MATH 516 FINAL EXAM

This is the take-home final for Math 516. It is due Wednesday December 4. This is an open book exam. You may discuss the problems with people in the class and you may consult books. However, the final write-up must be yours. You may not collaborate while writing the solutions and you may not use anything other than the text book and your course notes.

Problem 1. Do problems 7.15, 7.16, 7.17, 7.18, 7.19, 7.20 on page 383 of the book.

Problem 2. A cruise company operates three ships A,B,C continually on cruises from Florida to the Bahamas for 7, 13 and 29 day cruises, respectively. The company needs to hire extra hands on days when all three ships dock in Florida simultaneously. If these ships left Florida on Monday, Tuesday and Wednesday of this week, respectively, find all future times when they will be in Florida simultaneously. What would you suggest to the company if they do not want any of the two ships to be in Florida on the same day?

Problem 3. Let R be a commutative ring with unit. If for every $r \in R$, there exists an integer $n_r > 1$ such that $r^{n_r} = r$, prove that every prime ideal of R is maximal.

Problem 4. Let $\phi : \mathbb{Z}^2 \to \mathbb{Z}^3$ be the $\mathbb{Z}$-module homomorphism given by the matrix

$$\begin{pmatrix} 1 & 3 \\ 4 & 2 \\ 8 & 4 \end{pmatrix}.$$

Determine the cokernel of ϕ up to isomorphism.

Problem 5. Let R be a commutative ring with unit. Show that an ideal I is prime if and only if I satisfies the following two conditions

(1) If $I = I_1 \cap I_2$ for two ideals I_1, I_2 in R, then $I = I_1$ or $I = I_2$.
(2) If $a \in R$ and $a^n \in I$ for some positive integer n, then $a \in I$

Problem 6. Prove that any subring R of $\mathbb{Q}$ that contains 1 is a PID.

Problem 7. Prove that if I is an ideal in a Noetherian ring R, then there exists finitely many prime ideals $P_1, \dots, P_m$ such that $P_1 P_2 \cdots P_m \subset I$.

Problem 8. Describe the spectrum of $\mathbb{Z}[i]$ as explicitly as you can.

Problem 9. (1) Show that the ring $\mathbb{Z}[i\sqrt{2}] := \{a + bi\sqrt{2} \mid a, b \in \mathbb{Z}\}$ is a Euclidean domain. (Hint: Use the norm $a^2 + 2b^2$).
(2) Find all the integer solutions of the equation $y^2 + 2 = x^3$.

Problem 10. Let S be a multiplicative set and let I be an ideal disjoint from S. Prove that there exists a prime ideal containing I which is disjoint from S. (One corollary is that the nilradical of a ring is the intersection of all the prime ideals in the ring.)

Problem 11. (1) Prove that $\mathbb{Q}/\mathbb{Z}$ is torsion as a $\mathbb{Z}$ module.
(2) Show, however, that the annihilator ideal of $\mathbb{Q}/\mathbb{Z}$ in $\mathbb{Z}$ is zero.
(3) Deduce that $\mathbb{Q}/\mathbb{Z}$ is not a finitely generated $\mathbb{Z}$-module.

Problem 12. Let $A \in GL_n(\mathbb{C})$ be an element of finite order. Prove that A is diagonalizable. (Hint: What can you say about the minimal polynomial of A?)

Problem 13. (1) Let G be a finite group and let (ρ, V) be a finite dimensional complex representation of G. Show that there exists an inner product on V invariant under the action of G by ρ. (Hint: Start with any inner product, then make it invariant by averaging over G).

(2) Let G be a compact topological group G and let (ρ, V) be a finite dimensional complex representation of G. Show that there exists an inner product on V invariant under the action of G by ρ. (Hint: Average over the group using the Haar measure.)

(3) Show that $SL_2(\mathbb{R})$ acts on the vector space of homogeneous polynomials of degree n in two variables $\mathbb{C}[x, y]$ by change of variables. Prove that the resulting representation is an irreducible representation of dimension $n+1$. Show that these representations are not unitarizable. (Remark: In fact, the finite-dimensional representations of a non-compact Lie group (other than the trivial representation) are not unitarizable. There are infinite dimensional unitary representations of non-compact Lie groups.)

Problem 14 (Waring's problem for quadratic polynomials). Let $f = \sum_{i,j} a_{i,j} x_i x_j$ be a real quadratic form in $\mathbb{R}[x_1, \ldots, x_n]$.

(1) Show that there is a one-to-one correspondence between real quadratic forms f and real symmetric matrices M such that $f = x^T M x$, where x is the column vector with entries consisting of the variables. For example, $2x^2 + 4xy + y^2$ can be expressed as

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(2) Show that f can be written as a linear combination of squares of linear forms.

(3) Show that the minimal number of squares that are needed to express f is equal to the rank of the corresponding symmetric matrix M.

Problem 15. Compute the character table of $\mathfrak{S}_4$.

Problem 16. (1) Show that $M_{2\times 2}(\mathbb{C})$ has the structure of a $\mathbb{C}$ vector space and is isomorphic to $\mathbb{C}^4$. Hence, endow $M_{2\times 2}(\mathbb{C})$ with the Euclidean topology.

(2) Show that $GL_2(\mathbb{C})$ is an open subset in $\mathbb{C}^4$. Hence, it inherits the Euclidean topology from $\mathbb{C}^4$.

(3) Let $O \in M_{2\times 2}(\mathbb{C})$ be an orbit of the $GL_2(\mathbb{C})$ acting by conjugation. Classify all the orbits. (Hint: Think Jordan canonical form.)

(4) Show that an orbit O is closed in $M_{2\times 2}(\mathbb{C})$ if and only if every $A \in O$ is diagonalizable.

(5) Generalize to the conjugation action of $GL_n(\mathbb{C})$ on $M_{n\times n}(\mathbb{C})$.