Math 180: Calculus I **Fall 2014**

September 16

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1. Use the definition of derivative to find the slope of the tangent line at P, and the equation of the tangent line.

   (a) $f(x) = -3x^2 - 5x + 1 \quad P(1, -7)$
   SOLUTION:
   For polynomials, multiply out the numerator and the factor out an h.

$$\begin{aligned}
\lim_{h\to 0} f(1+h) - f(1)h &= \lim_{h\to 0} \frac{(-3(1+h)^2 - 5(1+h) + 1) - (-7)}{h} \\
&= \lim_{h\to 0} \frac{(-3(1+2h+h^2) - 5 - 5h + 1) + 7}{h} \\
&= \lim_{h\to 0} \frac{-7 - 11h - 3h^2 + 7}{h} \\
&= \lim_{h\to 0} \frac{(-11-3h)\cancel{h}}{\cancel{h}} \\
&= -11
\end{aligned}$$

   The tangent line is $(y - y_1) = m(x - x_1)$ which is $y + 7 = -11(x - 1)$.

   (b) $g(x) = \frac{1}{3-2x} \quad P(-1, \frac{1}{5})$
   SOLUTION:
   For rational functions like this, we first add the fractions in the numerator by getting common denominators.

$$\begin{aligned}
\lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} &= \lim_{h\to 0} \frac{\frac{1}{3-2(-1+h)} - \frac{1}{5}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{5-2h} - \frac{1}{5}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{5}{5(5-2h)} - \frac{5-2h}{5(5-2h)}\right) \\
&= \lim_{h\to 0} \frac{1}{\cancel{h}} \frac{2\cancel{h}}{25-10h} \\
&= \frac{2}{25}
\end{aligned}$$

   The tangent line is $y - \frac{1}{5} = \frac{2}{25}(x + 1)$.

   (c) $h(x) = \sqrt{x-1} \quad P(2, 1)$
   SOLUTION:

When dealing with roots, multiplying by the conjugate will usually work.

$$\begin{aligned}\lim_{h\to 0}\frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0}\frac{\sqrt{2+h-1}-1}{h}\\ &= \lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}\frac{(\sqrt{1+h}+1)}{(\sqrt{1+h}+1)}\\ &= \lim_{h\to 0}\frac{(1+h)-1}{h(\sqrt{1+h}+1)}\\ &= \lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}(\sqrt{1+h}+1)}\\ &= \frac{1}{2}\end{aligned}$$

The tangent line is $y-1=\frac{1}{2}(x-2)$.

2. Show the statement is true or find a counterexample

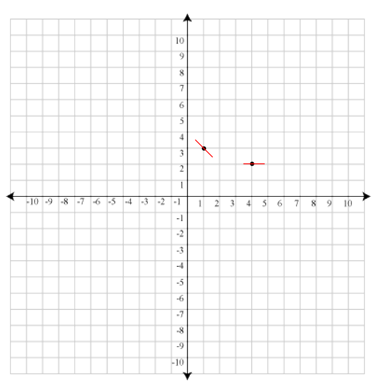
   (a) For linear functions, the slope of any secant line always equals the slope of any tangent line.
   TRUE: A secant line drawn between any two points on the graph will have the same slope m as the linear function, which is the same as the slope of the tangent line - the tangent line is in fact the same as the function itself.

   (b) The slope of the secant line between P and Q is less than the slope of the tangent line at P.
   FALSE: This is true for what are called concave functions, but there are many counter examples. $y=x^2$, with $P(0,0)$ and $Q(1,1)$ for example.

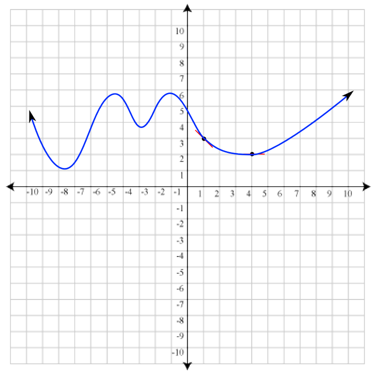
   (c) If f is differentiable for all values of x, then f is continuous for all values of x.
   TRUE: In order for a function to be differentiable at a point, it must be continuous at that point.

   (d) It is possible for the domain of f to be (a,b), but the domain of f' is $[a,b]$.
   FALSE: The function as described is not continuous at a or b, so the function could not be differentiable there.

3. Sketch a function f such that: $f(1)=3, f'(1)=-1, f(4)=2, f'(4)=0$.
   SOLUTION:
   You want to start by plotting the points $(1,3)$ and $(4,2)$ and draw the tangent line going through those with the slope equal to the derivative, like so:



Now just draw any curve (piece-wise, continuous or whatever) that is differentiable AT THOSE TWO POINTS (at least), though it could be differentiable everywhere if you want.



4. Show using the definition of a derivative that $f(x) = |x + 2|$ is not differentiable at $x = -2$.
SOLUTION: The If $f(x)$ was differentiable at $x = -2$ then we could calculate the following limit:

$$\lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h} = \lim_{h\to 0} \frac{|-2+h+2| - |-2+2|}{h}$$

But simplifying we get

$$\lim_{h\to 0} \frac{|h|}{h}$$

which as we know does not exist! (To verify, the left hand limit is -1, while the right hand limit is 1). So $f(x)$ is not differentiable at $x = -2$.

5. Calculate the following derivatives (no need to use the definition of a derivative).

(a) $f(x) = 4x^2 + 3x - 2$
SOLUTION: $f'(x) = (2)4x^{2-1} + (1)3x^{1-1} - 0 = 8x + 3$

(b) $g(x) = 7$
SOLUTION: $g'(x) = 0$ because g is a constant function.

(c) $h(x) = ax + b$, where a and b are constants.
SOLUTION: $h'(x) = (1)ax^{1-1} + 0 = a$

(d) $s(x) = \sqrt{x^5}$
SOLUTION: It's easier to write $s(x) = x^{5/2}$ and use the power rule.
$s'(x) = \frac{5}{2}x^{5/2-2/2} = \frac{5}{2}x^{3/2}$

(e) $t(x) = 9\sqrt{x} + \frac{1}{3x^2}$
SOLUTION: Again, it's easier to write $t(x) = 9x^{1/2} + \frac{1}{3}x^{-2}$ and use the power rule.
$t'(x) = \frac{1}{2}9x^{1/2-2/2} + (-2)\frac{1}{3}x^{-2-1} = \frac{9}{2}x^{-1/2} - \frac{2}{3}x^{-3}$.

(f) $u(x) = 7e^x + x^3$
SOLUTION: $u'(x) = 7e^x + 3x^2$ using the fact that the derivative of e^x is itself, and the power rule & sum rule.

(g) $v(x) = (x^2 + 1)^2$ (no product rule or chain rule unless it's been covered in lecture)
SOLUTION: Just multiply it out and use the power rule. $v(x) = x^4+2x^2+1$, so $v'(x) = 4x^3+4x$.

6. Find x values where the slope of $f(x) = 2x^3 - 3x^2 - 12x + 4$ is zero.
SOLUTION:
First we must find the first derivative. Use the power rule.

$$f'(x) = (3)2x^{3-1} - (2)3x^{2-1} - (1)12x^{1-1} + 0 = 6x^2 - 6x - 12$$

Now we set this equal to zero and solve for x.

$$\begin{aligned} 0 &= 6x^2 - 6x - 12 \\ &= 6(x^2 - x - 2) \\ &= 6(x - 2)(x + 1) \end{aligned}$$

So the slope of the function is zero at $x = 2$ and $x = -1$.