12.1 Expected Value

Definition 12.1. The **mean** of random variable X, denoted μ_X is the long-term average value of this random variable.

Example 12.2. Suppose you flip a fair coin 3 times and let X be the number of heads. What is the average number of heads (i.e. what is the mean number of heads)?

Definition 12.3. The **mean** or **expected value** of a random variable is defined as follows: For a discrete random variable X,

$$\mu_x = E(X) = \sum_x xf(x).$$

For a continuous random variable X,

$$\mu_x = E(X) = \int_{-\infty}^{\infty} xf(x)dx.$$

Example 12.4. An urn has 15 marbles: 6 blue and 9 red. A sample of 3 marbles is taken. What is the expected number of red marbles?

Example 12.5. Let X be a random variable with pdf

$$f(x) == \begin{cases} \frac{20000}{x^3} & x > 100 \\ 0 & \text{elsewhere} \end{cases}.$$

Find $E(X)$.

Theorem 12.6. *Let X be a random variable with pmf (or pdf) $f(x)$. The expected value of $g(X)$ is*

$$\mu_{g(X)} = E(g(X)) = \sum_x g(x)f(x) \quad \textit{(discrete)}$$

$$\mu_{g(X)} = E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx \quad \textit{(continuous).}$$

Example 12.7. Let X be a random variable with pdf

$$f(x) = \begin{cases} \frac{x^2}{3} & -1 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Find the expected value of $Y = 3X + 2$ and $W = X^2$.

12.2 Variance and Covariance

Definition 12.8. Let X be a random variable with pmf (pdf) $f(x)$ and mean μ. The variance of X, denoted $Var(X)$ or σ_X^2 (or simply σ^2) is

$$\sigma^2 = E\left[(X-\mu)^2\right] = \sum_x (x-\mu)^2 f(x), \quad (discrete)$$

$$\sigma^2 = E\left[(X-\mu)^2\right] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx, \quad (continuous).$$

$\sigma = \sqrt{\sigma^2}$ is called the **standard deviation** of X.

Example 12.9. Calculate the mean and variance of X and Y:

x	1	2	3
$f_X(x)$	0.3	0.4	0.3

y	0	1	2	3	4
$f_Y(y)$	0.2	0.1	0.3	0.3	0.1

Theorem 12.10. *The variance of random variable X is*

$$\sigma^2 = E(X^2) - \mu^2$$

This can be shown by just expanding $(X-\mu)^2$ in the original definition and working it out.

Theorem 12.11. *Let X be a random variable with pmf (pdf) $f(x)$. Let $g(X)$ be a function of X.*

$$\sigma_{g(X)}^2 = E\left[(g(X)-\mu_{g(X)})^2\right] = \sum_x (g(x)-\mu_{g(X)})^2 f(x), \quad (discrete)$$

$$\sigma_{g(X)}^2 = E\left[(g(X)-\mu_{g(X)})^2\right] = \int_{-\infty}^{\infty} (g(x)-\mu_{g(X)})^2 f(x)dx, \quad (discrete)$$

Example 12.12. Let X have the mass function given below. Find the variance of $g(X) = 2X+3$

x	0	1	2	3
$f(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$

Example 12.13. Let X be a continuous random variable with pdf $f(x) = 2.5(x+2)$ with support $[0,1]$. Let $g(X) = 2X^2 - 3$. Find $Var(g(X))$.

12.2.1 Calculator Tasks

- Mean of a discrete random variable
- Variance of a discrete random variable

12.3 Mean and Variance of Linear Combinations of Random Variables

12.3.1 Expected Value of Linear Combinations

Theorem 12.14. *For constants a, b,*

$$E(aX + b) = aE(X) + b$$

Corollary 12.15. *For any constant b,*

$$E(b) = b.$$

Corollary 12.16. *For any constant a,*

$$E(aX) = aE(X).$$

Theorem 12.17. *For any functions g and h,*

$$E[g(X) + h(X)] = E[g(X)] + E[h(X)].$$

Theorem 12.18. *If X and Y are random variables,*

$$E(aX + bY) = aE(X) + bE(Y).$$

Corollary 12.19. *If $X_1, X_2, \ldots, X_n$ are random variables with $a_1, \ldots, a_n$ constants, then*

$$E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E(X_i).$$

12.3.2 Variance of Linear Combinations

Theorem 12.20. *For any constants a, b,*

$$Var(aX + b) = a^2\sigma^2.$$

Corollary 12.21. *For any constant b,*

$$Var(b) = 0.$$

Theorem 12.22. *If X and Y are independent random variables with variances σ_X^2 and σ_Y^2 respectively,*

$$Var(aX + bY) = a^2\sigma_X^2 + b^2\sigma_Y^2.$$

Corollary 12.23. *If $X_1, X_2, \ldots, X_n$ are independent random variables with $a_1, \ldots, a_n$ constants, then*

$$Var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 \sigma_{X_i}^2.$$