Math 180: Calculus I **Fall 2014**

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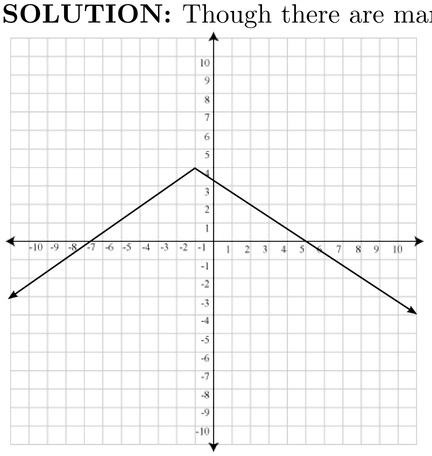
TA: Brian Powers

1. Sketch a function that is continuous on $(-\infty, \infty)$ with the following conditions: $f'(-1)$ is undefined; $f'(x) > 0$ on $(-\infty, -1)$; $f'(x) < 0$ on $(-1, \infty)$.
SOLUTION: Though there are many possible graphs, here is one.



2. On what intervals is $f(x)$ increasing / decreasing? On what intervals is the function concave up / concave down? Identify all critical points and inflection points. Use the second derivative test to classify critical points if possible.

(a) $f(x) = x^4 - 4x^3$
SOLUTION: First take the derivatives (first and second)

$$f'(x) = 4x^3 - 12x^2$$

$$f''(x) = 12x^2 - 24x$$

Set the first derivative equal to zero to find critical points.

$$4x^3 - 12x^2 = 0 \Rightarrow 4x^2(x-3) = 0$$

Critical points are $x = 0, 3$. The second derivative test gives us that $f''(0) = 0$ and $f''(3) > 0$, so $x = 3$ is a local minimum (where the function goes from decreasing to increasing), but it is inconclusive to classify $x = 0$. Test $f'(-1) = -4 + 12 > 0$ tells us that the function is decreasing on $(-\infty, 0)$ and $(0, 3)$ and increasing on $(3, \infty)$. Check concavity by setting the second derivative equal to zero.

$$12x^2 - 24x = 0 \Rightarrow 12x(x-2)$$

So at $x = 0, 2$ we have possible inflection points. We can check that the function is concave up on $(-\infty, 0), (2, \infty)$ and concave down on $(0, 2)$ by testing the sign of the 2nd derivative at points in these intervals. Therefore both $x = 0$ and $x = 2$ are inflection points.

(b) $f(x) = \cos^2 x$ on $[-\pi, \pi]$
SOLUTION:

$$f'(x) = 2\cos x(-\sin x)$$

$$f''(x) = 2\sin^2 x - 2\cos^2 x$$

Setting the first derivative equal to 0 we get

$$-2\cos x \sin x = 0 \Rightarrow \cos x = 0 \text{ or } \sin x = 0 \Rightarrow x = -\pi, \pi, 0, \frac{\pi}{2}, -\frac{\pi}{2}$$

$f(-\pi) = 1, f(-\frac{\pi}{2}) = 0, f(0) = 1, f(\frac{\pi}{2}) = 0, f(\pi) = 1$. So we have local minimums at $x = -\pi/2, \pi/2$ and a local maximum at $x = 0$. The function is increasing on $(-\frac{\pi}{2}, 0), (\frac{\pi}{2}, \pi)$ and decreasing on $(-\pi, -\frac{\pi}{2}), (0, \frac{\pi}{2})$.

We set the second derivative equal to zero for concavity intervals.

$$2\sin^2 x - 2\cos^2 x = 0 \Rightarrow \sin x = \pm\cos x \Rightarrow x = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$$

We can check $\pm\pi, \pm\frac{\pi}{2}$ and 0 in $f''(x)$ to get the intervals of concavity, but since we already know where the local extrama occur and because the first derivative is defined everywhere there are no cusps, we know that the intervals of upward concavity are $(-\frac{3\pi}{4}, -\frac{\pi}{4}), (\frac{\pi}{4}, \frac{3\pi}{4})$ so the other intervals are downward concavity: $(-\pi, -\frac{3\pi}{4}), (-\frac{\pi}{4}, \frac{\pi}{4}), (\frac{\pi}{4}, \pi)$.

(c) $f(x) = x^2 - 2\ln x$

SOLUTION:

$$f'(x) = 2x - \frac{2}{x}, f''(x) = 2 + \frac{2}{x^2}$$

Setting the first derivative equal to zero, we find critical points

$$0 = 2x - \frac{2}{x} \Rightarrow 0 = x^2 - 1 \Rightarrow x = \pm 1$$

And the derivative is undefined at $x = 0$ but of these points, only $x = 1$ is in the domain (notice the $\ln x$ in the function).

Because $f''(1) > 0$, $x = 1$ is a local minimum. So the function is decreasing on $(0, 1)$ and increasing on $(1, \infty)$. The second derivative is always positive, so the function is concave up on $(0, \infty)$.

3. Explain why the following statements are true, or provide a counterexample.

(a) If $f''(a) = 0$, then f has an inflection point at a.
SOLUTION: Inflection points are not the same thing as $f'' = 0$. For example, $f(x) = 2x + 1$ has a zero second derivative for all x, but these are not inflection points.

(b) If $f(x) = g(x) + c$ for some constant c, then f and g increase and decrease on the same intervals.
SOLUTION: Yes, because $f'(x) = g'(x)$, so they have the exact same first derivatives. Therefore f and g increase and decrease on the same intervals.

(c) If f and g both increase on an interval, then the product fg also increases on that interval.
SOLUTION: Say on (a, b) both f and g are increasing, i.e. their first derivatives are positive. Does this mean that the first derivative of fg is positive on the interval?

$$\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$$

We know that $f'(x) > 0$ and $g'(x) > 0$ for all $a < x < b$, but maybe $f(x) < 0$ and $g(x) < 0$, in which case the derivative will be negative. For example, Consider $f(x) = x - 5$ and $g(x) = 2x - 10$ on $(0, 2)$. $\frac{d}{dx} f(x)g(x) = 2x - 10 + 2x - 10 = 4x - 10$ Which is negative for all $0 < x < 2.5$.

4. Can a continuous function on $(-\infty, \infty)$ have exactly four zeros and two local extrema?
SOLUTION: No it cannot! Exactly 4 zeroes means it crosses the x axis 4 times, which means the derivative changes sign 3 times. At each point the derivative changes sign, we will get a local extrema (since the function is continious). So there are at least 3 local extrema.

5. For a general parabola $f(x) = ax^2 + bx + c$, for what values of a, b and c is the parabola concave up, and for what values is it concave down?
   SOLUTION: $f'(x) = 2ax + b, f''(x) = 2a$, so the function is concave up if $a > 0$ and concave down if $a < 0$.