Math 180: Calculus I **Fall 2014**

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1.1 Limits of Special Functions

1. Constant Functions: if $f(x) = c$, then $\lim_{x\to a} f(x) = c$
2. Linear, Polynomial, Sine, Cosine: $\lim_{x\to a} f(x) = f(a)$
3. Rational Functions: Factor, Cancel, Hope (that when you plug in a you're not dividing by zero). If you're not, then $\lim_{x\to a} f(x) = L$ for the value the reduced function at a. If you are dividing by zero, it may be that the limit does not exist (is $-\infty$ or ∞).

1.2 Techniques

Plug In Method: for nicely behaved functions as above.
Rational Functions: factor the numerator and denominator. Cancel common factors. If you are still dividing by zero when plugging in a, then the limit does not exist.
Conjugate Method: If the denominator is of the form $a - b$ then multiplying the numerator and denominator by $a + b$ may do the trick.
Combine Fractions Method: If the numerator has a nasty sum of fractions in it, then by adding the fractions by getting common denominators may lead you to a simplification of the function.

1.3 Limit Laws

Provided $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist,

Sum Law	$\lim\limits_{x\to a} [f(x) + g(x)] = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$
Difference Law	$\lim\limits_{x\to a} [f(x) - g(x)] = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$
Scalar Multple Law	$\lim\limits_{x\to a} [cf(x)] = c \lim\limits_{x\to a} f(x)$ for any constant c
Product Law	$\lim\limits_{x\to a} [f(x) \cdot g(x)] = \left[\lim\limits_{x\to a} f(x)\right] \cdot \left[\lim\limits_{x\to a} g(x)\right]$
Quotient Law	$\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ so long as $\lim\limits_{x\to a} g(x) \neq 0$
Power Law	$\lim\limits_{x\to a} [f(x)^n] = \left[\lim\limits_{x\to a} f(x)\right]^n$
Fractional Powers Law	$\lim\limits_{x\to a} \left[f(x)^{m/n}\right] = \left[\lim\limits_{x\to a} f(x)\right]^{m/n}$

1.4 Examples

Example 1.1 *If* $\lim_{x\to 1} f(x) = 8, lim_{x\to 1} g(x) = 3, \lim_{x\to 1} h(x) = 2$, *evaluate:*

1. $\lim_{x\to 1} f(x)h(x)$
2. $\lim_{x\to 1} \frac{f(x)}{g(x)-h(x)}$
3. $\lim_{x\to 1} \sqrt[3]{f(x)g(x)+3}$

$$\begin{aligned}\lim_{x\to 1} f(x)h(x) &= \lim_{x\to 1} f(x)\cdot\lim_{x\to 1} h(x) \text{ by the Product Law}\\ &= 8\cdot 2\\ &= 16\end{aligned}$$

$$\begin{aligned}\lim_{x\to 1}\frac{f(x)}{g(x)-h(x)} &= \frac{\lim_{x\to 1} f(x)}{\lim_{x\to 1}[g(x)-h(x)]} \text{ by the Quotient Law}\\ &= \frac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} g(x)-\lim_{x\to 1} h(x)} \text{ by the Subtraction Law}\\ &= \frac{8}{3-2}\\ &= 8\end{aligned}$$

$$\begin{aligned}\lim_{x\to 1}\sqrt[3]{f(x)g(x)+3} &= \sqrt[3]{\lim_{x\to 1}[f(x)g(x)+3]} \text{ by the Fractional Power Law}\\ &= \sqrt[3]{\lim_{x\to 1}[f(x)g(x)]+\lim_{x\to 1} 3} \text{ by the Sum Law}\\ &= \sqrt[3]{\lim_{x\to 1} f(x)\cdot\lim_{x\to 1} g(x)+\lim_{x\to 1} 3} \text{ by the Product Law}\\ &= \sqrt[3]{8\cdot 3+3} \text{ by substitution and limit of constant function}\\ &= \sqrt[3]{27}\\ &= 3\end{aligned}$$

Example 1.2 *True or False: if* $g(a) = 0$, *then* $\lim_{x\to a}\frac{f(a)}{g(a)}$ *does not exist*

FALSE: Just because $g(a) = 0$ does not mean $\lim_{x\to a} g(x) = 0$. As a counter-example, consider

$$g(x) = \begin{cases} 1 \text{ if } x \neq a\\ 0 \text{ if } x = a\end{cases}$$

$g(a) = 0$, but $\lim_{x\to a} g(x) = 1$.

Example 1.3 *True or False: If* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = L$ *for some number* L, *then* $f(a) = g(a)$.

FALSE: Again, the value of the function at a does not have to be the limit as $x \to a$. Simply consider $f(x) = 1$ and $g(x)$ as defined above. As $x \to a$ both functions have the limit 1, but the functions do not take the same value when $x = a$.

Example 1.4 *Evaluate:*

1. $\lim_{h\to 0} \frac{100}{(10h-1)^{11}+2}$
2. $\lim_{x\to c} \frac{x^2-2cx+x^2}{x-c}$
3. $\lim_{x\to 1} \frac{x-1}{\sqrt{4x+5}-3}$
4. $\lim_{x\to 2}(5x-6)^{3/2}$
5. $\lim_{h\to 0} \frac{(5+h)^2-25}{h}$

$$\begin{aligned}\lim_{h\to 0} \frac{100}{(10h-1)^{11}+2} &= \frac{100}{(10\cdot 0-1)^{11}+2}\\ &= \frac{100}{-1+2}\\ &= 100\end{aligned}$$

$$\begin{aligned}\lim_{x\to c} \frac{x^2-2cx+x^2}{x-c} &= \lim_{x\to c} \frac{\cancel{(x-c)}(x-c)}{\cancel{x-c}}\\ &= \lim_{x\to c} \frac{(x-c)}{1}\\ &= \frac{c-c}{1}\\ &= 0\end{aligned}$$

$$\begin{aligned}\lim_{x\to 1} \frac{x-1}{\sqrt{4x+5}-3} &= \lim_{x\to 1} \frac{x-1}{\sqrt{4x+5}-3}\frac{(\sqrt{4x+5}+3)}{(\sqrt{4x+5}+3)}\\ &= \lim_{x\to 1} \frac{(x-1)(\sqrt{4x+5}+3)}{(4x+5)-(9)}\\ &= \lim_{x\to 1} \frac{(x-1)(\sqrt{4x+5}+3)}{4x-4}\\ &= \lim_{x\to 1} \frac{\cancel{(x-1)}(\sqrt{4x+5}+3)}{4\cancel{(x-1)}}\\ &= \frac{\sqrt{4\cdot 1+5}+3}{4}\\ &= \frac{6}{4}\\ &= \frac{3}{2}\end{aligned}$$

$$\begin{aligned}\lim_{x\to 2}(5x-6)^{3/2} &= (5\cdot 2-6)^{3/2}\\ &= 4^{3/2}\\ &= 8\end{aligned}$$

$$\begin{aligned}\lim_{h\to 0}\frac{(5+h)^2-25}{h} &= \lim_{h\to 0}\frac{(25+10h+h^2)-25}{h}\\ &= \lim_{h\to 0}\frac{10h+h^2}{h}\\ &= \lim_{h\to 0}\frac{\not{h}(10+h)}{\not{h}}\\ &= \lim_{h\to 0} 10+h\\ &= 10\end{aligned}$$

Example 1.5 *Find two functions f and g so that $\lim_{x\to 1} f(x) = 0$ and $\lim_{x\to 1} f(x)g(x) = 5$.*

Certainly if $\lim_{x\to 1} g(x)$ exists and equals L for some finite number L, then the product law would have $\lim_{x\to 1} f(x)g(x) = 0$. So we can only consider functions g that do not have a limit as $x \to 1$. But somehow f and g must cancel each other out when multiplied to give us the limit of 5. One such possiblity is:

$$f(x) = x - 1 \text{ and } g(x) = \frac{5}{x-1}.$$

Example 1.6 *Show that $-|x| \le x\sin\frac{1}{x} \le |x|$ for $x \ne 0$. Use the Squeeze Theorem to show $\lim_{x\to 0} x\sin\frac{1}{x} = 0$.*

For $x \ne 0$ we know that

$$-1 \le \sin\frac{1}{x} \le 1 \tag{1.1}$$

If $x > 0$,then by multiplying through by x we get

$$-x \le x\sin\frac{1}{x} \le x$$

And because $x > 0$, $x = |x|$, so by substitution

$$-|x| \le x\sin\frac{1}{x} \le |x|.$$

Now consider if $x < 0$. When we multiply through inequalities (1.1) by x we get

$$-x \ge x\sin\frac{1}{x} \ge x$$

the direction of inequality changes because x is negative. And because $x < 0$, we have $|x| = -x$ so

$$|x| \ge x\sin\frac{1}{x} \ge -|x|$$

It is now fairly straightforward to verify that $\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0$. The squeeze theorem then gives us the conclusion that $\lim_{x\to 0} x\sin\frac{1}{x} = 0$.