Math 180, Exam 1, Fall 2007
Problem 1 Solution

1. Find the limit, $\lim_{x \to 1} \dfrac{x^2 + 2x - 3}{x^2 - x}$.

Solution: Upon substituting $x = 1$ into the function $f(x) = \dfrac{x^2 + 2x - 3}{x^2 - x}$ we find that

$$\frac{x^2 + 2x - 3}{x^2 - x} = \frac{1^2 + 2(1) - 3}{1^2 - 1} = \frac{0}{0}$$

which is indeterminate. We can resolve the indeterminacy by factoring the numerator and denominator of $f(x)$.

$$\lim_{x \to 1} \frac{x^2 + 2x - 3}{x^2 - x} = \lim_{x \to 1} \frac{(x+3)(x-1)}{x(x-1)} = \lim_{x \to 1} \frac{x+3}{x} = \frac{1+3}{1} = \boxed{4}$$

In the final step above we were able to plug in $x = 1$ by using the fact that the function $\dfrac{x+3}{x}$ is continuous at $x = 1$. In fact, $\dfrac{x+3}{x}$ is continuous at all values of x in its domain ($x \neq 0$).

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Problem 2 Solution

2. Find the derivatives of the following functions using the basic rules. Do not simplify your answer.

(a) $4x^3 - 5x^{1/3} + 3x^{-2}$ (b) $(x^2 - 3x)e^x$ (c) $\dfrac{x-3}{x^2+x+1}$

Solution:

(a) Use the Power Rule.

$$(4x^3 - 5x^{1/3} + 3x^{-2})' = \boxed{12x^2 - \frac{5}{3}x^{-2/3} - 6x^{-3}}$$

(b) Use the Product Rule.

$$\begin{aligned}[(x^2 - 3x)e^x]' &= (x^2 - 3x)(e^x)' + (x^2 - 3x)'e^x \\ &= \boxed{(x^2 - 3x)e^x + (2x - 3)e^x}\end{aligned}$$

(c) Use the Quotient Rule.

$$\begin{aligned}\left(\frac{x-3}{x^2+x+1}\right)' &= \frac{(x^2+x+1)(x-3)' - (x-3)(x^2+x+1)'}{(x^2+x+1)^2} \\ &= \boxed{\frac{(x^2+x+1) - (x-3)(2x+1)}{(x^2+x+1)^2}}\end{aligned}$$

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Problem 3 Solution

3. Find the equation of the tangent line to $y = x^3 - 3x$ at $x = 2$.

Solution: The derivative y' is found using the Power Rule.

$$y' = (x^3 - 3x)' = 3x^2 - 3$$

At $x = 2$ the values of y and y' are:

$$y(2) = 2^3 - 3(2) = 2$$
$$y'(2) = 3(2)^2 - 3 = 9$$

We now know that the point $(2, 2)$ is on the tangent line and that the slope of the tangent line is 9. Therefore, an equation for the tangent line in point-slope form is:

$$\boxed{y - 2 = 9(x - 2)}$$

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Problem 4 Solution

4. Let $f(x) = \sqrt{x}$.

(a) Find the average rate of change of $f(x)$ over the interval $4 \le x \le 9$.

(b) Find the instantaneous rate of change of $f(x)$ at $x = 4$.

Solution:

(a) The average rate of change formula is:

$$\text{average ROC} = \frac{f(b) - f(a)}{b - a}$$

Using $f(x) = \sqrt{x}$, $b = 9$, and $a = 4$ we have:

$$\text{average ROC} = \frac{\sqrt{9} - \sqrt{4}}{9 - 4} = \frac{3 - 2}{5} = \boxed{\frac{1}{5}}$$

(b) The instantaneous rate of change at $x = 4$ is $f'(4)$. The derivative $f'(x)$ is:

$$f'(x) = \frac{1}{2\sqrt{x}}$$

At $x = 4$ we have:

$$\text{instantaneous ROC} = f'(4) = \frac{1}{2\sqrt{4}} = \boxed{\frac{1}{4}}$$

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Problem 5 Solution

5. Let $f(x) = \dfrac{1}{x}$.

(a) Write the derivative, $f'(5)$, as the limit of the difference quotient.

(b) Evaluate this limit to find $f'(5)$.

Solution:

(a) There are two possible difference quotients we can use to evaluate $f'(5)$. One is:

$$f'(5) = \lim_{h\to 0} \frac{f(h+5) - f(5)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{h+5} - \dfrac{1}{5}}{h}.$$

The other is:

$$f'(5) = \lim_{x\to 5} \frac{f(x) - f(5)}{x-5} = \lim_{x\to 5} \frac{\dfrac{1}{x} - \dfrac{1}{5}}{x-5}$$

(b) Evaluating the first limit above we have:

$$\begin{aligned}
f'(5) &= \lim_{h\to 0} \frac{\dfrac{1}{h+5} - \dfrac{1}{5}}{h} \cdot \frac{5(h+5)}{5(h+5)} \\
&= \lim_{h\to 0} \frac{5-(h+5)}{5h(h+5)} \\
&= \lim_{h\to 0} \frac{-h}{5h(h+5)} \\
&= \lim_{h\to 0} \frac{-1}{5(h+5)} \\
&= \frac{-1}{5(0+5)} \\
&= \boxed{-\frac{1}{25}}
\end{aligned}$$

Evaluating the second limit we have:

$$\begin{aligned}
f'(5) &= \lim_{x\to 5} \frac{\frac{1}{x} - \frac{1}{5}}{x-5} \cdot \frac{5x}{5x} \\
&= \lim_{x\to 5} \frac{5-x}{5x(x-5)} \\
&= \lim_{x\to 5} \frac{-(x-5)}{5x(x-5)} \\
&= \lim_{x\to 5} \frac{-1}{5x} \\
&= \frac{-1}{5(5)} \\
&= \boxed{-\frac{1}{25}}
\end{aligned}$$

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Problem 6 Solution

6. Use the table below, which shows values of $f(x)$ for x near 2.5,

x	2.3	2.4	2.5	2.6	2.7
$f(x)$	1.41	1.40	1.38	1.35	1.31

to find the slope of a secant line that is an estimate for $f'(2.5)$. Why did you choose the line you did?

Solution: An approximate value for $f'(2.5)$ is

$$f'(2.5) \approx \frac{f(2.6) - f(2.5)}{2.6 - 2.5} = \frac{1.35 - 1.38}{0.1} = \boxed{-0.3}$$

This formula was used because the exact value of $f'(2.5)$ is:

$$f'(2.5) = \lim_{x \to 2.5} \frac{f(x) - f(2.5)}{x - 2.5}$$

As we approach $x = 2.5$ from the right, we can plug in either $x = 2.7$ or $x = 2.6$ to estimate the value of $f'(2.5)$. We used $x = 2.6$ because the estimate is generally more accurate as x gets closer and closer to 2.5.