < \epsilon$
$\Rightarrow \left|\frac{6n+9-(6n+4)}{9n+6}\right| < \epsilon$
$\Rightarrow \left|\frac{6n+9-6n-4)}{9n+6}\right| < \epsilon$
$\Rightarrow \left|\frac{5}{9n+6}\right| < \epsilon$
Since $n > 0$, the fraction is positive so we can remove the absolute value sign:
$\Rightarrow \frac{5}{9n+6} < \epsilon$
$\Rightarrow 5 < \epsilon(9n+6)$
$\Rightarrow 5 < 9\epsilon n + 6\epsilon$
$\Rightarrow 5 - 6\epsilon < 9\epsilon n$
$\Rightarrow \frac{5-6\epsilon}{9\epsilon} < n$
So as long as $n > N = \frac{5-6\epsilon}{9\epsilon}$, it will be the case that $\left|\frac{2n+3}{3n+2} - \frac{2}{3}\right| < \epsilon$.

3 Due Jan 25

1. Use the definition of the limit in order to justify that

$$\lim_{n\to+\infty} \frac{5n+1}{2n+1} = \frac{5}{2}$$

Let $\epsilon > 0$. Suppose $\left|\frac{5n+1}{2n+1} - \frac{5}{2}\right| < \epsilon$. This is equivalent to
$\left|\frac{2(5n+1)}{2(2n+1)} - \frac{5(2n+1)}{2(2n+1)}\right| < \epsilon$
$\Rightarrow \left|\frac{10n+2}{4n+2} - \frac{10n+5}{4n+2}\right| < \epsilon$
$\Rightarrow \left|\frac{10n+2-(10n+5)}{4n+2}\right| < \epsilon$
$\Rightarrow \left|\frac{10n+2-10n-5)}{4n+2}\right| < \epsilon$
$\Rightarrow \left|\frac{-3}{4n+2}\right| < \epsilon$

Since $n > 0$, the fraction is negative so we can remove the absolute value sign and change the numerator to 3:

$\Rightarrow \dfrac{3}{4n+2} < \epsilon$

$\Rightarrow 3 < \epsilon(4n+2)$

$\Rightarrow 3 < 4\epsilon n + 2\epsilon$

$\Rightarrow 3 - 2\epsilon < 4\epsilon n$

$\Rightarrow \dfrac{3-2\epsilon}{4\epsilon} < n$

So as long as $n > N = \dfrac{3-2\epsilon}{4\epsilon}$, it will be the case that $\left|\dfrac{5n+1}{2n+1} - \dfrac{5}{2}\right| < \epsilon$.

2. Use the definition of limit in order to justify that

$$\lim_{n\to+\infty} 4^n = +\infty$$

Fix $M > 0$ large. We want to show that for for some large N, $n > N$ implies $4^n > M$. Take a logarithm of each side.

$$\Rightarrow \ln(4^n) > \ln M$$

Solve for n:

$$\Rightarrow n \ln 4 > \ln M$$

$$\Rightarrow n > \frac{\ln M}{\ln 4}$$

So provided $n > N = \frac{\ln M}{\ln 4}$, it is true that $4^n > M$.

3. Find the limit of the sequence $a_n = \dfrac{2^{3n}}{3^{2n}}$.

This can be handled by using some properties of exponents. Recall the property that $a^{bc} = (a^b)^c$. Thus the sequence can be written

$$a_n = \frac{2^{3n}}{3^{2n}} = \frac{(2^3)^n}{(3^2)^n} = \frac{8^n}{9^n}$$

Furthermore, recall another property of exponents: $\frac{a^b}{c^b} = \left(\frac{a}{c}\right)^b$. So the sequence can be written

$$a_n = \left(\frac{8}{9}\right)^n$$

because $0 < \frac{8}{9} < 1$, as n gets bigger and bigger, a_n will get smaller and smaller, getting closer to 0. Lets prove that the limit is 0. Let $\epsilon > 0$. We want to show that for some large enough $n > N$, that $\left(\frac{8}{9}\right)^n < \epsilon$. Take a log of both sides:

$$\ln\left(\left(\frac{8}{9}\right)^n\right) < \ln \epsilon$$

$$\Rightarrow n \ln \frac{8}{9} < \ln \epsilon$$

before we divide both sides by $\ln \frac{8}{9}$ we have to take a moment and be careful. Because $\frac{8}{9} < 1$, its logarithm is negative. In fact, $\ln \frac{8}{9} = \ln 8 - \ln 9$. Lets put this into the inequality:

$$n(\ln 8 - \ln 9) < \ln \epsilon$$

When we divide both sides by $\ln 8 - \ln 9$ the direction of inequality will flip.

$$\Rightarrow n > \frac{\ln \epsilon}{\ln 8 - \ln 9}$$

Since ϵ is *small* (close to zero), $\ln \epsilon < 0$, so the fraction is a negative divided by another negative - it is a positive number N. Thus as long as $n > N = \frac{\ln \epsilon}{\ln 8 - \ln 9}$, it will be the case that $0 < a_n < \epsilon$.

4. Find the limit of the sequence $a_n = \dfrac{3n^2 + n + 1}{2n^2 + 3}$.
   The way we handle these rational functions (polynomial divided by a polynomial) is that we find the highest power of n (in this case n^2) and multiply the fraction by $\dfrac{1/n^2}{1/n^2}$. We get an equivalent expression for a_n but it will allow us to see the limit of the numerator and denominator separately:

$$a_n = \frac{3n^2 + n + 1}{2n^2 + 3} \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \frac{3 + \frac{1}{n} + \frac{1}{n^2}}{2 + \frac{3}{n^2}}$$

   Looking at the numerator, we can see that as n gets big the second and third term go to zero. In the denominator, as n gets big the second term goes to zero. Then the limit should be $\frac{3}{2}$.

4 Due Jan 27

1. By using a table of values, try to guess the limit

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$$

x	$\dfrac{x^2 - 9}{x - 3}$
3.1	6.1
3.01	6.01
3.001	6.001
3.0001	6.0001
2.9999	5.9999
2.999	5.999
2.99	5.99
2.9	5.9

   From the table it seems that the limit is 6.

2. By taking one-sided limits, show that the limit

$$\lim_{x \to 3} \frac{|x - 3|}{x - 3}$$

   does not exist.
   First the left side limit.

$$\lim_{x \to 3^-} \frac{|x - 3|}{x - 3}$$

In this case, $x < 3$ so $x - 3 < 0$, thus $|x-3| = -(x-3)$.

$$\lim_{x\to 3^-} \frac{|x-3|}{x-3} = \lim_{x\to 3^-} \frac{-(x-3)}{x-3} = \lim_{x\to 3^-} -1 = -1$$

Now the right-side limit.

$$\lim_{x\to 3^+} \frac{|x-3|}{x-3}$$

In this case, $x > 3$ so $x - 3 > 0$, thus $|x-3| = x - 3$.

$$\lim_{x\to 3^+} \frac{|x-3|}{x-3} = \lim_{x\to 3^+} \frac{x-3}{x-3} = \lim_{x\to 3^+} 1 = 1$$

But the two one-sided limits do not agree, so the limit does not exist.

3. Use synthetic division to simplify $\dfrac{x^4 - x^3 - 2}{x^3 - 3x + 2}$ by dividing both numerator and denominator by $x - 1$.

From the numerator we get:

$$\begin{array}{c|ccccc} & 1 & 1 & 0 & 0 & -2 \\ 1 & & 1 & 2 & 2 & 2 \\ \hline & 1 & 2 & 2 & 2 & 0 \end{array}$$

Thus $x^4 + x^3 - 2 = (x-1)(x^3 + 2x^2 + 2x + 2)$. For the denominator we get:

$$\begin{array}{c|cccc} & 1 & 0 & -3 & 2 \\ 1 & & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & 0 \end{array}$$

Thus $x^3 - 3x + 2 = (x-1)(x^2 + x - 2)$. Then we can simplify the fraction to $\dfrac{x^4 - x^3 - 2}{x^3 - 3x + 2} = \dfrac{x^3 + 2x^2 + 2x + 2}{x^2 + x - 2}$.

5 Due Jan 29

1. Compute the limit $\lim_{x\to 3} x^3 - 5x^2 + 6x - 1$.
   First try plugging in 3: $(3)^3 - 5(3)^3 + 6(3) - 1 = -1$.

2. Compute the limit $\lim_{x\to -2} \dfrac{x^3 - 7}{x^2 + x + 5}$.
   First try plugging in -2: $\dfrac{(-2)^3 - 7}{(-2)^2 + (-2) + 5} = \dfrac{-8-7}{4-2+5} = \dfrac{-15}{7}$.

3. Compute the limit $\lim_{x\to 1} \sqrt{x^3 + 5x^2 - 2x + 3}$.
   Try plugging in $x = 1$: $\sqrt{(1)^3 + 5(1)^2 - 2(1) + 3} = \sqrt{1 + 5 - 2 + 3} = \sqrt{7}$

4. Compute the limit $\lim_{x\to 1} \dfrac{x^4 + 3x^2 - 5x + 1}{x^3 + 5x^2 - 6}$. If we plug in $x = 1$ we will find the denominator is zero, so we cannot simply plug it in. For a rational, if the function is undefined at x, then x is either a vertical asymptote or a removable discontinuity (a hole). The limit won't exist at an asymptote, but it will at a removable

discontinuity. We have a removable discontinuity if x is a zero of the numerator and denominator. We can verify this: the numerator is indeed 0 when $x = 1$. So we can use synthetic division to factor out $(x-1)$ from the numerator and denominator. From the numerator we get:

	1	0	3	−5	1
1		1	1	4	−1
	1	1	4	−1	0

Thus $x^4 + 3x^2 - 5x + 1 = (x-1)(x^3 + x^2 + 4x - 1)$. For the denominator we get:

	1	5	0	−6
1		1	6	6
	1	6	6	0

Thus $x^3 + 5x^2 - 6 = (x-1)(x^2 + 6x + 6)$. Then we can simplify the fraction to $\dfrac{x^4 + 3x^2 - 5x + 1}{x^3 + 5x^2 - 6} = \dfrac{x^3 + x^2 + 4x - 1}{x^2 + 6x + 6}$. If we plug in $x = 1$ in this new fraction, we get $\dfrac{5}{13}$, so this is our limit.

5. Compute the limit $\lim\limits_{x \to -2} \dfrac{x^3 - x + 6}{x^2 - 6x - 16}$.

   If we plug in -2 we get $\dfrac{(-2)^3 - (-2) + 6}{(-2)^2 - 6(-2) - 16} = \dfrac{-8 + 2 + 6}{4 + 12 - 16} = \dfrac{0}{0}$. So our function has a removable discontinuity at $x = -2$. We can use synthetic division to factor $x + 2$ out from the numerator and denominator. From the numerator we get:

	1	0	−1	6
−2		−2	4	−6
	1	−2	3	0

Thus $x^3 - x + 6 = (x+2)(x^2 - 2x + 3)$. For the denominator we get:

	1	−6	−16
−2		−2	16
	1	−8	0

Thus $x^2 - 6x - 16 = (x+2)(x-8)$. Then we can simplify the fraction to $\dfrac{x^3 - x + 6}{x^2 - 6x - 16} = \dfrac{x^2 - 2x + 3}{x - 8}$. If we plug in $x = -2$ now, we get $\dfrac{(-2)^2 - 2(-2) + 3}{(-2) - 8} = \dfrac{4 + 4 + 3}{-2 - 8} = \dfrac{11}{-10} = -\dfrac{11}{10}$.

6. Compute the limit $\lim\limits_{x \to 1} \dfrac{\sqrt{x+3} - 2}{x^2 + x - 2}$.

   If we plug in $x = 1$, we get $\dfrac{\sqrt{(1) + 3} - 2}{(1)^2 + (1) - 2} = \dfrac{\sqrt{4} - 2}{1 + 1 - 2} = \dfrac{0}{0}$. We have to look for some common factor of the numerator and denominator, but the square root makes it tricky. Fist let's look at the denominator and factor it into $(x-1)(x+2)$. Next let's multiply the numerator and denominator by the conjugate of the numerator, namely $\sqrt{x+3}+2$.

$$\frac{\sqrt{x+3}-2}{x^2+x-2} = \frac{\sqrt{x+3}-2}{(x-1)(x+2)}\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2} = \frac{(x+3)-4}{(x-1)(x+2)(\sqrt{x+3}+2)}$$

The numerator is now $x-1$, and this is a factor of the denominator. So finally the fraction can be written as:

$$\frac{1}{(x+2)(\sqrt{x+3}+2)}$$

Plugging in $x=1$ now, we get

$$\frac{}{(1+2)(\sqrt{1+3}+2)} = \frac{1}{(3)(2+2)} = \frac{1}{12}$$

7. Compute the limit $\lim_{x\to 0} \dfrac{\sqrt{x+4}-2}{\sqrt{x+9}-3}$.

   If we plug in $x=0$ we get $\frac{0}{0}$, so we have to re-write the fraction. Again we use the trick of conjugates, but we have to multiply by both conjugates of the numerator AND denominator.

$$\frac{\sqrt{x+4}-2}{\sqrt{x+9}-3} = \frac{\sqrt{x+4}-2}{\sqrt{x+9}-3}\frac{(\sqrt{x+4}+2)(\sqrt{x+9}+3)}{(\sqrt{x+4}+2)(\sqrt{x+9}+3)}$$

   Don't bother multiplying it all out - just multiply the conjugate pairs.

$$= \frac{\big((x+4)-4\big)(\sqrt{x+9}+3)}{\big((x+9)-9\big)(\sqrt{x+4}+2)} = \frac{x(\sqrt{x+9}+3)}{x(\sqrt{x+4}+2)}$$

   Now we have x as a common factor of the numerator and denominator - we can cancel this and write our fraction as

$$\frac{\sqrt{x+9}+3}{\sqrt{x+4}+2}$$

   If we plug in $x=0$ now we get $\dfrac{\sqrt{9}+3}{\sqrt{4}+2} = \dfrac{3+3}{2+2} = \dfrac{3}{2}$.

6 Due Feb 1

1. Determine the limit $\lim_{x\to 3} \dfrac{x^2+1}{x^2-3}$.

   If we plug in $x=3$ we get $\dfrac{(3)^2+1}{(3)^2-3} = \dfrac{9+1}{9-3} = \dfrac{10}{6} = \dfrac{5}{3}$.

2. Determine the limit $\lim_{x\to 2} \dfrac{x+1}{x^2-4x+4}$.

   If we plug in $x=2$ we get $\dfrac{2+1}{(2)^2-4(2)+4} = \dfrac{3}{0}$, which is undefined. Let us factor the denominator into $(x-2)(x-2) = (x-2)^2$. Thus the denominator is NEVER negative. Near $x = 2$, the numerator is positive and the denominator is also positive, so both left-hand limit and right-hand limit is $+\infty$. Thus

$$\lim_{x\to 2} \frac{x+1}{x^2-4x+4} = +\infty$$

3. Determine the limit $\lim_{x\to 0} \frac{x^2-3x-5}{x^2+x}$.
   If we plug in $x = 0$ we get $\frac{-5}{0}$ which is undefined. Let us factor the denominator into $x(x+1)$. So we can write out limit as

$$\lim_{x\to 0} \frac{x^2-3x-5}{x+1} \cdot \frac{1}{x}$$

   By direct substitution, $\lim_{x\to 0} \frac{x^2-3x-5}{x+1} = -5$. But $\lim_{x\to 0} \frac{1}{x}$ doesn't exist. This is because $\frac{1}{x}$ is negative for $x < 0$ but positive for $x > 0$. Thus the left-hand and right-hand limits don't agree. For this reason, our limit does not exist.

7 Due Feb 3

1. Determine the limit $\lim_{x\to +\infty} \frac{2x^2+1}{3x^2-3}$.
   Since x^2 is the highest power of x, multiply the fraction by $\frac{1/x^2}{1/x^2}$ to get and equivalent limit:

$$\lim_{x\to +\infty} \frac{2x^2+1}{3x^2-3} \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to +\infty} \frac{2+\frac{1}{x^2}}{3-\frac{3}{x^2}}$$

   As x gets big, the terms divided by x^2 go to zero, so this limit is $\frac{2}{3}$.

2. Determine the limit $\lim_{x\to -\infty} \frac{x^3+1}{x^2-4x+4}$.
   Since x^3 is the highest power of x, we multiply by the fraction $\frac{1/x^3}{1/x^3}$ to get an equivalent limit:

$$\lim_{x\to -\infty} \frac{x^3+1}{x^2-4x+4} \frac{1/x^3}{1/x^3} = \lim_{x\to -\infty} \frac{1+\frac{1}{x^3}}{\frac{1}{x}-\frac{4}{x^3}+\frac{4}{x^4}}$$

   As x goes to $-\infty$, the numerator goes to 4. The denominator, however, consists of three terms which all get close to zero. The question is whether or not the denominator is negative or positive. In this case, however, $\frac{1}{x}$ is negative, $-\frac{4}{x^2}$ is negative, and $\frac{4}{x^3}$ are all negative when $x < 0$, so the denominator is negative. Thus the limit is $-\infty$.

3. Find the horizontal asymptotes of the function $f(x) = \dfrac{x^4 - 3x + 5}{2x^4 - 3}$.
We need to take the limit as $x \to +\infty$ and as $x \to -\infty$. First, however, we need to multiply by $\dfrac{1/x^4}{1/x^4}$ since x^4 is the highest power of x. By doing this, the limits become

$$\lim_{x\to+\infty} \frac{1 - \frac{3}{x^3} + \frac{5}{x^4}}{2 - \frac{3}{x^4}} = \frac{1}{2}$$

$$\lim_{x\to-\infty} \frac{1 - \frac{3}{x^3} + \frac{5}{x^4}}{2 - \frac{3}{x^4}} = \frac{1}{2}$$

So this function has a single horizontal asymptote, $y = \frac{1}{2}$.

4. Find the horizontal asymptotes of the function $f(x) = \dfrac{x+3}{|x|+1}$.
We need to take the limits as $x \to +\infty$ and $x \to -\infty$. In either case, $|x|$ will be x and $-x$ respectively. The limits are:

$$\lim_{x\to+\infty} \frac{x+3}{|x|+1} = \lim_{x\to+\infty} \frac{x+3}{x+1} = \lim_{x\to+\infty} \frac{1+\frac{3}{x}}{1+\frac{1}{x}} = \frac{1}{1} = 1$$

$$\lim_{x\to-\infty} \frac{x+3}{|x|+1} = \lim_{x\to-\infty} \frac{x+3}{-x+1} = \lim_{x\to-\infty} \frac{1+\frac{3}{x}}{-1+\frac{1}{x}} = \frac{1}{-1} = -1$$

So we have two horizontal asymptotes: $y = 1$ and $y = -1$.

8 Due Feb 5

1. Evaluate the limit $\lim_{x\to 0} \dfrac{\sin(5x)}{x}$. We will use the fact that $\lim_{u\to 0} \dfrac{\sin u}{u} = 1$, but we need this to be of the same form first. We can simply multiply by $\frac{5}{5}$:

$$\lim_{x\to 0} 5\frac{\sin(5x)}{5x} = 5\lim_{x\to 0} \frac{\sin(5x)}{(5x)} = 5(1) = 5$$

Note of course that as $x \to 0$, $5x \to 0$ as well.

2. Evaluate the limit $\lim_{x\to 0} \dfrac{\sin(5x)}{\sin(10x)}$.

As with 1, we need to get this fraction in the form of $\dfrac{\sin u}{u}$ before we can substitute in a 1. We need a $5x$ in the denominator and a $10x$ in the numerator. But we must counterbalance with another one, since we can't create them out of nowhere.

$$\lim_{x\to 0} \frac{\sin(5x)}{\sin(10x)} = \lim_{x\to 0} \frac{10x}{\sin(10x)} \frac{\sin(5x)}{5x} \frac{5x}{10x}$$

A limit of a product of 3 functions can be written as the product of 3 limits:

$$\left(\lim_{x\to 0} \frac{10x}{\sin(10x)}\right)\left(\lim_{x\to 0} \frac{\sin(5x)}{5x}\right)\left(\lim_{x\to 0} \frac{5x}{10x}\right) = (1)(1)\left(\frac{1}{2}\right) = \frac{1}{2}$$

3. Evaluate the limit $\lim_{x\to 1} \frac{\sin(x^2+x-2)}{x-1}$.

   In order to put this in the form of $\lim_{u\to 0} \frac{\sin u}{u}$, we need the denominator to be x^2+x-2. But notice that $x^2+x-2 = (x-1)(x+2)$. So we need to multiply this fraction by $\frac{x+2}{x+2}$. Then we get:

$$\lim_{x\to 1} \frac{\sin(x^2+x-2)}{x-1}\frac{x+2}{x+2} = \lim_{x\to 1}(x+2)\frac{\sin(x^2+x-2)}{x^2+x-2}$$

   This can be written as the product of two limits:

$$\left(\lim_{x\to 1} x+2\right)\left(\frac{\sin(x^2+x-2)}{x^2+x-2}\right) = (3)(1) = 3$$

   Since as $x \to 1$, $x^2+x-2 \to 0$.

4. Evaluate the limit $\lim_{x\to 0} \cos(1/x)$.

   This one uses the Squeeze Theorem. Observe first that $-1 \leq \cos(1/x) \leq 1$. Also, $-|x^3| \leq x^3 \leq |x^3|$. Therefore

$$-|x^3| \leq x^3\cos(1/x) \leq |x^3|$$

   It is straightforward to show by direct substitution that $\lim_{x\to 0} -|x^3| = \lim_{x\to 0} |x^3| = 0$, so by the squeeze theorem, the original limit is 0 as well.

5. Evaluate the limit $\lim_{x\to +\infty} \frac{\sin x + 3}{x^2}$.

   This uses the Squeeze Theorem as well. Since $-1 \leq \sin x \leq 1$, it follows that

$$2 \leq \sin x + 3 \leq 4$$

$$\Rightarrow \frac{2}{x^2} \leq \frac{\sin x + 3}{x^2} \leq \frac{4}{x^2}$$

   As $x \to \infty$, the limits of both the left-hand side and right-hand side of the inequality chain are both 0, so the original limit in question is 0 as well.

9 Due Feb 8

1. Determine whether the function

$$f(x) = \begin{cases} \frac{x^2-9}{x-3} & x \neq 3 \\ 5 & x = 3 \end{cases}$$

   is continuous at 3 or not.

   The function is continuous at 3 if $f(3)$ exists, $\lim_{x\to 3} f(x)$ exists and they are equal. In other words, we need to check if

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = 5$$

We need to factorize the numerator to find the limit:

$$\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}\frac{(x-3)(x+3)}{x-3}=\lim_{x\to 3}(x+3)=6$$

This is not 5, so the function is not continuous at 3.

2. Find c so that the function

$$f(x)=\begin{cases}3x^2-c & x\geq 0\\ \dfrac{\sin x}{x} & x<0\end{cases}$$

is continuous on the set of real numbers.
The two pieces of the function are continuous on their intervals. $f(0)=3(0)^3-c=-c$. For negative x,

$$\lim_{x\to 0^-}\frac{\sin x}{x}=1$$

Thus the function is only continuous if $f(0)=-c=1$, in other words, $c=-1$.

3. Show that the equation $x^5-x=3$ has a real root.
This will use the Intermediate Value Theorem. It says that if a function is continuous on $[a,b]$, then for any v between $f(a)$ and $f(b)$, there exists a c on the interval $[a,b]$ such that $f(c)=v$.
Consider the function $f(x)=x^5-x$. Note that $f(0)=(0)^5-(0)=0$ and $f(2)=(2)^5-(2)=32-2=30$. Since $f(x)$ is a polynomial, it is continuous on any interval, and $f(0)<3<f(2)$, so by the IVT, There exists SOME c between 0 and 2 where $f(c)=3$.

4. Show that the equation $\cos x=x^3$ has a real root.
Notice that

$$\cos 0=1>0=(0)^3$$

and

$$\cos\frac{\pi}{2}=0<\frac{\pi^3}{8}=\left(\frac{\pi}{2}\right)^3$$

Since $\cos x$ and x^3 are both continuous functions, by the intermediate value theorem there MUST be some value c between 0 and $\frac{\pi}{2}$ where $\cos c=c^3$.

5. Determine whether the function $f(x)=\dfrac{x^3-3x+3}{x^2-1}$ has any removable discontinuities or not.
To study the removable discontinuities of rational functions (polynomial divided by a polynomial) we look to see if the numerator and denominator have any common roots; if x is a root of both numerator AND denominator, it is a removable discontinuity. If it is a root of the denominator only, then it gives a vertical asymptote. Since both polynomials are quadratics, we can factor them without too much difficulty.

$$f(x)=\frac{(x-1)(x+2)}{(x-1)(x+1)}$$

The function is discontinuous at $x=1$ and $x=-1$; $x=1$ is a root of both numerator and denominator so it is a removable discontinuity, but $x=-1$ is only a root of the denominator, so it is a vertical asymptote.

10 Due Feb 10

1. Use the definition of the derivative in order to compute $f'(3)$, where $f(x) = x^2$.

$$\begin{aligned}
f'(3) &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} \\
&= \lim_{h\to 0} \frac{(3+h)^2 - (3)^2}{h} \\
&= \lim_{h\to 0} \frac{9 + 6h + h^2 - 9}{h} \\
&= \lim_{h\to 0} \frac{h(6+h)}{h} \qquad = 6
\end{aligned}$$

2. Use the definition of the derivative in order to compute $f'(2)$, where $f(x) = x^3 - x + 5$.

$$\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h\to 0} \frac{(2+h)^3 - (2+h) + 5 - (2^3 - 2 + 5)}{h} \\
&= \lim_{h\to 0} \frac{8 + 12h + 6h^2 + h^3 - 2 - h + 5 - 8 + 2 - 5}{h} \\
&= \lim_{h\to 0} \frac{12h + 6h^2 + h^3 - h}{h} \\
&= \lim_{h\to 0} \frac{h(11 + 6h + h^2)}{h} \\
&= 11
\end{aligned}$$

3. Use the definition of the derivative in order to compute $f'(2)$, where $f(x) = \frac{1}{x}$.

$$\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{2}{2(2+h)} - \frac{2+h}{2(2+h)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{2-2-h}{2(2+h)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\frac{-h}{2(2+h)} \\
&= \lim_{h\to 0} \frac{-1}{2(2+h)} \\
&= -\frac{1}{4}
\end{aligned}$$

4. Use the definition of the derivative in order to compute $f'(16)$, where $f(x) = \sqrt{x}$.

$$\begin{aligned}
f'(16) &= \lim_{h\to 0} \frac{f(16+h) - f(16)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{16+h} - \sqrt{16}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{16+h} - 4}{h} \left(\frac{\sqrt{16+h}+4}{\sqrt{16+h}+4}\right) \\
&= \lim_{h\to 0} \frac{(16+h) - (16)}{h(\sqrt{16+h}+4)} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{16+h}+4)} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{16+h}+4} \qquad = \frac{1}{8}
\end{aligned}$$

5. If $f(1) = 5$ and $f'(1) = -3$, find the equation of the tangent lne to the graph of f at the point $(1, 5)$.
A line going through (x_1, y_1) with slope m has equation

$$y - y_1 = m(x - x_1).$$

the derivative at 1, $f'(1) = -3$ is the slope, so the equation is

$$y - 5 = -3(x - 1)$$

11 Due Feb 17

1. Use the definition of the derivative in order to computer the derivative of the function $f(x) = x^2 - x$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{((x+h)^2 - (x+h)) - (x^2 - x)}{h} \\
&= \lim_{h\to 0} \frac{(x^2 + 2hx + h^2 - x - h) - x^2 + x}{h} \\
&= \lim_{h\to 0} \frac{2hx + h^2 - h}{h} \\
&= \lim_{h\to 0} \frac{h(2x + h - 1)}{h} \\
&= \lim_{h\to 0} 2x + h - 1 \\
&= 2x - 1
\end{aligned}$$

2. Use the definition of the derivative in order to compute the derivative of the function $f(x) = x^3 + x + 1$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{((x+h)^3 + (x+h) + 1) - (x^3 + x + 1)}{h} \\
&= \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 + x + h + 1) - x^3 - x - 1}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 + h}{h} \\
&= \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2 + 1)}{h} \\
&= \lim_{h\to 0} 3x^2 + 3xh + h^2 + 1 \\
&= 3x^2 + 1
\end{aligned}$$

3. Use the definition of the derivative in order to compute the derivative of the function $f(x) = \dfrac{1}{\sqrt{x}}$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{\sqrt{x}}{\sqrt{x}\sqrt{x+h}} - \frac{\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}\right)\left(\frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{x - (x+h)}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{-h}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}\right) \\
&= \lim_{h\to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})} \\
&= \frac{-1}{\sqrt{x}\sqrt{x}(\sqrt{x} + \sqrt{x})} \\
&= \frac{-1}{2x\sqrt{x}}
\end{aligned}$$

4. Let $y = \dfrac{1}{x+2}$. Compute $\dfrac{dy}{dx}$.
Letting $f(x) = y$,

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{\frac{1}{(x+h)+2}-\frac{1}{x+2}}{h}\\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{x+2}{(x+2)((x+h+2)}-\frac{x+h+2}{(x+2)(x+h+2)}\right)\\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{-h}{(x+2)(x+h+2)}\right)\\ &= \lim_{h\to 0}\frac{-1}{(x+2)(x+h+2)}\\ &= \frac{-1}{(x+2)^2}\end{aligned}$$

5. Let $y = \dfrac{1}{x^2}$. Compute $\left.\dfrac{dy}{dx}\right|_{x=1}$.
Letting $f(x) = y$,

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}\\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{x^2}{x^2(x+h)^2}-\frac{(x+h)^2}{x^2(x+h)^2}\right)\\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{x^2-(x^2+2hx+h^2)}{x^2(x+h)^2}\right)\\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{-2hx-h^2}{x^2(x+h)^2}\right)\\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{h(-2x-h)}{x^2(x+h)^2}\right)\\ &= \lim_{h\to 0}\frac{-2x-h}{x^2(x+h)^2}\\ &= \frac{-2x}{x^2(x)^2}\\ &= -\frac{2}{x^3}\end{aligned}$$

6. Use the definition of derivative in order to find the 2nd and 3rd derivatives of $f(x) = x^3 + x$.

First we need the first derivative:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{((x+h)^3+(x+h))-(x^3+x)}{h} \\
&= \lim_{h\to 0} \frac{(x^3+3x^2h+3xh^2+h^3+x+h)-x^3-x}{h} \\
&= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3+h}{h} \\
&= \lim_{h\to 0} \frac{h(3x^2+3xh+h^2+1)}{h} \\
&= \lim_{h\to 0} 3x^2+3xh+h^2+1 \\
&= 3x^2+1
\end{aligned}$$

Now we find $f''(x)$:

$$\begin{aligned}
f''(x) &= \lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h} \\
&= \lim_{h\to 0} \frac{(3(x+h)^2+1)-(3x^2+1)}{h} \\
&= \lim_{h\to 0} \frac{(3x^2+6hx+3h^2+1)-3x^2-1}{h} \\
&= \lim_{h\to 0} \frac{6hx+3h^2}{h} \\
&= \lim_{h\to 0} \frac{h(6x+3h)}{h} \\
&= \lim_{h\to 0} 6x+3h \\
&= 6x
\end{aligned}$$

Now we find $f'''(x)$.

$$\begin{aligned}
f'''(x) &= \lim_{h\to 0} \frac{f''(x+h)-f''(x)}{h} \\
&= \lim_{h\to 0} \frac{6(x+h)-6x}{h} \\
&= \lim_{h\to 0} \frac{6h}{h} \\
&= 6
\end{aligned}$$

7. Give all possible notations for the derivatives of $y = f(x)$ up to order 8.

Order					
1	$f'(x)$	$f^{(1)}(x)$	y'	$y^{(1)}$	$\frac{dy}{dx}$
2	$f''(x)$	$f^{(2)}(x)$	y''	$y^{(2)}$	$\frac{d^2y}{dx^2}$
3	$f'''(x)$	$f^{(3)}(x)$	y'''	$y^{(3)}$	$\frac{d^3y}{dx^3}$
4	$f''''(x)$	$f^{(4)}(x)$	y''''	$y^{(4)}$	$\frac{d^4y}{dx^4}$
5	$f'''''(x)$	$f^{(5)}(x)$	y'''''	$y^{(5)}$	$\frac{d^5y}{dx^5}$
6	$f''''''(x)$	$f^{(6)}(x)$	y''''''	$y^{(6)}$	$\frac{d^6y}{dx^6}$
7	$f'''''''(x)$	$f^{(7)}(x)$	y'''''''	$y^{(7)}$	$\frac{d^7y}{dx^7}$
8	$f''''''''(x)$	$f^{(8)}(x)$	y''''''''	$y^{(8)}$	$\frac{d^8y}{dx^8}$

12 Due Feb 19

1. Find the derivative of the function $f(x) = 5x^3 - 4x^2 + 7x + 2$.
   Use the power rule:

$$\begin{aligned} f'(x) &= 5(3x^2) - 4(2x^1) + 7 \\ &= 15x^2 - 8x \end{aligned}$$

2. Find the derivative of the function $f(x) = 2\sqrt[5]{x} + 1$.
   First express the root as a power: $f(x) = 2x^{1/5} + 1$.

$$\begin{aligned} f'(x) &= 2(\frac{1}{5}x^{1/5-1}) \\ &= \frac{2}{5}x^{-4/5} \end{aligned}$$

3. Find the derivative of the function $f(x) = x^2 - x + 3\sqrt{x}$.
   Re-write roots as exponents: $f(x) = x^2 - x + 3x^{1/2}$.

$$\begin{aligned} f'(x) &= 2x^1 - 1 + 3\left(\frac{1}{2}x^{1/2-1}\right) \\ &= 2x - 1 + \frac{3}{2}x^{-1/2} \end{aligned}$$

4. Find the derivative of the function $f(x) = x^{1/3} - x^{-1/3}$.
   $f'(x) = \frac{1}{3}x^{1/3-1} - \frac{-1}{3}x^{-1/3-1} = \frac{1}{3}x^{-2/3} + \frac{1}{3}x^{-4/3}$

5. Find the derivative of the function $f(x) = (3x^2 + 1)^3$.
   First we need to expand the function into a polynomial - use the fact that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.
   $f(x) = (3x^2)^3 + 3(3x^2)^2(1) + 3(3x^2)(1)^2 + (1)^3 = 27x^6 + 27x^4 + 9x^2 + 1$
   Now use the power rule:
   $f'(x) = 27(6x^5) + 27(4x^3) + 9(2x) = 162x^5 + 108x^3 + 18x$

6. Find the derivative of the function $f(x) = x(2x + 1)^2$.
   Expand the square and then distribute the x:
   $f(x) = x(4x^2 + 4x + 1) = 4x^3 + 4x^2 + x$
   Now take the derivative using the power rule:
   $f'(x) = 4(3x^2) + 4(2x) + 1 = 12x^2 + 8x + 1$

7. Find the derivative of the function $f(x) = \dfrac{x^2 + 3}{\sqrt{x}}$.
   First let's factor out $\frac{1}{\sqrt{x}}$ as $x^{-1/2)}$
   $f(x) = x^{-1/2}(x^2 + 3)$
   Now distribute: $f(x) = x^{3/2} + 3x^{-1/2}$.
   Now we can take the derivative using the power rule.
   $f'(x) = \dfrac{3}{2}x^{1/2} + 3\left(\dfrac{-1}{2}x^{-3/2}\right) = \dfrac{3}{2}x^{1/2} - \dfrac{3}{2}x^{-3/2}$.

13 Due Feb 23

1. Compute the derivative of $f(x) = 2^x x^2$.
   The function is in the form of $g(x) \cdot h(x)$ where $g(x) = 2^x$ and $h(x) = x^2$, so we use the Product Rule:
   $f'(x) = g'(x)h(x) + g(x)h'(x) = (2^x \ln 2)(x^2) + (2^x)(2x)$

2. Compute the derivative of $f(x) = x^3 e^x$.
   The function is in the form of $g(x) \cdot h(x)$ where $g(x) = x^3$ and $h(x) = e^x$, so we use the Product Rule:
   $f'(x) = g'(x)h(x) + g(x)h'(x) = (3x^2)(e^x) + (x^3)(e^x)$

3. Compute the derivative of $f(x) = \dfrac{x^2 - 3x + 5}{x^3 + 1}$.
   The function is in the form of $\dfrac{g(x)}{h(x)}$ where $g(x) = x^2 - 3x + 5$ and $h(x) = x^3 + 1$, so we use the Quotient Rule:
   $f'(x) = \dfrac{g'(x)h(x) - g(x)h'(x)}{h(x)^2} = \dfrac{(2x-3)(x^3+1) - (x^2-3x+5)(3x^2)}{(x^3+1)^2}$

4. Compute the derivative of $f(x) = e^{2x}$.
   We can either write the function as $f(x) = (e^2)^x$ and use the derivative of b^x, or we can write it as $f(x) = e^{x+x} = e^x \cdot e^x$ and use the product rule.
   First if $f(x) = (e^2)^x$,

$f'(x) = (e^2)^x \ln(e^2) = e^{2x}2 = 2e^{2x}$.
Alternatively, if $f(x) = e^x e^x$,
$f'(x) = (e^x)'(e^x) + (e^x)(e^x)' = e^{2x} + e^{2x} = 2e^{2x}$

5. Compute the derivative of $f(x) = \dfrac{e^x x^4}{x^2+1}$.
Since this function is in the form of $\dfrac{g(x)}{h(x)}$, we use the quotient rule, but to find $g'(x)$ we have to additionally use the product rule:
$f'(x) = \dfrac{g'(x)h(x) - g(x)h'(x)}{h(x)^2} = \dfrac{((e^x)(4x^3) + (e^x)(x^4))(x^2+1) - (e^x x^4)(2x)}{(x^2+1)^2}$

14 Due Feb 24

1. Compute the derivative of the function $f(x) = \dfrac{x^3 \sin x}{x^2 + \cos x}$
We use the quotient rule where $g(x) = x^3 \sin x$ and $h(x) = x^2 + \cos x$. Then to find $g'(x)$ we need to use the product rule:
$g'(x) = (x^3)'(\sin x) + (x^3)(\sin x)' = (3x^2)(\sin x) + (x^3)(\cos x) = 3x^2 \sin x + x^3 \cos x$.
$h'(x) = 2x - \sin x$.
So by the quotient rule:
$f'(x) = \dfrac{g'h - gh'}{h^2} = \dfrac{(3x^2 \sin x + x^3 \cos x)(x^2 + \cos x) - (x^3 \sin x)(2x - \sin x)}{(x^2 + \cos x)^2}$

2. Compute the derivative of the function $f(x) = 2^x \tan x$.
The derivative is found by the product rule, where $g(x) = 2^x$, $h(x) = \tan x$.
$g'(x) = 2^x \ln 2$, and $h'(x) = \sec^2 x$. Thus by the product rule:
$f'(x) = g'h + gh' = (2x \ln 2)(\tan x) + (2^x)(\sec^2 x)$

3. Compute the derivative of the function $f(x) = \dfrac{\tan x}{x \cos x}$.
This derivative is found by the quotient rule where $g(x) = \tan x$ and $h(x) = x \cos x$.
$g'(x) = (\tan x)' = \sec^2 x$, but for $h'(x)$ we need to use the product rule:
$h'(x) = (x)'(\cos x) + (x)(\cos x)' = (1)\cos x + x(-\sin x) = \cos x - x \sin x$
So by the quotient rule:
$f'(x) = \dfrac{g'h - gh'}{h^2} = \dfrac{(\sec^2 x)(x \cos x) - (\tan x)(\cos x - x \sin x)}{(x \cos x)^2}$

4. Find the tangent to the graph of the function $f(x) = \sin x + \cos x + x$ at the point $(\pi/4, \sqrt{2} + \pi/4)$.
We need the first derivative $f'(x)$, but this is pretty straight forward:
$f'(x) = \cos x - \sin x + 1$
So the slope of the tangent line is
$f'(\pi/2) = \cos(\frac{\pi}{2}) - \sin(\frac{\pi}{2}) + 1 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} + 1 = 1$
So the equation of the tangent line is

$$y - (\sqrt{2} + \frac{\pi}{4}) = 1(x - \frac{\pi}{4})$$

5. Compute the derivative of the function $f(x) = \dfrac{x^2 3^x + \sin x 4^x}{\cos x 5^x + \tan x 6^x}$.
We need to use the quotient rule here, but the derivative of the numerator and denominator need to be found, and both require the product rule. Let

$$g(x) = x^2 3^x + \sin x 4^x, \quad h(x) = \cos x 5^x + \tan x 6^x$$

$$\begin{aligned} g'(x) &= \left[(x^2)'(3^x) + (x^2)(3^x)'\right] + [(\sin x)'(4^x) + (\sin x)(4^x)'] \\ &= \left[(2x)(3^x) + (x^2)(3^x \ln 3)\right] + [(\cos x)(4^x) + (\sin x)(4^x \ln 4)] \\ &= 2x3^x + x^2 3^x \ln 3 + \cos x 4^x + \sin x 4^x \ln 4 \end{aligned}$$

$$\begin{aligned} h'(x) &= [(\cos x)'(5^x) + (\cos x)(5^x)'] + [(\tan x)'(6^x) + (\tan x)(6^x)'] \\ &= [(-\sin x)(5^x) + (\cos x)(5^x \ln 5)] + \left[(\sec^2 x)(6^x) + (\tan x)(6^x \ln 6)\right] \\ &= -\sin x 5^x + \cos x 5^x \ln 5 + \sec^2 x 6^x + \ln 6 \tan x 6^x \end{aligned}$$

Putting it all together,

$$f'(x) = \frac{g'h - gh'}{h^2}$$

$$= \frac{(2x3^x + x^2 3^x \ln 3 + \cos x 4^x + \sin x 4^x \ln 4)(\cos x 5^x + \tan x 6^x) - (x^2 3^x + \sin x 4^x)(-\sin x 5^x + \cos x 5^x}{(\cos x 5^x + \tan x 6^x)^2}$$

Additional practice problems: Pg 177, 1-6:

15 Due Feb 26

1. Compute the derivative of the function e^{x^2+1}.
Using the chain rule, $f(x) = e^{u(x)}$, where $u(x) = x^2 + 1$, and $u'(x) = 2x$, so

$$f'(x) = e^{u(x)} u'(x) = e^{x^2+1}(2x) = 2xe^{x^2+1}$$

2. Compute the derivative of the function $f(x) = \tan(\sqrt{x})$.
Using the chain rule, $f(x) = \tan(u(x))$, where $u(x) = \sqrt{x}, u'(x) = \dfrac{1}{2\sqrt{x}}$, so

$$f'(x) = \sec^2(u(x))u'(x) = \sec^2(\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right) = \frac{\sec^2 x}{2\sqrt{x}}$$

3. Compute the derivative of the function $f(x) = (\sin x + x)^{10}$.
Using the chain rule, $f(x) = (u(x))^{10}$, where $u(x) = \sin x + x, u'(x) = \cos x + 1$, so

$$f'(x) = 10(u(x))^9 u'(x) = 10(\sin x + x)^9(\cos x + 1)$$

4. Compute the derivative of the function $f(x) = \sqrt{x \sec x - 5}$.
Using the chain rule, $f(x) = \sqrt{u(x)}$, where $u(x) = x \sec x - 5$, and $u'(x) = \sec x + x \sec x \tan x$. Thus,

$$f'(x) = \frac{1}{2\sqrt{u(x)}} u'(x) = \frac{\sec x + x \sec x \tan x}{2\sqrt{x \sec x - 5}}$$

5. Compute the derivative of the function $f(x) = \sin\left(3^{x^2+1}\right)$.
Using the chain rule, $f(x) = \sin\left(u(x)\right)$, where $u(x) = 3^{v(x)}$, and $v(x) = x^2 + 1$. Then

$$v'(x) = 2x$$
$$u'(x) = 3^{v(x)} \ln 3 v'(x) = 2x3^{x^2+1} \ln 3$$
$$f'(x) = \cos(u(x))u'(x) = \cos\left(3^{x^2+1}\right) 2x3^{x^2+1} \ln 3$$

Additional practice problems on pg 172, 1-12

16 Due Feb 29

1. Compute the derivative of $f(x) = \cos^6(\ln x)$
$f(x)$ may explicitly be written

$$f(x) = (\cos(\ln x))^6$$

Using the chain rule and the derivative of a logarithm, we get

$$\begin{aligned} f'(x) &= 6(\cos(\ln x))^5 (\cos(\ln x))' \\ &= 6(\cos(\ln x))^5 \left(-\sin(\ln x)(\ln x)'\right) \\ &= 6(\cos(\ln x))^5 \left(-\sin(\ln x)\left(\frac{1}{x}\right)\right) \\ &= -\frac{6\cos^5(\ln x)\sin(\ln x)}{x} \end{aligned}$$

2. Compute the derivative of $f(x) = \sec\left(\sqrt{x^2+1}\right)$
Using the chain rule we have:

$$\begin{aligned} f'(x) &= \sec\left(\sqrt{x^2+1}\right)\tan\left(\sqrt{x^2+1}\right)\left(\sqrt{x^2+1}\right)' \\ &= \sec\left(\sqrt{x^2+1}\right)\tan\left(\sqrt{x^2+1}\right)\left(\frac{1}{2\sqrt{x^2+1}}(x^2+1)'\right) \\ &= \sec\left(\sqrt{x^2+1}\right)\tan\left(\sqrt{x^2+1}\right)\left(\frac{1}{2\sqrt{x^2+1}}(2x)\right) \\ &= \sec\left(\sqrt{x^2+1}\right)\tan\left(\sqrt{x^2+1}\right)\left(\frac{x}{\sqrt{x^2+1}}\right) \end{aligned}$$

3. Compute the derivative of $f(x) = \log_3(x^2+1)$
   Using the chain rule and derivative of a logarithm, we get

$$\begin{aligned} f'(x) &= \frac{1}{(x^2+1)\ln 3}(x^2+1)' \\ &= \frac{1}{(x^2+1)\ln 3}(2x) \\ &= \frac{2x}{(x^2+1)\ln 3} \end{aligned}$$

4. Compute the derivative of $f(x) = \sqrt{\ln(x^3+5)}$
   Using chain rule and derivative of a logarithm, we get:

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{\ln(x^3+5)}}\left(\ln(x^3+5)\right)' \\ &= \frac{1}{2\sqrt{\ln(x^3+5)}}\left(\frac{1}{x^3+5}(x^3+5)'\right) \\ &= \frac{1}{2\sqrt{\ln(x^3+5)}}\left(\frac{1}{x^3+5}(3x^2)\right) \\ &= \frac{3x^2}{2(x^3+5)\ln(x^3+5)} \end{aligned}$$

5. Compute the derivative of $f(x) = x^{\sin x}$
   To take the derivative of $x^{u(x)}$, we have to use the trick that $f(x) = e^{\ln(f(x))}$. Thus

$$x^{\sin x} = e^{\ln(x^{\sin x})} = e^{\sin x \ln x}$$

   The latter uses the Power Rule of logarithms. Now we can take the derivative using chain rule and product rule.

$$\begin{aligned} f'(x) &= e^{\sin x \ln x}(\sin x \ln x)' \\ &= e^{\sin x \ln x}\left((\sin x)' \ln x + \sin x (\ln x)'\right) \\ &= e^{\sin x \ln x}\left((\cos x)\ln x + \sin x\left(\frac{1}{x}\right)\right) \\ &= x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right) \end{aligned}$$

Additional practice problems on pg 192, 23-34

17 Due Mar 2

1. Compute the linear approximation of $f(x) = \dfrac{1}{1+x^3}$ at $a = 1$ and use this in order to estimate $f(1.2)$.
   The linear approximation is the tangent line at $x = a$. The formula is

$$L(x) = f'(a)(x-a) + f(a)$$

The first derivative is found using chain rule, but it is useful to write $f(x) = (1+x^3)^{-1}$:

$$\begin{aligned} f'(x) &= -(1+x^3)^{-2}(1+x^3)' \\ &= -(1+x^3)^{-2}(3x^2) \\ &= \frac{-3x^2}{(1+x^3)^2} \end{aligned}$$

Now $f'(a) = f'(1) = -\frac{3}{4}$ and $f(a) = f(1) = \frac{1}{2}$. So

$$L(x) = -\frac{3}{4}(x-1) + \frac{1}{2}$$

So the estimate is

$$f(1.2) \approx L(1.2) = -\frac{3}{4}(1.2-1) + \frac{1}{2} = -\frac{3}{4}\left(\frac{1}{5}\right) + \frac{1}{2} = -\frac{3}{20} + \frac{10}{20} = \frac{7}{20}$$

2. Compute the linear approximation of the function $f(x) = \sqrt[3]{x}$ at the point $a = 8$ and use this approximation in order to estimate $\sqrt[3]{8.5}$.
   As in the previous problem, the linear approximation is the tangent line at $x = a$. The formula is

$$L(x) = f'(a)(x-a) + f(a)$$

With $f(x) = x^{1/3}$, we have

$$f'(x) = \frac{1}{3}x^{-2/3}$$

so $f'(8) = \frac{1}{3}(8)^{-2/3} = \frac{1}{3}\frac{1}{4} = \frac{1}{12}$. Since $f(8) = 2$, our linear approximation is

$$L(x) = \frac{1}{12}(x-8) + 2$$

Thus

$$\sqrt[3]{8.5} \approx L(8.5) = \frac{1}{12}(8.5-8) + 2 = \frac{1}{24} + 2 = \frac{49}{24}$$

3. Estimate $\sqrt{62}$.
   We want to estimate based on a nearby a which is a perfect square: Let's use $a = 64$ since $\sqrt{64} = 8$. With $f(x) = \sqrt{x}$, and $x = a$ we can find the linear approximation. Note that $f'(x) = \frac{1}{2\sqrt{x}}$, so $f'(64) = \frac{1}{2(8)} = \frac{1}{16}$.

$$\sqrt{62} \approx L(62) = \frac{1}{16}(62-64) + 8 = \frac{1}{16}(-2) + 8 = -\frac{1}{8} + 8 = 7.875$$

4. Estimate $\sqrt[3]{26}$.
This is similar to number 2. We find a nearby a which is a perfect cube: $a = 27$ is an obvious choice. We let $f(x) = x^{1/3}$. As in problem 2, $f'(x) = \frac{1}{3}(x)^{-2/3}$. We have $f(27) = 3$ and

$$f'(27) = \frac{1}{3}(27)^{-2/3} = \frac{1}{3}\frac{1}{9} = \frac{1}{27}$$

The approximation is thus

$$\sqrt[3]{26} \approx L(26) = \frac{1}{27}(26 - 27) + 3 = -\frac{1}{27} + 3 = \frac{80}{27}$$

5. Estimate $\ln 2$.
We use a linear approximation of $f(x) = \ln x$ with $a = e$, since that is the only nearby x value where $f(x)$ gives a reasonably nice value. Note:

$$f'(x) = \frac{1}{x}$$

so $f'(e) = \frac{1}{e}$ and $f(e) = 1$. The approximation is thus:

$$\ln 2 \approx L(2) = \frac{1}{e}(2 - e) + 1 = \frac{2}{e}$$

6. Find the linear approximation of $y = \cos x \sin x$ at $a = \dfrac{\pi}{4}$.
By the product rule,

$$y' = (\sin x)\sin x + \cos x(-\cos x) = \sin^2 x - \cos^2 x$$

At the point $x = \frac{\pi}{4}$, the slope is

$$y'(\pi/4) = \sin^2(\pi/4) - \cos^2(\pi/4) = \left(\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2 = 0$$

Since at $x = \frac{\pi}{4}$, $y = \frac{1}{2}$,

$$L(x) = 0\left(x - \frac{\pi}{4}\right) + \frac{1}{2} = \frac{1}{2}$$

7. Find the linear approximation of $y = e^{\sqrt{x}}$ at $a = (\ln 2)^2$.
By chain rule,

$$y' = e^{\sqrt{x}}(\sqrt{x})' = e^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right)$$

$y(a) = e^{\ln 2} = 2$, and $y'(a) = e^{\ln 2}\left(\frac{1}{2\ln 2}\right) = 2\left(\frac{1}{2\ln 2}\right) = \frac{1}{\ln 2}$ So the linear approximation is

$$L(x) = \frac{1}{\ln 2}\left(x - (\ln 2)^2\right) + 2$$

18 Due March 4

1. Use the mean value theorem to show that if x and y are two numbers on $[1, +\infty)$, then

$$|\sqrt{x} - \sqrt{y}| \leq \frac{1}{2}|x - y|$$

First, let's assume $y \leq x$. So what we are trying to prove is equivalent to

$$\sqrt{x} - \sqrt{y} \leq \frac{1}{2}(x - y)$$

or, by dividing both sides by $(x - y)$ we could write

$$\frac{\sqrt{x} - \sqrt{y}}{x - y} \leq \frac{1}{2}$$

Consider the function $f(x) = \sqrt{x}$, where $f'(x) = \frac{1}{2\sqrt{x}}$. By the MVT, there must be a value c where $y < c < x$ such that

$$f'(c) = \frac{f(x) - f(y)}{x - y} = \frac{\sqrt{x} - \sqrt{y}}{x - y}$$

Also, notice that $f'(x)$ is decreasing; so

$$f'(x) \geq f'(x) \geq f'(c) =$$

The first expression is $\frac{1}{2}$, so we can write

$$\frac{1}{2} \geq \frac{\sqrt{x} - \sqrt{y}}{x - y}$$

2. Use the mean value theorem to show that if x and y are two numbers on $[0, 1]$, then

$$|e^x - e^y| \leq e|x - y|$$

First, let us assume $y \leq x$. Then what we are trying to prove can be written

$$e^x - e^y \leq e(x - y)$$

or

$$\frac{e^x - e^y}{x - y} \leq e$$

Consider the function $f(x) = e^x$, where $f'(x) = e^x$ which is increasing. By the MVT, there exists some c where $y < c < x$ such that

$$f'(c) = \frac{f(x) - f(y)}{x - y} = \frac{e^x - e^y}{x - y}$$

Since $f'(x)$ is increasing, we can write

$$f'(c) \leq f'(1)$$

Namely

$$\frac{e^x - e^y}{x - y} \leq e$$

3. Use the mean value theorem to show that if x and y are any two numbers, then

$$|\cos x - \cos y| \leq |x - y|$$

Let us assume $y \leq x$. So what we are trying to prove can be written as

$$\frac{|\cos x - \cos y|}{x - y} \leq 1$$

Consider the function $f(x) = \cos x$. By the mean value theorem, there exists some c between x and y where

$$f'(c) = \frac{f(x) - f(y)}{x - y}$$

in other words,

$$f'(c) = \frac{\cos x - \cos y}{x - y}$$

Since $f'(c) = -\sin x \leq 1$, we can say for sure that

$$\frac{\cos x - \cos y}{x - y} \leq \left| \frac{\cos x - \cos y}{x - y} \right| \leq 1$$

Or

$$|\cos x - \cos y| \leq |x - y|$$

4. Examine whether Rolle's theorem applies to the function $f(x) = x(1 - x)$ on the interval $[0, 1]$.
Since $f(x)$ is differentiable on $(0, 1)$ and $f(0) = 0 = f(1)$, Rolle's Theorem does apply.

5. Examine whether Rolle's theorem applies to the function $f(x) = \sin x$ on the interval $[0, \pi/2]$. How about the Mean Value Theorem?
Since $\sin x$ is differentiable, but $f(0) = 0$ and $f(\pi/2) = 1$, so Rolle's Theorem does not apply, but the Mean Value Theorem does.

6. Show that if a differential function that is defined on all of $\mathbb{R}$, has distinct roots, then its derivative has at least five distinct roots.
Say the five distinct roots, in increasing order are $x_1, x_2, x_3, x_4, x_5, x_6$. Since the value of the function is zero at each of these points, by Rolle's Theorem there exist distinct points $y_1, \ldots y_5$ where

$$x_1 < y_1 < x_2 < y_2 < x_3 < y_3 < x_4 < y_4 < x_5 < y_5 < x_6$$

where $f'(y_i) = 0$ for each $i = 1, \ldots, 5$.

19 Due March 7

1. Find the critical points of the function $f(x) = x \ln x$.
   The domain of the function is $(0, \infty)$.
   $f'(x) = \ln x + x \cdot \frac{1}{x} = \ln x + 1$. Setting this equal to zero
   $$\ln x + 1 = 0$$
   $$\Leftrightarrow \ln x = -1$$
   $$\Leftrightarrow e^{-1} = x$$
   This is the only critical point on the domain.

2. Find the critical points of the function $f(x) = x^3 - x + 1$.
   The domain of the function is all reals.
   $f'(x) = 3x^2 - 1$. Setting this equal to zero
   $$3x^2 - 1 = 0 \Leftrightarrow x^2 = \frac{1}{3}$$
   So the two critical points are $x = \sqrt{\frac{1}{3}}, -\sqrt{\frac{1}{3}}$.

3. Find the minimum and the maximum of the function $f(x) = e^x - x - 1$ on the interval $[-1, 1]$.
   $f'(x) = e^x - 1$. Setting this equal to zero we have
   $$e^x - 1 = 0 \Leftrightarrow e^x = 1 \Leftrightarrow x = 0$$
   We now evaluate the function at the endpoints of the domain and at $x = 0$
   $f(-1) = e^{-1} + 1 - 1 = \frac{1}{e}$
   $f(0) = e^0 - 0 - 1 = 0$ (minimum)
   $f(1) = e^1 - 1 - 1 = e - 2$ (maximum)

4. Find the minimum and the maximum of the function $f(x) = \cos x + x$ on the interval $[-\pi, \pi]$.
   $f'(x) = -\sin x + 1$, which is zero when $\sin x = 1$, i.e. $x = \frac{\pi}{2}$. We check the endpoints of the domain and the critical point:
   $f(-\pi) = -1 - \pi$ (minimum)
   $f(\frac{\pi}{2}) = 0 + \frac{\pi}{2} = \frac{\pi}{2}$
   $f(\pi) = -1 + \pi$ (maximum)

5. Find the minimum and the maximum of the function $f(x) = \dfrac{1 - x}{x^2 + 3x}$ on the interval $[1, 4]$.
   $f'(x) = \dfrac{-(x^2 + 3x) - (1 - x)(2x + 3)}{(x^2 + 3x)^2}$.
   Where we get a critical point when the numerator is zero, i.e.
   $$-x^2 - 3x - (2x - 2x^2 - 3x + 3) = 0$$
   $$\Leftrightarrow x^2 - 2x - 3 = (x - 3)(x + 1) = 0$$

Where roots are $x = -1, 3$. The only critical point in the interval is $x = 3$. Now we check the endpoints of the interval as well:
$f(1) = 0$ (maximum)
$f(3) = \dfrac{1-3}{18} = -\frac{1}{9}$ (minimum)
$f(4) = \dfrac{-3}{16+12} = -\dfrac{3}{28}$

6. Find the minimum and the maximum of the function $f(x) = x^5 - x$ on the interval $[0, 2]$.
$f'(x) = 5x^4 - 1$. Setting this equal to zero

$$5x^4 - 1 = 0 \Leftrightarrow x^4 = \frac{1}{5}$$

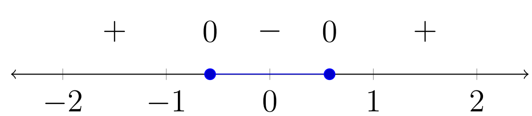
So the only critical point in our interval is $\dfrac{1}{\sqrt[4]{5}}$.
We check the endpoints and critical point to find min and max:
$f(0) = 0$ (maximum)
$f\left(\dfrac{1}{\sqrt[4]{5}}\right) = \dfrac{1}{\sqrt[4]{5}}\left(\dfrac{1}{5} - 1\right) = -\dfrac{4}{5\sqrt[4]{5}}$ (minimum)
$f(1) = 0$ (maximum)

20 Due March 9

1. Determine the intervals of monotonicity and the local extrema of the function $f(x) = x^3 - x$
$f'(x) = 3x^2 - 1$. If we set the first derivative equal to zero and solve for x we get

$$3x^2 - 1 = 0 \Leftrightarrow x^2 = \frac{1}{3}$$

So we get $x = -\sqrt{\frac{1}{3}}$ and $x = \sqrt{\frac{1}{3}}$. We can draw the sign graph of $f'(x)$:

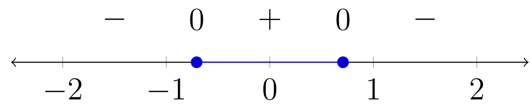


So the function is increasing on $(-\infty, -\sqrt{\frac{1}{3}}) \cup (\frac{\sqrt{13}}{,}\infty)$ and decreasing on $(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}})$.
Since the function goes from increasing to decreasing there, $x = -\sqrt{\frac{1}{3}}$ is a local maximum. Similarly, $x = \sqrt{\frac{1}{3}}$ is a local minimum.

2. Determine the intervals of monotonicity and the local extrema of the function $f(x) = xe^{-x^2}$
$f'(x) = e^{-x^2} + xe^{-x^2}(-2x) = x^{-x^2}(1 - 2x^2)$. Since $e^{-x^2} > 0$ always, we only need to set $1 - 2x^2 = 0$. We get

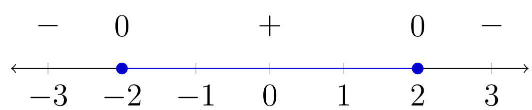
$$x^2 = \frac{1}{2}$$

So we have critical points at $x = -\sqrt{\frac{1}{2}}$ and $x = \sqrt{\frac{1}{2}}$. We can draw the sign graph of $f'(x)$:



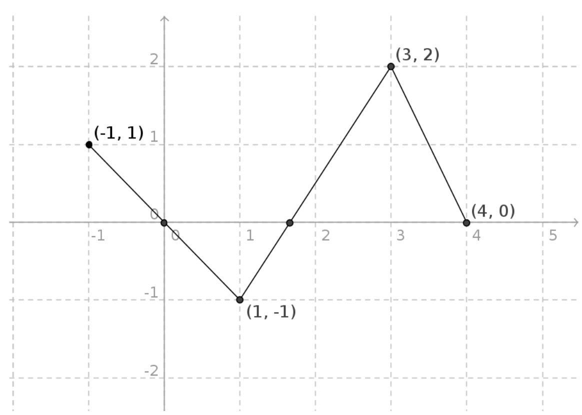
f is increasing on $(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}})$.
f is decreasing on $(-\infty, \sqrt{\frac{1}{2}}) \cup (\sqrt{\frac{1}{2}}, \infty)$.
$x = -\sqrt{\frac{1}{2}}$ is a local minimum and $x = \sqrt{\frac{1}{2}}$ is a local maximum.

3. Determine the intervals of monotonicity and the local extrema of the function $f(x) = \dfrac{x}{x^2+4}$
$f'(x) = \dfrac{(x^2+4) - x(2x)}{(x^2+4)^2} = \dfrac{-x^2+4}{(x^2+4)^2}$. The denominator is always positive, so this derivative is only zero when the numerator is zero, i.e. at $x = -2$ and $x = 2$. The sign graph of $f'(x)$ looks like:



f is increasing on $(-2, 2)$.
f is decreasing on $(-\infty, 2) \cup (2, \infty)$.
$x = -2$ is a local minimum and $x = 2$ is a local maximum.

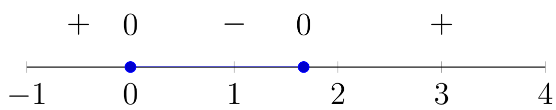
4. A differentiable function f is defined on the interval $[-1, 4]$. The graph of its derivative is a broken line with vertices at the points $(-1, 1), (1, -1), (3, 2), (4, 0)$. Determine the intervals of monotonicity of f and the local extrema of f on $(-1, 4)$.
First, it is helpful to have a sense of the picture of the derivative function.



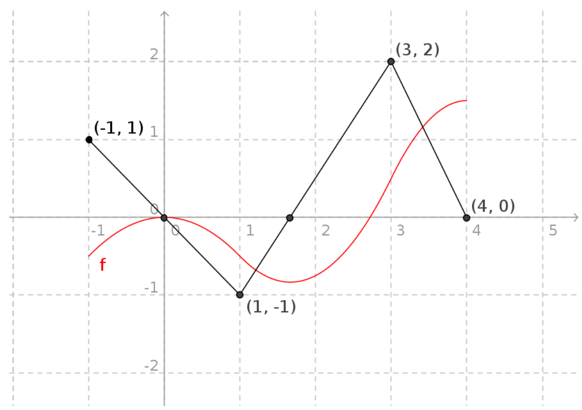
This is what $f'(x)$ looks like. The function is increasing when $f'(x) > 0$ and it is decreasing when $f'(x) < 0$. Local extrema happen when f goes from increasing to decreasing, or vice versa, so let's first find the intervals of monotonicity. From $(-1, 1)$ to $(1, -1)$ the slope of f' is $\dfrac{(-1) - (1)}{(1) - (-1)} = -1$. From the point $-1, 1)$ we can see that the line will hit the point $(0,0)$. From $(1,-1)$ to $(3,2)$, we have a slope of $\frac{(2)-(-1)}{(3)-(1)} = \frac{3}{2}$. The line in point-slope form is

$$y - 2 = \frac{3}{2}(x-3)$$

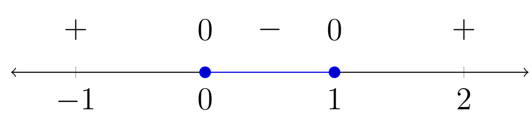
Plugging in $y = 0$ we solve for x and get $x = \frac{5}{3}$, so this line hits the point $(\frac{5}{3}, 0)$. The sign graph of $f'(x)$ can be given by



So the function is increasing on $(-1,0) \cup (\frac{5}{3}, 4)$.
f is decreasing on $(0, \frac{5}{3})$.
$x = 0$ is a local maximum and $x = \frac{5}{3}$ is a local minimum. Although not necessary, it is perhaps helpful to see what $f(x)$ may look like (in red):

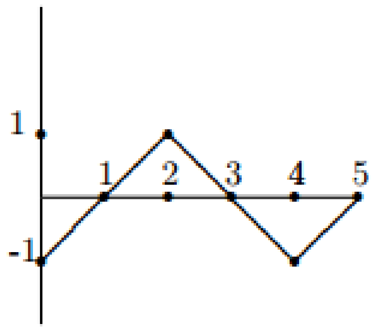


5. Determine the intervals of monotonicity and the local extrema of the function $f(x) = 2x^5 - 5x^2$
$f'(x) = 10x^4 - 10x = 10x(x^3 - 1)$.
So $x = 0$ and $x = 1$ are the two roots of $f'(x)$. The sign graph of $f'(x)$ is



f is increasing on $(-\infty, 0) \cup (1, \infty)$ and decreasing on $(0, 1)$.
$x = 0$ is a local maximum while $x = 1$ is a local minimum.

6. The graph of the derivative of a function $f(x)$ is given by

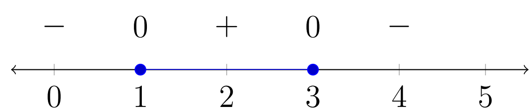


Determine the intervals on which f is

(a) increasing

(b) decreasing

Don't forget, this is the graph of $f'(x)$, not $f(x)$. The sign graph of $f'(x)$ is



From this we can get the intervals of increasing/decreasing:
Increasing on $(1, 3)$, decreasing on $(0, 1) \cup (3, 5)$.

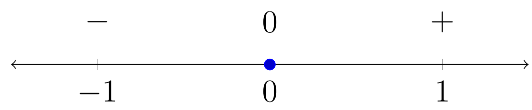
21 Due March 11

1. Use the second derivative test in order to identify the local extrema of the function $f(x) = x^3 - 3x$.
   $f'(x) = 3x^2 - 3$, so set $3(x^2 - 1) = 0$ gives us $x = -1, 1$ as critical points.
   $f''(x) = 6x$. $f''(-1) < 0$ so -1 is a local maximum. $f''(1) > 0$ so 1 is a local minimum.

2. Use the second derivative test in order to identify the local extrema of the function $f(x) = \dfrac{x}{x^2 + 1}$.
   $f'(x) = \dfrac{(x^2 + 1) - (2x^2)}{(x^2 + 1)^2} = \dfrac{-x^2 + 1}{(x^2 + 1)^2}$. Zeroes come from the numerator, $x = -1, 1$, so these are critical points. Note the denominator is always positive.
   $f''(x) = \dfrac{(-2x)(x^2 + 1)^2 - (-x^2 + 1)(2(x^2 + 1)(2x)}{(x^2 + 1)^4} = \dfrac{-2x\left((x^2 + 1) + 2(-x^2 + 1)\right)}{(x^2 + 1)^3}$
   $= \dfrac{-2x\left(-x^2 + 3\right)}{(x^2 + 1)^3}$.
   $f''(-1) = \dfrac{-2(-1)(2)}{2^3} > 0$ so -1 is a local minimum.
   $f''(1) = \dfrac{-2(1)(2)}{2^3} < 0$ so 1 is a local maximum.

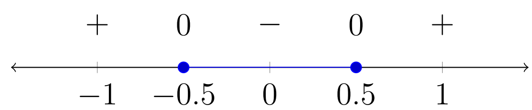
3. Determine the intervals of concavity and the inflection points of the function $f(x) = x^3 - x$.

$f'(x) = 3x^2 - 1$, and $f''(x) = 6x$ The sign graph of $f''(x)$ is



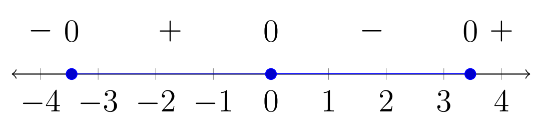
f is concave down on $(-\infty, 0)$, concave up on $(0, \infty)$ and 0 is an inflection point.

4. Determine the intervals of concavity and the inflection points of the function $f(x) = e^{-x^2}$. $f'(x) = -2xe^{-x^2}$, and $f''(x) = -e^{-x^2} + (4x^2)e^{-x^2} = e^{-x^2}(4x^2 - 1)$ which has zeroes at $x = \pm\frac{1}{2}$. The sign graph of $f''(x)$ is



f is concave down on $(-.5, .5)$, concave up on $(-\infty, -.5) \cup (.5, \infty)$. $-.5$ and $.5$ are inflection points.

5. Determine the intervals of concavity and the inflection points of the function $f(x) = \dfrac{x}{x^2 + 4}$.
$f'(x) = \dfrac{(x^2 + 4) - (2x^2)}{(x^2 + 4)^2} = \dfrac{-x^2 + 4}{(x^2 + 4)^2}$.
$f''(x) = \dfrac{(-2x)(x^2 + 4)^2 - (-x^2 + 4)(2(x^2 + 4)(2x)}{(x^2 + 4)^4} = \dfrac{-2x\left((x^2 + 4) + 2(-x^2 + 4)\right)}{(x^2 + 4)^3}$
$= \dfrac{-2x\left(-x^2 + 12\right)}{(x^2 + 4)^3}$.
$f''(x)$ has zeroes (from the numerator) at $x = 0, \pm\sqrt{12}$. The sign graph of $f''(x)$ is

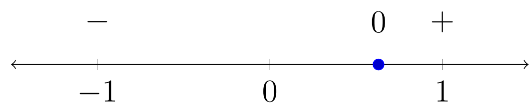


f is concave down on $(-\infty, -\sqrt{12}) \cup (0, \sqrt{12})$, concave up on $(-\sqrt{12}, 0) \cup (\sqrt{12}, \infty)$. $-\sqrt{12}, 0$ and $\sqrt{12}$ are inflection points.

6. A differentiable function f is defined on the interval $[-1, 4]$. The graph of its derivative is a broken line with vertices at the points $(-1, 1), (1, -1), (3, 2), (4, 0)$. Determine the intervals of concavity and the inflection points of f on $(-1, 4)$.
The slope of the derivative function is -1 on $(-1, 1)$, $+\frac{3}{2}$ on $(1, 3)$ and -2 on $(3, 4)$. Therefore, the function f is concave down on $(-1, 1) \cup (3, 4)$, concave up on $(1, 3)$ and the points $1, 3$ are inflection points.

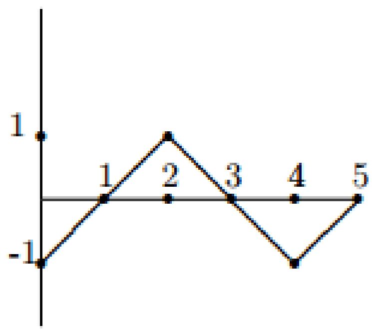
7. Determine the intervals of concavity and the inflection points of the function $f(x) = 2x^5 - 5x^2$
$f'(x) = 10x^4 - 10x$
$f''(x) = 40x^3 - 10 = 10(4x^3 - 1)$. There is only one zero, which is $\sqrt[3]{\dfrac{1}{4}}$. The sign

graph of $f''(x)$ is



So the function is concave down on $(-\infty, \sqrt[3]{\frac{1}{4}})$, concave up on $(\sqrt[3]{\frac{1}{4}}, \infty)$ and $\sqrt[3]{\frac{1}{4}}$ is the only inflection point.

8. The graph of the derivative of a function is given by:



Determine the intervals on which f is

(a) concave up

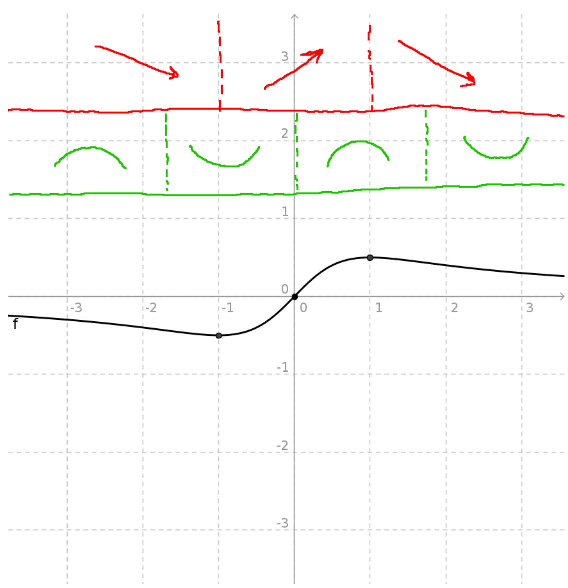
(b) concave down

The slope of the derivative function is 1 on $(1, 2)$, -1 on $(2, 4)$ and 1 on $(4, 5)$. Therefore, the function f is concave up on $(0, 2) \cup (4, 5)$, and concave down on $(2, 4)$.

22 Due March 14

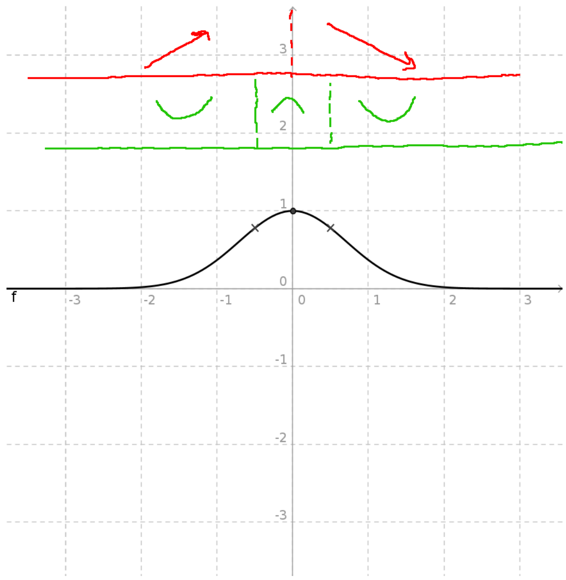
1. Sketch the graph of the function $f(x) = x^3 - 3x$.
$f(x) = x(x^2 - 3)$ so we have zeroes at $x = 0, \pm\sqrt{3}$.
$f'(x) = 3x^2 - 3$ which has zeroes at $x = -1, 1$. From homework 20 we have f increasing on $(-\infty, -1) \cup (1, \infty)$ and decreasing on $(-1, 1)$.
$f''(x) = 6x$. From homework 21 we have concave down on $(-\infty, 0)$, concave up on $(0, \infty)$ and an inflection point at 0.
There are no vertical asymptotes, and $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to+\infty} f(x) = +\infty$.
The sketch looks like this:

2. Sketch the graph of the function $f(x) = \dfrac{x}{x^2+1}$.
   f has a zero only at $x = 0$.
   From homework 20 we have f increasing on $(-1, -1)$ and decreasing on $(-\infty, -1) \cup (1, \infty)$.
   From homework 21 we have $f''(x) = \dfrac{-2x\,(-x^2+3)}{(x^2+1)^3}$. This has zeroes at $x = 0, \pm\sqrt{3}$, and we can show that it is concave down on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$, concave up on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$.
   There are no vertical asymptotes, and $\lim_{x\to-\infty} f(x) = 0$, $\lim_{x\to+\infty} f(x) = 0$.
   The sketch looks like this:

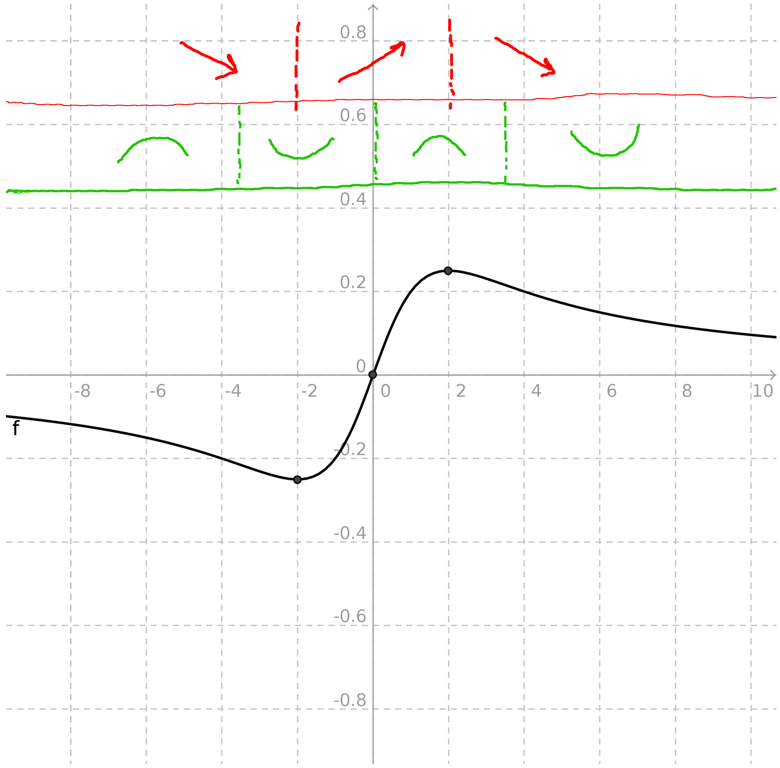


3. Sketch the graph of the function $f(x) = e^{-x^2}$.
   $f(x) > 0$ always, so there are no zeroes.

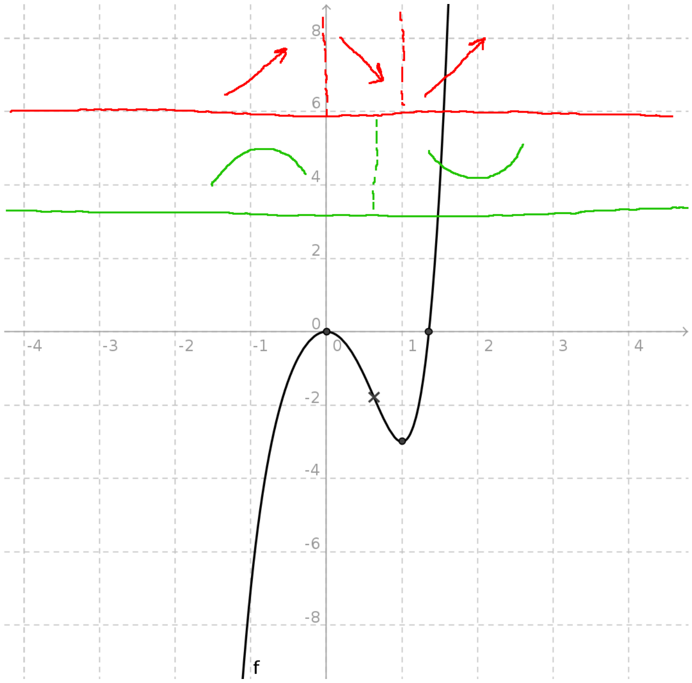
From homework 20 we have f increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.
From homework 21 we have concave down on $(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}})$, concave up on $(-\infty, -\sqrt{\frac{1}{2}}) \cup (\sqrt{\frac{1}{2}}, \infty)$ and inflection points at $-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}$.
There are no vertical asymptotes, and $\lim_{x \to -\infty} f(x) = 0$, $\lim_{x \to +\infty} f(x) = 0$.
The sketch looks like this:



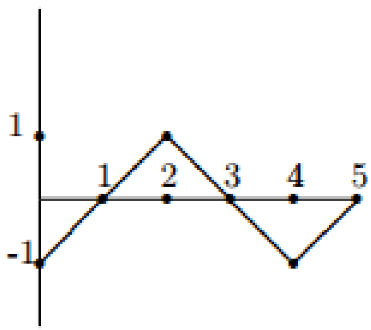
4. Sketch the graph of the function $f(x) = \dfrac{x}{x^2 + 4}$.
f has a zero only at $x = 0$.
From homework 20 we have f increasing on $(-2, -2)$ and decreasing on $(-\infty, -2) \cup (2, \infty)$.
From homework 21 we have concave down on $(-\infty, -\sqrt{12}) \cup (0, \sqrt{12})$, concave up on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$.
There are no vertical asymptotes, and $\lim_{x \to -\infty} f(x) = 0$, $\lim_{x \to +\infty} f(x) = 0$.
The sketch looks like this:



5. Sketch the graph of the function $f(x) = 2x^5 - 5x^2$.
   $f(x) = x^2(2x^3 - 5)$ so we have zeroes at $x = 0, \pm\sqrt[3]{\frac{5}{2}}$.
   From homework 20 we have f increasing on $(-\infty, 0) \cup (1, \infty)$ and decreasing on $(0, 1)$.
   From homework 21 we have concave down on $(-\infty, \sqrt[3]{\frac{1}{4}})$, concave up on $(\sqrt[3]{\frac{1}{4}}, \infty)$ and an inflection point at 0.
   There are no vertical asymptotes, and $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to+\infty} f(x) = +\infty$.
   The sketch looks like this:

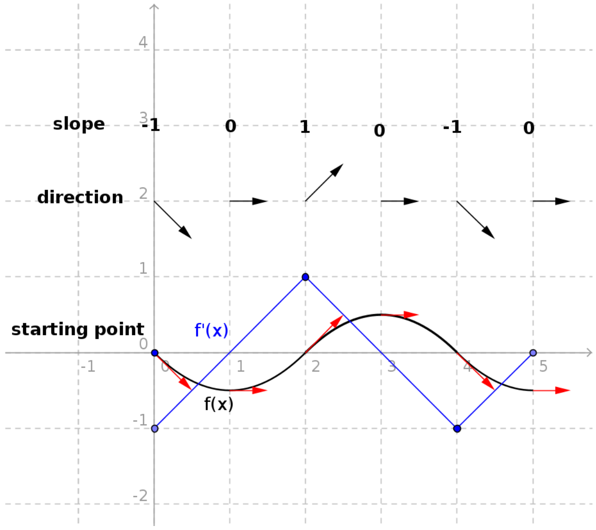


6. The graph of the derivative of a function is given by:



Try to sketch the graph of f.
We have f increasing on $(1, 3)$ only and decreasing on $(0, 1) \cup (3, 5)$.
From homework 21 we have concave up on $(0, 2) \cup (4, 5)$ and concave down on $(2, 4)$. Furthermore, we know that the curve has a derivative of $-1, 0, 1, 0, -1, 0$ at $x = 0, 1, 2, 3, 4, 5$ respectively. With this in mind, we can attempt a sketch of

the curve:



Note, however, that the starting point of the curve is arbitrary. I started it at $(0,0)$ but it could have started at any $y-$coordinate.

23 Due March 30

1. Use L'Hôpital's rule to compute the limit $\lim_{x\to 0}\frac{\sqrt{x+4}-2}{\sqrt{x+9}-3}$.
   We can see that both the numerator and denominator approach 0 as $x\to 0$, so we may use L'Hôpital's rule.

$$\lim_{x\to 0}\frac{\sqrt{x+4}-2}{\sqrt{x+9}-3}=\lim_{x\to 0}\frac{\frac{1}{2\sqrt{x+4}}}{\frac{1}{2\sqrt{x+9}}}=\lim_{x\to 0}\frac{1}{2\sqrt{x+4}}\frac{2\sqrt{x+9}}{1}=\frac{\sqrt{9}}{\sqrt{4}}=\frac{3}{2}$$

2. Use L'Hôpital's rule to compute the limit $\lim_{x\to 0}\frac{\sin(3x)}{\sin(5x)}$. As $x\to 0$, both numerator and denominator approach 0, so we may use L'Hôpital's rule.

$$\lim_{x\to 0}\frac{\sin(3x)}{\sin(5x)}=\lim_{x\to 0}\frac{\cos(3x)3}{\cos(5x)5}=\frac{3}{5}$$

3. Use L'Hôpital's rule to compute the limit $\lim_{x\to 0}\frac{\cos(2x)-1}{\cos(7x)-1}$.
   As $x\to 0$, both numerator and denominator approach 0, so we may use L'Hôpital's rule.

$$\lim_{x\to 0}\frac{\cos(2x)-1}{\cos(7x)-1}=\lim_{x\to 0}\frac{-\sin(2x)2}{-\sin(7x)7}$$

   But still both numerator and denominator approach 0, so we can use L'Hôpital's ruleagain:

$$=\lim_{x\to 0}\frac{\cos(2x)4}{\cos(7x)49}=\frac{4}{49}$$

4. Use L'Hôpital's rule to compute the limit $\lim_{x\to 0} \frac{e^{3x}-1}{e^{8x}-1}$.

Both numerator and denominator approach 0 as $x \to 0$, so we may use L'Hôpital's rule.

$$\lim_{x\to 0} \frac{e^{3x}-1}{e^{8x}-1} = \lim_{x\to 0} \frac{3e^{3x}}{8e^{8x}} = \frac{3}{8}$$

5. Use L'Hôpital's rule to compute the limit

(a) $\lim_{x\to 0} \frac{e^{7x}-1}{e^{6x}-1+x}$.

both numerator and denominaot approach 0 as $x \to 0$ so we may use L'Hôpital's rule.

$$\lim_{x\to 0} \frac{e^{7x}-1}{e^{6x}-1+x} = \lim_{x\to 0} \frac{7e^{7x}}{6e^{6x}+1} = \frac{7}{6+1} = 1$$

(b) $\lim_{x\to +\infty} \frac{\ln x}{\sqrt{x}}$.

Both numerator and denominator approach $+\infty$ as $x \to +\infty$, so we may use L'Hôpital's rule.

$$\lim_{x\to +\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to +\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to +\infty} \frac{2\sqrt{x}}{x} = \lim_{x\to +\infty} \frac{2}{\sqrt{x}} = 0$$

6. Use L'Hôpital's rule to compute the limit $\lim_{x\to 0^+} x^x$.

To handle a limit like this, we have to use a few tricks before employing L'Hôpital's rule. Let us call this limit L. The limit of $\ln(x^x)$ will be $\ln(L)$. It will be easier to find this limit, then to get L, we use the fact that $L = e^{\ln(L)}$.

$$\ln(L) = \lim_{x\to 0^+} \ln(x^x) = \lim_{x\to 0^+} x\ln(x)$$

Now we can use a trick, we can move x to the denominator as $1/x$, since $x = \frac{1}{\frac{1}{x}}$.

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}}$$

Now notice that the numerator approaches $-\infty$ and the denominator approaches $+\infty$, so we may use L'Hôpital's rule.

$$\lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{-x^2}{x} = \lim_{x\to 0^+} -x = 0$$

So the answer to the original limit is $e^0 = 1$.

7. Use L'Hôpital's ruleto compute the limit $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x$.
To handle a limit like this, we have to use a few tricks before employing L'Hôpital's rule. Let us call this limit L. The limit of $\ln\left(\left(1+\frac{1}{x}\right)^x\right)$ is $\ln(L)$. It will be easier to find this limit, then to get L we use the fact that $L = e^{\ln(L)}$.

$$\ln(L) = \lim_{x\to+\infty}\ln\left(\left(1+\frac{1}{x}\right)^x\right) = \lim_{x\to+\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to+\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

Using the same trick as in 6. Now we can see that both numerator and denominator approach 0, so we can use L'Hôpital's rule.

$$\lim_{x\to+\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to+\infty}\frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to+\infty}\frac{1}{1+\frac{1}{x}} = 1$$

So the answer to the original limit is $L = e^1 = e$. This is a famous limit, it is the definition of the special constant e.

8. Use L'Hôpital's ruleto compute the limit $\lim_{x\to 0}\frac{1}{x} - \frac{1}{\sin x}$.
We can't use L'Hôpital's rule immediately, we have to fanagle this limit so that it is a fraction. One way to do that is to multiply by $\frac{x}{x}$, where we distribute the numerator x and keep teh other one in the denominator:

$$\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right)\frac{x}{x} = \lim_{x\to 0}\frac{\frac{x}{x} - \frac{x}{\sin x}}{x} = \lim_{x\to 0}\frac{1-\frac{x}{\sin x}}{x}$$

Now observe that the numerator and denominator both approach 0, so we may use L'Hôpital's rule.

$$\lim_{x\to 0}\frac{1-\frac{x}{\sin x}}{x} = \lim_{x\to 0}\frac{\frac{\sin x + x\cos x}{\sin^2 x}}{1} = \lim_{x\to 0}\frac{\sin x + x\cos x}{\sin^2 x}$$

Once again, both numerator and denominator approach 0, so we use L'Hôpital's rule again.

$$\lim_{x\to 0}\frac{\sin x + x\cos x}{\sin^2 x} = \lim_{x\to 0}\frac{\cos x + \cos x - x\sin x}{2\sin x\cos x} = \lim_{x\to 0}\frac{-x\sin x}{2\sin x\cos x} = \lim_{x\to 0}\frac{-x}{2\cos x} = 0$$

24 Due April 1

1. Write the sum $3^3 + 5^3 + 7^3 + \cdots + 101^3$ in summation notation.
We want to sum up the odd numbers cubed from 3 to 101. The best way to index the odd numbers is $2n+1$. If we start $n = 1$ the last value of n should be 50. So this sum will be denoted

$$\sum_{n=1}^{50}(2n+1)^3$$

2. Evaluate the sum $\sum_{n=10}^{40}(2n+1)$

The first trick is to write this as a difference of sums starting at $n = 1$.

$$\sum_{n=10}^{40}(2n+1) = \sum_{n=1}^{40}(2n+1) - \sum_{n=1}^{9}(2n+1)$$

Next we want to split the constant part of each sum from the indexed part:

$$= \sum_{n=1}^{40}(2n) + 40 - \sum_{n=1}^{9}(2n) - 9 = 31 + \sum_{n=1}^{40}(2n) - \sum_{n=1}^{9}(2n)$$

Next we can factor out the 2:

$$= 31 + 2\sum_{n=1}^{40} n - 2\sum_{n=1}^{9} n$$

Finally we can use the summation formula

$$\sum_{n=1}^{k} n = \frac{(k)(k+1)}{2}$$

which will give us:

$$31 + 2\frac{(40)(41)}{2} - \frac{(9)(10)}{2} = 31 + 2(820) - 2(45) = 1581$$

3. Compute R_4 and L_4 for the function $f(x) = x^2 - x$ on the interval $[0, 4]$.

R_4 is the right-Riemann sum, splitting $[0, 4]$ into 4 sub-intervals, whereas L_4 is the left Riemann sum using the same 4 sub-intervals. The sub intervals are of course $[0, 1], [1, 2], [2, 3], [3, 4]$. Let's organize the info in a table:

x	0	1	2	3	4
$f(x)$	0	0	2	6	12

Since the width of each sub-interval is 1, we simply have $R_4 = 0+2+6+12 = 20$, while $L_4 = 0 + 0 + 2 + 6 = 8$.

4. Compute M_4 for the function $f(x) = x^2 + x + 1$ on the interval $[0, 8]$.

M_4 is the midpoint Riemann sum. We are splitting $[0, 8]$ into 4 sub-intervals $[0, 2], [2, 4], [4, 6], [6, 8]$, and we evaluate the function at the midpoint of each interval:

x	1	3	5	7
$f(x)$	3	13	31	57

Since the width of each sub-interval is 2, $M_4 = 2(3 + 13 + 31 + 57) = 208$

25 Due April 6

1. Use L_3 in order to estimate the integral $\int_0^3 x^3 - x + 1dx$.
   L_3 is the left-Riemann sum, splitting $[0, 3]$ into 3 sub-intervals: $[0, 1], [1, 2], [2, 3]$. Let's organize the info in a table:

x	0	1	2
$f(x)$	1	1	7

   Since the width of each sub-interval is 1, we simply have $L_3 = 1 + 1 + 7 = 9$.

2. Use M_4 in order to estimate the integral $\int_0^\pi \sin x dx$.
   M_4 is the midpoint-Riemann sum, splitting $[0, \pi]$ into 4 sub-intervals: $[0, \frac{\pi}{4}]$, $[\frac{\pi}{4}, \frac{\pi}{2}]$, $[\frac{\pi}{2}, \frac{3\pi}{4}]$, $[\frac{3\pi}{4}, \pi]$. We must evaluate $\sin x$ at the midpoint of each interval. Let's organize the info in a table:

x	$\frac{\pi}{8}$	$\frac{3\pi}{8}$	$\frac{5\pi}{8}$	$\frac{7\pi}{8}$
$f(x)$	.3827	.9239	.9239	.3827

   Since the width of each sub-interval is $\frac{\pi}{4}$, we simply have $M_4 = \frac{\pi}{4}(.3827 + .9239 + .9239 + .3827) = 2.0524$.

3. Compute the integral $\int_{-1}^1 \sin^5 x + 3dx$.
   We can first split the integral into two parts:

$$\int_{-1}^{1} \sin^5 x dx + \int_{-1}^{1} 3dx$$

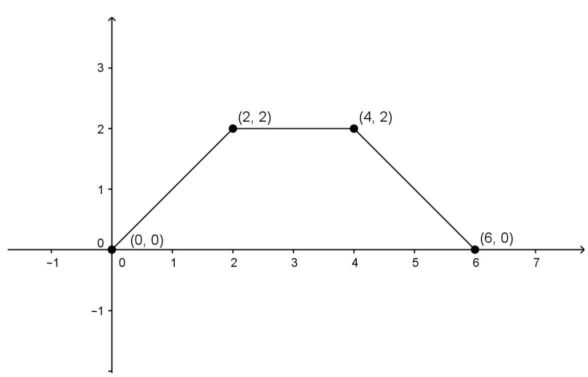
   The first integral is of an odd function from $-a$ to a so it is equal to zero. The second integral is of a constant function, so we measure the area of the rectangle with a height of 3 and a base length of $1 - (-1) = 2$, so we have $3 \cdot 2 = 6$.

4. Compute the integral $\int_{-3}^3 x^7 e^{x^2} dx$.
   We can check to see if this is an odd function or not by plugging in $-x$ in.

$$(-x)^7 e^{(-x)^2} = -x^7 e^{x^2}$$

   So yes, this is an odd function, and since it is integrated over the interval $[-1, 1]$, the integral is zero.

5. The function f defined on $[0, 6]$ has a graph which is a broken line with vertices $(0, 0), (2, 2), (4, 2)$ and $(6, 0)$. Compute the integral $\int_0^6 f(x)dx$.
   The function looks like this:



The integral can be calculated geometrically. This is a trapezoid with bases $b_1 = 6, b_2 = 2$ and height $h = 2$. The area of a trapezoid is

$$A = \frac{b_1 + b_2}{2} \cdot h = \frac{6+2}{2} \cdot 2 = 8$$

6. Show the integral $\int_0^2 x^3 dx$ cannot exceed 16.
Since x^3 is increasing on the interval $[0, 2]$, a right Riemann sum would OVER-estimate. Let's calculate R_1, which is simply $2 \cdot f(2) = 2 \cdot 8 = 16$. This is an over-estimate, so the integral cannot exceed 16.

26 Due April 8

1. Use R_4 in order to estimate $\int_0^2 x^2 - x + 1dx$
R_4 is the right-Riemann sum, splitting $[0, 2]$ into 4 sub-intervals: $[0, .5], [.5, 1], [1, 1.5], [1.5, 2.$
Let's organize the info in a table:

x	$.5$	1	1.5	2
$f(x)$	$.25 - .5 + 1 = .75$	1	$2.25 - 1.5 + 1 = 1.75$	3

Since the width of each sub-interval is .5, we simply have

$$R_4 = .5(.75 + 1 + 1.75 + 3) = .5(6.5) = 3.25$$

2. Compute the integral $\int_{-2}^2 \sin^3 x + 1dx$.
We can first split the integral into two parts:

$$\int_{-2}^{2} \sin^3 x dx + \int_{-2}^{2} 1dx$$

The first integral is of an odd function from $-a$ to a so it is equal to zero. The second integral is of a constant function, so we measure the area of the rectangle with a height of 1 and a base length of $2 - (-2) = 4$, so we have $1 \cdot 4 = 4$.

3. Compute the integral $\int_{-1}^1 x^5 e^{x^4} dx$.
We can check to see if this is an odd function or not by plugging in $-x$ in.

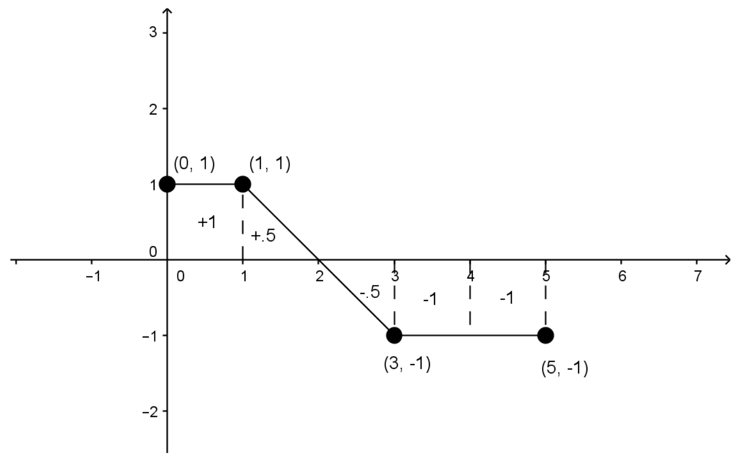
$$(-x)^5 e^{(-x)^4} = -x^5 e^{x^4}$$

So yes, this is an odd function, and since it is integrated over the interval $[-1, 1]$, the integral is zero.

4. The function f defined on $[0, 5]$ has a graph which is a broken line with vertices $(0, 1), (1, 1), (3, -1)$ and $(5, -1)$. Compute the following integrals

$$\int_0^1 f(x)dx \quad \int_1^3 f(x)dx \quad \int_2^4 f(x)dx \quad \int_3^5 f(x)dx \quad \int_0^5 f(x)dx$$

The function looks like this:



The integral over any interval is the net area, so we can calculate them by adding up the positive or negative triangular or square areas.

$$\int_0^1 f(x)dx = 1$$

$$\int_1^3 f(x)dx = .5 + -.5 = 0$$

$$\int_2^4 f(x)dx = -.5 + -1 = -1.5$$

$$\int_3^5 f(x)dx = -1 + -1 = -2$$

$$\int_0^5 f(x)dx = 1 + .5 + -.5 + -1 + -1 = -1$$

5. Show the integral $\int_0^\pi \sin x dx$ cannot exceed 4.
   The maximum and minimum values of $\sin x$ on the interval $[0, \pi]$ occurs at a critical point or at one of the endpoints. $(\sin x)' = \cos x = 0$ has only one solution on this interval, $x = \pi/2$. Since $\sin(\pi/2) = 1$ and $\sin(0) = \sin(\pi) = 0$, we can say for sure that $0 \leq \sin x \leq 1$ on this interval. So

$$\int_0^\pi \sin x dx \leq \pi \cdot 1 = \pi \leq 4$$

   This is why the integral cannot exceed 4.

27 Due April 11

1. Use the Newton-Leibniz formula to compute $\int_0^\pi \sin x dx$.
   Since $(-\cos x)' = \sin x$, by the FTC,

$$\int_0^\pi \sin x dx = (-\cos(\pi)) - (-\cos 0) = (- - 1) - (-1) = 2$$

2. Use the Newton-Leibniz formula to compute $\int_0^{\ln 2} e^x dx$.
Since $(e^x)' = e^x$, by the FTC,

$$\int_0^{\ln 2} e^x dx = e^{(\ln 2)} - e^0 = 2 - 1 = 1$$

3. Use the Newton-Leibniz formula to compute $\int_0^1 3x^2 dx$.
Since $(x^3)' = 3x^2$, by the FTC,

$$\int_0^1 3x^2 dx = (1)^3 - (0)^3 = 1$$

4. Compute the indefinite integral $\int 3x dx$.
Since $(x^2)' = 2x$, that means $\left(\frac{3}{2}x^2\right)' = 3x$, so

$$\int 3x dx = \frac{3}{2}x^2 + C$$

5. Compute the indefinite integral $\int e^{2x} dx$.
Since $(e^{2x})' = 2e^{2x}$, that means $\left(\frac{1}{2}e^{2x}\right)' = e^{2x}$. Therefore,

$$\int e^{2x} dx = \frac{1}{2}e^{2x} + C$$

6. Compute the indefinite integral $\int \sqrt{x} dx$.
Since $\sqrt{x} = x^{1/2}$, and $(x^{3/2})' = \frac{3}{2}x^{1/2}$, $(\frac{2}{3}x^{3/2} = x^{1/2}$. Therefore,

$$\int \sqrt{x} dx = \frac{2}{3}x^{3/2} + C$$

28 Due April 13

1. Use the Newton-Leibniz formula to compute $\int_{\pi/2}^{\pi} \cos x dx$.
Since $(\sin x)' = \cos x$,

$$\int_{\pi/2}^{\pi} \cos x dx = \sin(\pi) - \sin(\pi/2) = 0 - 1 = -1$$

2. Use the Newton-Leibniz formula to compute $\int_0^{\ln 3} e^{2x} dx$ and express your answer in the form $\frac{a}{b}$ where a and b are integer numbers.
since $(e^{2x})' = 2e^{2x}$, $(\frac{1}{2}e^{2x} = e^{2x}$, so

$$\int_0^{\ln 3} e^{2x} dx = \frac{1}{2}e^{2(\ln 3)} - \frac{1}{2}e^{2(0)} = \frac{e^{\ln 9}}{2} - \frac{1}{2} = \frac{9}{2} - \frac{1}{2} = \frac{8}{2} = 4$$

3. Use the Newton-Leibniz formula to compute $\int_0^1 4x^3 - xdx$.
Since $(x^4)' = 4x^3$ and $(x^2)' = 2x$, therefore $(x^4 - \frac{1}{2}x^2)' = 4x^3 - x$. Thus

$$\int_0^1 4x^3 - xdx = \left((1)^4 - \frac{1}{2}(1)^2\right) - \left((0)^4 - \frac{1}{2}(0)^2\right) = 1 - \frac{1}{2} = \frac{1}{2}$$

4. Compute the indefinite integral $\int \sin(3x)dx$.
Since $(\cos(3x))' = -3\sin(3x)$, $\left(-\frac{1}{3}\cos(3x)\right)' = \sin(3x)$. Therefore,

$$\int \sin(3x)dx = -\frac{1}{3}\cos(3x) + C$$

5. Compute the indefinite integral $\int xe^{5x^2}dx$.
Since $(e^{5x^2})' = e^{5x^2}(10x) = 10xe^{5x^2}$, $(\frac{1}{10}e^{5x^2})' = xe^{5x^2}$. Therefore,

$$\int xe^{5x^2}dx = \frac{1}{10}e^{5x^2} + C$$

6. Compute the indefinite integral $\int \sqrt[3]{x}dx$.
First write $\sqrt[3]{x} = x^{1/3}$. Since $(x^{4/3})' = \frac{3}{4}x^{1/3}$, $(\frac{3}{4}x^{4/3})' = x^{1/3}$. Therefore,

$$\int \sqrt[3]{x}dx = \frac{3}{4}x^{4/3} + C$$

29 Due April 15

1. Find $\frac{d}{dx}\int_1^x \ln(t^2 + 1)dt$
Let $f(t) = \ln(t^2 + 1)$. If $F(t) = \int f(t)dt$, then

$$\int_1^x \ln(t^2 + 1)dt = F(x) - F(1)$$

The derivative, with respect to x is simply $f(x)$, or $\ln(x^2 + 1)$.

2. Find $\frac{d}{dx}\int_0^x e^{t^3}dt$
Let $f(t) = e^{t^3}$. If $F(t) = \int f(t)dt$, then

$$\frac{d}{dx}\int_0^x e^{t^3}dt = F(x) = F(0)$$

So the derivative with respect to x is simply $f(x)$, or e^{x^3}.

3. Find the integral $\int(5x-1)^6dx$
Let $u = 5x-1$. Then $du = 5dx$, or $dx = \frac{1}{5}du$. The integral becomes

$$\int \frac{1}{5}u^6 du = \frac{1}{5}\frac{1}{7}u^7 + C = \frac{1}{35}u^7 + C$$

Substituting the x expression back in, we get

$$\frac{1}{25}(5x-1)^7 + C$$

4. Find the integral $\int \cos^3 x dx$
Taking the integral of cosine or sine to an odd power uses a clever trick. We use the trig identity that $\cos^2 x = 1 - \sin^2 x$ first. The integral becomes

$$\int \cos x(1-\sin^2 x)dx$$

We now use the substitution that $u = \sin x$ and $du = \cos x dx$. The substituted integral is

$$\int 1 - u^2 du = u - \frac{1}{3}u^3 + C$$

Substituting the x expression back in we get $\sin x - \frac{1}{3}\sin^3 x + C$.

5. Find the integral $\int \frac{dx}{1+e^{-x}}$
If we first multiply the numerator and denominator by e^x, we get

$$\int \frac{e^x}{e^x+1}dx$$

We now let $u = e^x + 1$, and then $du = e^x dx$. The integral is now

$$\int \frac{1}{u}du = \ln|u| + C$$

If we substitute the expression in x (which is always positive) our integral is

$$\ln(e^x+1) + C$$

6. Find the integral $\int x\sqrt[3]{3x+5}dx$
We make the substitution $u = 3x+5$, which means $x = \frac{u-5}{3}$ and $dx = \frac{1}{3}du$. The substituted integral is

$$\int \frac{u-5}{3}u^{1/3}\frac{1}{3}du = \frac{1}{9}\int u^{4/3} - 5u^{1/3}du$$

We can integrate this using the power rule. It is

$$\frac{1}{9}\left[\frac{3}{7}u^{7/3} - 5\frac{3}{4}u^{4/3}\right] + C = \frac{1}{21}u^{7/3} - \frac{5}{12}u^{4/3} + C$$

Substituting the x expression in, we get

$$\frac{1}{21}(3x+5)^{7/3} - \frac{5}{12}(3x+5)^{4/3} + C$$

7. Find the integral $\int x^3 e^{-x^4} dx$
Letting $u = -x^4$, and $du = -4x^3 dx$, we get $x^3 dx = -\frac{1}{4} du$, so the integral becomes

$$\int -\frac{1}{4} e^u du = -\frac{1}{4} e^u + C$$

Substituting x back in we get

$$-\frac{1}{4} e^{-x^4} + C$$

30 Due April 18

1. Compute $\displaystyle\int \tan(3x) dx$
First write tan as $\frac{\sin}{\cos}$:

$$\int \frac{\sin(3x)}{\cos(3x)} dx$$

Now we can use a substitution. Let $u = \cos(3x)$, $du = -3\sin(3x)dx$, so $\sin(3x)dx = -\frac{1}{3} du$. The integral becomes

$$\int -\frac{1}{3}\frac{1}{u} du = -\frac{1}{3} \ln|u| + C = -\frac{1}{3} \ln|3x| + C$$

2. Compute $\displaystyle\int \sin^5 x dx$
Taking the integral of cosine or sine to an odd power uses a clever trick. First let's write the factor of $(\sin^2 x)$ explicitly:

$$\int (\sin^2 x)^2 \sin x dx$$

We next use the trig identity that $\sin^2 x = 1 - \cos^2 x$ first. The integral becomes

$$\int (1 - \cos^2 x)^2 \sin x dx$$

We now use the substitution that $u = \cos x$ and $du = -\sin x dx$. The substituted integral is

$$-\int (1 - u^2)^2 du = -\int 1 - 2u^2 + u^4 du = -(u - \frac{2}{3}u^3 + \frac{1}{5}u^5) + C$$

Substituting the x expression back in we get $-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$.

3. Compute $\displaystyle\int_1^2 \frac{\ln x}{x} dx$
Letting $u = \ln x, du = \frac{1}{x} dx$, and converting the bounds of integration, we get

$$\int_{\ln 1}^{\ln 2} u du = \frac{1}{2} u^2 \Big|_0^{\ln 2} = \frac{1}{2}\left((\ln 2)^2\right)$$

4. Compute $\int \frac{x}{\sqrt[5]{3x-1}}dx$

   Letting $u = 3x - 1$, $du = 3dx$ and solving for x we get $x = \frac{u+1}{3}$, we make the substitution and get

$$\int \frac{u+1}{3u^{1/5}} \frac{1}{3} du = \frac{1}{9} \int \frac{u}{u^{1/5}} + \frac{1}{u^{1/5}} du = \frac{1}{9} \int u^{4/5} + u^{-1/5} du$$

   Integrating using the power rule we get

$$\frac{1}{9}\left(\frac{5}{9}u^{9/5} + \frac{5}{4}u^{4/5}\right) + C$$

5. Compute $\int_0^1 x^2 \sqrt[4]{x^3+2}dx$

   For this integral, we can use a substitution $u = x^3 + 2$, so $du = 3x^2dx$. Thus $x^2dx = \frac{1}{3}du$. The bounds of integration become $u(1) = 1^3 + 2 = 3$, $u(0) = 0^3 + 2 = 2$. The integral is now

$$\int_2^3 \frac{1}{3}u^{1/4}du = \frac{1}{3}\frac{4}{5}u^{5/4}\bigg|_2^3 = \frac{4}{15}\left(3\sqrt[4]{3} - 2\sqrt[4]{2}\right)$$

6. Compute $\int \frac{1}{1+e^{-2x}}dx$

   For this integral it will be useful to multiply the numerator and denominator by e^{2x}. This gives us

$$\int \frac{e^{2x}}{e^{2x}+1}dx$$

   Now if we use a substitution $u = e^{2x}+1$, this gives us $du = 2e^{2x}dx$, so $e^{2x}dx = \frac{1}{2}du$. The integral after substitution is

$$\int \frac{1}{2}\frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(e^{2x}+1)$$

31 Due April 20

1. Compute $\int x \sin x dx$

   For this integral we use integration by parts. We choose u such that its derivative is simpler and dv such that its anti-derivative is no worse; in other words, $u = x$ with $du = dx$ and $dv = \sin x dx$ with $v = -\cos x$. The integral after integration by parts is

$$x(-\cos x) - \int(-\cos x)dx = -x\cos x + \int \cos x dx = -x\cos x + \sin x + C$$

2. Compute $\int x \ln^2 x dx$

   If we were to choose $u = x$ then $dv = \ln^2 x dx$, and its anti-derivative is very ugly. Let's instead choose $u = \ln^2 x$ (with $du = 2\ln x \frac{1}{x} dx$) and $dv = xdx$ which gives $v = \frac{1}{2}x^2$. Integration by parts gives us

$$(\ln^2 x)\left(\frac{1}{2}x^2\right) - \int \frac{1}{2}x^2 \cdot 2\ln x \frac{1}{x} dx = \frac{1}{2}x^2 \ln^2 x - \int x \ln x dx$$

   Now we once again do integration by parts, with $u = \ln x$ and $dv = xdx$. Thus $du = \frac{1}{x}dx$ and $v = \frac{1}{2}x^2$. The integral becomes:

$$\begin{aligned}
&\frac{1}{2}x^2 \ln^2 x - \left(\frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2 \frac{1}{x} dx\right) \\
&= \frac{1}{2}x^2 \ln^2 x - \frac{1}{2}x^2 \ln x + \frac{1}{2}\int x dx \\
&= \frac{1}{2}x^2 \ln^2 x - \frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 + C
\end{aligned}$$

3. Compute $\int x^2 e^x dx$

   We perform integration by parts with $u = x^2$ and $dv = e^x dx$; thus $du = 2xdx$ and $v = e^x$. Integration by parts gives us

$$x^2 e^x - \int 2xe^x dx$$

   We need to perform integration by parts one more time, this time with $u = 2x, du = 2dx$. The choice of dv is the same. We get

$$= x^2 e^x - \left(2xe^x - \int 2e^x dx\right)$$

   which becomes

$$= x^2 e^x - 2xe^x + 2e^x + C = (x^2 - 2x + 2)e^x + C$$

4. Compute $\int e^{2x} \cos x dx$

   Here we will use integration by parts twice to finally recover the original integral, then solve for the integral with algebra. Let $u = e^{2x}$ and $dv = \cos x dx$ (so $du = 2e^{2x}dx, v = \sin x$). Integration by parts gives us

$$= e^{2x} \sin x - \int 2e^{2x} \sin x dx$$

   This time we choose $u = 2e^{2x}$ ($du = 4w^{2x}$) and $dv = \sin x dx$ and $v = -\cos x$. Integration by parts gives

$$= e^{2x} \sin x - \left(-2e^{2x} \cos x - \int (-\cos x) 4e^{2x} dx\right)$$

which simplifies to

$$= e^{2x}\sin x + 2e^{2x}\cos x - \int 4e^{2x}\cos x dx$$

We equate this with the original integral

$$\int e^{2x}\cos x dx = e^{2x}\sin x + 2e^{2x}\cos x - \int 4e^{2x}\cos x dx$$

We collect the integral terms to one side

$$5\int e^{2x}\cos x dx = e^{2x}\sin x + 2e^{2x}\cos x$$

We divide by 5 and then add the arbitrary constant

$$5\int e^{2x}\cos x dx = \frac{e^{2x}}{5}(\sin x + \cos x) + C$$

5. Compute $\int \sin(x^{1/3})dx$

Before we jump into integration by parts, we need to first do a substitution. Let $t = x^{1/3}$. Thus $t^3 = x$, so $dx = 3t^2 dt$. The integral becomes

$$\int 3t^2 \sin t dt$$

We can now proceed with integration by parts. Let $u = 3t^2$ and $dv = \sin t dt$. Thus, $du = 6t dt$ and $v = -\cos t$. Integration by parts gives us

$$= (3t^2)(-\cos t) - \int (-\cos t)(6t dt) = -3t^2\cos t + \int 6t\cos t dt$$

We perform integration by parts a second time, now $u = 6t$ and $dv = \cos t dt$. Thus $du = 6dt$ and $v = \sin t$. We have

$$= -3t^2\cos t + (6t)(\sin t) - \int (\sin t)(6dt)$$

$$= -3t^2\cos t + 6t\sin t - \int 6\sin t dt$$

$$= -3t^2\cos t + 6t\sin t + 6\cos t + C$$

$$= (6 - 3t^2)\cos t + 6t\sin t + C$$

We substitute $t = x^{1/3}$ again to get

$$= (6 - 3x^{2/3})\cos(x^{1/3}) + 6x^{1/3}\sin(x^{1/3}) + C$$