Math 180, Exam 1, Spring 2009
Problem 1 Solution

1. Evaluate the limits, if they exist:

(a) $\displaystyle\lim_{x\to 1} \frac{x+1}{x^2+1}$

(b) $\displaystyle\lim_{x\to 3} \frac{2x^2-7x+3}{3x-x^2}$

Solution:

(a) The function $f(x) = \dfrac{x+1}{x^2+1}$ is continuous at $x = 1$. In fact, $f(x)$ is continuous at all x in the interval $(-\infty, \infty)$. Therefore, we can evaluate the limit using substitution.

$$\lim_{x\to 1} \frac{x+1}{x^2+1} = \frac{1+1}{1^2+1} = \boxed{1}$$

(b) When substituting $x = 3$ into the function $f(x) = \dfrac{2x^2-7x+3}{3x-x^2}$ we find that

$$\frac{2x^2-7x+3}{3x-x^2} = \frac{2(3)^2-7(3)+3}{3(3)-3^2} = \frac{0}{0}$$

which is indeterminate. We can resolve this indeterminacy by factoring.

$$\begin{aligned}
\lim_{x\to 3} \frac{2x^2-7x+3}{3x-x^2} &= \lim_{x\to 3} \frac{(x-3)(2x-1)}{-x(x-3)} \\
&= \lim_{x\to 3} \frac{2x-1}{-x} \\
&= \frac{2(3)-1}{-3} \\
&= \boxed{-\frac{5}{3}}
\end{aligned}$$

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Problem 2 Solution

2. Use the Intermediate Value Theorem to show that the function

$$f(x) = xe^{x-1} - \frac{1}{2}$$

has a zero in the interval $[0, 1]$.

Solution: First we recognize that $f(x) = xe^{x-1} - \frac{1}{2}$ is continuous on the interval $[0, 1]$. In fact, $f(x)$ is continuous everywhere. Next, we evaluate $f(x)$ at the endpoints of the interval.

$$f(0) = 0 \cdot e^{0-1} - \frac{1}{2} = -\frac{1}{2}$$
$$f(1) = 1 \cdot e^{1-1} - \frac{1}{2} = \frac{1}{2}$$

Since $f(0) < 0$ and $f(1) > 0$, the Intermediate Value Theorem tells us that $f(c) = 0$ for some c in the interval $[0, 1]$.

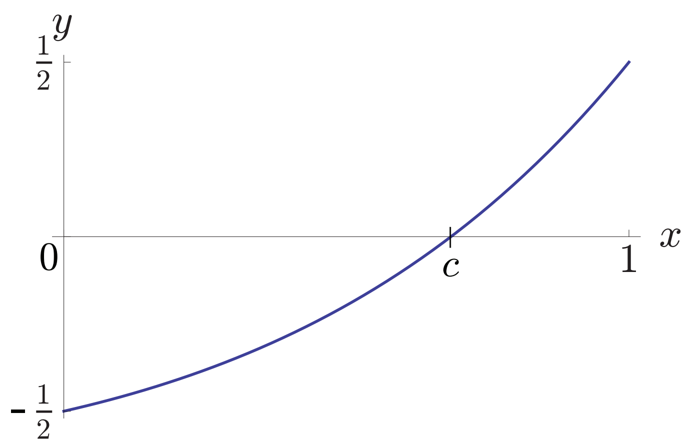

Figure 1: Graph of $f(x) = xe^{x-1} - \frac{1}{2}$ on the interval $[0, 1]$.

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Problem 3 Solution

3. Compute the derivatives of the following functions.

(a) $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$

(b) $f(x) = 3x^5 - 6x^{-4/3}$

(c) $f(x) = (x - 1)e^x$

Solution:

(a) Use the Quotient Rule.

$$f'(x) = \frac{(x^2+1)(x^2-1)' - (x^2-1)(x^2+1)'}{(x^2+1)^2}$$
$$= \boxed{\frac{2x(x^2+1) - 2x(x^2-1)}{(x^2+1)^2}}$$

(b) Use the Power Rule.

$$f'(x) = \boxed{15x^4 + 8x^{-7/3}}$$

(c) Use the Product Rule.

$$f'(x) = (x-1)(e^x)' + (x-1)'e^x$$
$$= \boxed{(x-1)e^x + e^x}$$

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Problem 4 Solution

4. Consider the function $f(x) = x^2 - 2x$.

(a) Use the definition of the derivative as a limit of a difference quotient to compute $f'(3)$.

(b) Write an equation for the line tangent to the graph of f at $x = 3$.

Solution:

(a) There are two possible difference quotients we can use to evaluate $f'(3)$. One is:

$$f'(3) = \lim_{h\to 0} \frac{f(h+3) - f(3)}{h} = \lim_{h\to 0} \frac{[(h+3)^2 - 2(h+3)] - [3^2 - 2(3)]}{h}.$$

The other is:

$$f'(3) = \lim_{x\to 3} \frac{f(x) - f(3)}{x-3} = \lim_{x\to 3} \frac{(x^2 - 2x) - [3^2 - 2(3)]}{x-3}$$

Evaluating the first limit above we have:

$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{[(h+3)^2 - 2(h+3)] - [3^2 - 2(3)]}{h} \\ &= \lim_{h\to 0} \frac{h^2 + 6h + 9 - 2h - 6 - 3}{h} \\ &= \lim_{h\to 0} \frac{h^2 + 4h}{h} \\ &= \lim_{h\to 0} (h+4) \\ &= 0 + 4 \\ &= \boxed{4} \end{aligned}$$

(b) The slope of the tangent line is $f'(3) = 4$. At $x = 3$ we have $f(3) = 3^2 - 2(3) = 3$ so $(3, 3)$ is a point on the line. Therefore, an equation for the tangent line is:

$$\boxed{y - 3 = 4(x - 3)}$$

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Problem 5 Solution

5. Consider the piecewise-defined function below:

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 4 - kx & \text{if } x \geq 1 \end{cases}$$

(a) Find the value of k for which $f(x)$ is continuous for all values of x. Justify your answer.

(b) For the value of k you found in part (a), is $f(x)$ differentiable at $x = 1$? Explain your answer.

Solution:

(a) The functions x^2 and $4-kx$ are continuous for all x. In order for $f(x)$ to be continuous for all x, we must select k so that $f(x)$ is continuous at $x = 1$. To do this, we must compute the one-sided limits at $x = 1$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1^2 = 1$$
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (4 - kx) = 4 - k(1) = 4 - k$$

In order to have continuity at $x = 1$, the one-sided limits must be equal there. Thus, we need:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1+} f(x)$$
$$1 = 4 - k$$
$$\boxed{k = 3}$$

Therefore, $\lim_{x\to 1} f(x) = 1$ for this value of k. Furthermore, we have $f(1) = 4-3(1) = 1$. Thus, since $\lim_{x\to 1} f(x) = f(1)$ we know that $f(x)$ is continuous at $x = 1$.

(b) $f(x)$ is differentiable at $x = 1$ if $f'(x)$ is continuous there. The derivative $f'(x)$ when $k = 3$ is:

$$f'(x) = \begin{cases} 2x & \text{if } x < 1 \\ -3 & \text{if } x > 1 \end{cases}$$

The one-sided limits of $f'(x)$ at $x = 1$ are:

$$\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^-} 2x = 2(1) = 2$$
$$\lim_{x\to 1^+} f'(x) = \lim_{x\to 1^+} -3 = -3$$

Therefore, since the one-sided limits are not equal at $x = 1$, $f(x)$ is not differentiable there.