STABLE MODEL CATEGORIES ARE CATEGORIES OF MODULES

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Abstract: A stable model category is a setting for homotopy theory where the suspension functor is invertible. The prototypical examples are the category of spectra in the sense of stable homotopy theory and the category of unbounded chain complexes of modules over a ring. In this paper we develop methods for deciding when two stable model categories represent 'the same homotopy theory'. We show that stable model categories with a single compact generator are equivalent to modules over a ring spectrum. More generally stable model categories with a set of generators are characterized as modules over a 'ring spectrum with several objects', i.e., as spectrum valued diagram categories. We also prove a Morita theorem which shows how equivalences between module categories over ring spectra can be realized by smashing with a pair of bimodules. Finally, we characterize stable model categories which represent the derived category of a ring. This is a slight generalization of Rickard's work on derived equivalent rings. We also include a proof of the model category equivalence of modules over the Eilenberg-Mac Lane spectrum HR and (unbounded) chain complexes of R-modules for a ring R.

1. Introduction

The recent discovery of highly structured categories of spectra has opened the way for a new wholesale use of algebra in stable homotopy theory. In this paper we use this new algebra of spectra to characterize stable model categories, the settings for doing stable homotopy theory, as categories of highly structured modules. This characterization also leads to a Morita theory for equivalences between categories of highly structured modules.

The motivation and techniques for this paper come from two directions, namely stable homotopy theory and homological algebra. Specifically, stable homotopy theory studies the classical stable homotopy category which is the category of spectra up to homotopy. For our purposes though, the homotopy category is inadequate because too much information is lost, for example the homotopy type of mapping spaces. Instead, we study the model category of spectra which captures the whole stable homotopy theory. More generally we study stable model categories, those model categories which share the main formal property of spectra, namely that the suspension functor is invertible up to homotopy. We list examples of stable model categories in Section 2.

The algebraic part of the motivation arises as follows. A classical theorem, due to Gabriel [17], characterizes categories of modules as the cocomplete abelian categories with a single small projective generator; the classical Morita theory for equivalences between module categories (see for example [1, §21, 22]) follows from this. Later Rickard [47, 48] developed a Morita theory for derived categories based on the notion of a tilting complex. In this paper we carry this line of thought one step further. Spectra are the homotopy theoretical generalization of abelian groups and stable model categories are the homotopy theoretic analogue of abelian categories (or rather their categories of chain complexes). Our generalization of Gabriel's theorem develops a Morita theory for stable model categories. Here the appropriate notion of a model category

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equivalence is a Quillen equivalence since these equivalences preserve the homotopy theory, not just the homotopy category, see 2.5.

We have organized our results into three groups:

Characterization of module categories. The model category of modules over a ring spectrum has a single compact generator, namely the free module of rank one. But module categories are actually characterized by this property. To every object in a stable model category we associate an endomorphism ring spectrum, see Definition 3.7.5. We show that if there is a single compact generator, then the given stable model category has the same homotopy theory as the modules over the endomorphism ring spectrum of the generator (Theorem 3.1.1). More generally in Theorem 3.3.3, stable model categories with a set of compact generators are characterized as modules over a 'ring spectrum with many objects,' or *spectral category*, see Definition 3.3.1. This is analogous to Freyd's generalization of Gabriel's theorem [16, 5.3H p.120]. Examples of these characterizations are given in 3.2 and 3.4.

Morita theory for ring spectra. In the classical algebraic context Morita theory describes equivalences between module categories in terms of bimodules, see e.g. [1, Thm. 22.1]. In Theorem 4.1.2 we present an analogous result which explains how a chain of Quillen equivalences between module categories over ring spectra can be replaced by a single Quillen equivalence given by smashing with a pair of bimodules.

Generalized tilting theory. In [47, 48], Rickard answered the question of when two rings are derived equivalent, i.e., when various derived module categories are equivalent as triangulated categories. Basically, a derived equivalence exists if and only if a so-called tilting complex exists. From our point of view, a tilting complex is a particular compact generator for the derived category of a ring. In Theorem 5.1.1 we obtain a generalized tilting theorem which characterizes stable model categories which are Quillen (or derived) equivalent to the derived category of a ring.

Another result which is very closely related to this characterization of stable model categories can be found in [57] where we give necessary and sufficient conditions for when a stable model category is Quillen equivalent to spectra, see also Example 3.2 (i). These uniqueness results are then developed further in [62, 55]. Moreover, the results in this paper form a basis for developing an algebraic model for any rational stable model category. This is carried out in [58] and applied in [61, 19].

In order to carry out our program it is essential to have available a highly structured model for the category of spectra which admits a symmetric monoidal and homotopically well behaved smash product before passing to the homotopy category. The first examples of such categories were the S-modules of [15] and the symmetric spectra of Jeff Smith [25]; by now several more such categories have been constructed [35, 38]. We work with symmetric spectra because we can replace stable model categories by Quillen equivalent ones which are enriched over symmetric spectra (Section 3.6). Also, symmetric spectra are reasonably easy to define and understand and several other model categories in the literature are already enriched over symmetric spectra. The full strength of our viewpoint comes from combining enriched (over symmetric spectra) category theory with the language of closed model categories. We give specific references throughout; for general background on model categories see Quillen's original article [45], a modern introduction [12], or [21] for a more complete overview.

We want to point out the conceptual similarities between the present paper and the work of Keller [31]. Keller uses differential graded categories to give an elegant reformulation (and generalization) of Rickard's results on derived equivalences for rings. Our approach is similar to Keller's, but where he considers categories whose hom-objects are chain complexes of abelian

groups, our categories have hom-objects which are spectra. Keller does not use the language of model categories, but the 'P-resolutions' of [31, 3.1] are basically cofibrant-fibrant replacements.

Notation and conventions: We use the symbol $\mathcal{S}_*$ to denote the category of *pointed* simplicial sets, and we use Sp^Σ for the category of symmetric spectra [25]. The letters $\mathcal{C}$ and $\mathcal{D}$ usually denote model categories, most of the time assumed to be simplicial and stable. For cofibrant or fibrant approximations of objects in a model category we use superscripts $(-)^{\mathrm{c}}$ and $(-)^{\mathrm{f}}$. For an object X in a pointed simplicial model category we use the notation ΣX and ΩX for the simplicial suspension and loop functors (i.e., the pointed tensor and cotensor of an object X with the pointed simplicial circle $S^1 = \Delta[1]/\partial\Delta[1]$); one should keep in mind that these objects may have the 'wrong' homotopy type if X is not cofibrant or fibrant respectively. Our notation for various kinds of morphism objects is as follows: the set of morphisms in a category $\mathcal{C}$ is denoted '$\hom_{\mathcal{C}}$'; the simplicial set of morphisms in a simplicial category is denoted 'map'; we use 'Hom' for the symmetric function spectrum in a spectral model category (Definition 3.5.1); square brackets '$[X,Y]^{\mathrm{Ho}(\mathcal{C})}$' denote the abelian group of morphisms in the homotopy category of a stable model category $\mathcal{C}$; and for objects X and Y in any triangulated category $\mathcal{T}$ we use the notation '$[X,Y]^{\mathcal{T}}_*$' to denote the *graded* abelian group of morphisms, i.e., $[X,Y]^{\mathcal{T}}_n = [X[n],Y]^{\mathcal{T}}$ for $n \in \mathbb{Z}$ and where $X[n]$ is the n-fold shift of X.

We want to write the evaluation of a morphism f on an element x as $f(x)$. This determines the following conventions about actions of rings and ring spectra: the endomorphism monoid, ring or ring spectrum $\mathrm{End}(X)$ acts on the object X from the *left*, and it acts on the set (group, spectrum) $\mathrm{Hom}(X,Y)$ from the *right*. A module will always be a right module; this way the left multiplication map establishes an isomorphism between a ring and the endomorphism ring of the free module of rank one. A T-R-bimodule is a $(T^{\mathrm{op}} \otimes R)$-module (or a $(T^{\mathrm{op}} \wedge R)$-module in the context of ring spectra).

Organization: In Section 2 we recall stable model categories and some of their properties, as well as the notions of compactness and generators, and we give an extensive list of examples. In Section 3 we prove the classification theorems (Theorems 3.1.1 and 3.3.3). In Section 3.6 we introduce the category $Sp(\mathcal{C})$ of symmetric spectra over a simplicial model category $\mathcal{C}$. Under certain technical assumptions we show in Theorem 3.8.2 that it is a stable model category with composable and homotopically well-behaved function symmetric spectra which is Quillen equivalent to the original stable model category $\mathcal{C}$. In Definition 3.7.5 we associate to an object P of a simplicial stable model category a symmetric *endomorphism ring spectrum* $\mathrm{End}(P)$. In Theorem 3.9.3 we then prove Theorem 3.3.3 for *spectral model categories* (Definition 3.5.1), such as for example $Sp(\mathcal{C})$. This will complete the classification results. In Section 4 we prove the Morita context (Theorem 4.1.2) and in Section 5 we prove the tilting theorem (Theorem 5.1.1). In two appendices we consider modules over spectral categories, the homotopy invariance of endomorphism ring spectra and the characterization of Eilenberg-Mac Lane spectral categories.

Title: For some time, this paper circulated as a preprint with the title "The classification of stable model categories". The referee convinced us that this title was misleading, and we think that the present and final title is more appropriate.

2. Stable model categories

In this section we recall stable model categories and some of their properties, as well as the notions of compactness and generators, and we give a list of examples.

2.1. **Structure on the homotopy category.** Recall from [45, I.2] of [21, 6.1] that the homotopy category of a pointed model category supports a suspension functor Σ with a right adjoint loop functor Ω.

Definition 2.1.1. A *stable model category* is a pointed closed model category for which the functors Ω and Σ on the homotopy category are inverse equivalences.

The homotopy category of a stable model category has a large amount of extra structure, some of which plays a role in this paper. First of all, it is naturally a triangulated category (cf. [64] or [24, A.1]). A complete reference for this fact can be found in [21, 7.1.6]; we sketch the constructions: by definition of 'stable' the suspension functor is a self-equivalence of the homotopy category and it defines the shift functor. Since every object is a two-fold suspension, hence an abelian co-group object, the homotopy category of a stable model category is additive. Furthermore, by [21, 7.1.11] the cofiber sequences and fiber sequences of [45, 1.3] coincide up to sign in the stable case, and they define the distinguished triangles. Since we required a stable model category to have all limits and colimits, its homotopy category has infinite sums and products. So such a homotopy category behaves like the unbounded derived category of an abelian category. This motivates thinking of a stable model category as a homotopy theoretic analog of an abelian category.

We recall the notions of compactness and generators in the context of triangulated categories:

Definition 2.1.2. Let $\mathcal{T}$ be a triangulated category with infinite coproducts. A full triangulated subcategory of $\mathcal{T}$ (with shift and triangles induced from $\mathcal{T}$) is called *localizing* if it is closed under coproducts in $\mathcal{T}$. A set $\mathcal{P}$ of objects of $\mathcal{T}$ is called a set of *generators* if the only localizing subcategory which contains the objects of $\mathcal{P}$ is $\mathcal{T}$ itself. An object X of $\mathcal{T}$ is *compact* (also called *small* of *finite*) if for any family of objects $\{A_i\}_{i\in I}$ the canonical map

$$\bigoplus_{i\in I}[X,A_i]^{\mathcal{T}} \longrightarrow [X,\coprod_{i\in I}A_i]^{\mathcal{T}}$$

is an isomorphism. Objects of a stable model category are called 'generators' or 'compact' if they are so when considered as objects of the triangulated homotopy category.

A triangulated category with infinite coproducts and a set of compact generators is often called *compactly generated*. We avoid this terminology because of the danger of confusing it with the terms 'cofibrantly generated' and 'compactly generated' in the context of model categories.

2.2. **Remarks.**

(i) There is a convenient criterion for when a set of *compact* objects generates a triangulated category. This characterization is well known, but we have been unable to find a reference which proves it in the form we need.

Lemma 2.2.1. *Let $\mathcal{T}$ be a triangulated category with infinite coproducts and let $\mathcal{P}$ be a set of compact objects. Then the following are equivalent:*
(i) The set $\mathcal{P}$ generates $\mathcal{T}$ in the sense of Definition 2.1.2.
(ii) An object X of $\mathcal{T}$ is trivial if and only if there are no graded maps from objects of $\mathcal{P}$ to X, i.e. $[P,X]_*=0$ for all $P\in\mathcal{P}$.

Proof. Suppose the set $\mathcal{P}$ generates $\mathcal{T}$ and let X be an object with the property that $[P,X]_*=0$ for all $P\in\mathcal{P}$. The full subcategory of $\mathcal{T}$ of objects Y satisfying $[Y,X]_*=0$ is localizing. Since it contains the set $\mathcal{P}$, it contains all of $\mathcal{T}$. Taking $Y=X$ we see that the identity map of X is trivial, so X is trivial.

The other implication uses the existence of Bousfield localization functors, which in this case is a *finite localization* first considered by Miller in the context of the stable homotopy category [41]. For every set $\mathcal{P}$ of compact objects in a triangulated category with infinite coproducts there exist functors $L_{\mathcal{P}}$ (localization) and $C_{\mathcal{P}}$ (colocalization)

and a natural distinguished triangle

$$C_{\mathcal{P}}X \longrightarrow X \longrightarrow L_{\mathcal{P}}X \longrightarrow C_{\mathcal{P}}X[1]$$

such that $C_{\mathcal{P}}X$ lies in the localizing subcategory generated by $\mathcal{P}$, and such that $[P, L_{\mathcal{P}}X]_* = 0$ for all $P \in \mathcal{P}$ and $X \in \mathcal{T}$; one reference for this construction is in the proof of [24, Prop. 2.3.17] see also 3.9.4. So if we assume condition (ii) then for all X the localization $L_{\mathcal{P}}X$ is trivial, hence X is isomorphic to the colocalization $C_{\mathcal{P}}X$ and thus contained in the localizing subcategory generated by $\mathcal{P}$. □

(ii) Our terminology for 'generators' is different from the use of the term in category theory; generators in our sense are sometimes called *weak generators* elsewhere. By Lemma 2.2.1, a set of generators detects if *objects* are trivial (or equivalently if maps in $\mathcal{T}$ are isomorphisms). This notion has to be distinguished from that of a categorical generator which detects if *maps* are trivial. For example, the sphere spectra are a set of generators (in the sense of Definition 2.1.2) for the stable homotopy category of spectra. Freyd's generating hypothesis conjectures that the spheres are a set of categorical generators for the stable homotopy category of *finite* spectra. It is unknown to this day whether the generating hypothesis is true or false.

(iii) An object of a triangulated category is compact if and only if its shifts (suspension and loop objects) are. Any finite coproduct or direct summand of compact objects is again compact. Compact objects are closed under extensions: if two objects in a distinguished triangle are compact, then so is the third one. In other words, the full subcategory of compact objects in a triangulated category is *thick*. There are non-trivial triangulated categories in which only the zero object is compact. Examples with underlying stable model categories arise for example as suitable Bousfield localizations of the category of spectra, see [26, Cor. B.13].

(iv) If a triangulated category has a set of generators, then the coproduct of all of them is a single generator. However, infinite coproducts of compact objects are in general not compact. So the property of having a single compact generator is something special. In fact we see in Theorem 3.1.1 below that this condition characterizes the module categories over ring spectra among the stable model categories. If generators exist, they are far from being unique.

(v) In the following we often consider stable model categories which are cofibrantly generated. Hovey has shown [21, Thm. 7.3.1] that a cofibrantly generated model category always has a set of generators in the sense of Definition 2.1.2 (the cofibers of any set of generating cofibrations will do). So having generators is not an extra condition in the situation we consider, although these generators may not be compact. See [21, Cor. 7.4.4] for conditions that guarantee a set of compact generators.

2.3. **Examples.**

(i) **Spectra.** As we mentioned in the introduction, one of our main motivating examples is the category of spectra in the sense of stable homotopy theory. The sphere spectrum is a compact generator. Many model categories of spectra have been constructed, for example by Bousfield and Friedlander [4]; Robinson [49, 'spectral sheaves']; Jardine [28, 'n-fold spectra']; Elmendorf, Kriz, Mandell and May [15, 'coordinate free spectra', '$\mathbb{L}$-spectra', 'S-modules']; Hovey, Shipley and Smith [25, 'symmetric spectra']; Lydakis [35, 'simplicial functors']; Mandell, May, Schwede and Shipley [38, 'orthogonal spectra', '$\mathcal{W}$-spaces'].

(ii) **Modules over ring spectra.** Modules over an S-algebra [15, VII.1] or modules over a symmetric ring spectrum [25, 5.4.2] form proper, cofibrantly generated, simplicial, stable model categories, see also [38, Sec. 12]. In each case a module is compact if and only if it is weakly equivalent to a retract of a finite cell module. The free module of rank one is a compact generator. More generally there are stable model categories of modules over 'symmetric ring spectra with several objects', or *spectral categories*, see Definition 3.3.1 and Theorem A.1.1.

(iii) **Equivariant stable homotopy theory.** If G is a compact Lie group, there is a category of G-equivariant coordinate free spectra [34] which is a stable model category. Modern versions of this model category are the G-equivariant orthogonal spectra of [37] and G-equivariant S-modules of [15]. In this case the equivariant suspension spectra of the coset spaces G/H_+ for all closed subgroups $H \subseteq G$ form a set of compact generators. This equivariant model category is taken up again in Examples 3.4 (i) and 5.1.2.

(iv) **Presheaves of spectra.** For every Grothendieck site Jardine [27] constructs a proper, simplicial, stable model category of presheaves of Bousfield-Friedlander type spectra; the weak equivalences are the maps which induce isomorphisms of the associated sheaves of stable homotopy groups. For a general site these stable model categories do not seem to have a set of compact generators.

(v) **The stabilization of a model category.** In principle every pointed model category should give rise to an associated stable model category by 'inverting' the suspension functor, i.e., by passage to internal spectra. This has been carried out for certain simplicial model categories in [52] and [23]. The construction of symmetric spectra over a model category (see Section 3.6) is another approach to stabilization.

(vi) **Bousfield localization.** Following Bousfield [3], localized model structures for modules over an S-algebra are constructed in [15, VIII 1.1]. Hirschhorn [20] shows that under quite general hypotheses the localization of a model category is again a model category. The localization of a stable model category is stable and localization preserves generators. Compactness need not be preserved, see Example 3.2 (iii).

(vii) **Motivic stable homotopy.** In [42, 66] Morel and Voevodsky introduced the $\mathbb{A}^1$-local model category structure for schemes over a base. An associated stable homotopy category of $\mathbb{A}^1$-local T-spectra (where $T = \mathbb{A}^1/(\mathbb{A}^1 - 0)$ is the 'Tate-sphere') is an important tool in Voevodsky's proof of the Milnor conjecture [65]. This stable homotopy category arises from a stable model category with a set of compact generators, see Example 3.4 (ii) for more details.

2.4. **Examples: abelian stable model categories.** Some examples of stable model categories are 'algebraic', i.e., the model category is also an abelian category. Most of the time the objects consist of chain complexes in some abelian category and depending on the choice of weak equivalences one gets a kind of derived category as the homotopy category. A different kind of example is formed by the stable module categories of Frobenius rings.

For algebraic examples as the ones below, our results are essentially covered by Keller's paper [31], although Keller does not use the language of model categories. Also there is no need to consider spectra when dealing with abelian model categories: the second author shows [63] that every cofibrantly generated, proper, abelian stable model category is Quillen equivalent to a DG-model category, i.e., a model category enriched over chain complexes of abelian groups.

(i) **Complexes of modules.** The category of unbounded chain complexes of left modules over a ring supports a model category structure with weak equivalences the quasi-isomorphisms and with fibrations the epimorphisms [21, Thm. 2.3.11] (this is called the *projective* model structure). Hence the associated homotopy category is the unbounded

derived category of the ring. A chain complex of modules over a ring is compact if and only if it is quasi-isomorphic to a bounded complex of finitely generated projective modules [7, Prop. 6.4]. We show in Theorem 5.1.6 that the model category of unbounded chain complexes of A-modules is Quillen equivalent to the category of modules over the symmetric Eilenberg-Mac Lane ring spectrum for A. This example can be generalized in at least two directions: one can consider model categories of chain complexes in an abelian category with enough projectives (see e.g. [8, 2.2] for a very general construction under mild smallness assumptions). On the other hand one can consider model categories of differential graded modules over a differential graded algebra, or even a 'DGA with many objects', alias DG-categories [31].

(ii) **Relative homological algebra.** In [8], Christensen and Hovey introduce model category structures for chain complexes over an abelian category based on a *projective class*. In the special case where the abelian category is modules over some ring and the projective class consists of all summands of free modules this recovers the (projective) model category structure of the previous example. Another special case of interest is the *pure derived category* of a ring. Here the projective class consists of all summands of (possibly infinite) sums of finitely generated modules, see also Example 5.1.3.

(iii) **Homotopy categories of abelian categories.** For any abelian category $\mathcal{A}$, there is a stable model category structure on the category of unbounded chain complexes in $\mathcal{A}$ with the *chain homotopy equivalences* as weak equivalences, see e.g., [8, Ex. 3.4]. The associated homotopy category is usually denoted $K(\mathcal{A})$. Such triangulated homotopy categories tend not to have a set of small generators; for example, Neeman [44, E.3.2] shows that the homotopy category of chain complexes of abelian groups $K(\mathbb{Z})$ does not have a set of generators of any sort.

(iv) **Quasi-coherent sheaves.** For a nice enough scheme X the derived category of quasi-coherent sheaves $\mathcal{D}(qc/X)$ arises from a stable model category and has a set of compact generators. More precisely, if X is quasi-compact and quasi-separated, then the so-called *injective* model structure exists. The objects of the model category are unbounded complexes of quasi-coherent sheaves of $\mathcal{O}_X$-modules, the weak equivalences are the quasi-isomorphisms and the cofibrations are the injections [22, Cor. 2.3 (b)]. If X is separated, then the compact objects of the derived category are precisely the *perfect complexes*, i.e., the complexes which are locally quasi-isomorphic to a bounded complex of vector bundles [43, 2.3, 2.5]. If X also admits an ample family of line bundles $\{\mathcal{L}_\alpha\}_{\alpha\in A}$, then the set of line bundles $\{\mathcal{L}_\alpha^{\otimes m} \,|\, \alpha \in A, m \in \mathbb{Z}\}$, considered as complexes concentrated in dimension zero, generates the derived category $\mathcal{D}(qc/X)$, see [43, 1.11]. This class of examples contains the derived category of a ring as a special case, but the injective model structure is different from the one mentioned in (i). Hovey [22, Thm. 2.2] generalizes the injective model structure to abelian Grothendieck categories.

(v) **The stable module category of a Frobenius ring.** A Frobenius ring is defined by the property that the classes of projective and injective modules coincide. Important examples are finite dimensional Hopf-algebras over a field and in particular group algebras of finite groups. The *stable module category* is obtained by identifying two module homomorphisms if their difference factors through a projective module. Fortunately the two different meanings of 'stable' fit together nicely; the stable module category is the homotopy category associated to an underlying stable model category structure [21, Sec. 2]. Every finitely generated module is compact when considered as an object of the stable module category. Compare also Example 3.2 (v).

(vi) **Comodules over a Hopf-algebra.** Suppose B is a commutative Hopf-algebra over a field. Hovey, Palmieri and Strickland introduce the category $\mathcal{C}(B)$ of chain complexes of injective B-comodules, with morphisms the chain homotopy classes of maps [24, Sec. 9.5]. Compact generators are given by injective resolutions of simple comodules (whose isomorphism classes form a set). In [21, Thm. 2.5.17], Hovey shows that there is a cofibrantly generated model category structure on the category of *all* chain complexes of B-comodules whose homotopy category is the category $\mathcal{C}(B)$.

2.5. **Quillen equivalences.** The most highly structured notion to express that two model categories describe the same homotopy theory is that of a *Quillen equivalence*. An adjoint functor pair between model categories is a *Quillen pair* if the left adjoint L preserves cofibrations and trivial cofibrations. An equivalent condition is to demand that the right adjoint R preserve fibrations and trivial fibrations. Under these conditions, the functors pass to an adjoint functor pair on the homotopy categories, see [45, I.4 Thm. 3], [12, Thm. 9.7 (i)] or [21, 1.3.10]. A Quillen functor pair is a *Quillen equivalence* if it induces an equivalence on the homotopy categories. A Quillen pair is a Quillen equivalence if and only if the following criterion holds [21, 1.3.13]: for every cofibrant object A of the source category of L and for every fibrant object X of the source category of R, a map $L(A) \longrightarrow X$ is a weak equivalence if and only if its adjoint $A \longrightarrow R(X)$ is a weak equivalence.

As pointed out in [12, 9.7 (ii)] and [45, I.4, Thm. 3], in addition to inducing an equivalence of homotopy categories, Quillen equivalences also preserve the homotopy theory associated to a model category, that is, the higher order structure such as mapping spaces, suspension and loop functors, and cofiber and fiber sequences. Note that the notions of compactness, generators, and stability are invariant under Quillen equivalences of model categories.

For convenience we restrict our attention to *simplicial* model categories (see [45, II.2]). This is not a big loss of generality; it is shown in [46] that every cofibrantly generated, proper, stable model category is in fact Quillen equivalent to a simplicial model category. In [11], Dugger obtains the same conclusion under somewhat different hypotheses. In both cases the candidate is the category of simplicial objects over the given model category endowed with a suitable localization of the Reedy model structure.

3. Classification theorems

3.1. **Monogenic stable model categories.** Several of the examples of stable model categories mentioned in 2.3 already come as categories of modules over suitable rings or ring spectra. This is no coincidence. In fact, our first classification theorem says that every stable model category with a single compact generator has the same homotopy theory as the modules over a symmetric ring spectrum (see [25, 5.4] for background on symmetric ring spectra). This is analogous to the classical fact [17, V1, p. 405] that module categories are characterized as those cocomplete abelian categories which posses a single small projective generator; the classifying ring is obtained as the endomorphism ring of the generator.

In Definition 3.7.5 we associate to every object P of a simplicial, cofibrantly generated, stable model category $\mathcal{C}$ a symmetric *endomorphism ring spectrum* $\mathrm{End}(P)$. The ring of homotopy groups $\pi_*\,\mathrm{End}(P)$ is isomorphic to the ring of graded self maps of P in the homotopy category of $\mathcal{C}$, $[P,P]_*^{\mathrm{Ho}(\mathcal{C})}$.

For the following theorem we have to make two technical assumptions. We need the notion of *cofibrantly generated model categories* from [13] which is reviewed in some detail in [56, Sec. 2] and [21, Sec. 2.1]. We also need properness (see [4, Def. 1.2] or [25, Def. 5.5.2]). A model category is *left proper* if pushouts across cofibrations preserve weak equivalences. A model category is

right proper if pullbacks over fibrations preserve weak equivalences. A *proper* model category is one which is both left and right proper.

Theorem 3.1.1. (Classification of monogenic stable model categories) *Let $\mathcal{C}$ be a simplicial, cofibrantly generated, proper, stable model category with a compact generator P. Then there exists a chain of simplicial Quillen equivalences between $\mathcal{C}$ and the model category of* $\mathrm{End}(P)$*-modules.*

$$\mathcal{C} \simeq_Q \text{mod-}\,\mathrm{End}(P)$$

This theorem is a special case of the more general classification result Theorem 3.3.3, which applies to stable model categories with a set of compact generators and which we prove in Section 3.6. Furthermore if in the situation of Theorem 3.1.1, P is a compact object but not necessarily a generator of $\mathcal{C}$, then $\mathcal{C}$ still 'contains' the homotopy theory of $\mathrm{End}(P)$-modules, see Theorem 3.9.3 (ii) for the precise statement. In the Morita context (Theorem 4.1.2) we also prove a partial converse to Theorem 3.1.1.

3.2. **Examples: stable model categories with a compact generator.**

(i) **Uniqueness results for stable homotopy theory.** The classification theorem above yields a characterization of the model category of spectra: a simplicial, cofibrantly generated, proper, stable model category is simplicially Quillen equivalent to the category of symmetric spectra if and only if it has a compact generator P for which the unit map of ring spectra $S \longrightarrow \mathrm{End}(P)$ is a stable equivalence. The paper [57] is devoted to other necessary and sufficient conditions for when a stable model category is Quillen equivalent to spectra – some of them in terms of the homotopy category of $\mathcal{C}$ and the natural action of the stable homotopy groups of spheres. In [55], this result is extended to a uniqueness theorem showing that the 2-local stable homotopy category has only one underlying model category up to Quillen equivalence. In both of these papers, we eliminate the technical conditions 'cofibrantly generated' and 'proper' by working with spectra in the sense of Bousfield and Friedlander [4], as opposed to the Quillen equivalent symmetric spectra and 'simplicial' by working with *framings* [21, Chpt. 5]. In another direction, the uniqueness result is extended to include the monoidal structure in [62].

(ii) **Chain complexes and Eilenberg-Mac Lane spectra.** Let A be a ring. Theorem 5.1.6 shows that the model category of chain complexes of A-modules is Quillen equivalent to the model category of modules over the symmetric Eilenberg-Mac Lane ring spectrum HA. This can be viewed as an instance of Theorem 3.1.1: the free A-module of rank one, considered as a complex concentrated in dimension zero, is a compact generator for the unbounded derived category of A. Since the homotopy groups of its endomorphism ring spectrum (as an object of the model category of chain complexes) are concentrated in dimension zero, the endomorphism ring spectrum is stably equivalent to the Eilenberg-Mac Lane ring spectrum for A (see Proposition B.2.1). This also shows that although the model category of chain complexes of A-modules is not simplicial it is Quillen equivalent to a simplicial model category. So although our classification theorems do not apply directly, they do apply indirectly.

(iii) **Smashing Bousfield localizations.** Let E be a spectrum and consider the E-local model category structure on some model category of spectra (see e.g. [15, VIII 1.1]). This is another stable model category in which the localization of the sphere spectrum $L_E S^0$ is a generator. This localized sphere is compact if the localization is *smashing*, i.e., if a certain natural map $X \wedge L_E S^0 \longrightarrow L_E X$ is a stable equivalence for all X. So for a smashing localization the E-local model category of spectra is Quillen equivalent

to modules over the ring spectrum $L_E S^0$ (which is the endomorphism ring spectrum of the localized sphere in the localized model structure).

(iv) **$K(n)$-local spectra.** Even if a Bousfield localization is not smashing, Theorem 3.1.1 might be applicable. As an example we consider Bousfield localization with respect to the n-th Morava K-theory $K(n)$ at a fixed prime. The localization of the sphere is still a generator, but for $n > 0$ it is not compact in the local category, see [24, 3.5.2]. However the localization of any finite type n spectrum F is a compact generator for the $K(n)$-local category [26, 7.3]. Hence the $K(n)$-local model category is Quillen equivalent to modules over the endomorphism ring $\mathrm{End}(L_{K(n)}F)$.

(v) **Frobenius rings.** As in Example 2.3 (iv) we consider a Frobenius ring and assume that the stable module category has a compact generator. Then we are in the situation of Theorem 3.1.1; however this example is completely algebraic, and there is no need to consider ring spectra to identify the stable module category as the derived category of a suitable 'ring'. In fact Keller shows [31, 4.3] that in such a situation there exists a differential graded algebra (DGA) and an equivalence between the stable module category and the unbounded derived category of the DGA.

A concrete example of this situation arises for group algebras of p-groups over a field k of characteristic p. In this case the trivial module is the only simple module, and it is a compact generator of the stable module category. More generally a result of Benson [5, Thm. 1.1] says that the trivial module generates the stable module category of the principal block of a group algebra kG if and only if the centralizer of every element of order p is p-nilpotent. So in this situation Keller's theorem applies and identifies the stable module category as the unbounded derived category of a certain DGA. The homology groups of this DGA are isomorphic (by construction) to the ring of graded self maps of the trivial module in the stable module category, which is just the Tate-cohomology ring $\widehat{\mathrm{H}}^*(G;k)$.

(vi) **Stable homotopy of algebraic theories.** Another motivation for this paper and an early instance of Theorem 3.1.1 came from the stabilization of the model category of algebras over an algebraic theory [54]. For every pointed algebraic theory T, the category of simplicial T-algebras is a simplicial model category so that one has a category $\mathcal{S}p(T)$ of (Bousfield-Friedlander type) spectra of T-algebras, a cofibrantly generated, simplicial stable model category [54, 4.3]. The free T-algebra on one generator has an endomorphism ring spectrum which is constructed as a Gamma-ring in [54, 4.5] and denoted T^s. Then [54, Thm. 4.4] provides a Quillen equivalence between the categories of connective spectra of T-algebras and the category of T^s-modules (the connectivity condition could be removed by working with symmetric spectra instead of Γ-spaces). This fits with Theorem 3.1.1 because the suspension spectrum of the free T-algebra on one generator is a compact generator for the category $\mathcal{S}p(T)$. See [54, Sec. 7] for a list of ring spectra that arise from algebraic theories in this fashion.

Remark 3.2.1. The notion of a compact generator and the *homotopy groups* of the endomorphism ring spectrum only depend on the homotopy category, and so they are invariant under equivalences of triangulated categories. However, the *homotopy type* of the endomorphism ring spectrum depends on the model category structure. The following example illustrates this point. Consider the n-th Morava K-theory spectrum $K(n)$ for a fixed prime and some number $n > 0$. This spectrum admits the structure of an A_∞-ring spectrum [51]. Hence it also has a model as an S-algebra or a symmetric ring spectrum and the category of its module spectra is a stable model category. The ring of homotopy groups of $K(n)$ is the graded field $\mathbb{F}_p[v_n, v_n^{-1}]$ with v_n of

dimension $2p^n - 2$. Hence the homotopy group functor establishes an equivalence between the homotopy category of $K(n)$-module spectra and the category of graded $\mathbb{F}_p[v_n, v_n^{-1}]$-modules.

Similarly the homology functor establishes an equivalence between the derived category of differential graded modules over the graded field $\mathbb{F}_p[v_n, v_n^{-1}]$ and the category of graded $\mathbb{F}_p[v_n, v_n^{-1}]$-modules. So the two stable model categories of $K(n)$-module spectra and DG-modules over $\mathbb{F}_p[v_n, v_n^{-1}]$ have equivalent triangulated homotopy categories (including the action of the stable homotopy groups of spheres — all elements in positive dimension act trivially in both cases). But the endomorphism ring spectra of the respective free rank one modules are the Morava K-theory ring spectrum on the one side and the Eilenberg-Mac Lane ring spectrum for $\mathbb{F}_p[v_n, v_n^{-1}]$ on the other side, which are not stably equivalent. Similarly the two model categories are not Quillen equivalent since for DG-modules all function spaces are products of Eilenberg-Mac Lane spaces, but for $K(n)$-modules they are not.

3.3. **Multiple generators.** There is a generalization of Theorem 3.1.1 to the case of a stable model category with a set of compact generators (as opposed to a single compact generator).

Let us recall the algebraic precursors of this result: a *ringoid* is a category whose hom-sets are abelian groups with bilinear composition. Ringoids are sometimes called *pre-additive categories* or *rings with several objects*. Indeed a ring in the traditional sense is the same as a ringoid with one object. A (right) *module* over a ringoid is defined to be a contravariant additive functor to the category of abelian groups. These more general module categories have been identified as the cocomplete abelian categories which have a set of small projective generators [16, 5.3H, p. 120]. An analogous theory for derived categories of DG categories has been developed by Keller [31].

Our result is very much in the spirit of Freyd's or Keller's, with spectra substituting for abelian groups or chain complexes. A symmetric ring spectrum can be viewed as a category with one object which is enriched over symmetric spectra; the module category then becomes the category of enriched (spectral) functors to symmetric spectra. So we now look at 'ring spectra with several objects' which we call *spectral categories*. This is analogous to pre-additive, differential graded or simplicial categories which are enriched over abelian groups, chain complexes or simplicial sets respectively.

Definition 3.3.1. A *spectral category* is a category $\mathcal{O}$ which is enriched over the category Sp^Σ of symmetric spectra (with respect to smash product, i.e., the monoidal closed structure of [25, 2.2.10]). In other words, for every pair of objects o, o' in $\mathcal{O}$ there is a morphism symmetric spectrum $\mathcal{O}(o, o')$, for every object o of $\mathcal{O}$ there is a map from the sphere spectrum S to $\mathcal{O}(o, o)$ (the 'identity element' of o), and for each triple of objects there is an associative and unital composition map of symmetric spectra

$$\mathcal{O}(o', o'') \ \wedge \ \mathcal{O}(o, o') \longrightarrow \mathcal{O}(o, o'') \ .$$

An $\mathcal{O}$-module M is a contravariant spectral functor to the category Sp^Σ of symmetric spectra, i.e., a symmetric spectrum $M(o)$ for each object of $\mathcal{O}$ together with coherently associative and unital maps of symmetric spectra

$$M(o) \ \wedge \ \mathcal{O}(o', o) \longrightarrow M(o')$$

for pairs of objects o, o' in $\mathcal{O}$. A morphism of $\mathcal{O}$-modules $M \longrightarrow N$ consists of maps of symmetric spectra $M(o) \longrightarrow N(o)$ strictly compatible with the action of $\mathcal{O}$. We denote the category of $\mathcal{O}$-modules by mod-$\mathcal{O}$. The *free* (or 'representable') module F_o is given by $F_o(o') = \mathcal{O}(o', o)$.

Remark 3.3.2. In Definition 3.3.1 we are simply spelling out what it means to do enriched category theory over the symmetric monoidal closed category Sp^Σ of symmetric spectra with respect to the smash product and the internal homomorphism spectra. Kelly's book [32] is an

introduction to enriched category theory in general; the spectral categories, modules over these (spectral functors) and morphisms of modules as defined above are the Sp^Σ-categories, Sp^Σ-functors and Sp^Σ-natural transformations in the sense of [32, 1.2]. So the precise meaning of the coherence and compatibility conditions in Definition 3.3.1 can be found in [32, 1.2].

We show in Theorem A.1.1 that for any spectral category $\mathcal{O}$ the category of $\mathcal{O}$-modules is a model category with objectwise stable equivalences as the weak equivalences. There we also show that the set of free modules $\{F_o\}_{o\in\mathcal{O}}$ is a set of compact generators. If $\mathcal{O}$ has a single object o, then the $\mathcal{O}$-modules are precisely the modules over the symmetric ring spectrum $\mathcal{O}(o,o)$, and the model category structure is the one defined in [25, 5.4.2].

In Definition 3.7.5 we associate to every set $\mathcal{P}$ of objects of a simplicial, cofibrantly generated stable model category $\mathcal{C}$ a spectral *endomorphism category* $\mathcal{E}(\mathcal{P})$ whose objects are the members of the set $\mathcal{P}$ and such that there is a natural, associative and unital isomorphism

$$\pi_*\,\mathcal{E}(\mathcal{P})(P,P') \;\cong\; [P,P']_*^{\mathrm{Ho}(\mathcal{C})}\ .$$

For a set with a single element this reduces to the notion of the endomorphism ring spectrum.

Theorem 3.3.3. (Classification of stable model categories) *Let $\mathcal{C}$ be a simplicial, cofibrantly generated, proper, stable model category with a set $\mathcal{P}$ of compact generators. Then there exists a chain of simplicial Quillen equivalences between $\mathcal{C}$ and the model category of $\mathcal{E}(\mathcal{P})$-modules.*

$$\mathcal{C} \simeq_Q \text{mod-}\mathcal{E}(\mathcal{P})$$

There is an even more general version of Theorem 3.3.3 which also provides information if the set $\mathcal{P}$ does not generate the whole homotopy category, see Theorem 3.9.3 (ii). This variant implies that for any set $\mathcal{P}$ of compact objects in a proper, cofibrantly generated, simplicial, stable model category the homotopy category of $\mathcal{E}(\mathcal{P})$-modules is triangulated equivalent to the localizing subcategory of $\mathrm{Ho}(\mathcal{C})$ generated by the set $\mathcal{P}$.

The proof of Theorem 3.3.3 breaks up into two parts. In order to mimic the classical proof for abelian categories we must consider a situation where the hom functor $\mathrm{Hom}_{\mathcal{C}}(P,-)$ takes values in the category of modules over a suitable endomorphism ring spectrum of P. In Section 3.6 we show how this can be arranged, given the technical conditions that $\mathcal{C}$ is cofibrantly generated, proper and simplicial. We introduce the category $Sp(\mathcal{C})$ of symmetric spectra over $\mathcal{C}$ and show in Theorem 3.8.2 that it is a stable model category with composable and homotopically well-behaved function symmetric spectra which is Quillen equivalent to the original stable model category $\mathcal{C}$.

In Theorem 3.9.3 we prove Theorem 3.3.3 under the assumption that $\mathcal{C}$ is a *spectral model category* (Definition 3.5.1), i.e., a model category with composable and homotopically well-behaved function symmetric spectra. Since the model category $Sp(\mathcal{C})$ is spectral and Quillen equivalent to $\mathcal{C}$ (given the technical assumptions of Theorem 3.3.3), this will complete the classification results.

3.4. **Examples: stable model categories with a set of generators.**

(i) **Equivariant stable homotopy.** Let G be a compact Lie group. As mentioned in Example 2.3(iii) there are several versions of model categories of G-equivariant spectra. In [37], G-equivariant orthogonal spectra are shown to form a cofibrantly generated, topological (hence simplicial), proper model category (the monoidal structure plays no role for our present considerations). For every closed subgroup H of G the equivariant suspension spectrum of the homogeneous space G/H_+ is compact and the set $\mathcal{G}$ of these spectra for all closed subgroups H generates the G-equivariant stable homotopy category, see [34, I 4.4] or [24, 9.4].

Recall from [34, V §9] that a *Mackey functor* is a module over the stable homotopy orbit category, i.e., an additive functor from the homotopy orbit category $\pi_0\,\mathcal{E}(\mathcal{G})$ to the category of abelian groups; by [34, V Prop. 9.9] this agrees with the original algebraic definition of Dress [10] in the case of finite groups. The spectral endomorphism category $\mathcal{E}(\mathcal{G})$ is a spectrum valued lift of the stable homotopy orbit category. Theorem 3.3.3 shows that the category of G-equivariant spectra is Quillen equivalent to a category of *topological Mackey functors*, i.e., the category of modules over the stable orbit category $\mathcal{E}(\mathcal{G})$. Note that the homotopy type of each morphism spectrum of $\mathcal{E}(\mathcal{G})$ depends on the universe $\mathcal{U}$.

After rationalization the Mackey functor analogy becomes even more concrete: for *finite* groups G we will see in Example 5.1.2 that the model category of rational G-equivariant spectra is in fact Quillen equivalent to the model category of chain complexes of rational Mackey functors. For certain non-finite compact Lie groups, our approach via 'topological Mackey functors' is used in [61] and [19] as an intermediate step in forming algebraic models for rational G-equivariant spectra.

(ii) **Motivic stable homotopy of schemes.** In [42, 66] Morel and Voevodsky introduce the $\mathbb{A}^1$-local model category structure for schemes over a base. The objects of their category are sheaves of sets in the Nisnevich topology on smooth schemes of finite type over a fixed base scheme. The weak equivalences are the $\mathbb{A}^1$-local equivalences – roughly speaking they are generated by the projection maps $X \times \mathbb{A}^1 \longrightarrow X$ for smooth schemes X, where $\mathbb{A}^1$ denotes the affine line.

Voevodsky [66, Sec. 5] introduces an associated stable homotopy category by inverting smashing with the 'Tate-sphere' $T = \mathbb{A}^1/(\mathbb{A}^1 - 0)$. The punch-line is that theories like algebraic K-theory or motivic cohomology are represented by objects in this stable homotopy category [66, Sec. 6], at least when the base scheme is the spectrum of a field.

In [30], Jardine provides the details of the construction of model categories of T-spectra over the spectrum of a field k. He constructs two Quillen equivalent proper, simplicial model categories of Bousfield-Friedlander type and symmetric $\mathbb{A}^1$-local T-spectra [30, 2.11, 4.18]. Since T is weakly equivalent to a suspension (of the multiplicative group scheme), this in particular yields a stable model category. A set of compact generators for this homotopy category is given by the T-suspension spectra $\Sigma^\infty_T(\mathrm{Spec}R)_+$ when R runs over smooth k-algebras of finite type. So if k is countable then this is a countable set of compact generators, compare [66, Prop. 5.5].

(iii) **Algebraic examples.** Again the classification theorem 3.3.3 has an algebraic analogue and precursor, namely Keller's theory of derived equivalences of DG categories [31]. The bottom line is that if an example of a stable model category is algebraic (such as derived or stable module categories in Examples 2.4), then it is not necessary to consider spectra and modules over spectral categories, but one can work with chain complexes and differential graded categories instead. As an example, Theorem 4.3 of [31] identifies the stable module category of a Frobenius ring with the unbounded derived category of a certain differential graded category.

3.5. **Prerequisites on spectral model categories.** A spectral model category is analogous to a simplicial model category, [45, II.2], but with the category of simplicial sets replaced by symmetric spectra. Roughly speaking, a spectral model category is a pointed model category which is compatibly enriched over the stable model category of symmetric spectra. The compatibility is expressed by the axiom (SP) below which takes the place of [45, II.2 SM7]. For the precise meaning of 'tensors' and 'cotensors' over symmetric spectra see e.g. [32, 3.7]. A spectral model category is the same as a 'Sp^Σ-model category' in the sense of [21, Def. 4.2.18], where the

category of symmetric spectra is endowed with the stable model structure of [25, 3.4.4]. Condition two of [21, 4.2.18] is automatic since the unit S for the smash product of symmetric spectra is cofibrant. Examples of spectral model categories are module categories over a symmetric ring spectrum, module categories over a spectral category (Theorem A.1.1) and the category of symmetric spectra over a suitable simplicial model category (Theorem 3.8.2).

Definition 3.5.1. A *spectral model category* is a model category $\mathcal{C}$ which is tensored, cotensored and enriched (denoted $\mathrm{Hom}_{\mathcal{C}}$) over the category of symmetric spectra with the closed monoidal structure of [25, 2.2.10] such that the following compatibility axiom (SP) holds:
(SP) For every cofibration $A \longrightarrow B$ and every fibration $X \longrightarrow Y$ in $\mathcal{C}$ the induced map

$$\mathrm{Hom}_{\mathcal{C}}(B, X) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, X) \times_{\mathrm{Hom}_{\mathcal{C}}(A,Y)} \mathrm{Hom}_{\mathcal{C}}(B, Y)$$

is a stable fibration of symmetric spectra. If in addition one of the maps $A \longrightarrow B$ or $X \longrightarrow Y$ is a weak equivalence, then the resulting map of symmetric spectra is also a stable equivalence. We use the notation $K \wedge X$ and X^K to denote the tensors and cotensors for $X \in \mathcal{C}$ and K a symmetric spectrum.

In analogy with [45, II.2 Prop. 3] the compatibility axiom (SP) in Definition 3.5.1 of a spectral model category can be cast into two adjoint forms, one of which will be of use for us. Given a categorical enrichment of a model category $\mathcal{C}$ over the category of symmetric spectra, then axiom (SP) is equivalent to (SPb) below. The equivalence of conditions (SP) and (SPb) is a consequence of the adjointness properties of the tensor and cotensor functors, see [21, Lemma 4.2.2] for the details.

(SPb) For every cofibration $A \longrightarrow B$ in $\mathcal{C}$ and every stable cofibration $K \longrightarrow L$ of symmetric spectra, the canonical map (*pushout product map*)

$$L \wedge A \cup_{K \wedge A} K \wedge B \longrightarrow L \wedge B$$

is a cofibration; the pushout product map is a weak equivalence if in addition $A \longrightarrow B$ is a weak equivalence in $\mathcal{C}$ or $K \longrightarrow L$ is a stable equivalence of symmetric spectra.

In a spectral model category the levels of the symmetric function spectra $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ can be rewritten as follows. The adjunctions give an isomorphism of simplicial sets

$$\mathrm{Hom}_{\mathcal{C}}(X, Y)_n \;\cong\; \mathrm{map}_{\mathcal{S}p}(F_n S^0, \mathrm{Hom}_{\mathcal{C}}(X, Y)) \;\cong\; \mathrm{map}_{\mathcal{C}}(F_n S^0 \wedge X, Y) \;\cong\; \mathrm{map}_{\mathcal{C}}(X, Y^{F_n S^0})$$

where $F_n S^0$ is the free symmetric spectrum generated at level n by the 0-sphere (see [25, 2.2.5] or Definition 3.6.5).

Lemma 3.5.2. *A spectral model category is in particular a* simplicial *and* stable *model category. For X a cofibrant and Y a fibrant object of a spectral model category $\mathcal{C}$ there is a natural isomorphism of graded abelian groups $\pi_*^s \mathrm{Hom}_{\mathcal{C}}(X, Y) \cong [X, Y]_*^{\mathrm{Ho}(\mathcal{C})}$.*

Proof. The tensor and cotensor of an object of $\mathcal{C}$ with a pointed simplicial set K is defined by applying the tensor and cotensor with the symmetric suspension spectrum $\Sigma^\infty K$. The homomorphism simplicial set between two objects of $\mathcal{C}$ is the 0-th level of the homomorphism symmetric spectrum. The necessary adjunction formulas and the compatibility axiom [45, II.2 SM7] hold because the suspension spectrum functor $\Sigma^\infty \colon \mathcal{S}_* \longrightarrow \mathcal{S}p$ from the category of pointed simplicial sets to symmetric spectra is the left adjoint of a Quillen adjoint functor pair and preserves the smash product (i.e., it is strong symmetric monoidal). In order to see that $\mathcal{C}$ is stable we recall [21, 7.1.6] that the shift functor in the homotopy category of $\mathcal{C}$ is the suspension functor. For cofibrant objects suspension is represented on the model category level by the smash product with the one-dimensional sphere spectrum $\Sigma^\infty S^1$. This sphere spectrum is invertible, up to stable equivalence of symmetric spectra, with inverse the (-1)-dimensional sphere spectrum (modeled as a symmetric spectrum by $F_1 S^0$). Since the action of symmetric spectra on $\mathcal{C}$ is

associative up to coherent isomorphism this implies that suspension is a self-equivalence of the homotopy category of $\mathcal{C}$. This in turn implies that the right adjoint loop functor has to be an inverse equivalence. If X is cofibrant and Y fibrant in $\mathcal{C}$, then by the compatibility axiom (SP) the symmetric spectrum $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is stably fibrant, i.e., an Ω-spectrum. So for $n \geq 0$ we have isomorphisms

$$\pi_n \,\mathrm{Hom}_{\mathcal{C}}(X,Y) \;\cong\; \pi_n \,\mathrm{map}_{\mathcal{C}}(X,Y) \;\cong\; \pi_0 \,\mathrm{map}_{\mathcal{C}}(X,\Omega^n Y) \;\cong\; [X,\Omega^n Y]^{\mathrm{Ho}(\mathcal{C})} \;\cong\; [\Sigma^n X, Y]^{\mathrm{Ho}(\mathcal{C})} \;;$$

and for $n \leq 0$ we have the isomorphisms

$$\pi_n \,\mathrm{Hom}_{\mathcal{C}}(X,Y) \;\cong\; \pi_0 \,\mathrm{Hom}_{\mathcal{C}}(X,Y)_{-n} \;\cong\; \pi_0 \,\mathrm{map}_{\mathcal{C}}(F_{-n}S^0 \wedge X, Y) \;\cong\; [\Sigma^n X, Y]^{\mathrm{Ho}(\mathcal{C})} \;.$$

□

3.6. **Symmetric spectra over a category.** Throughout this section we assume that $\mathcal{C}$ is a cocomplete category which is tensored and cotensored over the category $\mathcal{S}_*$ of pointed simplicial sets, with this action denoted by $\otimes$ and morphism simplicial sets denoted $\mathrm{map}_{\mathcal{C}}$. We let $S^1 = \Delta[1]/\partial\Delta[1]$ be our model for the simplicial circle and we set $S^n = (S^1)^{\wedge n}$ for $n > 1$; the symmetric group on n letters acts on S^n by permuting the coordinates.

Definition 3.6.1. Let $\mathcal{C}$ be a category which is tensored over the category of pointed simplicial sets. A *symmetric sequence over* $\mathcal{C}$ is a sequence of objects $X = \{X_n\}_{n\geq 0}$ in $\mathcal{C}$ together with a left action of the symmetric group Σ_n on X_n for all $n \geq 0$. A *symmetric spectrum over* $\mathcal{C}$ is a symmetric sequence in $\mathcal{C}$ with coherently associative $\Sigma_p \times \Sigma_q$-equivariant morphisms

$$S^p \;\otimes\; X_q \longrightarrow X_{p+q}$$

(for all $p, q \geq 0$). A morphism of symmetric sequences or symmetric spectra $X \longrightarrow Y$ consists of a sequence of Σ_n-equivariant morphisms $X_n \longrightarrow Y_n$ which commute with the structure maps. We denote the category of symmetric sequences by $\mathcal{C}^\Sigma$ and the category of symmetric spectra by $Sp(\mathcal{C})$.

Since $\mathcal{C}$ is simplicial, the action of $\mathcal{S}_*$ on $\mathcal{C}$ extends to an action of $\mathcal{S}_*^\Sigma$, the category of symmetric sequences over $\mathcal{S}_*$, on $\mathcal{C}^\Sigma$:

Definition 3.6.2. Given X a symmetric sequence over $\mathcal{C}$, and K a symmetric sequence over $\mathcal{S}_*$, we define their *tensor product*, $K \otimes X$, by the formula

$$(K \otimes X)_n = \bigvee_{p+q=n} \Sigma_n^+ \otimes_{\Sigma_p \times \Sigma_q} (K_p \otimes X_q) \;.$$

The symmetric sequence $S = (S^0, S^1, \cdots, S^n, \cdots)$ of simplicial sets is a commutative monoid in the symmetric monoidal category $(\mathcal{S}_*^\Sigma, \otimes)$. The unit map here is the identity in the first spot and the base point elsewhere, $\eta\colon (S^0, *, *, \cdots) = u \longrightarrow S$. In this language a symmetric spectrum over $\mathcal{C}$ can be redefined as a left S-module in the category of symmetric sequences over $\mathcal{C}$.

Definition 3.6.3. Let X be an object in $Sp(\mathcal{C})$ and K a symmetric spectrum in Sp^Σ. Define their *smash product*, $K \wedge X$, as the coequalizer of the two maps

$$K \otimes S \otimes X \;\rightrightarrows\; K \otimes X$$

induced by the action of S on X and K respectively. So $Sp(\mathcal{C})$ is tensored over the symmetric monoidal category of symmetric spectra. Dually, we define a symmetric spectrum valued morphism object $\mathrm{Hom}_{Sp(\mathcal{C})}(X,Y) \in Sp^\Sigma$ for $X, Y \in Sp(\mathcal{C})$. As a preliminary step, define a shifting down functor, $\mathrm{sh}_n\colon Sp(\mathcal{C}) \to Sp(\mathcal{C})$, by $(\mathrm{sh}_n X)_m = X_{n+m}$ where Σ_m acts via the inclusion into Σ_{n+m}. Note there is a leftover action of Σ_n on $\mathrm{sh}_n X$. Define $\mathrm{Hom}_\Sigma(X,Y) \in \mathcal{S}_*^\Sigma$ for X, Y objects in $\mathcal{C}^\Sigma$ by $\mathrm{Hom}_\Sigma(X,Y)_n = \mathrm{map}(X, \mathrm{sh}_n Y)$, the simplicial mapping space given

by the simplicial structure on $\mathcal{C}^\Sigma$ with the Σ_n action given by the leftover action of Σ_n on $\mathrm{sh}_n Y$ as mentioned above. Then $\mathrm{Hom}_{Sp(\mathcal{C})}(X,Y) \in Sp^\Sigma$ is the equalizer of the two maps $\mathrm{Hom}_\Sigma(X,Y) \longrightarrow \mathrm{Hom}_\Sigma(S\otimes X, Y)$.

Using this spectrum valued hom functor, for K in Sp^Σ and Y in $Sp(\mathcal{C})$, we define $Y^K \in Sp(\mathcal{C})$ as the adjoint of the functor $K\otimes -$. That is, for any $X \in Sp(\mathcal{C})$ define Y^K such that

$$\mathrm{Hom}_{Sp(\mathcal{C})}(K\otimes X, Y) \cong \mathrm{Hom}_{Sp(\mathcal{C})}(X, Y^K) \cong \mathrm{Hom}_{Sp^\Sigma}(K, \mathrm{Hom}_{Sp(\mathcal{C})}(X,Y))\ . \tag{3.6.4}$$

Definition 3.6.5. The nth evaluation functor $\mathrm{Ev}_n : Sp(\mathcal{C}) \longrightarrow \mathcal{C}$ is given by $\mathrm{Ev}_n(X) = X_n$, ignoring the action of the symmetric group. The functor Ev_n has a left adjoint $F_n\colon \mathcal{C} \longrightarrow Sp(\mathcal{C})$ which has the form $(F_nX)_m = \Sigma^+_m \otimes_{\Sigma_{m-n}} S^{m-n}\otimes X$ where $S^n = *$ for $n<0$. We use Σ^∞ as another name for F_0 and call it the *suspension spectrum*.

3.7. **The level model structure on $Sp(\mathcal{C})$.** There are two model category structures on symmetric spectra over $\mathcal{C}$ which we consider; the level model category which we discuss in this section, and the stable model category (see Section 3.8). The level model category is a stepping stone for defining the stable model category, but it also allows us to define endomorphism ring spectra (Definition 3.7.5).

Definition 3.7.1. Let $f : X \longrightarrow Y$ be a map in $Sp(\mathcal{C})$. The map f is a *level equivalence* if each $f_n : X_n \longrightarrow Y_n$ is a weak equivalence in $\mathcal{C}$, ignoring the Σ_n action. It is a *level fibration* if each f_n is a fibration in $\mathcal{C}$. It is a *cofibration* if it has the left lifting property with respect to all level trivial fibrations.

Proposition 3.7.2. *For any simplicial, cofibrantly generated model category $\mathcal{C}$, $Sp(\mathcal{C})$ with the level equivalences, level fibrations, and cofibrations described above forms a cofibrantly generated model category referred to as the* level model category*, and denoted by $Sp(\mathcal{C})^{\mathrm{lv}}$. Furthermore the following level analogue of the spectral axiom* (SP) *holds:*

(SP$^{\mathrm{lv}}$) *for every cofibration $A \longrightarrow B$ and every level fibration $X \longrightarrow Y$ in $Sp(\mathcal{C})$ the induced map*

$$\mathrm{Hom}_{Sp(\mathcal{C})}(B,X) \longrightarrow \mathrm{Hom}_{Sp(\mathcal{C})}(A,X) \times_{\mathrm{Hom}_{Sp(\mathcal{C})}(A,Y)} \mathrm{Hom}_{Sp(\mathcal{C})}(B,Y)$$

is a level fibration of symmetric spectra. If in addition one of the maps $A \longrightarrow B$ or $X \longrightarrow Y$ is a level equivalence, then the resulting map of symmetric spectra is also a level equivalence.

There are various theorems in the model category literature which are useful in establishing model category structures. These theorems separate formal considerations that tend to show up routinely from the properties which require special arguments in each specific case. Since we will construct model category structures several times in this paper, we recall one such result that we apply in our cases. We work with the concept of cofibrantly generated model categories, introduced by Dwyer, Hirschhorn and Kan [13]; see also Section 2.1 of Hovey's book [21] for a detailed treatment of this concept.

We use the same terminology as [21, Sec. 2.1]. Let I be a set of maps in a category. A map is a *relative I-cell complex* if it is a (possibly transfinite) composition of cobase changes of maps in I. An *I-injective* map is a map with the right lifting property with respect to every map in I. An *I-cofibration* is a map with the left lifting property with respect to I-injective maps. For the definition of smallness relative to a set of maps see [21, 2.1.3].

One of the main properties of cofibrantly generated model categories is that they admit an abstract version of Quillen's small object argument [45, II 3.4].

Lemma 3.7.3. [13], [21, 2.1.14, 2.1.15] *Let $\mathcal{C}$ be a cocomplete category and I a set of maps in $\mathcal{C}$ whose domains are small relative to the relative I-cell complexes. Then*

- *there is a functorial factorization of any map f in $\mathcal{C}$ as $f = qi$ with q an I-injective map and i a relative I-cell complex, and thus*

- *every I-cofibration is a retract of a relative I-cell complex.*

Theorem 3.7.4. [13], [21, Thm. 2.1.19] *Let $\mathcal{C}$ be a complete and cocomplete category and I and J two sets of maps of $\mathcal{C}$ such that the domains of the maps in I and J are small with respect to the relative I-cell complexes and the relative J-cell complexes respectively. Suppose also that a subcategory of $\mathcal{C}$ is specified whose morphisms are called 'weak equivalences'.*

Then there is a cofibrantly generated model structure on $\mathcal{C}$ with the given class of weak equivalences, with I a set of generating cofibrations, and with J a set of generating trivial cofibrations if the following conditions hold:

1. *if f and g are composable morphisms such that two of the three maps f, g and gf are weak equivalences, then the third is also a weak equivalence.*
2. *every relative J-cell complex is an I-cofibration and a weak equivalence.*
3. *the I-injectives are precisely the maps which are both J-injective and weak equivalences.*

Proof of Proposition 3.7.2. Let $I_{\mathcal{C}}$ and $J_{\mathcal{C}}$ be sets of generators for the cofibrations and trivial cofibrations of $\mathcal{C}$. We define sets of generators for the level model category by $FI_{\mathcal{C}} = \{F_n I_{\mathcal{C}}\}_{n\geq 0}$ and $FJ_{\mathcal{C}} = \{F_n J_{\mathcal{C}}\}_{n\geq 0}$, i.e., F_n applied to the generators of $\mathcal{C}$ for each n. Then the $FI_{\mathcal{C}}$-injectives are precisely the levelwise trivial fibrations and the $FJ_{\mathcal{C}}$-injectives are precisely the level fibrations. We claim that every relative $FI_{\mathcal{C}}$-cell complex is levelwise a cofibration in $\mathcal{C}$, and similarly every relative $FJ_{\mathcal{C}}$-cell complex is levelwise a trivial cofibration in $\mathcal{C}$. We show this for relative $FJ_{\mathcal{C}}$-cell complexes; the argument for $FI_{\mathcal{C}}$ is the same. Since level evaluation preserves colimits it suffices to check the claim for the generating cofibrations, $F_n A \longrightarrow F_n B$ for $A \longrightarrow B \in J_{\mathcal{C}}$. But the mth level of this map is a coproduct of $m!/(m-n)!$ copies of the map $S^{m-n} \wedge A \longrightarrow S^{m-n} \wedge B$. By the simplicial compatibility axiom [45, II.2 SM7], smashing with a simplicial sphere preserves trivial cofibrations, so we are done.

Now we apply Theorem 3.7.4 to the sets $FI_{\mathcal{C}}$ and $FJ_{\mathcal{C}}$ with the level equivalences as weak equivalences. Checking that the maps in $FI_{\mathcal{C}}$ are small with respect to relative $FI_{\mathcal{C}}$-cell complexes comes down (by adjointness) to checking that the domains of the maps in $I_{\mathcal{C}}$ are small with respect to the levels of relative $FI_{\mathcal{C}}$-cell complexes, i.e. the cofibrations in $\mathcal{C}$. By [20, 14.2.14], since the domains in $I_{\mathcal{C}}$ are small with respect to the relative $I_{\mathcal{C}}$-cell complexes they are also small with respect to all cofibrations. The argument for $FJ_{\mathcal{C}}$ is the same. Conditions (1) and (3) of Theorem 3.7.4 hold and condition (2) follows from the above claim. So we indeed have a cofibrantly generated level model structure.

To prove the property ($\mathrm{SP}^{\mathrm{lv}}$) it suffices to check its adjoint pushout product form, i.e., the level analogue of condition (SPb) of Section 3.5; it is enough to show that the pushout product of two generating cofibrations is a cofibration, and similarly when one of the maps is a trivial cofibration (see [56, 2.3 (1)] or [21, Cor. 4.2.5]). So let $i \in I_{\mathcal{S}_*}$ and let $j \in I_{\mathcal{C}}$. Then the product $F_n i \wedge F_m j$ is isomorphic to $F_{n+m}(i \wedge j)$ and the result follows since the free functors preserve cofibrations and trivial cofibrations. □

We can now introduce endomorphism ring spectra and endomorphism categories.

Definition 3.7.5. Let $\mathcal{P}$ be a set of objects of a simplicial and cofibrantly generated model category $\mathcal{C}$. We assume the objects in $\mathcal{P}$ are cofibrant; if not, take cofibrant replacements instead. For every object $P \in \mathcal{P}$ let $\Sigma^\infty_{\mathrm{f}} P$ be a fibrant replacement of the symmetric suspension spectrum of P in the level model structure on $Sp(\mathcal{C})$ of Proposition 3.7.2. We define the *endomorphism category* $\mathcal{E}(\mathcal{P})$ as the full spectral subcategory of $Sp(\mathcal{C})$ with objects $\Sigma^\infty_{\mathrm{f}} P$ for $P \in \mathcal{P}$. To simplify notation, we usually denote objects of $\mathcal{E}(\mathcal{P})$ by P instead of $\Sigma^\infty_{\mathrm{f}} P$. If $\mathcal{P}$ has a single object P we also refer to the symmetric ring spectrum $\mathcal{E}(\mathcal{P})(P,P) = \mathrm{Hom}_{Sp(\mathcal{C})}(\Sigma^\infty_{\mathrm{f}} P, \Sigma^\infty_{\mathrm{f}} P)$ as the *endomorphism ring spectrum* of the object P.

Up to stable equivalence, the definition of the endomorphism category does not depend on the choices of fibrant replacements.

Lemma 3.7.6. *Let $\mathcal{C}$ be a simplicial and cofibrantly generated model category and $\mathcal{P}$ a set of cofibrant objects. Suppose $\{\Sigma^\infty_{\mathrm{f}} P\}_{P\in\mathcal{P}}$ and $\{\overline{\Sigma^\infty_{\mathrm{f}} P}\}_{P\in\mathcal{P}}$ are two sets of level fibrant replacements of the symmetric suspension spectra. Then the two full spectral subcategories of $Sp(\mathcal{C})$ with objects $\{\Sigma^\infty_{\mathrm{f}} P\}_{P\in\mathcal{P}}$ and $\{\overline{\Sigma^\infty_{\mathrm{f}} P}\}_{P\in\mathcal{P}}$ respectively are stably equivalent.*

Proof. The proof uses the notion of *quasi-equivalence*, see Definition A.2.1. For every $P\in\mathcal{P}$ we choose a level equivalence $\phi_P : \overline{\Sigma^\infty_{\mathrm{f}} P} \longrightarrow \Sigma^\infty_{\mathrm{f}} P$. We define a $\mathcal{E}(\mathcal{P})$-$\overline{\mathcal{E}(\mathcal{P})}$-bimodule M by the rule

$$M(P,P') \;=\; \mathrm{Hom}_{Sp(\mathcal{C})}(\overline{\Sigma^\infty_{\mathrm{f}} P}, \Sigma^\infty_{\mathrm{f}} P)\ .$$

Because of the property ($\mathrm{SP}^{\mathrm{lv}}$) of the homomorphism spectra in $Sp(\mathcal{C})$ the bimodule M is a quasi-equivalence with respect to the maps ϕ_P, and the result follows from Lemma A.2.3. □

3.8. **The stable model structure on $Sp(\mathcal{C})$.** In this section we provide the details of the stable model category structure for symmetric spectra over $\mathcal{C}$; the result is summarized as Theorem 3.8.2. We use the level model category to define the stable model category structures on $Sp(\mathcal{C})$. The stable model category is more difficult to establish than the level model category, and we need to assume that $\mathcal{C}$ is a simplicial, cofibrantly generated, proper, stable model category. The proof of the stable model structure for $Sp(\mathcal{C})$ is similar to the proof of the stable model structure for Sp^Σ in [25, 3.4], except for one point in the proof of Proposition 3.8.8 where we use the stability of $\mathcal{C}$ instead of the fact that fiber sequences and cofiber sequences of spaces are stably equivalent.

Categories of symmetric spectrum objects over a model category have been considered more generally by Hovey in [23]. Hovey relies on the general localization machinery of [20]. Theorem 3.8.2 below should be compared to [23, Thms. 8.11 and 9.1] which are more general but have slightly different technical assumptions.

Definition 3.8.1. Let $\lambda : F_1S^1 \longrightarrow F_0S^0 \cong S$ be the stable equivalence of symmetric spectra which is adjoint to the identity map on the first level. A spectrum Z in $Sp(\mathcal{C})$ is an *Ω-spectrum* if Z is fibrant on each level and the map $Z \cong Z^{F_0S^0} \longrightarrow Z^{F_1S^1}$ induced by λ is a level equivalence.

A map $g : A \longrightarrow B$ in $Sp(\mathcal{C})$ is a *stable equivalence* if the induced map

$$\mathrm{Hom}_{Sp(\mathcal{C})}(g^c, Z) \;:\; \mathrm{Hom}_{Sp(\mathcal{C})}(A^c, Z) \longrightarrow \mathrm{Hom}_{Sp(\mathcal{C})}(B^c, Z)$$

is a level equivalence of symmetric spectra for any Ω-spectrum Z; here $(-)^c$ denotes a cofibrant replacement functor in the level model category structure. A map is a *stable cofibration* if it has the left lifting property with respect to each level trivial fibration, i.e., if it is a cofibration in the level model category structure. A map is a *stable fibration* if it has the right lifting property with respect to each map which is both a stable cofibration and a stable equivalence.

The above definition of Ω-spectrum is just a rewrite of the usual one since the n-th level of the spectrum $Z^{F_1S^1}$ is isomorphic to ΩZ_{n+1}. This form is more convenient here, though. Lemma 3.8.7 below, combined with the fact that the stable fibrations are the J-injective maps shows that the stably fibrant objects are precisely the Ω-spectra.

Theorem 3.8.2. *Let $\mathcal{C}$ be a simplicial, cofibrantly generated, proper, stable model category. Then $Sp(\mathcal{C})$ supports the structure of a spectral model category – referred to as the* stable model structure *– such that the adjoint functors Σ^∞ and evaluation Ev_0, are a Quillen equivalence between $\mathcal{C}$ and $Sp(\mathcal{C})$ with the stable model structure.*

We deduce the theorem about the stable model structure on $Sp(\mathcal{C})$ from a sequences of lemmas and propositions.

Lemma 3.8.3. *Let K be a cofibrant symmetric spectrum, A a cofibrant spectrum in $Sp(\mathcal{C})$ and Z an Ω-spectrum in $Sp(\mathcal{C})$. Then the symmetric function spectrum $\mathrm{Hom}_{Sp(\mathcal{C})}(A,Z)$ is a symmetric Ω-spectrum and the function spectrum Z^K is an Ω-spectrum in $Sp(\mathcal{C})$.*

Proof. By the adjunctions between smash products and function spectra (see 3.6.4) we can rewrite the symmetric function spectrum $\mathrm{Hom}_{Sp(\mathcal{C})}(A,Z)^{F_1S^1}$ as $\mathrm{Hom}_{Sp(\mathcal{C})}(A,Z^{F_1S^1})$ in such a way that the map $\mathrm{Hom}_{Sp(\mathcal{C})}(A,Z)^\lambda$ is isomorphic to the map $\mathrm{Hom}_{Sp(\mathcal{C})}(A,Z^\lambda)$; since

$$Z^\lambda \ : \ Z \cong Z^{F_0S^0} \longrightarrow Z^{F_1S^1}$$

is a level equivalence between level fibrant objects, the first claim follows from property ($\mathrm{SP}^{\mathrm{lv}}$) of Proposition 3.7.2.

Similarly we can rewrite the spectrum $(Z^K)^{F_1S^1}$ as $(Z^{F_1S^1})^K$ in such a way that the map $(Z^K)^\lambda$ is isomorphic to the map $(Z^\lambda)^K$; since Z^λ is a level equivalence between level fibrant objects, the second claim follows from the adjoint form ($\mathrm{SP}^{\mathrm{lv}}$(a)) of property ($\mathrm{SP}^{\mathrm{lv}}$) of Proposition 3.7.2, see [45, II.2 SM7(a)]. □

Lemma 3.8.4. *Let $\mathcal{C}$ be a simplicial, cofibrantly generated and left proper model category. Then a cofibration $A \longrightarrow B$ is a stable equivalence if and only if for every Ω-spectrum Z the symmetric function spectrum $\mathrm{Hom}_{Sp(\mathcal{C})}(B/A,Z)$ is level contractible.*

Proof. Choose a factorization of the functorial level cofibrant replacement $A^c \longrightarrow B^c$ of the given cofibration as a cofibration $i : A^c \longrightarrow \bar{B}$ followed by a level equivalence $q : \bar{B} \longrightarrow B^c$. Then q is a level equivalence between cofibrant objects, so for every Ω-spectrum Z, the induced map $\mathrm{Hom}_{Sp(\mathcal{C})}(q,Z)$ is a level equivalence. Hence f is a stable equivalence if and only if

$$\mathrm{Hom}_{Sp(\mathcal{C})}(i,Z) \ : \ \mathrm{Hom}_{Sp(\mathcal{C})}(A^c,Z) \longrightarrow \mathrm{Hom}_{Sp(\mathcal{C})}(\bar{B},Z)$$

is a level equivalence for every Ω-spectrum Z.

The symmetric spectrum $\mathrm{Hom}_{Sp(\mathcal{C})}(\bar{B}/A^c,Z)$ is the fiber of the level fibration $\mathrm{Hom}_{Sp(\mathcal{C})}(i,Z)$ (by ($\mathrm{SP}^{\mathrm{lv}}$)) between symmetric Ω-spectra (by Lemma 3.8.3). Hence the given map is a stable equivalence if and only if $\mathrm{Hom}_{Sp(\mathcal{C})}(\bar{B}/A^c,Z)$ is level contractible.

Since $A^c \longrightarrow A$ and $\bar{B} \longrightarrow B$ are level equivalences and $\mathcal{C}$ is left proper, the induced map on the cofibers $\bar{B}/A^c \longrightarrow B/A$ is a level equivalence between level cofibrant objects. So for every Ω-spectrum Z the induced map $\mathrm{Hom}_{Sp(\mathcal{C})}(B/A,Z) \longrightarrow \mathrm{Hom}_{Sp(\mathcal{C})}(\bar{B}/A^c,Z)$ is a level equivalence (by ($\mathrm{SP}^{\mathrm{lv}}$)) between symmetric Ω-spectra (by Lemma 3.8.3). Hence the given map is a stable equivalence if and only if $\mathrm{Hom}_{Sp(\mathcal{C})}(B/A,Z)$ is level contractible, which proves the lemma. □

We now show that $Sp(\mathcal{C})$ satisfies (SPb) of Section 3.5, an adjoint form of (SP) from Definition 3.5.1. This shows that $Sp(\mathcal{C})$ is a spectral model category as soon as the stable model structure on $Sp(\mathcal{C})$ is established.

Proposition 3.8.5. *Let $\mathcal{C}$ be a simplicial, cofibrantly generated and left proper model category. Let $i : A \longrightarrow B$ be a cofibration in $Sp(\mathcal{C})$ and $j : K \longrightarrow L$ a stable cofibration of symmetric spectra. Then the pushout product map*

$$j \,\square\, i \ : \ L \wedge A \cup_{K\wedge A} K \wedge B \longrightarrow L \wedge B$$

is a cofibration in $Sp(\mathcal{C})$; the pushout product map is a stable equivalence if in addition i is a stable equivalence in $Sp(\mathcal{C})$ or j is a stable equivalence of symmetric spectra.

Proof. Since the cofibrations coincide in the level and the stable model structures for $Sp(\mathcal{C})$ and for symmetric spectra, we know by property ($\mathrm{SP}^{\mathrm{lv}}$) of Proposition 3.7.2 that $j \,\square\, i$ is again a cofibration in $Sp(\mathcal{C})$. Now suppose that one of the maps is in addition a stable equivalence. The pushout product map $j \,\square\, i$ is a cofibration with cofiber isomorphic to $(L/K) \wedge (B/A)$. So by

Lemma 3.8.4 it suffices to show that $\mathrm{Hom}_{Sp(\mathcal{C})}((L/K)\wedge(B/A),Z)$ is level contractible for every Ω-spectrum Z. If i is a stable acyclic cofibration, then we can rewrite this function spectrum as

$$\mathrm{Hom}_{Sp(\mathcal{C})}((L/K)\wedge(B/A),Z) \;\cong\; \mathrm{Hom}_{Sp(\mathcal{C})}(B/A,Z^{(L/K)})\;;$$

the latter spectrum is level contractible by Lemma 3.8.4 since $Z^{(L/K)}$ is an Ω-spectrum by Lemma 3.8.3 and i is a cofibration and stable equivalence. If j is a stable acyclic cofibration, then we similarly rewrite the spectrum as

$$\mathrm{Hom}_{Sp(\mathcal{C})}((L/K)\wedge(B/A),Z) \;\cong\; \mathrm{Hom}_{Sp^\Sigma}(L/K,\mathrm{Hom}_{Sp(\mathcal{C})}(B/A,Z))\;;$$

the latter spectrum is level contractible by [25, 5.3.9] since $\mathrm{Hom}_{Sp(\mathcal{C})}(B/A,Z)$ is a symmetric Ω-spectrum by Lemma 3.8.3 and L/K is stably contractible. □

We use Theorem 3.7.4 to verify the stable model category structure on $Sp(\mathcal{C})$. We first define two sets I and J of maps in $Sp(\mathcal{C})$ which will be generating sets for the cofibrations and stable trivial cofibrations. Since the stable cofibrations are the same class of maps as the cofibrations in the level model structure we let I be the generating set $FI_{\mathcal{C}}$ which was used in Proposition 3.7.2 to construct the level model structure. With this choice the I-injectives are precisely the level trivial fibrations.

The generating set for the stable trivial cofibrations is the union $J = FJ_{\mathcal{C}}\cup K$, where $FJ_{\mathcal{C}}$ is the generating set of trivial cofibrations for the level model category (see the proof of Proposition 3.7.2) and K is defined as follows. In the category of symmetric spectra over simplicial sets there is a map $\lambda : F_1S^1 \longrightarrow F_0S^0 = S$ which is adjoint to the identity map on the first level; this map was also used in defining an Ω-spectrum in Definition 3.8.1. Let $M\lambda$ be the mapping cylinder of this map, formed by taking the mapping cylinder of simplicial sets on each level. So $\lambda = r\kappa$ with $\kappa : F_1S^1 \longrightarrow M\lambda$ a stable equivalence and stable cofibration and $r : M\lambda \longrightarrow S$ a simplicial homotopy equivalence, see [25, 3.4.9]. Then K is the set of maps

$$K \;=\{\kappa\,\square\, FI_{\mathcal{C}}\} \;=\; \{\kappa\,\square\, F_ni | i\in I_{\mathcal{C}}\}\;,$$

where for $i : A \longrightarrow B$,

$$\kappa\,\square\, F_ni : (F_1S^1\wedge F_nB)\cup_{F_1S^1\wedge F_nA}(M\lambda\wedge F_nA) \longrightarrow M\lambda\wedge F_nB\;.$$

Here we only use the pushout product, $\square$, as a convenient way of naming these maps, see also [25, 5.3]. Now we can verify condition (2) of Theorem 3.7.4.

Proposition 3.8.6. *Let $\mathcal{C}$ be a simplicial, cofibrantly generated and left proper model category. Then every relative J-cell complex is an I-cofibration and a stable equivalence.*

Proof. All maps in J are cofibrations in the level model structure on $Sp(\mathcal{C})$ of Proposition 3.7.2, hence the relative J-cell complexes are contained in the I-cofibrations.

We claim that for every J-cofibration $A \longrightarrow B$ and every Ω-spectrum Z, the map

$$\mathrm{Hom}_{Sp(\mathcal{C})}(B,Z) \longrightarrow \mathrm{Hom}_{Sp(\mathcal{C})}(A,Z)$$

is a level trivial fibration of symmetric spectra. Hence the fiber, the symmetric spectrum $\mathrm{Hom}_{Sp(\mathcal{C})}(B/A,Z)$, is level contractible and $A \longrightarrow B$ is a stable equivalence by Lemma 3.8.4.

The property of inducing a trivial fibration after applying $\mathrm{Hom}_{Sp(\mathcal{C})}(-,Z)$ is closed under pushout, transfinite composition and retract, so by the small object argument 3.7.3 it suffices to check this for the generating maps in $J = FJ_{\mathcal{C}}\cup K$. The generating cofibrations in $FJ_{\mathcal{C}}$ are level trivial cofibrations, so for these the claim holds by the compatibility axiom ($\mathrm{SP}^{\mathrm{lv}}$). A map in the set K is of the form $\kappa\,\square\, F_ni$ where $\kappa : F_1S^1 \longrightarrow M\lambda$ is a stable trivial cofibration of symmetric spectra and F_ni is a cofibration in $Sp(\mathcal{C})$; hence the map $\kappa\,\square\, F_ni$ is a stable trivial cofibration between cofibrant objects by Proposition 3.8.5. So the induced map of symmetric

spectra $\mathrm{Hom}_{Sp(\mathcal{C})}(\kappa \square F_n i, Z)$ is a level fibration (by ($\mathrm{SP}^{\mathrm{lv}}$)) between Ω-spectra by Lemma 3.8.3. In addition the fiber of the map $\mathrm{Hom}_{Sp(\mathcal{C})}(\kappa \square F_n i, Z)$ is level contractible by Lemma 3.8.4, so the map is indeed a level trivial fibration. □

Before turning to property (3) of Theorem 3.7.4, we need the following lemma.

Lemma 3.8.7. *Let $\mathcal{C}$ be a simplicial, cofibrantly generated model category and X a symmetric spectrum over $\mathcal{C}$. Then the map $X \longrightarrow *$ is J-injective if and only if X is an Ω-spectrum.*

Proof. The maps in $FJ_{\mathcal{C}}$ generate the trivial cofibrations in the level model structure of Proposition 3.7.2, so $X \longrightarrow *$ is $FJ_{\mathcal{C}}$-injective if an only if X is levelwise fibrant. Now we assume that X is levelwise fibrant and show the map $X \longrightarrow *$ is K-injective if and only if X is an Ω-spectrum. By adjointness $X \longrightarrow *$ is K-injective if and only if the map $X^{\kappa} : X^{M\lambda} \longrightarrow X^{F_1 S^1}$ is a level trivial fibration. The projection $r : M\lambda \longrightarrow S$ is a simplicial homotopy equivalence, so it induces a level equivalence $X \longrightarrow X^{M\lambda}$. So $X \longrightarrow *$ is K-injective if and only if the map X^{λ} is a level equivalence, which precisely means that X is an Ω-spectrum. □

Proposition 3.8.8. *Let $\mathcal{C}$ be a simplicial, cofibrantly generated, right proper, stable model category. Then a map is J-injective and a stable equivalence if and only if it is a level trivial fibration.*

Proof. Every level equivalence is a stable equivalence and the level trivial fibrations are precisely the I-injectives. Since the J-cofibrations are contained in the I-cofibrations, these I-injectives are also J-injective. The converse is more difficult to prove.

Since J in particular contains maps of the form $F_n A \longrightarrow F_n B$ where n runs over the natural numbers and $A \longrightarrow B$ runs over a set of generating trivial cofibrations for $\mathcal{C}$, J-injective maps are level fibrations. So we show that a J-injective stable equivalence, $E \longrightarrow B$, is a level equivalence. Let F denote the fiber and choose a cofibrant replacement $F^{\mathrm{c}} \longrightarrow F$ in the level model category structure. Then choose a factorization in the level model category structure

$$F^{\mathrm{c}} \rightarrowtail E^{\mathrm{c}} \xrightarrow{\mathrm{lv}\sim} E$$

of the composite map $F^{\mathrm{c}} \longrightarrow E$ as a cofibration followed by a level equivalence. Since $\mathcal{C}$ is right proper, each level of $F \longrightarrow E \longrightarrow B$ is a homotopy fibration sequence in $\mathcal{C}$. Each level of $F^{\mathrm{c}} \longrightarrow E^{\mathrm{c}} \longrightarrow E^{\mathrm{c}}/F^{\mathrm{c}}$ is a homotopy cofibration sequence in $\mathcal{C}$ (left properness is not needed here since each object is cofibrant). So since $\mathcal{C}$ is stable, we see that $E^{\mathrm{c}}/F^{\mathrm{c}} \longrightarrow B$ is a level equivalence. Thus $E^{\mathrm{c}} \longrightarrow E^{\mathrm{c}}/F^{\mathrm{c}}$ is a stable equivalence. For any Ω-spectrum Z there is a fiber sequence of symmetric Ω-spectra

$$\mathrm{Hom}_{\mathcal{C}}(E^{\mathrm{c}}/F^{\mathrm{c}}, Z) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(E^{\mathrm{c}}, Z) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(F^{\mathrm{c}}, Z)$$

in which the left map is a level equivalence and the right map is a level fibration. Hence the symmetric spectrum $\mathrm{Hom}_{\mathcal{C}}(F^{\mathrm{c}}, Z)$ is level contractible which means that F is stably contractible.

Since $F \longrightarrow *$ is the pull back of the map $E \longrightarrow B$, it is a J-injective map. So F is an Ω-spectrum by Proposition 3.8.7. Since F is both stably contractible and an Ω-spectrum, the spectrum $\mathrm{Hom}_{\mathcal{C}}(F^{\mathrm{c}}, F)$ is level contractible, so F is level equivalent to a point. But this means that $E^{\mathrm{c}} \longrightarrow E^{\mathrm{c}}/F^{\mathrm{c}}$, and thus also $E \longrightarrow B$ is a level equivalence. □

Proof of Theorem 3.8.2. We apply Theorem 3.7.4 to show that the I-cofibrations, stable equivalences, and J-injectives form a cofibrantly generated model category on $Sp(\mathcal{C})$. Since the I-cofibrations are exactly the stable cofibrations this implies that the J-injectives are the maps with the right lifting property with respect to the stable trivial cofibrations, i.e., the stable fibrations as defined before the statement of the theorem. The 2-out-of-3 condition, part (1) in Theorem 3.7.4, is clear from the definition of stable equivalences. Condition (2) is verified

in Proposition 3.8.6 and condition (3) is verified in Proposition 3.8.8 (since the I-injectives are precisely the levelwise trivial fibrations). So to conclude that the sets I and J generate a model structure with the stable equivalences as weak equivalence it is enough to verify that the domains of the generators are small with respect to the level cofibrations. This has already been checked in the proof of Proposition 3.7.2 for the generators in $FI_{\mathcal{C}}$ and $FJ_{\mathcal{C}}$. So only the generators in K remain. Since a pushout of objects which are small is small, we only need to check that $F_kA \wedge M\lambda$ and $F_kA \wedge F_1S^1$ are small with respect to the level cofibrations. Here A is small with respect to relative $I_{\mathcal{C}}$-cell complexes and hence also the cofibrations by [20, 14.2.14]. F_kA is small with respect to relative I-cell complexes in $Sp(\mathcal{C})$ by adjointness and F_1S^1 is small with respect to all of Sp^Σ. So by various adjunctions $F_1S^1 \wedge F_kA$ is small with respect to level cofibrations. Since $M\lambda$ is the pushout of small objects, similar arguments show that $F_kA \wedge M\lambda$ is also small with respect to the level cofibrations in $Sp(\mathcal{C})$.

The spectral compatibility axiom is verified in Proposition 3.8.5 in its adjoint form (SPb). Thus, it remains to show that the adjoint functors Σ^∞ and Ev_0 are a Quillen equivalence. The suspension spectrum functor Σ^∞ takes (trivial) cofibrations to (trivial) cofibrations in the level model structure. Hence Σ^∞ also preserves (trivial) cofibrations with respect to the stable model structure. So the adjoint functors Ev_0 and Σ^∞ are a Quillen pair between $\mathcal{C}$ and $Sp(\mathcal{C})$.

To show that the functors are a Quillen equivalence it suffices to show (see [21, Cor. 1.3.16]) that Ev_0 reflects stable equivalences between stably fibrant objects and that for every cofibrant object A of $\mathcal{C}$ the map $A \longrightarrow \mathrm{Ev}_0\, R(\Sigma^\infty A)$ is a weak equivalence where R denotes any stably fibrant replacement in $Sp(\mathcal{C})$. So suppose that $f : X \longrightarrow Y$ is a map between Ω-spectra with the property that $f_0 : X_0 \longrightarrow Y_0$ is a weak equivalence in $\mathcal{C}$. Since X is an Ω-spectrum, $X_0 \longrightarrow \Omega^n X_n$ is a weak equivalence, and similarly for Y. Hence the map $\Omega^n f_n : \Omega^n X_n \longrightarrow \Omega^n Y_n$ is a weak equivalence in $\mathcal{C}$. Since $\mathcal{C}$ is stable, the loop functor is a self-Quillen equivalence, so it reflects weak equivalences between fibrant objects, and so $f_n : X_n \longrightarrow Y_n$ is a weak equivalence in $\mathcal{C}$. Hence f is a level, and thus a stable equivalence of spectra over $\mathcal{C}$.

Since $\mathcal{C}$ is stable the spectrum $\Sigma^\infty_{\mathrm{f}} A$ (the fibrant replacement of the suspension spectrum in the level model structure) is an Ω-spectrum, and thus stably fibrant. Hence we may take $\Sigma^\infty_{\mathrm{f}} A$ as the stably fibrant replacement $R(\Sigma^\infty A)$, which proves that $A \longrightarrow \mathrm{Ev}_0\, R(\Sigma^\infty A)$ is a weak equivalence in $\mathcal{C}$. □

3.9. **The Quillen equivalence.** In this section we prove Theorem 3.3.3, i.e., we show that a suitable model category with a set of compact generators is Quillen equivalent to the modules over the spectral endomorphism category of the generators.

In Theorem 3.9.3 we first formulate the result for spectral model categories; this gives a more general result since the conditions about cofibrant generation and properness in $\mathcal{C}$ are not needed. We then combine this with the fact that every suitable stable model category is Quillen equivalent to a spectral model category to prove our main classification theorem.

Definition 3.9.1. Let $\mathcal{G}$ be a set of objects in a spectral model category $\mathcal{D}$. We denote by $\mathcal{E}(\mathcal{G})$ the full spectral subcategory of $\mathcal{D}$ with objects $\mathcal{G}$, i.e., $\mathcal{E}(\mathcal{G})(G, G') = \mathrm{Hom}_{\mathcal{D}}(G, G')$. We let

$$\mathrm{Hom}(\mathcal{G}, -) \,:\, \mathcal{D} \longrightarrow \text{mod-}\mathcal{E}(\mathcal{G})$$

denote the tautological functor given by $\mathrm{Hom}(\mathcal{G}, Y)(G) = \mathrm{Hom}_{\mathcal{D}}(G, Y)$.

We want to stress the reassuring fact that the stable equivalence type of the spectral endomorphism category $\mathcal{E}(\mathcal{G})$ only depends on the weak equivalence types of the objects in the set $\mathcal{G}$, as long as these are all fibrant and cofibrant, see Corollary A.2.4. This is not completely obvious since taking endomorphisms is not a functor.

The earlier Definition 3.7.5 of the endomorphism ring spectrum and endomorphism category of objects in a simplicial stable model category $\mathcal{C}$ is a special case of Definition 3.9.1 with

$\mathcal{D} = Sp(\mathcal{C})$ and $\mathcal{G}$ the level fibrant replacements of the suspension spectra of the chosen objects in $\mathcal{C}$. Again, if $\mathcal{G} = \{G\}$ has a single element then $\mathcal{E}(\mathcal{G})$ is determined by the single symmetric ring spectrum, $\mathrm{End}_{\mathcal{D}}(G) = \mathrm{Hom}_{\mathcal{D}}(G, G)$.

Definition 3.9.2. Let $\mathcal{C}$ and $\mathcal{D}$ be spectral model categories. A *spectral Quillen pair* is a Quillen adjoint functor pair $L : \mathcal{C} \longrightarrow \mathcal{D}$ and $R : \mathcal{D} \longrightarrow \mathcal{C}$ together with a natural isomorphism of symmetric homomorphism spectra

$$\mathrm{Hom}_{\mathcal{C}}(A, RX) \;\cong\; \mathrm{Hom}_{\mathcal{D}}(LA, X)$$

which on the vertices of the 0-th level reduces to the adjunction isomorphism. A spectral Quillen pair is a *spectral Quillen equivalence* if the underlying Quillen functor pair is an ordinary Quillen equivalence.

In the terminology of [21, Def. 4.2.18] a spectral Quillen pair would be called a 'Sp^{Σ}-Quillen functor'.

Theorem 3.9.3. *Let $\mathcal{D}$ be a spectral model category and $\mathcal{G}$ a set of cofibrant and fibrant objects.*
(i) *The tautological functor*

$$\mathrm{Hom}(\mathcal{G}, -) \;:\; \mathcal{D} \longrightarrow \text{mod-}\mathcal{E}(\mathcal{G})$$

is the right adjoint of a spectral Quillen functor pair. The left adjoint is denoted $- \wedge_{\mathcal{E}(\mathcal{G})} \mathcal{G}$.
(ii) *If all objects in $\mathcal{G}$ are compact, then the total derived functors of* $\mathrm{Hom}(\mathcal{G}, -)$ *and $- \wedge_{\mathcal{E}(\mathcal{G})} \mathcal{G}$ restrict to a triangulated equivalence between the homotopy category of $\mathcal{E}(\mathcal{G})$-modules and the localizing subcategory of* $\mathrm{Ho}(\mathcal{D})$ *generated by $\mathcal{G}$.*
(iii) *If $\mathcal{G}$ is a set of compact generators for $\mathcal{D}$, then the adjoint functor pair* $\mathrm{Hom}(\mathcal{G}, -)$ *and $- \wedge_{\mathcal{E}(\mathcal{G})} \mathcal{G}$ form a spectral Quillen equivalence.*

Proof. (i) For an $\mathcal{E}(\mathcal{G})$-module M the object $M \wedge_{\mathcal{E}(\mathcal{G})} \mathcal{G}$ is given by an enriched coend [32, 3.10]. This means that $M \wedge_{\mathcal{E}(\mathcal{G})} \mathcal{G}$ is the coequalizer of the two maps

$$\bigvee_{G, G' \in \mathcal{G}} M(G') \wedge \mathcal{E}(\mathcal{G})(G, G') \wedge G \;\Longrightarrow\; \bigvee_{G \in \mathcal{G}} M(G) \wedge G \;.$$

One map in the diagram is induced by the evaluation map $\mathcal{E}(\mathcal{G})(G, G') \wedge G \longrightarrow G'$ and the other is induced by the action map $M(G') \wedge \mathcal{E}(\mathcal{G})(G, G') \longrightarrow M(G)$. The tautological functor $\mathrm{Hom}(\mathcal{G}, -)$ preserves fibrations and trivial fibrations by the compatibility axiom (SP) of Definition 3.5.1, since all objects of $\mathcal{G}$ are cofibrant. So together with its left adjoint it forms a spectral Quillen pair.

(ii) Since the functors $\mathrm{Hom}(\mathcal{G}, -)$ and $- \wedge_{\mathcal{E}(\mathcal{G})} \mathcal{G}$ are a Quillen pair, they have adjoint total derived functors on the level of homotopy categories [45, I.4]; we denote these derived functors by $\mathrm{RHom}(\mathcal{G}, -)$ and $- \wedge^L_{\mathcal{E}(\mathcal{G})} \mathcal{G}$ respectively. The functor $- \wedge^L_{\mathcal{E}(\mathcal{G})} \mathcal{G}$ commutes with suspension and preserves cofiber sequences, and the functor $\mathrm{RHom}(\mathcal{G}, -)$ commutes with taking loops and preserves fiber sequences [45, I.4 Prop. 2]. In the homotopy category of a stable model category, the cofiber and fiber sequences coincide up to sign and they constitute the distinguished triangles. So both total derived functors preserve shifts and triangles, i.e., they are exact functors of triangulated categories.

For every $G \in \mathcal{G}$ the $\mathcal{E}(\mathcal{G})$-module $\mathrm{Hom}(\mathcal{G}, G)$ is isomorphic to the free module $F_G = \mathcal{E}(\mathcal{G})(-, G)$ by inspection and $F_G \wedge_{\mathcal{E}(\mathcal{G})} \mathcal{G}$ is isomorphic to G since they represent the same functor on $\mathcal{D}$. As a left adjoint, the functor $- \wedge^L_{\mathcal{E}(\mathcal{G})} \mathcal{G}$ preserves coproducts. We claim that the right adjoint $\mathrm{RHom}(\mathcal{G}, -)$ also preserves coproducts. Since the free modules F_G form a set of

compact generators for the category of $\mathcal{E}(\mathcal{G})$-modules (see Theorem A.1.1), it suffices to show that for all $G \in \mathcal{G}$ and for every family $\{A_i\}_{i\in I}$ of objects of $\mathcal{D}$ the natural map

$$\bigoplus_{i\in I} [F_G, \mathrm{RHom}(\mathcal{G}, A_i)]_*^{\mathrm{Ho}(\text{mod-}\mathcal{E}(\mathcal{G}))} \;\cong\; [F_G, \coprod_{i\in I} \mathrm{RHom}(\mathcal{G}, A_i)]_*^{\mathrm{Ho}(\text{mod-}\mathcal{E}(\mathcal{G}))} \longrightarrow$$
$$[F_G, \mathrm{RHom}(\mathcal{G}, \coprod_{i\in I} A_i)]_*^{\mathrm{Ho}(\text{mod-}\mathcal{E}(\mathcal{G}))}$$

is an isomorphism. By the adjunctions and the identification $F_G \wedge^L_{\mathcal{E}(\mathcal{G})} \mathcal{G} \cong G$ this map is isomorphic to the natural map

$$\bigoplus_{i\in I} [G, A_i]_*^{\mathrm{Ho}(\mathcal{D})} \longrightarrow [G, \coprod_{i\in I} A_i]_*^{\mathrm{Ho}(\mathcal{D})} .$$

But this last map is an isomorphism since G was assumed to be compact.

Both derived functors preserve shifts, triangles and coproducts; since they match up the free $\mathcal{E}(\mathcal{G})$-modules F_G with the objects of $\mathcal{G}$, they restrict to adjoint functors between the localizing subcategories generated by the free modules on the one side and the objects of $\mathcal{G}$ on the other side. We consider the full subcategories of those $M \in \mathrm{Ho}(\text{mod-}\mathcal{E}(\mathcal{G}))$ and $X \in \mathrm{Ho}(\mathcal{D})$ respectively for which the unit of the adjunction

$$\eta \;:\; M \longrightarrow \mathrm{RHom}(\mathcal{G}, M \wedge^L_{\mathcal{E}(\mathcal{G})} \mathcal{G})$$

or the counit of the adjunction

$$\nu \;: \mathrm{RHom}(\mathcal{G}, X) \wedge^L_{\mathcal{E}(\mathcal{G})} X \longrightarrow X$$

are isomorphisms. Since both derived functors are exact and preserve coproducts, these are localizing subcategories. Since $F_G \wedge^L_{\mathcal{E}(\mathcal{G})} \mathcal{G} \cong G$ and $\mathrm{RHom}(\mathcal{G}, G) \cong F_G$, the map η is an isomorphism for every free module, and the map ν is an isomorphism for every object of $\mathcal{G}$. Since the free modules F_G generate the homotopy category of $\mathcal{E}(\mathcal{G})$-modules, the claim follows.

(iii) Now the localizing subcategory generated by $\mathcal{G}$ is the entire homotopy category of $\mathcal{D}$, so part (ii) of the theorem implies that the total derived functors of $\mathrm{Hom}(\mathcal{G}, -)$ and $- \wedge_{\mathcal{E}(\mathcal{G})} \mathcal{G}$ are inverse equivalences of homotopy categories. Hence this pair is a Quillen equivalence. □

Now we can finally give the

Proof of Theorem 3.3.3: We can combine Theorem 3.8.2 and Theorem 3.9.3 (iii) to obtain a diagram of model categories and Quillen equivalences

$$\mathcal{C} \quad \underset{\mathrm{Ev}_0}{\overset{\Sigma^\infty}{\rightleftarrows}} \quad Sp(\mathcal{C}) \quad \underset{\mathrm{Hom}(\mathcal{G},-)}{\overset{-\wedge_{\mathcal{E}(\mathcal{G})}\mathcal{G}}{\rightleftarrows}} \quad \text{mod-}\mathcal{E}(\mathcal{G})$$

(the left adjoints are on top). First, we may assume that each object in the set $\mathcal{P}$ of compact generators for $\mathcal{C}$ is cofibrant. Since the left Quillen functor pair above induces an equivalence of homotopy categories, the suspension spectra of the objects in $\mathcal{P}$ form a set of compact cofibrant generators for $Sp(\mathcal{C})$.

We denote by $\mathcal{G}$ the set of level fibrant replacements $\Sigma^\infty_{\mathrm{f}} P$ of the given generators in $\mathcal{C}$. The spectral endomorphism category $\mathcal{E}(\mathcal{G})$ in the sense of Definition 3.9.1 is equal to the endomorphism category $\mathcal{E}(\mathcal{P})$ associated to $\mathcal{P}$ by Definition 3.7.5. Since $\mathcal{C}$ is stable, the spectra $\Sigma^\infty_{\mathrm{f}} P$ are Ω-spectra, hence they are both fibrant (by Lemma 3.8.7) and cofibrant in the stable model structure on $Sp(\mathcal{C})$. So we can apply Theorem 3.9.3 (iii) to get the second Quillen equivalence. □

Remark 3.9.4. Finite localization and $\mathcal{E}(\mathcal{P})$-modules. Suppose $\mathcal{P}$ is a set of compact objects of a triangulated category $\mathcal{T}$ with infinite coproducts. Then there always exists an idempotent localization functor $L_{\mathcal{P}}$ on $\mathcal{T}$ whose acyclics are precisely the objects of the localizing subcategory generated by $\mathcal{P}$ (compare [41] or the proofs of Lemma 2.2.1 or [24, Prop. 2.3.17]). These localizations are often referred to as *finite Bousfield localizations* away from $\mathcal{P}$.

Theorem 3.9.3 gives a lift of finite localization to the model category level. Suppose $\mathcal{C}$ is a stable model category with a set $\mathcal{P}$ of compact objects, and let $L_{\mathcal{P}}$ denote the associated localization functor on the homotopy category of $\mathcal{C}$. By Theorem 5.3.2 (ii) the acyclics for $L_{\mathcal{P}}$ are equivalent to the homotopy category of $\mathcal{E}(\mathcal{P})$-modules, the equivalence arising from a Quillen adjoint functor pair. Furthermore the counit of the derived adjunction

$$\mathrm{Hom}(\mathcal{P}, X) \wedge^{L}_{\mathcal{E}(\mathcal{P})} \mathcal{P} \longrightarrow X$$

is the acyclicization map and its cofiber is a model for the localization $L_{\mathcal{P}}X$.

4. Morita context

In the classical algebraic context there is a characterization of equivalences of module categories in terms of bimodules, see for example [1, §22]. We provide an analogous result for module categories over ring spectra. As usual, here instead of actual equivalences of module categories one obtains Quillen equivalences of model categories. We state the Morita context for symmetric ring spectra and spectral Quillen equivalences (see Definition 3.9.2).

Definition 4.1.1. If R is a symmetric ring spectrum and $\mathcal{C}$ a spectral model category, then an R-$\mathcal{C}$-bimodule is an object X of $\mathcal{C}$ on which R acts through $\mathcal{C}$-morphisms, i.e., a homomorphism of symmetric ring spectra from R to the endomorphism ring spectrum of X.

If $\mathcal{C}$ is the category of modules over another symmetric ring spectrum T, then this notion of bimodule specializes to the usual one, i.e., an R-(T-mod)-bimodule is the same as a right $R^{\mathrm{op}} \wedge T$-module. In the following Morita theorem, the implication (3) $\Longrightarrow$ (2) is a special case of the classification of monogenic stable model categories (Theorem 3.1.1); hence the implication (2) $\Longrightarrow$ (3) is a partial converse to that classification result. The implication (2) $\Longrightarrow$ (1) says that certain chains of Quillen equivalences can be rectified into a single Quillen equivalence whose left adjoint is given by smashing with a bimodule.

Theorem 4.1.2. (Morita context) *Consider the following statements for a symmetric ring spectrum R and a spectral model category $\mathcal{C}$.*

1. *There exists an R-$\mathcal{C}$-bimodule M such that smashing with M over R is the left adjoint of a Quillen equivalence between the category of R-modules and $\mathcal{C}$.*
2. *There exists a chain of spectral Quillen equivalences through spectral model categories between the category of R-modules and $\mathcal{C}$.*
3. *The category $\mathcal{C}$ has a compact, cofibrant and fibrant generator M such that R is stably equivalent to the endomorphism ring spectrum of M.*

Then conditions (2) *and* (3) *are equivalent, and condition* (1) *implies both conditions* (2) *and* (3)*. If R is cofibrant as a symmetric ring spectrum, then all three conditions are equivalent.*

Furthermore, if $\mathcal{C}$ is the category of modules over another symmetric ring spectrum T which is cofibrant as a symmetric spectrum, then condition (1) *is equivalent to the condition*

(4) *There exists an R-T-bimodule M and a T-R-bimodule N which are cofibrant as right modules such that*

- *$M \wedge_T N$ is stably equivalent to R as an R-bimodule and*
- *$N \wedge_R M$ is stably equivalent to T as a T-bimodule.*

Remark 4.1.3. The cofibrancy conditions in the Morita theorem can always be arranged since every symmetric ring spectrum has a stably equivalent cofibrant replacement in the model category of symmetric ring spectra [25, 5.4.3]; furthermore the underlying symmetric spectrum of a cofibrant ring spectrum is always cofibrant ([56, 4.1] or [25, 5.4.3]). It should not be surprising that cofibrancy conditions have to be imposed in the Morita theorem. In the algebraic context the analogous conditions show up in Rickard's paper [48]: when trying to realize derived equivalences between k-algebras by derived tensor product with bimodule complexes, he has to assume that the algebras are flat over the ground ring k, see [48, Sec. 3].

Proof of the Morita theorem. Condition (1) is a special case of (2).

(2) $\Longrightarrow$ (3): This implication follows from the homotopy invariance of endomorphism ring spectra under spectral Quillen equivalences, see Corollary A.2.4. We choose a chain of spectral Quillen equivalences through spectral model categories. Then we choose a trivial cofibration $R \longrightarrow R^{\mathrm{f}}$ of R-modules such that R^{f} is fibrant; then R is stably equivalent to the endomorphism ring spectrum of R^{f}. We define an object M of $\mathcal{C}$ by iteratively applying the functors in the chain of Quillen equivalences, starting with R^{f}. In addition we take a fibrant or cofibrant replacement after each application depending on whether we use a left or right Quillen functor. By a repeated application of Corollary A.2.4 the endomorphism ring spectra of these objects, including the final one, M, are all stably equivalent to R. By construction the object M is isomorphic in the homotopy category of $\mathcal{C}$ to the image of the free R-module of rank one under the equivalence of homotopy categories induced by the Quillen equivalences. Hence M is also a compact generator for $\mathcal{C}$ and satisfies condition (3).

(3) $\Longrightarrow$ (2): This is essentially a special case of Theorem 3.9.3 (iii). More precisely, that theorem constructs a spectral Quillen equivalence between $\mathcal{C}$ and the category of modules over the endomorphism ring spectrum of the generator M. Furthermore, restriction and extension of scalars are spectral Quillen equivalences for two stably equivalent ring spectra [25, Thm. 5.4.5], see also [38, Thm. 11.1] or Theorem A.1.1, which establishes condition (2).

(3) $\Longrightarrow$ (1), provided R is cofibrant as a symmetric ring spectrum: Since M is fibrant and cofibrant the endomorphism ring spectrum $\mathrm{End}_{\mathcal{C}}(M)$ is fibrant. Since R is cofibrant as a symmetric ring spectrum any isomorphism between R and $\mathrm{End}_{\mathcal{C}}(M)$ in the homotopy category of symmetric ring spectra can be lifted to a stable equivalence $\eta : R \longrightarrow \mathrm{End}_{\mathcal{C}}(M)$. In particular the stable equivalence η makes M into an R-$\mathcal{C}$-bimodule. The functor $X \mapsto X \wedge_R M$ is left adjoint to the functor $Y \mapsto \mathrm{Hom}_{\mathcal{C}}(M, Y)$ from $\mathcal{C}$ to the category of R-modules. To show that these adjoint functors form a Quillen equivalence we note that they factor as the composite of two adjoint functor pairs

$$\mathcal{C} \quad \underset{\mathrm{Hom}_{\mathcal{C}}(M,-)}{\overset{-\wedge_{\mathrm{End}_{\mathcal{C}}(M)} M}{\rightleftarrows}} \quad \text{mod-}\,\mathrm{End}_{\mathcal{C}}(M) \quad \underset{\eta^*}{\overset{\eta_*}{\rightleftarrows}} \quad \text{mod-}R$$

(the left adjoints are on top). Since M is a cofibrant and fibrant compact generator for $\mathcal{C}$, the left pair is a Quillen equivalence by Theorem 3.9.3 (iii). The other adjoint functor pair is restriction and extension of scalars along the stable equivalence of ring spectra $\eta : R \longrightarrow \mathrm{End}_{\mathcal{C}}(M)$, which is a Quillen equivalence by [25, Thm. 5.4.5] or [38, Thm. 11.1].

For the rest of the proof we assume that $\mathcal{C}$ is the category of modules over a symmetric ring spectrum T which is cofibrant as a symmetric spectrum.

(1) $\Longrightarrow$ (4): Since smashing with M over R is a left Quillen functor and since the free R-module of rank one is cofibrant, $M \cong R \wedge_R M$ is cofibrant as a right T-module. We choose a fibrant replacement $T \longrightarrow T^{\mathrm{f}}$ of T in the category of T-bimodules and we let N be a cofibrant replacement of the T-R-bimodule $\mathrm{Hom}_T(M, T^{\mathrm{f}})$. The forgetful functor from T-R-bimodules to

right R-modules preserves cofibrations since its right adjoint

$$\mathrm{Hom}_{Sp^\Sigma}(T,-) : \text{mod-}R \longrightarrow T\text{-mod-}R$$

preserves trivial fibrations (because T is cofibrant as a symmetric spectrum). In particular the bimodule N is cofibrant as a right R-module. We will exhibit two chains of stable equivalences of bimodules

$$N \wedge_R M \xrightarrow{\sim} T^{\mathrm{f}} \xleftarrow{\sim} T \quad \text{and} \quad M \wedge_T N \xrightarrow{\sim} \mathrm{Hom}_T(M, (M \wedge_T T^{\mathrm{f}})^{\mathrm{f}}) \xleftarrow{\sim} R$$

where $M \wedge_T T^{\mathrm{f}} \longrightarrow (M \wedge_T T^{\mathrm{f}})^{\mathrm{f}}$ is a fibrant approximation in the category of R-T-bimodules. This will establish condition (4).

Since $- \wedge_R M$ was assumed to be a left Quillen equivalence and the approximation map $N \longrightarrow \mathrm{Hom}_T(M, T^{\mathrm{f}})$ is a weak equivalence, so is its adjoint $N \wedge_R M \longrightarrow T^{\mathrm{f}}$; but this adjoint is even a map of T-bimodules. The equivalence $T \longrightarrow T^{\mathrm{f}}$ was chosen in the beginning. For the next equivalence we smash the T-bimodule equivalence $N \wedge_R M \longrightarrow T^{\mathrm{f}}$ with the right-cofibrant bimodule M to get a stable equivalence of R-T-bimodules

$$M \wedge_T N \wedge_R M \xrightarrow{\sim} M \wedge_T T^{\mathrm{f}} .$$

We then compose with the approximation map and obtain a stable equivalence of R-T-bimodules $M \wedge_T N \wedge_R M \longrightarrow (M \wedge_T T^{\mathrm{f}})^{\mathrm{f}}$. Since M and N are cofibrant as right modules, the R-bimodule $M \wedge_T N$ is cofibrant as a right R-module. Since $- \wedge_R M$ is a left Quillen equivalence, the adjoint $M \wedge_T N \longrightarrow \mathrm{Hom}_T(M, (M \wedge_T T^{\mathrm{f}})^{\mathrm{f}})$ is thus a stable equivalence of R-bimodules. For the same reason the adjoint of the composite stable equivalence

$$R \wedge_R M \;\cong\; M \wedge_T T \xrightarrow{\sim} M \wedge_T T^{\mathrm{f}} \xrightarrow{\sim} (M \wedge_T T^{\mathrm{f}})^{\mathrm{f}}$$

gives the final stable equivalence of R-bimodules $R \longrightarrow \mathrm{Hom}_T(M, (M \wedge_T T^{\mathrm{f}})^{\mathrm{f}})$.

(4) $\Longrightarrow$ (1): Let M and N be bimodules which satisfy the conditions of (4). The functor $X \mapsto X \wedge_R M$ is left adjoint to the functor $Y \mapsto \mathrm{Hom}_T(M, Y)$ from the category of T-modules to the category of R-modules. Since M is cofibrant as a right T-module, this right adjoint preserves fibrations and trivial fibrations by the spectral axiom (SP). So $- \wedge_R M$ and $\mathrm{Hom}_T(M, -)$ form a Quillen functor pair. By condition (4) the left derived functor of $- \wedge_R M$ is an equivalence of derived categories (with inverse the left derived functor of smashing with the bimodule N). So the functor $- \wedge_R M$ is indeed the left adjoint of a Quillen equivalence. □

5. A generalized tilting theorem

In this section we state and prove a generalization of Rickard's "Morita theory for derived categories", [47]. Rickard studies the question of when two rings are derived equivalent, i.e., when there exists a triangulated equivalence between various derived categories of the module categories. He shows [47, Thm. 6.4] that a necessary and sufficient condition for such a derived equivalence is the existence of a tilting complex. A tilting complex for a pair of rings Γ and Λ is a bounded complex X of finitely generated projective Γ-modules which generates the derived category and whose graded ring of self extension groups $[X, X]_*^{\mathcal{D}(\Gamma)}$ is isomorphic to Λ, concentrated in dimension zero.

We generalize the result in two directions. First, we allow the input to be a stable model category (which generalizes categories of chain complexes of modules). Second, we allow for a set of special generators, as opposed to a single tilting complex. The compact objects in the unbounded derived category of a ring are precisely the perfect complexes, i.e., those chain complexes which are quasi-isomorphic to a bounded complex of finitely generated projective modules [7, Prop. 6.4]. In our context we thus define a set of *tiltors* in a stable model category to be a set $\mathbb{T}$ of compact generators such that for any two objects $T, T' \in \mathbb{T}$ the graded homomorphism

group $[T,T']_*^{\mathrm{Ho}(\mathcal{C})}$ in the homotopy category is concentrated in dimension zero. Then Theorem 5.1.1 shows that the existence of a set of tiltors is necessary and sufficient for a stable model category to be Quillen equivalent or derived equivalent to the category of chain complexes over a *ringoid* (ring with several objects). Recall that a ringoid is a small category whose hom-sets carry an abelian group structure for which composition is bilinear. A *module* over a ringoid is a contravariant additive functor to the category of abelian groups.

Of course an interesting special case is that of a stable model category with a single tiltor, i.e., a single compact generator whose graded endomorphism ring in the homotopy category is concentrated in dimension zero. Then ringoids simplify to rings. In particular when $\mathcal{C}$ is the model category of chain complexes of Γ-modules for some ring Γ, then a single tiltor is the same (up to quasi-isomorphism) as a tilting complex, and the equivalence of conditions (2') and (3) below recovers Rickard's 'Morita theory for derived categories' [47, Thm. 6.4].

Condition (1) in the tilting theorem refers to a standard model structure on the category of chain complexes of $\underline{A}$-modules. The model structure we have in mind is the *projective* model structure: the weak equivalences are the quasi-isomorphisms and the fibrations are the epimorphisms. Every cofibration in this model structure is in particular a monomorphism with degreewise projective cokernel, but for unbounded complexes this condition is not sufficient to characterize a cofibration. In the single object case, i.e., for modules over a ring, the projective model structure on complexes is established in [21, Thm. 2.3.11]. For modules over a ringoid the arguments are very similar, just that the free module of rank one has to be replaced by the set of free (or representable) modules $F_a = \underline{A}(-,a)$ for $a \in \underline{A}$. This model structure can be established using a version of Theorem A.1.1 where the enrichment over Sp^Σ is replaced by one over chain complexes. The projective model structure for complexes of $\underline{A}$-modules is also a special case of [8, Thm. 5.1]. Indeed, the projective (in the usual sense) $\underline{A}$-modules together with the epimorphisms form a projective class (in the sense of [8, Def. 1.1]), and this class is determined (in the sense of [8, Sec. 5.2]) by the set of small, free modules $\{F_a\}_{a\in\underline{A}}$.

Theorem 5.1.1. (Tilting theorem) *Let $\mathcal{C}$ be a simplicial, cofibrantly generated, proper, stable model category and $\underline{A}$ a ringoid. Then the following conditions are equivalent:*

(1) *There is a chain of Quillen equivalences between $\mathcal{C}$ and the model category of chain complexes of $\underline{A}$-modules.*

(2) *The homotopy category of $\mathcal{C}$ is triangulated equivalent to $\mathcal{D}(\underline{A})$, the unbounded derived category of the ringoid $\underline{A}$.*

(2') *$\mathcal{C}$ has a set of compact generators and the full subcategory of compact objects in* $\mathrm{Ho}(\mathcal{C})$ *is triangulated equivalent to $K^b(\text{proj-}\underline{A})$, the homotopy category of bounded chain complexes of finitely generated projective $\underline{A}$-modules.*

(3) *The model category $\mathcal{C}$ has a set of tiltors whose endomorphism ringoid in the homotopy category of $\mathcal{C}$ is isomorphic to $\underline{A}$.*

Example 5.1.2. Let G be a finite group. As in Example 3.4 (i) the category of G-equivariant orthogonal spectra [37] based on a complete universe $\mathcal{U}$ is a simplicial, stable model category, and the equivariant suspension spectra of the homogeneous spaces G/H^+ form a set of compact generators as H runs through the subgroups of G. Rationalization is a smashing Bousfield localization so the rationalized suspension spectra form a set of compact generators of the rational G-equivariant stable homotopy category. The homotopy groups of the function spectra between the various generators are torsion in dimensions different from zero [18, Prop. A.3], so the rationalized suspension spectra form a set of tiltors. Modules over the associated ringoid are nothing but rational Mackey functors, so the tilting theorem 5.1.1 shows that the rational G-equivariant stable homotopy category is equivalent to the derived category of rational Mackey

functors. In turn, since these rational Mackey functors are all projective and injective, the derived category is equivalent to the graded category. So this recovers the Theorem A.1 in [18].

For a non-complete universe, one considers rational $\mathcal{U}$-Mackey functors [33] which are modules over the endomorphism ringoid of the rationalized suspension spectra of G/H^+. For example, for the trivial universe $\mathcal{U}$, these rational $\mathcal{U}$-Mackey functors are rational coefficient systems. The rational G-equivariant stable homotopy category based on a non-complete universe $\mathcal{U}$ is then equivalent to the derived category of the associated rational $\mathcal{U}$-Mackey functors.

Example 5.1.3. Let A be a ring and consider the *pure projective* model category structure in the sense of Christensen and Hovey [8, 5.3] on the category of chain complexes of A-modules (see also Example 2.3 (xiii)). A map $X \longrightarrow Y$ of complexes is a weak equivalence if and only if for every finitely generated A-module M the induced map of mapping complexes $\mathrm{Hom}_A(M, X) \longrightarrow \mathrm{Hom}_A(M, Y)$ is a quasi-isomorphism. Let $\mathcal{G}$ be a set of representatives of the isomorphism classes of indecomposable finitely generated A-modules. Then $\mathcal{G}$ forms a set of compact generators for the pure derived category $\mathcal{D}_{\mathcal{P}}(A)$. Since furthermore every finitely generated module is pure projective, maps in the pure derived category between modules in $\mathcal{G}$ are concentrated in dimension zero. In other words, the indecomposable finitely generated A-modules form a set of tiltors. So Theorem 5.1.1 implies that the pure projective model category of A is Quillen equivalent to the modules over the ringoid given by the full subcategory of A-modules with objects $\mathcal{G}$.

Remark 5.1.4. We want to emphasize one special feature of the tilting situation. For general stable model categories the notion of Quillen equivalence is considerably stronger than triangulated equivalence of homotopy categories (see Remark 3.2.1 for an example). Hence it is somewhat remarkable that for chain complexes of modules over ringoids the two notions are in fact equivalent. In general the homotopy category determines the homotopy groups of the spectral endomorphism category, but not its homotopy type. The real reason behind the equivalences of conditions (1) and (2) above is the fact that in contrast to arbitrary ring spectra or spectral categories, Eilenberg-Mac Lane objects are determined by their homotopy groups, see Proposition B.2.1.

As a tool for proving the generalized tilting theorem we introduce the *Eilenberg-Mac Lane spectral category* $H\underline{A}$ of a ringoid $\underline{A}$. This is simply the many-generator version of the symmetric Eilenberg-Mac Lane ring spectrum [25, 1.2.5]. The key property is that module spectra over the Eilenberg-Mac Lane spectral category $H\underline{A}$ are Quillen equivalent to chain complexes of $\underline{A}$-modules.

Definition 5.1.5. Let $\underline{A}$ be a ringoid. The Eilenberg-Mac Lane spectral category $H\underline{A}$ is defined by

$$H\underline{A} \;=\; \underline{A} \otimes H\mathbb{Z}\;,$$

where $H\mathbb{Z}$ is the symmetric Eilenberg-Mac Lane ring spectrum of the integers [25, 1.2.5]. In more detail, $H\underline{A}$ has the same set of objects as $\underline{A}$, and the morphism spectra are defined by

$$H\underline{A}(a,b)_p \;=\; \underline{A}(a,b) \otimes \widetilde{\mathbb{Z}}[S^p]\;.$$

Here $\widetilde{\mathbb{Z}}[S^p]$ denotes the reduced simplicial free abelian group generated by the pointed simplicial set $S^p = S^1 \wedge \ldots \wedge S^1$ (p factors), and the symmetric group permutes the factors. Composition is given by the composite

$$\begin{aligned} H\underline{A}(b,c)_p \wedge H\underline{A}(a,b)_q \quad &= \quad (\underline{A}(b,c) \otimes \widetilde{\mathbb{Z}}[S^p]) \wedge (\underline{A}(a,b) \otimes \widetilde{\mathbb{Z}}[S^q]) \\ &\xrightarrow{\text{shuffle}} \quad \underline{A}(b,c) \otimes \underline{A}(a,b) \otimes \widetilde{\mathbb{Z}}[S^p] \otimes \widetilde{\mathbb{Z}}[S^q] \\ &\xrightarrow{\circ \otimes \cong} \quad \underline{A}(a,c) \otimes \widetilde{\mathbb{Z}}[S^{p+q}] \;=\; H\underline{A}(a,c)_{p+q}\;. \end{aligned}$$

The unit map

$$S^p \longrightarrow \underline{A}(a,a) \otimes \widetilde{\mathbb{Z}}[S^p] \;=\; H\underline{A}(a,a)_p$$

is the inclusion of generators.

We prove the following result in Appendix B.

Theorem 5.1.6. *For any ringoid $\underline{A}$, the category of complexes of $\underline{A}$-modules and the category of modules over the Eilenberg-Mac Lane spectral category $H\underline{A}$ are Quillen equivalent,*

$$\text{mod-}H\underline{A} \simeq_Q \text{Ch}_{\underline{A}}$$

Proof of Theorem 5.1.1. Every Quillen equivalence of stable model categories induces an equivalence of triangulated homotopy categories, so condition (1) implies condition (2). Any triangulated equivalence restricts to an equivalence between the respective subcategories of compact objects. By the same argument as [7, Prop. 6.4] (which deals with the special case of complexes of modules over a ring), $K^b(\text{proj-}\underline{A})$ is equivalent to the full subcategory of compact objects in $\mathcal{D}(\underline{A})$. Since the derived category of a ringoid has a set of compact generators, so does any equivalent triangulated category. Hence condition (2) implies condition (2').

Now we assume condition (2') and we choose a triangulated equivalence between $K^b(\text{proj-}\underline{A})$ and the full subcategory of compact objects in $\text{Ho}(\mathcal{C})$. For $a \in \underline{A}$ we let T_a be a representative in $\mathcal{C}$ of the image of the representable $\underline{A}$-module $F_a = \underline{A}(-,a)$, viewed as a complex concentrated in dimension zero. Since the collection of modules $\{F_a\}_{a\in\underline{A}}$ is a set of tiltors for the derived category, the set $\{T_a\}_{a\in\underline{A}}$ has all the properties of a set of tiltors, except that it may not generate the full homotopy category of $\mathcal{C}$. However the localizing subcategory generated by the T_a's coincides with the localizing subcategory generated by all compact objects since on the other side of the equivalence the complexes F_a generate the category $K^b(\text{proj-}\underline{A})$. In general the compact objects might not generate all of $\text{Ho}(\mathcal{C})$ (see [26, Cor. B.13] for some extreme cases where the zero object is the only compact object), but here this is assumed in (2'). So the T_a's generate $\mathcal{C}$, hence they are a set of tiltors, and so condition (3) holds.

If on the other hand $\mathcal{C}$ has a set of tiltors $\mathbb{T}$, then $\mathbb{T}$ is in particular a set of compact generators, so by Theorem 3.1.1, $\mathcal{C}$ is Quillen equivalent to the category of modules over the endomorphism category $\text{End}(\mathbb{T})$. In this special case the homotopy type of the spectral category $\text{End}(\mathbb{T})$ is determined by its homotopy groups: since the homotopy groups of $\text{End}(\mathbb{T})$ are concentrated in dimension 0, $\text{End}(\mathbb{T})$ is stably equivalent to $H\underline{A}$, the Eilenberg-Mac Lane spectral category of its component ringoid $\underline{A}$ by Proposition B.2.1. Thus the categories of $\text{End}(\mathbb{T})$-modules and $H\underline{A}$-modules are Quillen equivalent by Theorem A.1.1. Theorem 5.1.6 gives the final step in the chain of Quillen-equivalences

$$\mathcal{C} \quad \simeq_Q \quad \text{mod-}\,\text{End}(\mathbb{T}) \quad \simeq_Q \quad \text{mod-}H\underline{A} \quad \simeq_Q \quad \text{Ch}\underline{A}\,.$$

□

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Appendix A. Spectral categories

In this appendix we develop some general theory of modules over spectral categories (Definition 3.3.1). The arguments are very similar to the case of spectral categories with one object, i.e., symmetric ring spectra.

A.1. **Model structures for modules over spectral categories.** A *morphism* $\Psi : \mathcal{O} \longrightarrow \mathcal{R}$ of spectral categories is simply a spectral functor. The *restriction of scalars*

$$\Psi^* \ : \ \text{mod-}\mathcal{R} \longrightarrow \text{mod-}\mathcal{O} \quad , \quad M \ \longmapsto \ M \circ \Psi$$

has a left adjoint functor Ψ_*, also denoted $- \wedge_{\mathcal{O}} \mathcal{R}$, which we refer to as *extension of scalars*. As usual it is given by an enriched coend, i.e., for an $\mathcal{O}$-module N the $\mathcal{R}$-module $\Psi_* N = N \wedge_{\mathcal{O}} \mathcal{R}$ is given by the coequalizer of the two $\mathcal{R}$-module homomorphisms

$$\bigvee_{o,p \in \mathcal{O}} N(p) \wedge \mathcal{O}(o,p) \wedge F_{\Psi(o)} \rightrightarrows \bigvee_{o \in \mathcal{O}} N(o) \wedge F_{\Psi(o)} \ ,$$

where $F_{\Psi(o)} = \mathcal{R}(-, \Psi(o))$ is the free $\mathcal{R}$-module associated to the object $\Psi(o)$. We call $\Psi : \mathcal{O} \longrightarrow \mathcal{R}$ a *stable equivalence* of spectral categories if it is a bijection on objects and if for all objects o, o' in $\mathcal{O}$ the map

$$\Psi_{o,o'} \ : \ \mathcal{O}(o,o') \longrightarrow \mathcal{R}(\Psi(o), \Psi(o'))$$

is a stable equivalence of symmetric spectra.

Next we establish the model category structure for $\mathcal{O}$-modules, show its invariance under restriction of scalars along a stable equivalence of spectral categories and exhibit a set of compact generators.

Theorem A.1.1. (i) *Let $\mathcal{O}$ be a spectral category. Then the category of $\mathcal{O}$-modules with the objectwise stable equivalences, objectwise stable fibrations, and cofibrations is a cofibrantly generated spectral model category.*

(ii) *The free modules $\{F_o\}_{o \in \mathcal{O}}$ given by $F_o = \mathcal{O}(-, o)$ form a set of compact generators for the homotopy category of $\mathcal{O}$-modules.*

(iii) *Assume $\Psi : \mathcal{O} \longrightarrow \mathcal{R}$ is a stable equivalence of spectral categories. Then restriction and extension of scalars along Ψ form a spectral Quillen equivalence of the module categories.*

Proof. We use [56, 2.3] to lift the stable model structure from (families of) symmetric spectra to $\mathcal{O}$-modules. Let $S_{\mathcal{O}}$ denote the spectral category with the same set of objects as $\mathcal{O}$, but with morphism spectra given by

$$S_{\mathcal{O}}(o,o') \ = \ \begin{cases} S & \text{if } o = o', \\ * & \text{else.} \end{cases}$$

An $S_{\mathcal{O}}$-module is just a family of symmetric spectra indexed by the objects of $\mathcal{O}$. Hence it has a cofibrantly generated model category structure in which the cofibrations, fibrations and weak equivalences are defined objectwise on underlying symmetric spectra. Here the generating trivial cofibrations are maps between modules concentrated at one object, i.e. of the form A_o with $A_o(o) = A$ and $A_o(o') = *$ if $o \neq o'$.

The unit maps give a morphism of spectral categories $S_{\mathcal{O}} \longrightarrow \mathcal{O}$, which in turn induces adjoint functors of restriction and extension of scalars between the module categories. This produces a triple $- \wedge_{S_{\mathcal{O}}} \mathcal{O}$ on $S_{\mathcal{O}}$-modules with the algebras over this triple the $\mathcal{O}$-modules. Then the generating trivial cofibrations for $\mathcal{O}$-modules are maps between modules of the form $A_o \wedge_{S_{\mathcal{O}}} \mathcal{O} = A \wedge \mathcal{O}(-, o) = A \wedge F_o$. Hence the monoid axiom [25, 5.4.1] applies to show that the new generating trivial cofibrations and their relative cell morphisms are weak equivalences. Thus, since all symmetric spectra, hence all $S_{\mathcal{O}}$-modules are small, the model category structure follows by criterion (1) of [56, 2.3]. We omit the verification of the spectral axiom (SP), which then implies stability by Lemma 3.5.2.

The proof of (ii) uses the adjunction defined above between $S_{\mathcal{O}}$-modules and $\mathcal{O}$-modules. Since $F_o = S_o \wedge_{S_{\mathcal{O}}} \mathcal{O}$,

$$[F_o, M]_*^{\text{Ho}(\text{mod-}\mathcal{O})} \cong [S_o, M]_*^{\text{Ho}(\text{mod-}S_{\mathcal{O}})} \cong [S, M(o)]_*^{\text{Ho}(Sp^\Sigma)}.$$

Thus, since S is a generator for Sp^Σ and an $\mathcal{O}$-module is trivial if and only if it is objectwise trivial, the set of free $\mathcal{O}$-modules is a set of generators. The argument that F_o is compact is similar because the map

$$\bigoplus_{i\in I}[F_o, M_i]^{\mathrm{Ho}(\text{mod-}\mathcal{O})} \longrightarrow [F_o, \coprod_{i\in I} M_i]^{\mathrm{Ho}(\text{mod-}\mathcal{O})}$$

is isomorphic to the map

$$\bigoplus_{i\in I}[S, M_i(o)]^{\mathrm{Ho}(Sp^\Sigma)} \longrightarrow [S, \coprod_{i\in I} M_i(o)]^{\mathrm{Ho}(Sp^\Sigma)}$$

and S is compact.

The proof of (iii) follows as in [56, 4.3]. The restriction functor Ψ^* preserves objectwise fibrations and objectwise equivalences, so restriction and extension of scalars form a Quillen adjoint pair. For every cofibrant right $\mathcal{O}$-module N, the induced map

$$N \;\cong\; N\wedge_{\mathcal{O}}\mathcal{O} \;\longrightarrow\; N\wedge_{\mathcal{O}}\mathcal{R}$$

is an objectwise stable equivalence, by a similar 'cell induction' argument as for ring spectra [25, 5.4.4] or [38, 12.7]. Thus if M is any right $\mathcal{R}$-module, an $\mathcal{O}$-module map $N \longrightarrow \Psi^*M$ is an objectwise stable equivalence if and only if the adjoint $\mathcal{R}$-module map $\Psi_*N = N\wedge_{\mathcal{O}}\mathcal{R} \longrightarrow M$ is an objectwise stable equivalence. □

A.2. **Quasi-equivalences.** This section introduces *quasi-equivalences* which are a bookkeeping device for producing stable equivalences between symmetric ring spectra or spectral categories, see Lemma A.2.3 below. The name is taken from [31, Summary, p. 64], where this notion is discussed in the context of differential graded algebras. Every (stable) equivalence of ring spectra gives rise to a quasi-equivalence; conversely the proof of Lemma A.2.3 shows that a single quasi-equivalence encodes a zig-zag of four stable equivalences relating two ring spectra or spectral categories. One place where quasi-equivalences arise 'in nature' is the proof that weakly equivalent objects in a model category have weakly equivalent endomorphism monoids, see Corollary A.2.4.

If $\mathcal{R}$ and $\mathcal{O}$ are spectral categories, their *smash product* $\mathcal{R}\wedge\mathcal{O}$ is the spectral category whose set of objects is the cartesian product of the objects of $\mathcal{R}$ and $\mathcal{O}$ and whose morphism objects are defined by the rule

$$\mathcal{R}\wedge\mathcal{O}((r,o),(r',o')) \;=\; \mathcal{R}(r,r')\wedge\mathcal{O}(o,o')\ .$$

An *$\mathcal{R}$-$\mathcal{O}$-bimodule* is by definition an $\mathcal{R}^{\mathrm{op}}\wedge\mathcal{O}$-module. Since modules for us are always contravariant functors, an $\mathcal{R}$-$\mathcal{O}$-bimodule translates to a *covariant* spectral functor from $\mathcal{O}^{\mathrm{op}}\wedge\mathcal{R}$ to Sp^Σ.

Definition A.2.1. Let $\mathcal{R}$ and $\mathcal{O}$ be two spectral categories *with the same set I of objects*. Then a *quasi-equivalence* between $\mathcal{R}$ and $\mathcal{O}$ is an $\mathcal{R}$-$\mathcal{O}$-bimodule M together with a collection of 'elements' $\varphi_i \in M(i,i)$ (i.e., morphisms $S \longrightarrow M(i,i)$) for all $i\in I$ such that the following holds: for all pairs i and j of objects the right multiplication with φ_i and the left multiplication with φ_j,

$$\mathcal{R}(i,j) \;\xrightarrow{\cdot\varphi_i}\; M(i,j) \;\xleftarrow{\varphi_j\cdot}\; \mathcal{O}(i,j)$$

are stable equivalences.

Remark A.2.2. In the important special case of spectral categories with a single object, i.e., for two symmetric ring spectra R and T, a quasi-equivalence is an R-T-bimodule M together

with an element $\varphi \in M$ (i.e., a vertex of the 0-th level of M or equivalently a map $S \longrightarrow M$ of symmetric spectra) such that the left and right multiplication maps with φ,

$$R \xrightarrow{\cdot\varphi} M \xleftarrow{\varphi\cdot} T$$

are stable equivalences of symmetric spectra.

If $\Psi : \mathcal{O} \longrightarrow \mathcal{R}$ is a stable equivalence of spectral categories, then the target $\mathcal{R}$ becomes an $\mathcal{R}$-$\mathcal{O}$-bimodule if $\mathcal{O}$ acts on the right via Ψ. Furthermore the identity elements in $\mathcal{R}(i,i)$ for all objects i of $\mathcal{R}$ make the bimodule $\mathcal{R}$ into a quasi-equivalence between $\mathcal{R}$ and $\mathcal{O}$. The following lemma shows conversely that quasi-equivalent spectral categories are related by a chain of weak equivalences:

Lemma A.2.3. *Let $\mathcal{R}$ and $\mathcal{O}$ be two spectral categories with the same set of objects. If a quasi-equivalence exists between $\mathcal{R}$ and $\mathcal{O}$, then there is a chain of stable equivalences between $\mathcal{R}$ and $\mathcal{O}$.*

Proof. i) Special case: suppose there exists a quasi-equivalence $(M, \{\varphi_i\}_{i\in I})$ for which all of the right multiplication maps $\cdot\varphi_i : \mathcal{R}(i,j) \longrightarrow M(i,j)$ are trivial fibrations. In this case we define a new spectral category $\mathcal{E}(M,\varphi)$ with objects I as the pullback of $\mathcal{R}$ and $\mathcal{O}$ over M. More precisely for every pair $i,j \in I$ the homomorphism object $\mathcal{E}(M,\varphi)(i,j)$ is defined as the pullback in Sp^Σ of the diagram

$$\mathcal{R}(i,j) \xrightarrow{\cdot\varphi_i} M(i,j) \xleftarrow{\varphi_j\cdot} \mathcal{O}(i,j) \ .$$

Using the universal property of the pullback there is a unique way to define composition and identity morphisms in $\mathcal{E}(M,\varphi)$ in such a way that the maps $\mathcal{E}(M,\varphi) \longrightarrow \mathcal{O}$ and $\mathcal{E}(M,\varphi) \longrightarrow \mathcal{R}$ are homomorphisms of spectral categories.

Since M is a quasi-equivalence, all the maps in the defining pullback diagrams are weak equivalences. By assumption the horizontal ones are even trivial fibrations, so both base change maps $\mathcal{E}(M,\varphi) \longrightarrow \mathcal{O}$ and $\mathcal{E}(M,\varphi) \longrightarrow \mathcal{R}$ are pointwise equivalences of spectral categories. The same argument works if instead of the right multiplication maps all the left multiplication maps $\varphi_j\cdot : \mathcal{O}(i,j) \longrightarrow M(i,j)$ are trivial fibrations.

ii) General case: taking fibrant replacement if necessary we can assume that the bimodule M is objectwise fibrant. The 'element' φ_j of $M(j,j)$ corresponds to a map $F_j = \mathcal{O}(-,j) \longrightarrow M(-,j)$ from the free $\mathcal{O}$-module; the map is left multiplication by φ_j and is an objectwise equivalence since M is a quasi-equivalence. We factor this $\mathcal{O}$-module equivalence as a trivial cofibration $\alpha_j : F_j \longrightarrow N_j$ followed by a trivial fibration $\psi_j : N_j \longrightarrow M(-,j)$; in particular, the objects N_j so obtained are cofibrant and fibrant. We let $\mathcal{E}(N)$ denote the endomorphism spectral category of the cofibrant-fibrant replacements, i.e., the full spectral subcategory of the category of $\mathcal{O}$-modules with objects N_j for $j \in I$. Now we appeal twice to the special case that we already proved, obtaining a chain of four stable equivalences of spectral categories

$$\mathcal{O} \xleftarrow{\sim} \mathcal{E}(W,\alpha) \xrightarrow{\sim} \mathcal{E}(N) \xleftarrow{\sim} \mathcal{E}(V,\psi) \xrightarrow{\sim} \mathcal{R} \ .$$

In more detail, we define a $\mathcal{E}(N)$-$\mathcal{O}$-bimodule W by the rule

$$W(i,j) \ = \ \mathrm{Hom}_{\text{mod-}\mathcal{O}}(F_i, N_j) \ \cong \ N_j(i) \ .$$

The bimodule W is a quasi-equivalence with respect to the maps α_j. Moreover, the right multiplication map $\cdot\alpha_i$ is the restriction map

$$\alpha_i^* \ : \ \mathrm{Hom}_{\text{mod-}\mathcal{O}}(N_i, N_j) \longrightarrow \mathrm{Hom}_{\text{mod-}\mathcal{O}}(F_i, N_j) \ .$$

So α_i^* is a trivial fibration since α_i is a trivial cofibration of $\mathcal{O}$-modules and N_j is a fibrant module. Case i) above then provides a chain of stable equivalences between $\mathcal{O}$ and $\mathcal{E}(N)$, passing through $\mathcal{E}(W,\alpha)$.

Now we define an $\mathcal{R}$-$\mathcal{E}(N)$-bimodule V by the rule

$$V(i,j) \;=\; \mathrm{Hom}_{\text{mod-}\mathcal{O}}(N_i, M(-,j)).$$

The bimodule V is a quasi-equivalence with respect to the maps ψ_j. The left multiplication map $\psi_j\cdot$ is the composition

$$(\psi_j)_* \;:\; \mathrm{Hom}_{\text{mod-}\mathcal{O}}(N_i, N_j) \longrightarrow \mathrm{Hom}_{\text{mod-}\mathcal{O}}(N_i, M(-,j))\;.$$

This time $(\psi_j)_*$ is a trivial fibration since ψ_j is a trivial fibration of $\mathcal{O}$-modules and N_i is a cofibrant module. Furthermore the right multiplication map

$$\cdot\psi_i : \mathcal{R}(i,j) \longrightarrow \mathrm{Hom}_{\text{mod-}\mathcal{O}}(N_i, M(-,j))$$

is an equivalence because its composite with the map

$$\alpha_i^* \;:\; \mathrm{Hom}_{\text{mod-}\mathcal{O}}(N_i, M(-,j)) \longrightarrow \mathrm{Hom}_{\text{mod-}\mathcal{O}}(F_i, M(-,j)) \;\cong\; M(i,j)$$

is right multiplication by ψ_i, an equivalence by assumption. Recall M is objectwise fibrant, so α_i^* is a weak equivalence. So case i) gives a chain of pointwise equivalences between $\mathcal{R}$ and $\mathcal{E}(N)$, passing through $\mathcal{E}(V,\psi)$. □

As a corollary we obtain the homotopy invariance of endomorphism spectral categories under spectral Quillen equivalences.

Corollary A.2.4. *Suppose $\mathcal{C}$ and $\mathcal{D}$ are spectral model categories and $L : \mathcal{C} \longrightarrow \mathcal{D}$ is the left adjoint of a spectral Quillen equivalence. Suppose I is a set, $\{P_i\}_{i\in I}$ and $\{Q_i\}_{i\in I}$ are sets of cofibrant-fibrant objects of $\mathcal{C}$ and $\mathcal{D}$ respectively, and that for all $i \in I$, LP_i is weakly equivalent to Q_i in $\mathcal{D}$. Then the spectral endomorphism categories of $\{P_i\}_{i\in I}$ and $\{Q_i\}_{i\in I}$ are stably equivalent. In particular the spectral endomorphism category of $\{P_i\}_{i\in I}$ depends up to pointwise equivalence only on the weak equivalence type of the objects P_i.*

Proof. Since the object LP_i is cofibrant and weakly equivalent to the fibrant object Q_i, we can choose a weak equivalence $\varphi_i : LP_i \longrightarrow Q_i$ for every $i \in I$. We claim that the collection of homomorphism objects $\mathrm{Hom}_{\mathcal{D}}(LP_i, Q_j)$ forms a quasi-equivalence for the endomorphism spectral categories of $\{P_i\}_{i\in I}$ and $\{Q_i\}_{i\in I}$ with respect to the equivalences φ_i. Indeed the endomorphism category of $\{Q_i\}_{i\in I}$ acts on the left by composition; also right multiplication by φ_j is a stable equivalence since Q_j is fibrant and φ_j is a weak equivalence between cofibrant objects. If R denotes the right adjoint of L, then $\mathrm{Hom}_{\mathcal{D}}(LP_i, Q_j)$ is isomorphic to $\mathrm{Hom}_{\mathcal{C}}(P_i, RQ_j)$, so the endomorphism category of $\{P_i\}_{i\in I}$ acts on the right by composition. Since R and L form a spectral Quillen equivalence, the adjoints $\widehat{\varphi}_i : P_i \longrightarrow RQ_i$ are weak equivalences between fibrant objects; so left multiplication by φ_i is a stable equivalence since P_i is cofibrant. The last statement is the special case where $\mathcal{D} = \mathcal{C}$ and L is the identity functor. □

Appendix B. Eilenberg-Mac Lane spectra and chain complexes

The proof of the generalized tilting theorem in Section 5 uses the Eilenberg-Mac Lane spectral category $H\underline{A}$ of a ringoid $\underline{A}$. Recall that a *ringoid* is a small category whose hom-sets carry an abelian group structure for which composition is bilinear. A right *module* over a ringoid is a contravariant additive functor to the category of abelian groups. The Eilenberg-Mac Lane spectral category $H\underline{A}$ of a ringoid $\underline{A}$ is defined in 5.1.5. In this appendix we provide some general facts about Eilenberg-Mac Lane spectral categories. The main results are that module spectra over the Eilenberg-Mac Lane spectral category $H\underline{A}$ are Quillen equivalent to chain complexes of $\underline{A}$-modules (Theorem 5.1.6) and that Eilenberg-Mac Lane spectral categories are determined up to stable equivalence by their coefficient ringoid (Theorem B.2.1). These properties are not unexpected, and variations have been proved for the special case of ring spectra in different

frameworks. Indeed the Quillen equivalence of Theorem 5.1.6 is a generalization and strengthening of the fact first proved in [50] that the unbounded derived category of modules over a ring R is equivalent to the homotopy category of HR-modules, see also [15, IV Thm. 2.4] in the context of S-algebras.

B.1. **Chain complexes and module spectra.** Throughout this section we fix a ringoid $\underline{A}$, and we want to prove Theorem 5.1.6 relating the modules over the Eilenberg-Mac Lane spectral category $H\underline{A}$ to complexes of $\underline{A}$-modules by a chain of Quillen equivalences. We do not know of a Quillen functor pair which does the job in a single step. Instead, we compare the two categories through the intermediate model category of *naive $H\underline{A}$-modules*, obtaining a chain of Quillen equivalences

$$\text{mod-}H\underline{A} \quad \underset{U}{\overset{L}{\rightleftarrows}} \quad \text{Nvmod-}H\underline{A} \quad \underset{\mathcal{H}}{\overset{\Lambda}{\rightleftarrows}} \quad \text{Ch}\,\underline{A}$$

(the left adjoints are on top), see Corollary B.1.8 and Theorem B.1.11. We mention here that an analogous statement holds for differential graded modules over a differential graded ring and modules over the associated generalized Eilenberg-Mac Lane spectrum, but the proof becomes more complicated; see Remark B.1.10 and [58].

Definition B.1.1. Let $\mathcal{O}$ be a spectral category. A *naive $\mathcal{O}$-module* M consists of a collection $\{M(o)\}_{o\in\mathcal{O}}$ of $\mathbb{N}$-graded, pointed simplicial sets together with associative and unital action maps

$$M(o)_p \ \wedge\ \mathcal{O}(o',o)_q \longrightarrow M(o')_{p+q}$$

for pairs of objects o, o' in $\mathcal{O}$ and for natural numbers $p, q \geq 0$. A morphism of naive $\mathcal{O}$-modules $M \longrightarrow N$ consists of maps of graded spaces $M(o) \longrightarrow N(o)$ strictly compatible with the action of $\mathcal{O}$. We denote the category of naive $\mathcal{O}$-modules by Nvmod-$\mathcal{O}$.

Note that a naive module M has *no symmetric group action* on $M(o)_n$, and hence there is no equivariance condition for the action maps. A naive $\mathcal{O}$-module has strictly less structure than a genuine $\mathcal{O}$-module, so there is a forgetful functor

$$U \ :\ \text{mod-}\mathcal{O} \longrightarrow \text{Nvmod-}\mathcal{O}\ .$$

The *free naive $\mathcal{O}$-module* F_o at an object $o \in \mathcal{O}$ is given by the graded spaces $F_o(o') = \mathcal{O}(o', o)$ with action maps

$$F_o(o')_p \ \wedge\ \mathcal{O}(o'',o')_q \ =\ \mathcal{O}(o',o)_p \ \wedge\ \mathcal{O}(o'',o')_q \longrightarrow \mathcal{O}(o'',o)_{p+q} \ =\ F_o(o'')_{p+q}$$

given by composition in $\mathcal{O}$. In other words, the forgetful functor takes the free, genuine $\mathcal{O}$-module to the free, naive $\mathcal{O}$-module. The free naive module F_o represents evaluation at the object $o \in \mathcal{O}$, i.e., there is an isomorphism of simplicial sets

$$\text{map}_{\text{Nvmod-}\mathcal{O}}(F_o, M) \ \cong\ M(o)_0 \tag{B.1.2}$$

which is natural for naive $\mathcal{O}$-modules M.

If M is a naive $\mathcal{O}$-module, then at every object $o \in \mathcal{O}$, $M(o)$ has an underlying spectrum in the sense of Bousfield-Friedlander [4, §2] (except that in [4], the suspension coordinates appear on the left, whereas we get suspension coordinates acting from the right). Indeed, using the unital structure map $S^1 \longrightarrow \mathcal{O}(o,o)_1$ of the spectral category $\mathcal{O}$, the graded space $M(o)$ gets suspension maps via the composite

$$M(o)_p \ \wedge\ S^1 \longrightarrow M(o)_p \ \wedge\ \mathcal{O}(o,o)_1 \longrightarrow M(o)_{p+1}\ .$$

A morphism of naive $\mathcal{O}$-modules $f : M \longrightarrow N$ is an *objectwise π_*-isomorphism* if for all $o \in \mathcal{O}$ the map $f(o) : M(o) \longrightarrow N(o)$ induces an isomorphism of stable homotopy groups. The map f is an *objectwise stable fibration* if each $f(o)$ is a stable fibration of spectra in the sense of [4,

Thm. 2.3]). A morphism of naive $\mathcal{O}$-modules is a *cofibration* if it has the left lifting properties for maps which are objectwise π_*-isomorphisms and objectwise stable fibrations.

Theorem B.1.3. *Let $\underline{A}$ be a ringoid.*

(i) *The category of naive $H\underline{A}$-modules with the objectwise π_*-isomorphisms, objectwise stable fibrations, and cofibrations is a cofibrantly generated, simplicial, stable model category.*
(ii) *The collection of free $H\underline{A}$-modules $\{F_a\}_{a\in\underline{A}}$ forms a set of compact generators for the homotopy category of naive $H\underline{A}$-modules.*
(iii) *Let $\mathcal{C}$ be a stable model category and consider a Quillen adjoint functor pair*

$$\mathcal{C} \quad \underset{\rho}{\overset{\lambda}{\rightleftarrows}} \quad \text{Nvmod-}H\underline{A}$$

where ρ is the right adjoint. Then (λ,ρ) is a Quillen equivalence, provided that
(a) *for every object $a\in\underline{A}$, the object $\lambda(F_a)$ is fibrant in $\mathcal{C}$*
(b) *for every object $a\in\underline{A}$, the unit of the adjunction $F_a\longrightarrow\rho\lambda(F_a)$ is an objectwise π_*-isomorphism, and*
(c) *the objects $\{\lambda(F_a)\}_{a\in\underline{A}}$ form a set of compact generators for the homotopy category of $\mathcal{C}$.*

Proof. (i) We use Theorem 3.7.4 to establish the model category structure. The category of naive $H\underline{A}$-modules is complete and cocomplete and every naive $H\underline{A}$-module is small. The objectwise π_*-isomorphisms are closed under the 2-out-of-3 condition (Theorem 3.7.4 (1)).

As generating cofibrations I we use the collection of maps

$$(\partial\Delta^i)^+\wedge F_a[n]\ \longrightarrow\ (\Delta^i)^+\wedge F_a[n]$$

for all $i,n\geq 0$ and $a\in\underline{A}$. Here Δ^i denotes the simplicial i-simplex and $\partial\Delta^i$ is its boundary; the square bracket $[n]$ means shifting (reindexing) of a naive $H\underline{A}$-modules and smashing of a module and a pointed simplicial set is levelwise. Since the free modules represent evaluation at an object (see (B.1.2) above), the I-injectives are precisely the maps which are objectwise level acyclic fibrations.

As generating acyclic cofibrations J we use the union $J=J^{\mathrm{lv}}\cup J^{\mathrm{st}}$. Here J^{lv} is the set of maps

$$(\Lambda^i_k)^+\wedge F_a[n]\ \xrightarrow{\sim}\ (\Delta^i)^+\wedge F_a[n]$$

for $i,n\geq 0$, $0\leq k\leq i$ and $a\in\underline{A}$, where $\Lambda^{i,k}$ is the k-th horn of the i-simplex. The J^{lv}-injectives are the objectwise level fibrations. Finally, J^{st} consists of the mapping cylinder inclusions of the maps

$$S^1\wedge(\Delta^i)^+\wedge F_a[n+1]\ \cup_{S^1\wedge(\partial\Delta^i)^+\wedge F_a[n+1]}\ (\partial\Delta^i)^+\wedge F_a[n]\longrightarrow(\Delta^i)^+\wedge F_a[n]\ . \tag{B.1.4}$$

Here the mapping cylinders are formed on each simplicial level, just as in [25, 3.1.7]. Every map in J is an I-cofibration, hence every relative J-cell complex is too; we claim that in addition, every map in J is an objectwise injective π_*-isomorphism. Since this property is closed under infinite wedges, pushout, sequential colimit and retracts this implies that every relative J-cell complex is an objectwise injective π_*-isomorphism and so condition (2) of Theorem 3.7.4 holds.

The maps in J^{lv} are even objectwise injective level-equivalences, so it remains to check the maps in J^{st}. These maps are defined as mapping cylinder inclusions, so they are injective, and we need only check that the maps in (B.1.4) above are objectwise π_*-isomorphisms. This in turn follows once we know that the maps

$$S^1\wedge F_a[n+1]\longrightarrow F_a[n] \tag{B.1.5}$$

are objectwise π_*-isomorphisms. At level $p \geq n+1$ and an object $b \in \underline{A}$, this map is given by the inclusion

$$S^1 \wedge (\underline{A}(b,a) \otimes \widetilde{\mathbb{Z}}[S^{p-n-1}]) \longrightarrow \underline{A}(b,a) \otimes \widetilde{\mathbb{Z}}[S^{p-n}]$$

whose adjoint is a weak equivalence. This map is roughly $2(p-n)$-connected, so in the limit we indeed obtain a π_*-isomorphism.

It remains to check condition (3) of Theorem 3.7.4, namely that the I-injectives coincide with the maps which are both J-injective and objectwise π_*-isomorphisms. Every map in J is an I-cofibration, so I-injectives are J-injective. Since I-injectives are level acyclic fibrations, they are also objectwise π_*-isomorphisms. Conversely, suppose $f : M \longrightarrow N$ is an objectwise π_*-isomorphism of naive $H\underline{A}$-modules which is also J-injective. Since f is J^{lv}-injective, it is an objectwise level fibration. Since f is J^{lv}-injective, at every object $a \in \underline{A}$, the underlying map of spectra $f(a) : M(a) \longrightarrow N(a)$ has the right lifting property for the maps

$$S^1 \wedge (\Delta^i)^+ \wedge S[n+1] \cup_{S^1 \wedge (\partial\Delta^i)^+ \wedge S[n+1]} (\partial\Delta^i)^+ \wedge S[n] \longrightarrow (\Delta^i)^+ \wedge S[n] \ ,$$

where S is the sphere spectrum. But then $f(a)$ is a stable fibration of spectra [54, A.3], so f is an objectwise stable fibration and π_*-isomorphism. By [4, A.8 (ii)], f is then an objectwise level fibration, so it is I-injective. So conditions (1)-(3) of Theorem 3.7.4 are satisfied and this theorem provides the model structure. We omit the verification that the model structure for naive $H\underline{A}$-modules is simplicial and stable; the latter is a consequence of the fact that stable equivalences of $H\underline{A}$-modules are defined objectwise and spectra form a stable model category.

(ii) The stable model structure for naive $H\underline{A}$-modules is defined so that evaluation at $a \in \underline{A}$ is a right Quillen functor to the stable model category of Bousfield-Friedlander type spectra. Moreover, evaluation at $a \in \underline{A}$ has a left adjoint which takes the sphere spectrum S to the free module F_a. So the derived adjunction provides an isomorphism of graded abelian groups

$$[F_a, M]_*^{\mathrm{Ho}(\text{Nvmod-}H\underline{A})} \ \cong \ [S, M(a)]_*^{\mathrm{Ho}(\mathcal{S}p)} \ \cong \ \pi_* M(a) \ .$$

This implies that in the homotopy category the free modules detect objectwise π_*-isomorphisms, so they form a set of generators. It also implies that the representable modules are compact, because evaluation at $a \in \underline{A}$ and homotopy groups commute with infinite sums.

(iii) We have to show that the derived adjunction on the level of homotopy categories

$$\mathrm{Ho}(\mathcal{C}) \quad \underset{R\rho}{\overset{L\lambda}{\rightleftarrows}} \quad \mathrm{Ho}(\text{Nvmod-}H\underline{A})$$

yields equivalences of (homotopy) categories. The right adjoint $R\rho$ detects isomorphisms: if $f : X \longrightarrow Y$ is a morphism in $\mathrm{Ho}(\mathcal{C})$ such that $R\rho(f)$ is an isomorphism in the homotopy category of naive $H\underline{A}$-modules, then for every $a \in \underline{A}$, the map f induces an isomorphism on $[L\lambda(F_a), -]$ by adjointness. Since the objects $L\lambda(F_a)$ generate the homotopy category of $\mathcal{C}$, f is an isomorphism. It remains to show that the unit of the derived adjunction $\eta_M : M \longrightarrow R\rho(L\lambda(M))$ on the level of homotopy categories is an isomorphism for every $H\underline{A}$-module M. For the free $H\underline{A}$-modules F_a this follows from assumptions (a) and (b): by (a), $\lambda(F_a)$ is fibrant in $\mathcal{C}$, so the point set level adjunction unit $F_a \longrightarrow \rho\lambda(F_a)$ models the derived adjunction unit, then by (b) η_M is an isomorphism. The composite derived functor $R\rho \circ L\lambda$ is exact; the functor $R\rho$ commutes with coproducts (a formal consequence of (ii)), hence so does $R\rho \circ L\lambda$ since $L\lambda$ is a left adjoint. Hence the full subcategory of those $H\underline{A}$-modules M for which the derived unit η_M is an isomorphism is a localizing subcategory. Since it also contains the generating representable modules, it coincides with the full homotopy category of naive $H\underline{A}$-modules. □

Remark B.1.6. The reader may wonder why we do not state Theorem B.1.3 for a general spectral category $\mathcal{O}$. The reason is that already the analog of part (i), the existence of the stable

model structure for naive $\mathcal{O}$-modules, can fail without some hypothesis on $\mathcal{O}$. The problem can be located: one needs that the analog of the map (B.1.5),

$$S^1 \wedge F_o[n+1] \longrightarrow F_o[n]$$

which is given by the action of the suspension coordinates from the left, induces an isomorphism of homotopy groups, taken with respect to suspension on the right. But in general, the effects of left and right suspension on homotopy groups can be related in a complicated way. We hope to return to these questions elsewhere.

As a corollary, we use the criteria in part (iii) of the previous theorem to establish the Quillen equivalence between the model category of (right) $H\underline{A}$-modules of symmetric spectra and the model category of (right) naive $H\underline{A}$-modules. These criteria are also used to establish the Quillen equivalence between naive $H\underline{A}$-modules and chain complexes of $\underline{A}$-modules, see Theorem B.1.11 below.

First we recall a general categorical criterion for the existence of left adjoints. Recall from [2, Def. 1.1, 1.17] that an object K of a category $\mathcal{C}$ is *finitely presentable* if the hom functor $\mathrm{Hom}_{\mathcal{C}}(K,-)$ preserves filtered colimits. A category $\mathcal{C}$ is called *locally finitely presentable* if it is cocomplete and there exists a set A of finitely presentable objects such that every object of $\mathcal{C}$ is a filtered colimit of objects in A. The condition 'locally finitely presentable' implies that every object is small in the sense of [21, 2.1.3]. For us the point of this definition is that every functor between locally finitely presentable categories which commutes with limits and filtered colimits has a left adjoint (this is a special case of [2, 1.66]). We omit the proof of the following lemma.

Lemma B.1.7. *Let $\underline{A}$ be a ringoid. Then the categories of complexes of $\underline{A}$-modules, of (genuine) $H\underline{A}$-modules and of naive $H\underline{A}$-modules are locally finitely presentable.*

Corollary B.1.8. *The forgetful functor from $H\underline{A}$-modules to naive $H\underline{A}$-modules is the right adjoint of a Quillen equivalence.*

Proof. The forgetful functor U from $H\underline{A}$-modules to naive $H\underline{A}$-modules preserves limits and filtered colimits. Since source and target category are locally finitely presentable, U has a left adjoint L by [2, 1.66]. The forgetful functor from symmetric spectra to (non-symmetric) spectra is the right adjoint of a Quillen functor pair, see [25, 4.2.4]. So the forgetful functor U from $H\underline{A}$-modules to naive $H\underline{A}$-modules preserves objectwise stable equivalences and objectwise stable fibrations. Thus U and L form a Quillen pair, and we can apply part (iii) of Theorem B.1.3. The left adjoint L sends the naive free modules F_a to the genuine free modules, so the relevant adjunction unit in condition (b) is even an *isomorphism*. For every pair of objects $a, b \in \underline{A}$, the symmetric spectrum $(LF_a)(b) = H\underline{A}(b,a)$ is a symmetric Ω-spectrum, hence stably fibrant, which gives condition (a). The free modules form a set of compact generators for the homotopy category of genuine $H\underline{A}$-modules, by Theorem A.1.1, so condition (c) is satisfied. □

To finish the proof of Theorem 5.1.6 we now construct a Quillen-equivalence between naive $H\underline{A}$-modules and complexes of $\underline{A}$-modules. We define another *Eilenberg-Mac Lane functor*

$$\mathcal{H} \;:\; \mathrm{Ch}\underline{A} \longrightarrow \mathrm{Nvmod}\text{-}H\underline{A}$$

from the category of chain complexes of $\underline{A}$-modules to the category of naive modules over $H\underline{A}$.

For any simplicial set K we denote by NK the normalized chain complex of the free simplicial abelian group generated by K. So NK is a non-negative dimensional chain complex which in dimension n is isomorphic to the free abelian group on the non-degenerate n-simplices of K. A functor W from the category of chain complexes $\mathrm{Ch}_{\mathbb{Z}}$ to the category of simplicial abelian groups is defined by

$$(WC)_k \;=\; \hom_{\mathrm{Ch}_{\mathbb{Z}}}(N\Delta[k], C)\ .$$

For non-negative dimensional complexes, W is just the inverse to the normalized chain functor in the Dold-Kan equivalence between simplicial abelian groups and non-negative dimensional chain complexes [9, 1.9]. For an arbitrary complex C there is a natural chain map $NWC \longrightarrow C$ which is an isomorphism in positive dimensions and which expresses NWC as the (-1)-connected cover of C.

For a chain complex of abelian groups C we define a graded space by the formula

$$(\mathcal{H}C)_n \;=\; W(C[n])$$

where $C[n]$ denotes the n-fold shift suspension of the complex C. To define the module structure maps we use the Alexander-Whitney map, see [14, 2.9] or [40, 29.7]. This map is a natural, associative and unital transformation of simplicial abelian groups

$$AW \;:\; W(C) \;\otimes\; W(D) \longrightarrow W(C \otimes D)\;.$$

Here the left tensor product is the dimensionwise tensor product of simplicial abelian groups, whereas the right one is the tensor product of chain complexes. The Alexander-Whitney map is neither commutative, nor an isomorphism. By our conventions the p-sphere S^p is the p-fold smash product of the simplicial circle $S^1 = \Delta[1]/\partial\Delta[1]$, so the reduced free abelian group generated by S^p is the p-fold tensor product of the simplicial abelian group $\widetilde{\mathbb{Z}}[S^1] = W(\mathbb{Z}[1])$ (where $\mathbb{Z}[1]$ is the chain complex which contains a single copy of the group $\mathbb{Z}$ in dimension 1). Since the p-th space in the Eilenberg-Mac Lane spectrum $H\underline{A}(a,b)$ is given by $H\underline{A}(a,b)_p = \underline{A}(a,b) \otimes \widetilde{\mathbb{Z}}[S^p]$, for every chain complex D of $\underline{A}$-modules the Alexander-Whitney map gives a map

$$\begin{aligned}
\mathcal{H}(D(b))_p \wedge H\underline{A}(a,b)_q \quad &\longrightarrow \quad \mathcal{H}(D(b))_p \otimes H\underline{A}(a,b)_q \\
&\xrightarrow{\cong} \quad W(D(b)[p]) \otimes \underline{A}(a,b) \otimes \underbrace{W(\mathbb{Z}[1]) \otimes \cdots \otimes W(\mathbb{Z}[1])}_{q}) \\
&\xrightarrow{AW} \quad W\left(D(b)[p] \otimes \underline{A}(a,b) \otimes \underbrace{\mathbb{Z}[1] \otimes \cdots \otimes \mathbb{Z}[1]}_{q} \right) \\
&\longrightarrow \quad W(D(a)[p+q]) \;=\; \mathcal{H}(D(a))_{p+q}\;.
\end{aligned}$$

These maps make $\mathcal{H}D$ into a naive $H\underline{A}$-module. The spectra underlying $\mathcal{H}D(a)$ are always Ω-spectra and the stable homotopy groups of $\mathcal{H}D(a)$ are naturally isomorphic to the homology groups of the chain complex $D(a)$,

$$\pi_*\,\mathcal{H}D \;\cong\; H_*D \tag{B.1.9}$$

as graded $\underline{A}$-modules.

Remark B.1.10. The functor $\mathcal{H}$ should not be confused with the Eilenberg-Mac Lane functor H of Definition 5.1.5. The functor H takes values in symmetric spectra, but it cannot be extended in a reasonable way to chain complexes; the functor $\mathcal{H}$ is defined for complexes, but it only takes values in *naive* $H\underline{A}$-modules.

The essential difference between the two functors can already be seen for an abelian group A. The simplicial abelian group $(\mathcal{H}A)_n = W(A[n])$ is the minimal model of an Eilenberg-Mac Lane space of type $K(A,n)$ and it is determined by the property that its normalized chain complex consists only of one copy of A in dimension n. The simplicial abelian group $(HA)_n = A \otimes \widetilde{\mathbb{Z}}[S^n]$ is another Eilenberg-Mac Lane space of type $K(A,n)$, but it has non-degenerate simplices in dimensions smaller than n. The Alexander-Whitney map gives a weak equivalence of simplicial abelian groups $A \otimes \widetilde{Z}[S^n] \longrightarrow W(A[n])$.

However, the Alexander-Whitney map is *not* commutative, and for $n \geq 2$ there is no Σ_n-action on the minimal model $W(A[n])$ which admits an equivariant weak equivalence from $\widetilde{Z}[S^n] \otimes A$. More generally, the graded space $\mathcal{H}A$ *cannot* be made into a symmetric spectrum which is level equivalent to the symmetric spectrum HA. This explains why the comparison between HA-modules and complexes of A-modules has to go through the category of naive HA-modules.

Theorem B.1.11. *Let $\underline{A}$ be a ringoid. Then the Eilenberg-Mac Lane functor $\mathcal{H}$ is the right adjoint of a Quillen equivalence between chain complexes of $\underline{A}$-modules and naive $H\underline{A}$-modules.*

Proof. The functor $\mathcal{H}$ commutes with limits and filtered colimits, and since source and target category of $\mathcal{H}$ are locally finitely presentable, a left adjoint Λ exists by [2, 1.66]. The Eilenberg-Mac Lane functor takes values in the category of Ω-spectra, which are the (stably) fibrant objects in the category of naive $H\underline{A}$-modules. Moreover, it takes objectwise fibrations of chain complexes (i.e., epimorphisms) to objectwise level fibrations. Since level fibrations between Ω-spectra are stable fibrations, $\mathcal{H}$ preserves fibrations. Because of the isomorphism labeled (B.1.9), $\mathcal{H}$ takes objectwise quasi-isomorphisms of $\underline{A}$-modules to objectwise stable equivalences of $H\underline{A}$-modules, so it also preserves acyclic fibrations. Thus $\mathcal{H}$ and Λ form a Quillen adjoint functor pair.

Now we apply criterion (iii) of Theorem B.1.3. Every chain complex of $\underline{A}$-modules is fibrant in the projective model structure, so condition (a) holds. If we consider the free $\underline{A}$-module $\underline{A}(-,a)$, as a complex in dimension 0, then the identity element in $\underline{A}(a,a) \cong \mathcal{H}\underline{A}(-,a)(a)_0$ is represented by a map of naive $H\underline{A}$-modules $\kappa : F_a \longrightarrow \mathcal{H}(\underline{A}(-,a))$. By the adjunction and representability isomorphisms

$$\hom_{\mathrm{Ch}\underline{A}}(\Lambda(F_a), D) \;\cong\; \hom_{\text{Nvmod-}H\underline{A}}(F_a, \mathcal{H}D) \;\cong\; (\mathcal{H}D(a))_0 \;\cong\; \hom_{\mathrm{Ch}\underline{A}}(\underline{A}(-,a), D)\ ,$$

so the complexes $\Lambda(F_a)$ and $\underline{A}(-,a)$ represent the same functor. Thus, the adjoint of κ is an isomorphism from $\Lambda(F_a)$ to $\underline{A}(-,a)$. The adjunction unit relevant for condition (b) is the map $\kappa : F_a \longrightarrow \mathcal{H}(\underline{A}(-,a)) \cong \mathcal{H}\Lambda(F_a)$. At an object $b \in \underline{A}$ and in dimension p, the map κ specializes to the Alexander-Whitney map

$$F_a(b)_p \;=\; \underline{A}(b,a) \otimes \widetilde{\mathbb{Z}}[S^p] \longrightarrow W(\underline{A}(b,a)[p]) \;=\; \mathcal{H}(\underline{A}(b,a))_p\ .$$

Both sides of this map are Eilenberg-Mac Lane spaces of type $K(\underline{A}(b,a), p)$, the target being the minimal model. The map is a weak equivalence, so condition (b) of Theorem B.1.3 (iii) holds. The free modules $\underline{A}(-,a)$ (viewed as a complexes in dimension 0) form a set of compact generators for the derived category of $\underline{A}$-modules, so condition (c) is satisfied. □

B.2. **Characterization of Eilenberg-Mac Lane spectra.** In this section we show that Eilenberg-Mac Lane spectral categories are determined up to stable equivalence by the property that their homotopy groups are concentrated in dimension zero.

Proposition B.2.1. *Let R be a spectral category all of whose morphism spectra are stably fibrant and have homotopy groups concentrated in dimension zero. Then there exists a natural chain of stable equivalences of spectral categories between R and $H\underline{\pi_0 R}$.*

The proposition is a special case of the following statement. Here we call a stably fibrant spectrum *connective* if the negative dimensional stable homotopy groups vanish.

Lemma B.2.2. *Let I be any set. There are functors M and E from the category of spectral categories with object set I to itself and natural transformations*

$$\mathrm{Id} \xrightarrow{\ \alpha\ } M \xleftarrow{\ \beta\ } E \xrightarrow{\ \gamma\ } H\underline{\pi_0}$$

with the following properties: for every spectral category R with connective stably fibrant morphism spectra the maps α_R and β_R are stable equivalences and the map γ_R induces the canonical isomorphism on component ringoids.

Proof. The strategy of proof is to transfer the corresponding statement from the category of Gamma-rings (where it is easy to prove) to the category of symmetric ring spectra and extend it to the 'multiple object case'. We use in a crucial way Bökstedt's $\mathrm{hocolim}_I$ construction [6]. The functors M, E as well as an intermediate functor D all arise as lax monoidal functors from the category of symmetric spectra to itself, and the natural maps between them are monoidal transformations. This implies that when we apply them to the morphism spectra of a spectral category, then the outcome is again a spectral category in a natural way, and the transformations assemble into spectral functors.

The two functors M and D from the category of symmetric spectra to itself are defined in [60, Sec. 3]. The n-th space of the symmetric spectrum MX is defined as the homotopy colimit

$$(MX)_n \;=\; \mathrm{hocolim}_{\mathbf{k}\in I}\, \Omega^k \mathrm{Sing}|X_{k+n}| \;.$$

Here I is a skeleton of the category of finite sets and injections with objects $\mathbf{k} = \{0, 1, \ldots, k\}$; for the precise definition and the structure maps making this a symmetric spectrum see [60, Sec. 3]. The map $\alpha : X \longrightarrow MX$ is induced by the inclusion of X into the colimit diagram at $k = 0$. In the proof of [60, Prop. 3.1.9] it is shown that the map α is a stable equivalence (even a level equivalence) for every stably fibrant symmetric spectrum X.

The n-th level of the functor D (for 'detection' - it detects stable equivalences of symmetric spectra) is defined as

$$(DX)_n \;=\; \mathrm{hocolim}_{\mathbf{k}\in I}\, \Omega^k \mathrm{Sing}|X_k \wedge S^n| \;,$$

see [60, Def. 3.1.1]. Also in the proof of [60, Prop. 3.1.9] a natural map $DX \longrightarrow MX$ is constructed which we denote β^1_X and which is a stable equivalence (even a level equivalence) for every stably fibrant spectrum X.

The symmetric spectrum DX in fact arises from a simplicial functor QX. The value of QX at a pointed simplicial set K is given by

$$QX(K) \;=\; \mathrm{hocolim}_{\mathbf{k}\in I}\, \Omega^k \mathrm{Sing}|X_k \wedge K| \;.$$

A simplicial functor F can be evaluated on the simplicial spheres to give a symmetric spectrum, which we denote $F(\mathbb{S})$. In the situation at hand we thus have $DX = QX(\mathbb{S})$. If we restrict the simplicial functor QX to the category Γ^{op} of finite pointed sets we obtain a Γ-space [59, 4] denoted ρQX. Every Γ-space can be prolonged to a simplicial functor defined on the category of pointed simplicial sets [4, §4]. Prolongation is left adjoint to the restriction functor ρ, and we denote it by P. We then set $EX = (P\rho QX)(\mathbb{S})$. The unit $P\rho QX \longrightarrow QX$ of the adjunction between restriction and prolongation, evaluated at the spheres, gives a map of symmetric spectra

$$\beta^2_X : EX = (P\rho QX)(\mathbb{S}) \longrightarrow QX(\mathbb{S}) = DX \;.$$

We claim that if X is a connective symmetric Ω-spectrum, then β^2_X is a level equivalence between connective symmetric Ω-spectra. Indeed, if X is a symmetric Ω-spectrum, it is in particular convergent in the sense of [39, 2.1]. By [39, Thm. 2.3] and the remark thereafter, the natural map $QX(K) \longrightarrow \Omega(QX(\Sigma K))$ is then a weak equivalence for all pointed simplicial sets K. This implies (see e.g. [38, Lemma 17.9]) that QX is a *linear* functor, i.e., that it takes homotopy cocartesian squares to homotopy cartesian squares. In particular, QX converts wedges to products, up to weak equivalence, and takes values in infinite loop spaces, so the restricted Γ-space ρQX is very special [4, p. 97]. By [4, Thm. 4.2], $EX = (P\rho QX)(\mathbb{S})$ is a connected Ω-spectrum. Since both EX and DX are connected Ω-spectra and the map β^2_X is an isomorphism at level 0, β^2_X is in fact a level equivalence. The map $\beta_X : EX \longrightarrow MX$ is defined as the composite of the maps $\beta^2_X : EX \longrightarrow DX$ and $\beta^1_X : DX \longrightarrow MX$; if X is stably fibrant and connective, then both of these are level equivalences, hence so is β_X.

Every Γ-space Y has a natural monoidal map $Y \longrightarrow H\pi_0 Y$ to the Eilenberg-Mac Lane Γ-space ([59, §0], [53, Sec. 1]) of its component group which induces the canonical isomorphism on π_0, see [53, Lemma 1.2]. (This map is in fact the unit of another monoidal adjunction, namely, between the Eilenberg-Mac Lane Γ-space functor and the π_0-functor.) In particular there is such a map of Γ-spaces $\rho QX \longrightarrow H\pi_0(\rho QX)$. From this we get the map

$$\gamma_X : EX = (P\rho QX)(\mathbb{S}) \longrightarrow (PH\pi_0(\rho QX))(\mathbb{S}) = H\pi_0 X$$

by prolongation and evaluation of the adjunction unit on spheres. Whenever X is a stably fibrant, the component groups $\pi_0 X$ and $\pi_0(\rho QX)$ are isomorphic. The symmetric spectrum associated to the Eilenberg-Mac Lane Γ-space by prolongation and then restriction to spheres is the Eilenberg-Mac Lane model of Definition 5.1.5. □

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