BASIC INTERSECTION THEORY ON THE MODULI SPACE OF CURVES

Let E be a vector bundle of rank r. To E, we associate the Chern polynomial

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_r(E).$$

The Chern roots of E are the formal roots of $c(E)$, that is

$$c(E) = \prod_{i=1}^{r}(1+\alpha_i).$$

The Chern character of E is defined by

$$ch(E) = \sum_{i=1}^{r} e^{\alpha_i}.$$

Since the Chern character is symmetric in the Chern roots, it can be expressed in terms of the Chern classes. Simple manipulations with power series show that

$$\begin{aligned} ch(E) &= r + c_1(E) + \frac{c_1^2(E) - 2c_2(E)}{2} + \frac{c_1^3(E) - 3c_1(E)c_2(E) + 3c_3(E)}{6} \\ &+ \frac{c_1^4(E) + 4c_1(E)c_3(E) - 4c_1^2(E)c_2(E) + 2c_2^2(E) - 4c_4(E)}{24} + \cdots \end{aligned}$$

The Chern character is a homomorphism from the Grothendieck K-group to cohomology. It is easy to see that it satisfies

$$ch(E \otimes F) = ch(E)ch(F).$$

The Todd class is similarly defined as a formal power series in the Chern roots

$$Td(E) = \prod_{i=1}^{r} \frac{\alpha_i}{1-e^{-\alpha_i}}.$$

Since the Todd class is also symmetric in the Chern roots, it has an expression in terms of the Chern classes.

$$\begin{aligned} Td(E) &= 1 + \frac{c_1(E)}{2} + \frac{c_1^2(E) + c_2(E)}{12} + \frac{c_1(E)c_2(E)}{24} \\ &+ \frac{-c_1^4(E) + 4c_1^2(E)c_2(E) + c_1(E)c_3(E) + 3c_2^2(E) - c_4(E)}{720} + \cdots \end{aligned}$$

The Todd class of a variety is the Todd class of its tangent bundle $Td(X) = Td(T_X)$. The Todd class is multiplicative for short each sequences. If

$$0 \to E \to F \to G \to 0,$$

then $Td(F) = Td(E)Td(G)$.

Exercise 0.1. Verify the formula for the Todd class. Using the Euler sequence, calculate the Todd class of $\mathbb{P}^n$.

Let $\pi : X \to Y$ be a proper morphism. Let E be a vector bundle on X. Recall that

$$\pi_!(E) = \sum_i (-1)^i R^i \pi_*(E).$$

Theorem 0.2 (Grothendieck-Riemann-Roch)**.** *Let $\pi : X \to Y$ be a proper morphism. Let E be a vector bundle and assume that Y is smooth. Then*

$$ch(\pi_!(E)) \cdot Td(Y) = \pi_*(ch(E) \cdot Td(X)).$$

As a warm up, we calculate the number of lines on a general cubic surface in $\mathbb{P}^3$. Consider the incidence correspondence

$$I = \{(p, l) \mid p \in l\} \subset \mathbb{P}^3 \times \mathbb{G}(1, 3).$$

The incidence correspondence admits two natural projections π_1 and π_2 to $\mathbb{P}^3$ and $\mathbb{G}(1, 3)$, respectively. We would like to calculate the Chern classes of the bundle $\pi_{2*}\pi_1^*\mathcal{O}_{\mathbb{P}^3}(3)$. More generally, for $n \geq 0$, let

$$\mathcal{F}_n = \pi_{2*}\pi_1^*\mathcal{O}_{\mathbb{P}^3}(n).$$

Let us calculate the Chern classes of the bundles $\mathcal{F}_n$.

Let S be the tautological bundle on $\mathbb{G}(1, 3)$. The incidence correspondence $I = \mathbb{P}S$ is the two-step flag variety. Let U be the tautological line bundle on I. Then we have the universal sequence

$$0 \to U \to \pi_2^* S \to Q \to 0.$$

The relative tangent bundle of I over $\mathbb{G}(1, 3)$ is given by $T_{I/\mathbb{G}(1,3)} = U^* \otimes Q$. Let $c_1(U) = -h$. Then

$$c(T_{I/\mathbb{G}(1,3)}) = 1 + 2h - \sigma_1.$$

We conclude that

$$Td(T_{I/\mathbb{G}(1,3)}) = 1 + \frac{2h - \sigma_1}{2} + \frac{(2h - \sigma_1)^2}{12} - \frac{(2h - \sigma_1)^4}{720}.$$

In the cohomology of the flag variety I, we have the relations

$$h^2 = h\sigma_1 - \sigma_{1,1}$$

. Therefore,

$$h^3 = h\sigma_2 - \sigma_{2,1}$$

and $h^4 = 0$. Simplifying the expression for the Todd class, we get

$$Td(T_{I/\mathbb{G}(1,3)}) = 1 - \frac{\sigma_1}{2} - \frac{\sigma_{1,1}}{4} + \frac{\sigma_2}{12} - \frac{\sigma_{2,2}}{72} + h.$$

We have that

$$ch(\pi_1^*(\mathcal{O}_{\mathbb{P}^3}(n)) = 1 + nh + \frac{n^2h^2}{2} + \frac{n^3h^3}{6}.$$

Simplifying this expression, we get

$$ch(\pi_1^*(\mathcal{O}_{\mathbb{P}^3}(n)) = 1 - \frac{n^2\sigma_{1,1}}{2} - \frac{n^3\sigma_{2,1}}{6} + (n + \frac{n^2\sigma_1}{2} + \frac{n^3\sigma_2}{6})h.$$

When $n \geq 0$, $\pi_1^*\mathcal{O}_{\mathbb{P}^3}(n)$ has no higher cohomology on the fibers of π_2. Therefore,

$$\pi_{2!}\pi_1^*\mathcal{O}_{\mathbb{P}^3}(n) = \pi_{2*}\pi_1^*\mathcal{O}_{\mathbb{P}^3}(n).$$

Hence, by the Grothendieck-Riemann-Roch Theorem, we conclude that

$$ch(\mathcal{F}_n) = \pi_{2*}(ch(\pi_1^*(\mathcal{O}_{\mathbb{P}^3}(n)) \cdot Td(T_{I/\mathbb{G}(1,3)})).$$

Multiplying out

$$\left(1 - \frac{n^2\sigma_{1,1}}{2} - \frac{n^3\sigma_{2,1}}{6} + (n + \frac{n^2\sigma_1}{2} + \frac{n^3\sigma_2}{6})h\right)\left(1 - \frac{\sigma_1}{2} - \frac{\sigma_{1,1}}{4} + \frac{\sigma_2}{12} - \frac{\sigma_{2,2}}{72} + h\right)$$

and simplifying and taking the Gysin image, we obtain that the Chern character of $\mathcal{F}_n$ is given by

$$ch(\mathcal{F}_n) = n+1+\frac{n^2+n}{2}\sigma_1 - \frac{n^2+n}{4}\sigma_{1,1} + \frac{2n^3+3n^2+n}{12}\sigma_2 - \frac{n^3+n^2}{12}\sigma_{2,1} - \frac{n^3-n}{72}\sigma_{2,2}.$$

We can now solve for the Chern classes of $\mathcal{F}_n$ successively. As expected, the rank of $\mathcal{F}_n$ is $n+1$.

$$c_1(\mathcal{F}_n) = \frac{n^2+n}{2}\sigma_1.$$

Exercise 0.3. Calculate the higher Chern classes of $\mathcal{F}_n$. Show that $\mathcal{F}_1 = S^*$ and in that case we recover that $c_1(S^*) = \sigma_1$ and $c_2(S^*) = \sigma_{1,1}$. Show that more generally $\mathcal{F}_n = Sym^n(S^*)$. Show that $c_1(\mathcal{F}_3) = 6\sigma_1$, $c_2(\mathcal{F}_3) = 21\sigma_{1,1} + 11\sigma_2$, $c_3(\mathcal{F}_3) = 43\sigma_{2,1}$ and $c_4(\mathcal{F}_3) = 27\sigma_{2,2}$.

We now apply the Grothendieck-Riemann-Roch formula to obtain relations among classes on the moduli space of curves. First, suppose that $\pi : X \to B$ is a smooth one parameter family of stable curves of genus g. Let $\gamma = c_1(\omega_{X/B})$. The relative tangent bundle is the dual of the dualizing sheaf. Hence,

$$Td(\omega^*_{X/B}) = 1 - \frac{\gamma}{2} + \frac{\gamma^2}{12} + \cdots$$

By the Grothendieck-Riemann-Roch formula,

$$\begin{aligned} ch(\pi_!\omega_{X/B}) &= \pi_*(1 - \frac{\gamma}{2} + \frac{\gamma^2}{12} + \cdots)(1 + \gamma + \frac{\gamma^2}{2} + \cdots) \\ &= \pi_*(1 + \frac{\gamma}{2} + \frac{\gamma^2}{12} + \cdots). \end{aligned}$$

$R^1\pi_*(\omega_{X/B})$ is the trivial bundle. Hence,

$$ch(\pi_!\omega_{X/B}) = ch(\Lambda) - 1.$$

Equating the two sides, we see that

$$c_1(\Lambda) = \frac{\kappa}{12}, \quad c_2(\Lambda) = \frac{\kappa^2}{288}.$$

If the family is not smooth, the calculation has to be slightly altered. Suppose that $\pi : X \to B$ is a one-parameter family of stable curves. Resolve any A_k singularity by blowing up $\nu : Y \to X$, to obtain a semi-stable family with smooth total space $\phi : Y \to B$. Since the singularities of X are canonical, $\nu^*(\omega_{X/B}) = \omega_{Y/B}$. Let

$$\delta = \delta_0 + \delta_1 + \cdots + \delta_{\lfloor \frac{g}{2} \rfloor}.$$

Let Z in Y be the locus of nodes in the fibers of ϕ. Then $\phi_*([Z) = \delta \cdot B$. We have to calculate the contribution of the nodes in Y to the relative dualizing sheaf. The local equations at the node are $t = xy$. The map $\phi^*T^*_B \to T^*_Y$ is very explicitly given by $\mathcal{O}_Y(dt) \to \mathcal{O}_Y(dx, dy)$ sending

$$dt \mapsto xdy + ydx.$$

The cokernel is the relative cotangent sheaf

$$\Omega_{Y/B} = \frac{\mathcal{O}_Y(dx, dy)}{< xdy + ydx >}.$$

The relative dualizing sheaf is the locally free rank one sheaf whose restriction to $Y \backslash Z$ is isomorphic to the relative cotangent bundle.

Hence, $\Omega_{Y/B} = I_Z \otimes \omega_{Y/B}$. Let η be the class of Z. Applying Grothendieck-Riemann-Roch to the inclusion $i : Z \to Y$, we get that

$$ch(i_*\mathcal{O}_Z) = i_*(ch(\mathcal{O}_Z) \cdot Td(T_Z - i * T_Y)) = i_*(\eta).$$

Using the standard exact sequence

$$0 \to I_Z \to \mathcal{O}_Y \to \mathcal{O}_Z \to 0,$$

we get that

$$ch(I_Z) = 1 - \eta.$$

Combining these results, we have that

$$ch(\Omega) = ch(\omega)ch(I_Z) = 1 + \gamma + (\frac{\gamma^2}{2} - \eta) + \cdots$$

$$Td(Y/B) = 1 - \frac{\gamma}{2} + \frac{\gamma^2 + \eta}{12} + \cdots$$

Finally, we get that $\kappa = 12\lambda - \delta$.

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