Math 180, Exam 1, Fall 2013
Problem 1 Solution

1. Calculate each limit below.

(a) $\lim_{x \to 7} \left(\frac{14}{x^2 - 7x} - \frac{2}{x-7} \right)$

(b) $\lim_{x \to \infty} \frac{19x^4 + 2x - 1}{3x^4 + 16x^2 + 100}$

Solution:

(a) The least common denominator of the function is $x^2 - 7x$. Thus, the function can be written as follows:

$$f(x) = \frac{14}{x^2 - 7x} - \frac{2}{x-7} = \frac{14}{x(x-7)} - \frac{2x}{x(x-7)} = \frac{14 - 2x}{x(x-5)} = \frac{-2(x-7)}{x(x-7)} = -\frac{2}{x}$$

provided that $x \neq 7$. Therefore, the limit of $f(x)$ as $x \to 7$ is

$$\boxed{\lim_{x \to 7} \left(\frac{14}{x^2 - 7x} - \frac{2}{x-7} \right) = \lim_{x \to 7} \left(-\frac{2}{x} \right) = -\frac{2}{7}}$$

(b) The function is rational and the degrees of the numerator and denominator are the same. Therefore, the limit of f as $x \to \infty$ is the ratio of the leading coefficients.

$$\boxed{\lim_{x \to \infty} \frac{19x^4 + 2x - 1}{3x^4 + 16x^2 + 100} = \frac{19}{3}}.$$

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Problem 2 Solution

2. If $f(x) = \sqrt{3x+1}$, calculate

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

Solution: It is easiest to calculate the limit by recognizing that, by definition,

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x).$$

Given that $f(x) = \sqrt{2x+1}$, we can use the Chain Rule:

$$f'(x) = \frac{d}{dx}\sqrt{3x+1}$$
$$f'(x) = \frac{1}{2\sqrt{3x+1}} \cdot \frac{d}{dx}(3x+1)$$
$$\boxed{f'(x) = \frac{1}{2\sqrt{3x+1}} \cdot 3}$$

The other method which almost every student used was to set up and evaluate the limit directly. Here's the calculation:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{3(x+h)+1}-\sqrt{3x+1}}{h}$$
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{3x+3h+1}-\sqrt{3x+1}}{h} \cdot \frac{\sqrt{3x+3h+1}+\sqrt{3x+1}}{\sqrt{3x+3h+1}+\sqrt{3x+1}}$$
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(3x+3h+1)-(3x+1)}{h(\sqrt{3x+3h+1}+\sqrt{3x+1})}$$
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{3h}{h(\sqrt{3x+3h+1}+\sqrt{3x+1})}$$
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{3}{\sqrt{3x+3h+1}+\sqrt{3x+1}}$$
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \frac{3}{\sqrt{3x+3(0)+1}+\sqrt{3x+1})}$$
$$\boxed{\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \frac{3}{2\sqrt{3x+1}}}$$

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Problem 3 Solution

3.

(a) Let $y = e^{2x}\cos(x)$. Find y''. **You do not need to simplify your answers!**

(b) Rewrite $\tan(x)$ in terms of $\sin(x)$ and $\cos(x)$ and use the quotient rule to show that $\frac{d}{dx}\tan(x) = \sec^2(x)$.

(c) Find $\frac{d}{d\theta}\cot(\sin\theta + 3\theta^4)$.

Solution:

(a) Using the Product and Chain Rules, the first derivative is

$$\begin{aligned}
y' &= e^{2x} \cdot \frac{d}{dx}\cos(x) + \cos(x) \cdot \frac{d}{dx}e^{2x} \\
y' &= e^{2x} \cdot (-\sin(x)) + \cos(x) \cdot (2e^{2x}) \\
y' &= -e^{2x} \cdot \sin(x) + 2e^{2x} \cdot \cos(x) \\
&\boxed{y' = e^{2x} \cdot (-\sin(x) + 2\cos(x))}
\end{aligned}$$

Another application of the Product and Chain Rules yields the second derivative:

$$\begin{aligned}
y'' &= e^{2x} \cdot \frac{d}{dx}(-\sin(x) + 2\cos(x)) + (-\sin(x) + 2\cos(x)) \cdot \frac{d}{dx}e^{2x} \\
y' &= e^{2x} \cdot (-\cos(x) - 2\sin(x)) + (-\sin(x) + 2\cos(x)) \cdot (2e^{2x}) \\
y' &= -e^{2x} \cdot \cos(x) - 2e^{2x} \cdot \sin(x) - 2e^{2x} \cdot \sin(x) + 4e^{2x} \cdot \cos(x) \\
&\boxed{y' = -2e^{-x}\cos(x)}
\end{aligned}$$

(b) By definition,

$$\tan(x) = \frac{\sin(x)}{\cos(x)}.$$

Using the Quotient Rule yields

$$\begin{aligned}
\frac{d}{dx}\tan(x) &= \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) \\
\frac{d}{dx}\tan(x) &= \frac{\cos(x) \cdot \frac{d}{dx}\sin(x) - \sin(x) \cdot \frac{d}{dx}\cos(x)}{\cos^2(x)} \\
\frac{d}{dx}\tan(x) &= \frac{\cos(x) \cdot \cos(x) - \sin(x) \cdot (-\sin(x))}{\cos^2(x)} \\
\frac{d}{dx}\tan(x) &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\
\frac{d}{dx}\tan(x) &= \frac{1}{\cos^2(x)} \\
\frac{d}{dx}\tan(x) &= \sec^2(x)
\end{aligned}$$

(c) Using the Chain Rule we have:

$$\frac{d}{d\theta}\cot(\sin\theta + 3\theta^4) = -\csc^2(\sin\theta + 3\theta^4)\cdot\frac{d}{d\theta}(\sin\theta + 3\theta^4)$$
$$\frac{d}{d\theta}\cot(\sin\theta + 3\theta^4) = -\csc^2(\sin\theta + 3\theta^4)\cdot(\cos\theta + 12\theta^3)$$

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Problem 4 Solution

4. Let f be defined by

$$f(x) = \begin{cases} x^4 + (1+A)e^x, & \text{if } x < 0 \\ -B, & \text{if } x = 0 \\ \sin(x), & \text{if } x > 0 \end{cases}$$

where A and B are constants. Find values for A and B such that f is continuous on $(-\infty, \infty)$ or state that no such constants exist. Justify your answer.

Solution: First, the function $x^4 + (1+A)e^x$ is continuous on $x < 0$ for any value A. Second, the function $\sin(x)$ is continuous on $x > 0$.

We must ensure that f is continuous at $x = 0$. That is, we must select A and B so that

$$\lim_{x \to 0} f(x) = f(0)$$

The limit exists when the one-sided limits are the same.

$$\begin{aligned} \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} \sin(x) = \sin(0) = 0 \\ \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} (x^4 + (1+A)e^x) = 0^4 + (1+A)e^0 = 1 + A \end{aligned}$$

These limits are the same when $A = -1$ and in both cases, the limit is 0. Since $f(0) = -B$ we must then have $B = 0$ for continuity at $x = 0$.

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Problem 5 Solution

5. Assume the tangent line to the graph of f at $x = 1$ is given by

$$y = 4x + 2.$$

(a) Find $f(1)$.

(b) Find $f'(1)$.

(c) Now assume that a function g is defined by $g(x) = f(x^3)$. Find $g(1)$ and $g'(1)$.

Solution:

(a) When $x = 1$, the y-coordinate of the point on the tangent line is

$$y = 4(1) + 2 = 6$$

Since the line is tangent to the graph of f at $x = 1$, we know that the point $(1, 6)$ is common to both graphs. Thus, $\boxed{f(1) = 6}$.

(b) The quantity $f'(1)$ is the slope of the tangent line. Thus, $\boxed{f'(1) = 4}$.

(c) We know that $\boxed{g(1) = f(1^3) = f(1) = 6}$ (see part (a)).

To obtain $g'(1)$ we begin by writing an expression for $g'(x)$ using the Chain Rule.

$$g'(x) = \frac{d}{dx} f(x^3) = f'(x^3) \cdot \frac{d}{dx} x^3 = f'(x^3) \cdot 3x^2$$

When $x = 1$ we have

$$\boxed{g'(1) = f'(1^3) \cdot 3(1)^2 = f'(1) \cdot 3 = 4 \cdot 3 = 12}$$