Math 180, Exam 1, Fall 2011
Problem 1 Solution

1. Evaluate the following limits. Show your work.

(a) $\lim_{x\to 0} \dfrac{2\cos x}{\sqrt{x+1}-2}$

(b) $\lim_{x\to +\infty} \dfrac{2x^2+4x-1}{x^3+1}$

Solution:

(a) The function $f(x) = \frac{2\cos x}{\sqrt{x+1}-2}$ is continuous for all $x \in (-1,3)\cup(3,+\infty)$. Therefore, since $f(x)$ is continuous at $x=0$, we can evaluate the limit using substitution.

$$\lim_{x\to 0} \frac{2\cos x}{\sqrt{x+1}-2} = \frac{2\cos 0}{\sqrt{0+1}-2} = \boxed{-2}$$

(b) This is a limit at infinity of a rational function. Our approach is to multiply the function by $\frac{1}{x^3}$ divided by itself and simplify:

$$\begin{aligned}\lim_{x\to +\infty} \frac{2x^2+4x-1}{x^3+1} &= \lim_{x\to +\infty} \frac{2x^2+4x-1}{x^3+1} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} \\ &= \lim_{x\to +\infty} \frac{\frac{2}{x}+\frac{4}{x^2}-\frac{1}{x^3}}{1+\frac{1}{x^3}}\end{aligned}$$

Using the fact that $\lim_{x\to+\infty} \frac{1}{x^n} = 0$ for $n>0$, we find that:

$$\begin{aligned}\lim_{x\to +\infty} \frac{2x^2+4x-1}{x^3+1} &= \lim_{x\to +\infty} \frac{\frac{2}{x}+\frac{4}{x^2}-\frac{1}{x^3}}{1+\frac{1}{x^3}} \\ &= \frac{0+0-0}{1+0} \\ &= \boxed{0}\end{aligned}$$

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Problem 2 Solution

2. Compute the derivatives of the following functions AND state where the derivative does not exist. Show your work and do not simplify your answers.

(a) $\dfrac{x^2+1}{x}$

(b) $|x|$

(c) $e^{\sin(3x)}$

Solution:

(a) We begin by rewriting the function as follows:

$$\frac{x^2+1}{x} = \frac{x^2}{x} + \frac{1}{x} = x + \frac{1}{x}$$

We now use the Power Rule to compute the derivative:

$$\left(\frac{x^2+1}{x}\right)' = \left(x + \frac{1}{x}\right)' = 1 - \frac{1}{x^2}$$

The derivative exists for all $x \neq 0$.

(b) By definition, the absolute value function is the piecewise defined function:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The derivative of $|x|$ is then:

$$(|x|)' = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

The derivative exists for all $x \neq 0$. It does not exist at $x = 0$ because the limit

$$\lim_{h \to 0} \frac{|0+h| - |0|}{h},$$

which defines the derivative of $|x|$ at $x = 0$, does not exist (the one-sided limits are 1 and -1).

(c) We use the Chain Rule.

$$\left(e^{\sin(3x)}\right)' = e^{\sin(3x)} \left(\sin(3x)\right)' = e^{\sin(3x)} \cdot 3\cos(3x)$$

The derivative exists for all x.

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Problem 3 Solution

3. (a) Find an equation of the tangent line at $x_0 = 1$ to the graph of the following function:

$$f(x) = x^4 - x^2 + 1$$

(b) Find all those points x_0 where the tangent line to the graph is horizontal. Show your work.

Solution: (a) The slope of the tangent line is the derivative $f'(1)$ and a point on the tangent line is $(1, f(1))$. The derivative of $f(x)$ is $f'(x) = 4x^3 - 2x$. Therefore, $f'(1) = 2$. We also have $f(1) = 1$. Thus, the equation of the tangent line in point-slope form is:

$$\boxed{y - 1 = 2(x - 1)}$$

(b) A horizontal line has a slope of 0. Therefore, we seek the values of x satisfying $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ 4x^3 - 2x &= 0 \\ 2x(2x^2 - 1) &= 0 \end{aligned}$$

We either have $2x = 0$ or $2x^2 - 1 = 0$. The first equation gives us $\boxed{x = 0}$ while the second equation gives us $\boxed{x = \pm\frac{1}{\sqrt{2}}}$.

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Problem 4 Solution

4. Use the Intermediate Value Theorem to show that there exists a solution to the equation $\cos x = x$ on the interval $[0, \frac{\pi}{2}]$. Show your work.

Solution: Let $f(x) = \cos(x) - x$. First we recognize that $f(x)$ is continuous everywhere. Next, we must show that $f(0)$ and $f(\frac{\pi}{2})$ have opposite signs.

$$f(0) = \cos(0) - 0 = 1$$
$$f(1) = \cos(\tfrac{\pi}{2}) - \tfrac{\pi}{2} = -\tfrac{\pi}{2}$$

Since $f(0) > 0$ and $f(\frac{\pi}{2}) < 0$, the Intermediate Value Theorem tells us that $f(c) = 0$ for some c in the interval $(0, \frac{\pi}{2})$.

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Problem 5 Solution

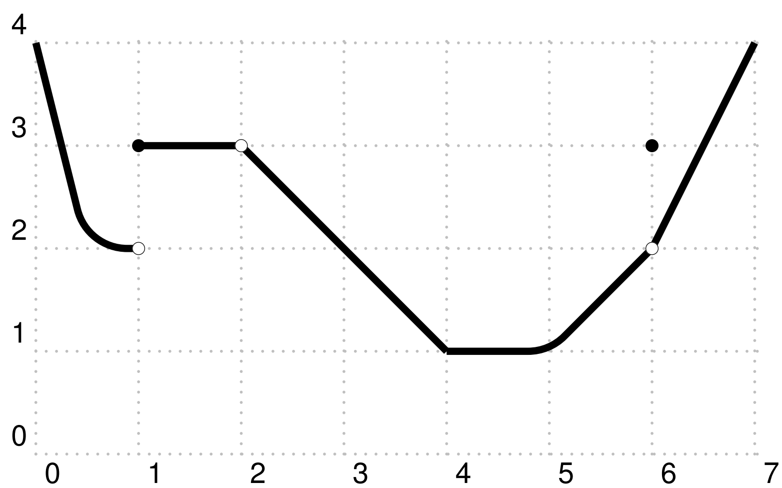
5. Consider the function f whose graph appears below and answer the following questions. **You must justify all answers.**

(a) (i) Is $f(1)$ defined? If so, what is it?
(ii) Does $\lim_{x\to 1} f(x)$ exist? If so, what is it?
(iii) Is f continuous at 1?

(b) (i) Is $f(2)$ defined? If so, what is it?
(ii) Does $\lim_{x\to 2} f(x)$ exist? If so, what is it?
(iii) Is f continuous at 2?

(c) (i) Is $f(4)$ defined? If so, what is it?
(ii) Does $\lim_{x\to 4} f(x)$ exist? If so, what is it?
(iii) Is f continuous at 4?

(d) (i) Is $f(6)$ defined? If so, what is it?
(ii) Does $\lim_{x\to 6} f(x)$ exist? If so, what is it?
(iii) Is f continuous at 6?



Solution:

(a) (i) $f(1) = 3$
(ii) $\lim_{x\to 1} f(x)$ does not exist because the one-sided limits are not the same ($\lim_{x\to 1^+} f(x) = 3$ but $\lim_{x\to 1^-} f(x) = 2$).
(iii) f is not continuous at 1 because $\lim_{x\to 1} f(x) \neq f(1)$

(b) (i) $f(2)$ is not defined
(ii) $\lim_{x \to 2} f(x) = 3$ because the one-sided limits are the same ($\lim_{x \to 2^+} f(x) = 3$ and $\lim_{x \to 2^-} f(x) = 3$).
(iii) f is not continuous at 2 because $\lim_{x \to 2} f(x) \neq f(2)$

(c) (i) $f(4) = 1$
(ii) $\lim_{x \to 4} f(x) = 1$ because the one-sided limits are the same ($\lim_{x \to 4^+} f(x) = 1$ and $\lim_{x \to 4^-} f(x) = 1$).
(iii) f is continuous at 4 because $\lim_{x \to 4} f(x) = f(4)$

(d) (i) $f(6) = 3$
(ii) $\lim_{x \to 6} f(x) = 2$ because the one-sided limits are the same ($\lim_{x \to 6^+} f(x) = 2$ and $\lim_{x \to 6^-} f(x) = 2$).
(iii) f is not continuous at 6 because $\lim_{x \to 6} f(x) \neq f(6)$