PRACTICE SECOND MIDTERM MATH 18.100B, ANALYSIS I

You may freely use Rudin's book Principles of Mathematical Analysis, your problem sets and your class notes. However, you may not use any other materials. In order to receive full credit on the problems you must prove any assertion that is not proved in Rudin or in the class notes. You may freely quote any theorems proved in Rudin or in class.

Problem 1. Let $r > 0$ be a real number. Consider the function

$$x^r : [0, \infty) \to [0, \infty).$$

Prove that x^r is a continuous function on $[0, \infty)$. Determine, with proof, the values of r for which x^r is uniformly continuous on $[0, \infty)$.

Problem 2. Prove that a topological space X is connected if and only if the only subsets that are both open and closed in X are the empty set and X itself.

Problem 3. Problem 15 on page 115 of Rudin.

Problem 4. A topological space X is called normal if given any two disjoint closed sets C_1, C_2 in X, there exists two open sets U_1, U_2 such that $C_1 \subset U_1$, $C_2 \subset U_2$ and $U_1 \cap U_2 = \emptyset$. Prove that metric spaces are normal. (Hint: Recall problem 20 on page 101 of Rudin and see problem 22 on the same page.)

Problem 5. Let f be a continuous real-valued function on $[-1, 1]$. Suppose that $f > 0$ if $x \neq 0$ and $f(0) = 0$.

(1) Show that the function $\sqrt{f(x)}$ is continuous.

(2) Suppose in addition that f is twice continuously differentiable on $[-1, 1]$. Prove that f may be expressed as

$$f = a_0 + a_1 x + a_2 x^2 g(x)$$

for some continuous function g on $[-1, 1]$. (Hint: Define g using Taylor's Theorem.)

(3) Suppose f is twice continuously differentiable on $[-1, 1]$ and that $f(0) = 0$, $f'(0) = 0$ and $f''(0) = 1$. Show that there exists a continuous function h on $[-1, 1]$ such that $f(x) = (xh(x))^2$.

Problem 6. Use Liouville's theorem about Diophantine approximations to show that

$$\sum_{n=1}^{\infty} \frac{1}{10^{10^{10^n}}}, \quad \sum_{n=1}^{\infty} \frac{1}{10^{10^{n!}}}$$

are not algebraic numbers. Use a similar construction to construct uncountably many non-algebraic numbers. The original number

$$\sum_{n=1}^{\infty} \frac{1}{10^{10^n}}$$

is also not algebraic. You can prove that as well, but this requires a stronger estimate.

Problem 7. Do problem 17 on page 116 of Rudin.