Math 180, Exam 1, Fall 2012
Problem 1 Solution

1. Compute the derivatives of the following functions:

(a) $y = \dfrac{x}{1+x^3}$

(b) $y = \sin(\sqrt{x}+1)$

(c) $y = x \cdot \sqrt{\sin(x)+1}$

Solution:

(a) The derivative may be computed using the Quotient Rule.

$$\frac{dy}{dx} = \frac{(1+x^3)(x)' - (x)(1+x^3)'}{(1+x^3)^2}$$
$$\frac{dy}{dx} = \frac{(1+x^3)(1) - (x)(3x^2)}{(1+x^3)^2}$$
$$\frac{dy}{dx} = \frac{1-2x^3}{(1+x^3)^2}$$

(b) The derivative calculation here requires the Chain Rule.

$$\frac{dy}{dx} = \cos(\sqrt{x}+1)\,\frac{d}{dx}\left(\sqrt{x}+1\right)$$
$$\frac{dy}{dx} = \cos(\sqrt{x}+1)\cdot\frac{1}{2\sqrt{x}}$$

(c) Both the Product and Chain Rules are necessary to compute this derivative.

$$\frac{dy}{dx} = (x)\left(\sqrt{\sin(x)+1}\right)' + (x)'\sqrt{\sin(x)+1}$$
$$\frac{dy}{dx} = x\cdot\frac{1}{2\sqrt{\sin(x)+1}}\cdot\frac{d}{dx}(\sin(x)+1) + 1\cdot\sqrt{\sin(x)+1}$$
$$\frac{dy}{dx} = x\cdot\frac{1}{2\sqrt{\sin(x)+1}}\cdot\cos(x) + \sqrt{\sin(x)+1}$$

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Problem 2 Solution

2. Find the equation of the tangent line to $y = \sqrt{x+1}$ at $x = 3$.

Solution: To find an equation for the tangent line we need a point on the line and the slope of the line. The point has an x-coordinate of 3 and a y-coordinate of

$$y(3) = \sqrt{3+1} = 2$$

The slope of the tangent line is the value of the derivative $\frac{dy}{dx}$ at $x = 3$.

$$\left.\frac{dy}{dx}\right|_{x=3} = \left.\frac{1}{2\sqrt{x+1}}\right|_{x=3} = \frac{1}{2\sqrt{3+1}} = \frac{1}{4}$$

Thus, the equation of the tangent line in point-slope form is

$$\boxed{y - 2 = \frac{1}{4}(x-3)}$$

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Problem 3 Solution

3. Evaluate the following limits, or show that they do not exist.

(a) $\lim_{x \to 1} \frac{x^2 - 1}{x^2 - 2x + 1}$

(b) $\lim_{x \to 2} \frac{|x^2 - 3|}{x^2 - 1}$

(c) $\lim_{x \to \infty} \frac{x + \sqrt{x^2 + 1}}{2x - 1}$

Solution:

(a) The limit has the indeterminate form $\frac{0}{0}$, which is apparent upon substituting $x = 1$ into the function. However, the numerator and denominator factor nicely enough to allow us to simplify the limit as follows:

$$\begin{aligned} \lim_{x \to 1} \frac{x^2 - 1}{x^2 - 2x + 1} &= \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)^2} \\ &= \lim_{x \to 1} \frac{x+1}{x-1} \end{aligned}$$

The corresponding one-sided limits are

$$\begin{aligned} \lim_{x \to 1^+} \frac{x+1}{x-1} &= \infty, \\ \lim_{x \to 1^-} \frac{x+1}{x-1} &= -\infty \end{aligned}$$

Since the one-sided limits are not the same, we say that the limit does not exist.

(b) The function is continuous at $x = 2$. Therefore, we may evaluate the limit using substitution.

$$\lim_{x \to 2} \frac{|x^2 - 3|}{x^2 - 1} = \frac{|2^2 - 3|}{2^2 - 1} = \frac{1}{3}$$

(c) We multiply the numerator and denominator of the function by $\frac{1}{x}$ to obtain

$$\lim_{x \to \infty} \frac{x + \sqrt{x^2 + 1}}{2x - 1} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x \to \infty} \frac{1 + \sqrt{1 + \frac{1}{x^2}}}{2 - \frac{1}{x}} = \frac{1 + \sqrt{1 + 0}}{2 - 0} = 1$$

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Problem 4 Solution

4. Consider the equation $x^2 - \cos(\pi x) - 1 = 0$.

(a) Use the Intermediate Value Theorem to show that it has a solution in the interval $[0, 1]$.

(b) Find an interval of length $\frac{1}{2}$ that contains a solution of this equation.

Solution:

(a) Let $f(x) = x^2 - \cos(\pi x) - 1$. Since $f(0) = -2$ and $f(1) = 1$ have opposite signs and f is continuous on $(0, 1)$, by the Intermediate Value Theorem we know that there is at least one number c in $(0, 1)$ such that $f(c) = 0$.

(b) The interval $(0, 1)$ has length 1. To find an interval of length $\frac{1}{2}$ that contains a solution to the equation, we evaluate f at the midpoint of $(0, 1)$ and determine its sign. We have

$$f(\tfrac{1}{2}) = -\tfrac{3}{4}$$

which is negative. Thus, since $f(1) = 1$ is positive we know that there is a solution on the interval $(\frac{1}{2}, 1)$. This interval has length $\frac{1}{2}$.

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Problem 5 Solution

5. Find the horizontal and vertical asymptotes (if they exist) for

$$f(x) = \frac{x}{x^2 - 3x + 2}$$

Solution:

- The line $x = c$ is a vertical asymptote of $f(x)$ if

$$\lim_{x \to c^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to c^-} f(x) = \pm\infty$$

  Since the roots of the denominator of $f(x) = \dfrac{x}{(x-1)(x-2)}$ are 1 and 2, we know that f has infinite discontinuities there. That is,

$$\lim_{x \to 1^+} \frac{x}{(x-1)(x-2)} = \frac{1}{(+\text{SMALL})(-1)} = -\infty$$

  and

$$\lim_{x \to 2^+} \frac{x}{(x-1)(x-2)} = \frac{2}{(1)(+\text{SMALL})} = +\infty$$

  Thus, $x = 1$ and $x = 2$ are vertical asymptotes of f.

- The line $y = c$ is a horizontal asymptote of $f(x)$ if

$$\lim_{x \to +\infty} f(x) = c \quad \text{or} \quad \lim_{x \to -\infty} f(x) = c$$

  Since

$$\lim_{x \to \pm\infty} \frac{x}{x^2 - 3x + 2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x \to \pm\infty} \frac{\frac{1}{x}}{1 - \frac{3}{x} + \frac{2}{x^2}} = \frac{0}{1 - 0 + 0} = 0$$

  we know that $y = 0$ is a horizontal asymptote of f.

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Problem 6 Solution

6. Use the definition of the derivative as a limit of a difference quotient to compute $f'(3)$ for the function $f(x) = x^2 + 2x - 5$.

Solution: The derivative $f'(3)$ may be computed as follows:

$$\begin{aligned}
f'(3) &= \lim_{x\to 3} \frac{f(x) - f(3)}{x - 3} \\
&= \lim_{x\to 3} \frac{x^2 + 2x - 5 - 10}{x - 3} \\
&= \lim_{x\to 3} \frac{x^2 + 2x - 15}{x - 3} \\
&= \lim_{x\to 3} \frac{(x - 3)(x + 5)}{x - 3} \\
&= \lim_{x\to 3} (x + 5) \\
&= 3 + 5 \\
&= 8
\end{aligned}$$