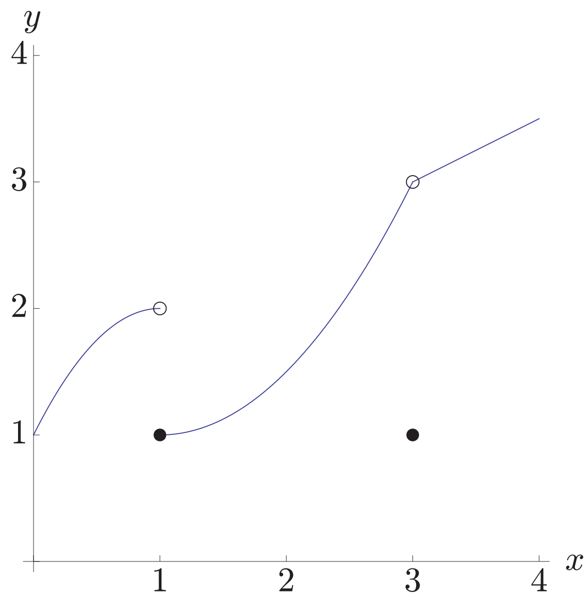
Math 180, Exam 1, Spring 2008
Problem 1 Solution

1. The graph of a function $f(x)$ is shown below.

   (a) At which values of x is f discontinuous? Which of these discontinuities are removable? Which are jump discontinuities?

   (b) Determine the limit at each removable discontinuity.

   (c) Determine the left and right limits at each jump discontinuity.



Solution:

(a) f is discontinuous at $x = 1$ and $x = 3$. f has a removable discontinuity at $x = 3$ because, although both $\lim_{x\to 3} f(x) = 3$ and $f(3) = 1$ exist, they are not equal to each other. f has a jump discontinuity at $x = 1$ because the one-sided limits $\lim_{x\to 1^+} f(x) = 1$ and $\lim_{x\to 1^-} f(x) = 2$ are not equal.

(b) At the removable discontinuity $x = 3$, we have $\lim_{x\to 3} f(x) = 3$.

(c) At the jump discontinuity $x = 1$, we have $\lim_{x\to 1^-} f(x) = 2$ and $\lim_{x\to 1^+} f(x) = 1$.

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Problem 2 Solution

2. Evaluate the following limits, or show they do not exist.

(a) $\lim_{x\to\pi} 3\cos(x+\pi)$

(b) $\lim_{x\to -2} \frac{x^2-4}{x-2}$

(c) $\lim_{x\to 9} \frac{2-\sqrt{x-5}}{x-9}$

Solution:

(a) The function $f(x) = 3\cos(x+\pi)$ is continuous at $x = \pi$. In fact, $f(x)$ is continuous at all values of x in the interval $(-\infty, \infty)$. Therefore, we can evaluate the limit using substitution.

$$\lim_{x\to\pi} 3\cos(x+\pi) = 3\cos(\pi+\pi) = 3\cos(2\pi) = \boxed{3}$$

(b) The function $f(x) = \frac{x^2-4}{x-2}$ is continuous at $x = -2$. In fact, $f(x)$ is continuous for all $x \neq 2$. Therefore, we can evaluate the limit using substitution.

$$\lim_{x\to -2} \frac{x^2-4}{x-2} = \frac{(-2)^2-4}{-2-2} = \boxed{0}$$

(c) When substituting $x = 9$ into the function $f(x) = \frac{2-\sqrt{x-5}}{x-9}$, we find that

$$\frac{2-\sqrt{x-5}}{x-9} = \frac{2-\sqrt{9-5}}{9-9} = \frac{0}{0}$$

which is indeterminate. We can resolve the indeterminacy by multiplying $f(x)$ by the "conjugate" of the numerator divided by itself.

$$\begin{aligned}
\lim_{x\to 9} \frac{2-\sqrt{x-5}}{x-9} &= \lim_{x\to 9} \frac{2-\sqrt{x-5}}{x-9} \cdot \frac{2+\sqrt{x-5}}{2+\sqrt{x-5}} \\
&= \lim_{x\to 9} \frac{4-(x-5)}{(x-9)(2+\sqrt{x-5})} \\
&= \lim_{x\to 9} \frac{-(x-9)}{(x-9)(2+\sqrt{x-5})} \\
&= \lim_{x\to 9} \frac{-1}{2+\sqrt{x-5}} \\
&= \frac{-1}{2+\sqrt{9-5}} \\
&= \boxed{-\frac{1}{4}}
\end{aligned}$$

We evaluated the limit above by substituting $x = 9$ into the function $\dfrac{-1}{2+\sqrt{x-5}}$. This is possible because the function is continuous at $x = 9$. In fact, the function is continuous at all $x \geq 5$.

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Problem 3 Solution

3. Let $f(x) = 2x^2 + 1$.

(a) Express $f'(3)$ as the limit of the difference quotient, as in the definition of the derivative.

(b) Evaluate the limit in part (a).

Solution:

(a) There are two possible difference quotients we can use to evaluate $f'(3)$. One is:

$$f'(3) = \lim_{h \to 0} \frac{f(h+3) - f(3)}{h} = \lim_{h \to 0} \frac{[2(h+3)^2 + 1] - [2(3)^2 + 1]}{h}.$$

The other is:

$$f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3} \frac{(2x^2 + 1) - [2(3)^2 + 1]}{x - 3}$$

(b) Evaluating the first limit above we have:

$$\begin{aligned} f'(3) &= \lim_{h \to 0} \frac{[2(h+3)^2 + 1] - [2(3)^2 + 1]}{h} \\ &= \lim_{h \to 0} \frac{2(h^2 + 6h + 9) + 1 - 19}{h} \\ &= \lim_{h \to 0} \frac{2h^2 + 12h}{h} \\ &= \lim_{h \to 0} (2h + 12) \\ &= 2(0) + 12 \\ &= \boxed{12} \end{aligned}$$

Evaluating the second limit we have:

$$\begin{aligned} f'(3) &= \lim_{x \to 3} \frac{(2x^2 + 1) - [2(3)^2 + 1]}{x - 3} \\ &= \lim_{x \to 3} \frac{2x^2 + 1 - 19}{x - 3} \\ &= \lim_{x \to 3} \frac{2(x^2 - 9)}{x - 3} \\ &= \lim_{x \to 3} \frac{2(x+3)(x-3)}{x - 3} \\ &= \lim_{x \to 3} 2(x + 3) \\ &= 2(3 + 3) \\ &= \boxed{12} \end{aligned}$$

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Problem 4 Solution

4. Use the differentiation laws to find the derivative of each of these functions. Show each step and do not simplify your answers.

(a) $g(x) = x^3 e^x$

(b) $h(x) = \dfrac{3x}{1+\sqrt{x}}$

Solution:

(a) Use the Product Rule.

$$\begin{aligned} g'(x) &= x^3(e^x)' + (x^3)'e^x \\ &= \boxed{x^3e^x + 3x^2e^x} \end{aligned}$$

(b) Use the Quotient Rule.

$$\begin{aligned} h'(x) &= \frac{(1+\sqrt{x})(3x)' - (3x)(1+\sqrt{x})'}{(1+\sqrt{x})^2} \\ &= \boxed{\frac{3(1+\sqrt{x}) - (3x)\left(\frac{1}{2\sqrt{x}}\right)}{(1+\sqrt{x})^2}} \end{aligned}$$

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Problem 5 Solution

5. The table below shows values of a function $g(x)$ for x near 0. Use these data to estimate $g'(0)$ and give a complete explanation of how you arrived at your estimate.

x	-0.2	-0.1	0	0.1	0.2
$g(x)$	3.5	4.6	5.6	6.7	7.9

Solution: An approximate value for $g'(0)$ is

$$g'(0) \approx \frac{g(0.1) - g(0)}{0.1 - 0} = \frac{6.7 - 5.6}{0.1} = \boxed{11}$$

This formula was used because the exact value of $g'(0)$ is:

$$g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0}.$$

As we approach $x = 0$ from the right, we can plug in either $x = 0.2$ or $x = 0.1$ to estimate the value of $g'(0)$. We used $x = 0.1$ because the estimate is generally more accurate as x gets closer and closer to 0.