HOMEWORK 7:GRADER'S NOTES AND SELECTED SOLUTIONS

Chapter 5, p 116, no. 43

Show that A_5 has 24 elements of erder 5, 20 elements of order 3, and 15 elements of order 2.

Proof.

Note that we can decompose any permutation into a product of disjiont cycles, in S_5, since disjiont cycles commute, an element must have one of the following forms:

(1) $(abcde)$ - even [1]
(2) $(abc)(de)$ - odd (even permutation × odd permutation)
(3) (abc) - even
(4) $(ab)(cd)$ - even (odd permutation × odd permutation)
(5) (ab) - odd
(6) e - even

□

So permutations in A_5 have form (1),(3),(4), or (6), we know that when a permutation is written as disjiont cycles it's order is the least common multiple of the lengths of the cycles so:

(1) $(abcde)$ has order 5
(3) (abc) has order 3
(4) $(ab)(cd)$ has order 2
(6) e has order 1

So the only permutations in A_5 that have order 5 are of the form (1). There are 5! distinct expressions for a cycle of the form $(abcde)$ where all the a, b, c, d, e are distinct, there are 5 choices for a, then 4 choices for b, then 3 choices for c, ... However, not all of these expressions represent distinct permutations, in particular there are 5 equivalent notations for any permutations:

$$(abcde) = (bcdea) = (cdeab) = (deabc) = (eabcd)$$

are all equivalent, and any other re-ordering of the symbols is non-equivalent, so we divide the total number of notations, 5!, to get the that the total number of permutations of order 5 is $5!/5 = 4(3(2)) = 24$.

To show that there are 20 permutations of order 3 in A_5 the argument is very similar, since the only permutations in A_5 of order 3 have from (abc), so there are $5(4(3))$ different ways to write such a cycle and:

$$(abc) = (bca) = (cab)$$

Specify which different notations represent the same permutation. So there $5(4(3))/3 = 20$ permutations of order 3 in A_5.

Even permutations of order 2 are a little bit trickier to count, they must be of from $(ab)(cd)$ and there are $5(4(3(2)))$ ways to write such a permutations, however, since disjiont cycles commute there are 8 different ways that differently represented cycles actually produce the same permutation:

$$(ab)(cd) = (ba)(cd) = (ab)(dc) = (ba)(dc) = (dc)(ab) = (dc)(ba) = (cd)(ab) = (cd)(ba)$$

So there are $5(4(3(2)))/8 = 15$ permutations of order 2 in A_5.

The clever and/or lazy mathematician, however, will note that there are $5!/2 = 60$ permutations in A_5 and that exactly one of these has order 1, and every permutationin A_5 has order 5,3,2 or 1. So once you know there are 24 permutations of order 5, 20 permutations of order 3, and one permutation of order 1, you know that 60 - 1 -24 -20 = 15, so there must 15 permuations of order 2.

[1] recall an n-cycle is even $\iff$ n is an even integer

Chapter 5, p 116, no 44

Find a cyclic subgroup of A_8 that has order 4

Proof.

It is enough to find an even permutation, $\sigma \in S_8$ that has order 4, then $\langle\sigma\rangle$ will give the desired subgroup; since any (sub)group G is always closed under the operation it follows that if $g \in G$ then $\langle g\rangle G$. $\langle g\rangle$ is, by definition, cylic. So we can choose any permutation of order 4 that is also even. Some examples are:

$$\begin{aligned}\sigma &= (1234)(5678)\\ \sigma &= (12)(1234)\end{aligned}$$

In fact if a, b, c, d, e, f, g, h are distinct symbols then any permutation σ of either the form $(abcd)(efhg)$ or $(abcd)(ef)$, will work. A permutation of from $(abcd)$ has order four, but is not even, and other even permutations of S_8 do not have order 4.

Important Note: It is, however, not sufficient simply to find σ of order 4 and say nothing about subgroups, the problem is about finding a cyclic sub-group, so you must find one (e.g. $\langle\sigma\rangle$).

Note: The careful reader will note that I've glossed over one particular detail above. □

Chapter 6, page 133, np. 6

Prove that the notion of group isomorphism is transitive. That is, if G, H, and K are groups and $G \cong H$ and $H \cong K$ then $G \cong K$.

Proof.

If $G \cong H$ and $H \cong K$ then there exists functions $f : G \to H$, and $g : H \to K$, such that f and g are bijective (1-1 and onto) and for all $a, b \in G$, $f(ab) = f(a)f(b)$ and for all $a, b \in H$, $g(ab) = g(a)(b)$.

Let $h = g \circ f$, to show h is an isomorphism we need to show that h is bijective and respects the group operation, that is for all $a, b \in G$, $h(ab) = h(a)h(b)$. Note that a composition of bijections is a bijection, so h is bijective. If you've not seen this result before, it is easy enough to prove.

Now let $a, b \in G$, observe that:

$$\begin{aligned}h(ab) &= g(f(ab))\\ &= g(f(a)f(b)) \quad \text{since } f \text{ is an isomorphism}\\ &= g(f(a))g(f(b)) \quad \text{since } g \text{ is an isomorphism}\\ &= h(a)h(b)\end{aligned}$$

□

Chapter 6, page 1335, no 27

Show that $\mathbb{Z}$ is not isomorphic to $\mathbb{Q}$ under addition.

Proof.

One of the theorems of the chapter states that if $G \cong \bar{G}$ then G is cyclic $\iff$ $\bar{G}$ is cyclic, since $\mathbb{Z}$ is cyclic, it is enough to show that $\mathbb{Q}$ is *not* cyclic.

Suppose for some $x \in \mathbb{Q}$, $\mathbb{Q} = \langle x\rangle$, note that since $\mathbb{Q} \neq \{0\}$, it follows $x \neq 0$, but then $x/2 \in \mathbb{Q}$ and $x/2 \notin \langle x\rangle = \{nx : x^n\}$. So it cannot be the case that for some $x \in \mathbb{Q}$, $\mathbb{Q} = \langle x\rangle$.

□