Exam 1 STAT 381, Applied Statistical Methods I, Spring 2015

Name:

For full credit you must show your work.

1. (10 points) If $P(B) = 0.4, P(A \cup B) = 0.6$ and $P(A|B) = 0.5$, are A and B *independent*, *mutually exclusive* or *neither*? Justify.

   $P(A|B) = P(A \cap B)/P(B) \Rightarrow .5 = P(A \cap B)/.4 \Rightarrow P(A \cap B) = .2$, so A and B are not mutually exclusive.
   $P(A \cup B) = P(A) + P(B) - P(A \cap B) \Rightarrow .6 = P(A) + .4 - .2 \Rightarrow P(A) = .4 \neq P(A|B)$ so A and B are not independent.
   (or)
   $P(A)P(B) = (.4)(.4) = .16 \neq P(A \cap B)$
   (or)
   $P(B|A) = P(B \cap A)/P(A) = .2/.4 = .5 \neq P(B)$

   +5 points for correctly stating and justifying that they are not mutually exclusive
   +5 points for correctly stating and justifying that they are not independent

2. (10 points) Laptops made from factories A,B and C are defective with respective probabilities 0.01, 0.02 and 0.05. Factory A produces 70% of the laptops, factory B produces 20% and factory C produces the remaining 10%. If a consumer gets a defective laptop, what is the probability it came from factory C?

$P(D|A) = .01, P(D|B) = .02, P(D|C) = .05, P(A) = .7, P(B) = .2, P(C) = .1$
By Bayes' Rule:

$$\begin{aligned} P(C|D) &= \frac{P(C)P(D|C)}{P(A)P(D|A) + P(B)P(D|B) + P(C)P(D|C)} \\ &= \frac{(.1)(.05)}{(.7)(.01) + (.2)(.02) + (.1)(.05)} \\ &= \frac{.005}{.007 + .004 + .005} \\ &= \frac{5}{16} \end{aligned}$$

+3 points - correctly interpret and name the probabilities from the problem
+3 points - set up the formula correctly
+4 points - calculate the final answer
-2 points for each arithmetic error

OR if setting up a tree diagram:
+3 for a correct tree diagram (The final probabilities for $P(D' \cap A), B$ or C are not required)
+3 for setting up the conditional probability formula
+4 for correctly calculating final answer
-2 points for each arithmetic mistake.

3. You flip a coin. If heads you then roll a 4-sided die, otherwise you roll a 6-sided die. Let R be the number you roll.

   (a) (10 points) Find the pmf of R

r	$f(r)$
1	$P(1) = .5(\frac{1}{4}) + .5(\frac{1}{6}) = \frac{5}{24} \approx .2083$
2	$P(2) = .5(\frac{1}{4}) + .5(\frac{1}{6}) = \frac{5}{24} \approx .2083$
3	$P(3) = .5(\frac{1}{4}) + .5(\frac{1}{6}) = \frac{5}{24} \approx .2083$
4	$P(4) = .5(\frac{1}{4}) + .5(\frac{1}{6}) = \frac{5}{24} \approx .2083$
5	$P(5) = .5(\frac{1}{6}) = \frac{2}{24} \approx .0833$
6	$P(6) = .5(\frac{1}{6}) = \frac{2}{24} \approx .0833$

   10 points (fractions or decimals to 4 digits are fine).
   -2 for each arithmetic error.

   (b) (10 points) find $P(R > 3)$.

$$\begin{aligned} P(R > 3) &= P(R = 4 \cup R = 5 \cup R = 6) \\ &= f(4) + f(5) + f(6) \\ &= \frac{5}{24} + \frac{2}{24} + \frac{2}{24} \\ &= \frac{9}{24} \\ &= \frac{3}{8} \\ &= .375 \end{aligned}$$

   3 points only If they find $P(R \geq 3)$ instead
   3 points only if they find $P(R = 3)$ instead.
   -2 points for arithmetic error.

4. (10 points) Find the constant c for the following cdf of X.

$$F(x) = \begin{cases} 0 & x < 0 \\ c\sqrt{x} & 0 \le x \le 5 \\ 1 & 5 < x \end{cases}$$

Because the function $F(x)$ must be right-continuous at $x = 5$ we need $F(5) = 1$, so

$$c\sqrt{5} = 1$$

thus $c = 1/\sqrt{5}$.

10 points if they get this right.
5 points only if they say that c can take any value in $[0, 1/\sqrt{5}]$.

5. The pdf of Y is given by

$$f(y) = \begin{cases} \frac{1}{2}\sin y & 0 < y < \pi \\ 0 & \text{elsewhere} \end{cases}$$

(a) (10 points) Find the cdf $F(y)$.
For $0 \le y \le \pi$,

$$F(y) = \frac{1}{2}\int_0^y \sin t dt = \frac{1}{2}\left[-\cos t\right]_0^y = \frac{1}{2}\left(-\cos y - -\cos 0\right) = \frac{1}{2} - \frac{1}{2}\cos y$$

Thus

$$F(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{2} - \frac{1}{2}\cos y & 0 \le y < \pi \\ 1 & \pi \ge y \end{cases}$$

-1 point if the bounds of each interval do not complement each other.
-3 points if they forget to define $F(y)$ on the intervals $x < 0$ and $x \ge \pi$.

(b) (15 points) Calculate $P(\frac{\pi}{3} < Y < \frac{2\pi}{3})$.

$$P\left(\frac{\pi}{3} < Y < \frac{2\pi}{3}\right) = F\left(\frac{2\pi}{3}\right) - F\left(\frac{\pi}{3}\right) = \left(\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\pi}{3}\right)\right) - \left(\frac{1}{2} - \frac{1}{2}\cos\left(\frac{\pi}{3}\right)\right)$$
$$= \left(\frac{1}{2} + \frac{1}{4}\right) - \left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}$$

(c) (15 points) Find $E(Y)$.

$$E(Y) = \int_0^\pi \frac{y\sin y}{2} dy = \left.\frac{-y\cos y}{2}\right|_0^\pi - \int_0^\pi -\frac{\cos y}{2} dy$$
$$= \frac{-\pi\cos\pi}{2} - \frac{-0\cos 0}{2} - \left[\frac{-\sin y}{2}\right]_0^\pi$$
$$= \frac{\pi}{2} - \left[\frac{-\sin\pi}{2} + \frac{\sin 0}{2}\right] = \frac{\pi}{2}$$

Or if they can argue that the pdf is symmetric over $y = \pi/2$, this is a proper justification (they must show that $f(\frac{\pi}{2} + y) = f(\frac{\pi}{2} - y)$.

6. (10 points) The pdf of W is

$$f(w) = \begin{cases} \frac{|w|}{4} & -2 < w < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Given that $\mu_W = 0$, find $Var(W)$.
We have that $E(W) = 0$, so $Var(W) = E(W^2) - 0^2$.

$$Var(W) = \frac{1}{4}\int_{-2}^{2} w^2|w|dw = 2\frac{1}{4}\int_{0}^{2} w^3 dw = \frac{1}{2}\left[\frac{1}{4}w^4\right]_0^4 = \frac{1}{8}2^4 = 2$$

Extra Credit: (10 points) Based on 4,5,6 and 7, calculate $E(5R+4X-3Y-2W+1)$.

$E(R) = 3$ (has to be calculated)
$E(X) = \frac{5}{3}$ (has to be calculated)
$E(Y) = \frac{\pi}{2}$
$E(W) = 0$
So

$$\begin{aligned} E(5R+4X-3Y+2W+1) &= 5(3) + 4(\frac{5}{3}) - 3(\frac{\pi}{2}) + 2(0) + 1 \\ &= 15 + \frac{20}{3} - \frac{3\pi}{2} + 1 \\ &= \frac{136 - 9\pi}{6} \end{aligned}$$