Math 180: Calculus I **Fall 2014**

November 13

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A few useful facts

$$\sum_{k=1}^{n} c = cn \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 \frac{n^2(n+1)^2}{4}$$

1. Approximate the displacement by using left-Riemann sums and the specified number of intervals.

$$v = \frac{1}{2t+1}, 0 \le t \le 8, n = 4$$

SOLUTION: First, $\Delta t = (8-0)/4 = 2$, so we have the starting time for each interval is $0, 2, 4, 6$. Our approximation of the displacement is therefore

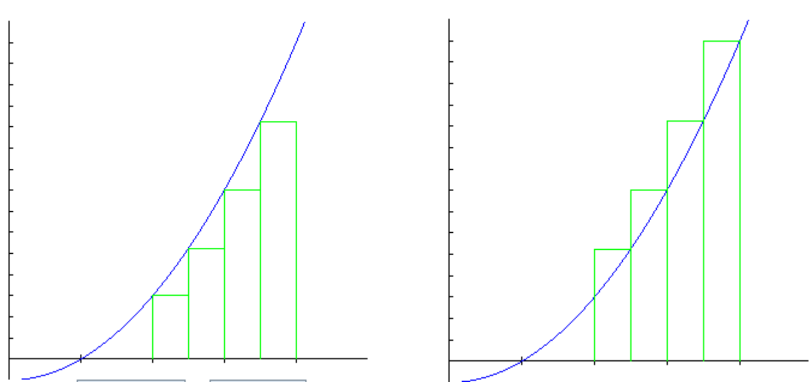
$$2v(0) + 2v(2) + 2v(4) + 2v(6) = 2\left(1 + \frac{1}{5} + \frac{1}{9} + \frac{1}{13}\right)$$

No need to simplify further at this point.

2. For the following functions, do the following:

- Sketch the graph of the function on the interval
- Calculate Δx and the grid points $x_0, x_1, ..., x_n$
- Illustrate left and right Riemann sums, determine which sum underestimates the area under the curve.
- Calculate both left and right Riemann sums.

(a) $f(x) = x^2 - 1$ on $[2, 4], n = 4$
SOLUTION: $\Delta x = (4-2)/4 = 0.5$, this is simple enough to find. Then we have $x_0 = 2, x_1 = 2.5, x_2 = 3, x_3 = 3.5, x_4 = 4$. The graphs of the left and right Riemann sums look like this:



(a) left Riemann Sum (b) right Riemann Sum

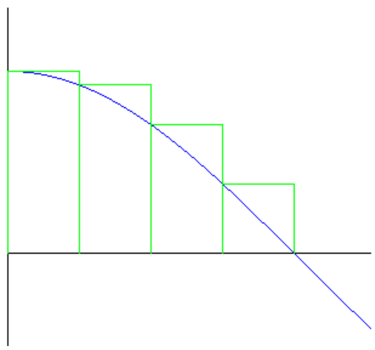
It is clear that the left Riemann sum is underestimating the area under the curve. The left Riemann sum is

$$.5f(2) + .5f(2.5) + .5f(3) + .5f(3.5) = .5(3 + 5.25 + 8 + 11.25) = 13.75$$

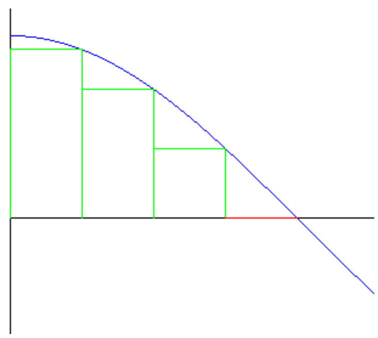
The Right Riemann sum is

$$.5f(2.5) + .5f(3) + .5f(3.5) + .5f(4) = .5(5.25 + 8 + 11.25 + 15) = 19.75$$

(b) $f(x) = \cos x$ on $[0, \pi/2], n = 3$
SOLUTION: $\Delta x = \frac{\pi}{2}/3 = \pi/6$, so $x_0 = 0, x_1 = \pi/6, x_2 = \pi/3, x_3 = \pi/2$. The graphs of the left and right Riemann sums will look like this:



(c) left Riemann Sum



(d) right Riemann Sum

And in this case it is clear that the right Riemann sum is underestimating the area under the curve. The left Riemann sum is

$$\frac{\pi}{6}\left(f(0) + f(\frac{\pi}{6}) + f(\frac{\pi}{3})\right) = \frac{\pi}{6}\left(1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\right) = \frac{(3+\sqrt{3})\pi}{12}$$

3. Express the following in sigma notation

(a) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$
SOLUTION: Usually, we start with $k = 1$ unless there's a good reason to start with k at another value. The formula for each value of k in this instance is pretty clear, we have $\frac{1}{k}$ for each term. And since the last term has $k = 4$, we have this in summation form is

$$\sum_{k=1}^{4} \frac{1}{k}$$

(b) $3 + 8 + 13 + \cdots + 63$
SOLUTION: Starting with $k = 1$, we notice that this summation has the pattern that each term is 5 more than the one before it. So for every increment of k we need to increase our terms by 5. So we must have $5k$ in the formula for the kth term. However, this would give us $5 + 10 + 15 + \cdots$. These are all two units too large. A simple correction is to subtract two, so the formula is $5k - 2$. Finally, to dertermine the final value for k, we set $5k - 2 = 63$ to get $k = 13$ is the last value of k. So our summation looks like this:

$$\sum_{k=1}^{1} 3(5k - 2)$$

(c) $\frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \cdots + \frac{100}{101}$
SOLUTION: Rather than starting with $k = 1$ it makes morse sense to start with $k = 2$. The formula for each term would involve $\frac{k}{k+1}$, but this does not handle the alternating sign. A handy trick to alternate sign would be to use $(-1)^k$ or $(-1)^{k+1}$, depending on the indexing. In our example, when $k = 2$ we need to take (-1) to an even power, so $(-1)^k$ will do the trick. the summation is thus:

$$\sum_{k=2}^{100} \frac{k}{k+1}(-1)^k$$

(d) $\ln 2 + \sqrt{3} + \ln 4 + \sqrt{5} + \ln 6 \cdots + \ln 24 + \sqrt{25}$ (*Hint*: What values does $|\sin(k\pi/2)|$ take? Or what values does $\frac{1+(-1)^k}{2}$ take?)
SOLUTION: There are a couple of ways we could handle this. If we take the indexing to be $k = 1$ to 25, then we would want to somehow write our summation as

$$\sum_{k=2}^{25} \sqrt{k} \text{ if } k \text{ is odd, } \ln k \text{ if } k \text{ is even}$$

But we're unable to use such verbiage in the summation. We we want to do is make use of a trick similar to that above where we alternated -1/1. Based on the hint, observe the following

k	1	2	3	4	5	6
$\sin^2(k\pi/2)$	1	0	1	0	1	0
$\cos^2(k\pi/2)$	0	1	0	1	0	1
$\frac{1+(-1)^k}{2}$	0	1	0	1	0	1

so we could write

$$\sum_{k=2}^{25} \sqrt{k} + \cdot \sin^2\left(\frac{k\pi}{2}\right) + \ln k \cdot \cos^2\left(\frac{k\pi}{2}\right)$$

But this is still pretty ugly. We can't always make it pretty, but in this case we could if we changed our setup from the beginning. What if we instead group terms together, say the first term is $\ln 2 + \sqrt{3}$, the second term is $\ln 4 + \sqrt{5}$ and so on. Then we just need to be sure we're taking natural log of even numbers and square roots of teh odd numbers. We observe that $2k$ is even for any integer k, $2k + 1$ is odd. So we may write

$$\sum_{k=1}^{1} 2\ln(2k) + \sqrt{2k+1}$$

4. Evaluate the following expressions, and also expand it out.

(a) $\sum_{k=1}^{10} k$
SOLUTION:

$$\sum_{k=1}^{10} k = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$$

can be calculated easily. To evaluate using the formula though, we have

$$\sum_{k=1}^{10} k = \frac{10(11)}{2} = 55$$

(b) $\sum_{k=1}^{6}(2k+1)$
SOLUTION:

$$\sum_{k=1}^{6}(2k+1) = 3+5+7+9+11+13 = 48$$

and if we evaluate using the theorems we get

$$\begin{aligned}
\sum_{k=1}^{6}(2k+1) &= \sum_{k=1}^{6} 2k + \sum_{k=1}^{6} 1 && \text{Addition rule} \\
&= 2\sum_{k=1}^{6} k + 6 && \text{factor out the 2 and evaluate the second sum} \\
&= 2\frac{6(7)}{2} + 6 && \text{use the formula to evaluate} \\
&= 48
\end{aligned}$$

(c) $\sum_{p=1}^{5}\left(2p+p^2\right)$
SOLUTION:

$$\sum_{p=1}^{5}\left(2p+p^2\right) = 3+8+15+24+35 = 85$$

Otherwise, by theorem we have

$$\begin{aligned}
\sum_{p=1}^{5}\left(2p+p^2\right) &= \sum_{p=1}^{5} 2p + \sum_{p=1}^{5} p^2 && \text{Addition rule} \\
&= 2\sum_{p=1}^{5} p + \frac{5(6)(11)}{6} && \text{factor the 2, and use formula} \\
&= 2\frac{5(6)}{2} + 55 \\
&= 85
\end{aligned}$$